Lecture 20. PLANAR KINEMATIC-PROBLEM EXAMPLES

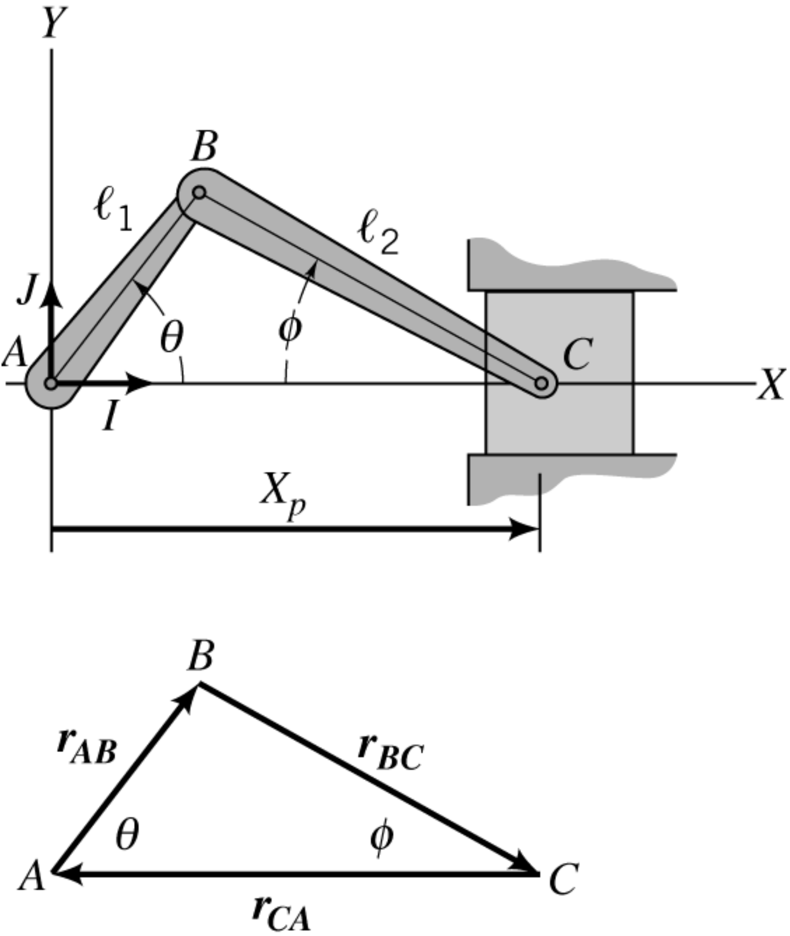


Figure 4.15 Slider-crank mechanism.

TASK: *For a given constant rotation rate $\dot{\theta} = \omega$, find the velocity $\dot{X}_P, \dot{\varphi}$ and acceleration $\ddot{X}_P, \ddot{\varphi}$ terms of the piston for one cycle of* θ.

Geometric Approach: There are three variables (θ, φ, and X_p) but only one degree of freedom. The following (constraint) relationships may be obtained by inspection:

$$X_P = l_1 \cos\theta + l_2 \cos\varphi \quad (X components)$$

$$l_1 \sin\theta = l_2 \sin\varphi \qquad (Y components) \ . \tag{4.16}$$

With θ as the input (known) variable, these equations can be easily solved for the output variables X_P, φ. The vector diagram in figure 4.15 shows the position vectors $\boldsymbol{r}_{AB}$, $\boldsymbol{r}_{BC}$, and $\boldsymbol{r}_{CA}$. For these vectors,

$$\boldsymbol{r}_{AB} + \boldsymbol{r}_{BC} + \boldsymbol{r}_{CA} = 0 \ . \tag{4.17}$$

Substituting,

$$\boldsymbol{r}_{AB} = \boldsymbol{I} l_1 \cos\theta + \boldsymbol{J} l_1 \sin\theta$$

$$\boldsymbol{r}_{BC} = \boldsymbol{I} l_2 \cos\varphi - \boldsymbol{J} l_2 \sin\varphi$$

$$\boldsymbol{r}_{CA} = -\boldsymbol{I} X_P \ ,$$

gives :

$$\boldsymbol{I}: \quad l_1 \cos\theta + l_2 \cos\varphi - X_p = 0$$

$$\boldsymbol{J}: \quad l_1 \sin\theta - l_2 \sin\varphi = 0 \ .$$

Differentiating Eq.(4.16) w.r.t. time gives:

$$\dot{X}_P + l_2 \sin\varphi\, \dot{\varphi} = -l_1 \sin\theta\, \dot{\theta} = -l_1\, \omega \sin\theta$$

$$l_2 \cos\varphi\, \dot{\varphi} = l_1 \cos\theta\, \dot{\theta} = l_1\, \omega \cos\theta \quad . \tag{4.18a}$$

Differentiating again gives:

$$\begin{aligned} \ddot{X}_P + l_2 \sin\varphi\, \ddot{\varphi} &= -l_1 \sin\theta\, \ddot{\theta} - l_1 \cos\theta\, \dot{\theta}^2 - l_2 \cos\varphi\, \dot{\varphi}^2 \\ &= -l_1\, \omega^2 \cos\theta - l_2 \cos\varphi\, \dot{\varphi}^2 \\ l_2 \cos\varphi\, \ddot{\varphi} &= l_1 \cos\theta\, \ddot{\theta} - l_1 \sin\theta\, \dot{\theta}^2 + l_2 \sin\varphi\, \dot{\varphi}^2 \\ &= -l_1\, \omega^2 \sin\theta + l_2 \sin\varphi\, \dot{\varphi}^2 \quad . \end{aligned} \tag{4.18b}$$

Matrix equations of unknowns

$$\begin{bmatrix} 1 & \sin\varphi \\ 0 & \cos\varphi \end{bmatrix} \begin{Bmatrix} \dot{X}_P \\ l_2 \dot{\varphi} \end{Bmatrix} = l_1 \omega \begin{Bmatrix} -\sin\theta \\ \cos\theta \end{Bmatrix} . \tag{4.19a}$$

$$\begin{aligned} \begin{bmatrix} 1 & \sin\varphi \\ 0 & \cos\varphi \end{bmatrix} \begin{Bmatrix} \ddot{X}_P \\ l_2 \ddot{\varphi} \end{Bmatrix} &= -l_1 \omega^2 \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \\ &\quad + l_2 \dot{\varphi}^2 \begin{Bmatrix} -\cos\varphi \\ \sin\varphi \end{Bmatrix}. \end{aligned} \tag{4.19b}$$

The engineering-analysis tasks are accomplished by the following steps:

1. Vary θ over the range of [0, 2π], yielding discrete values θ_i .

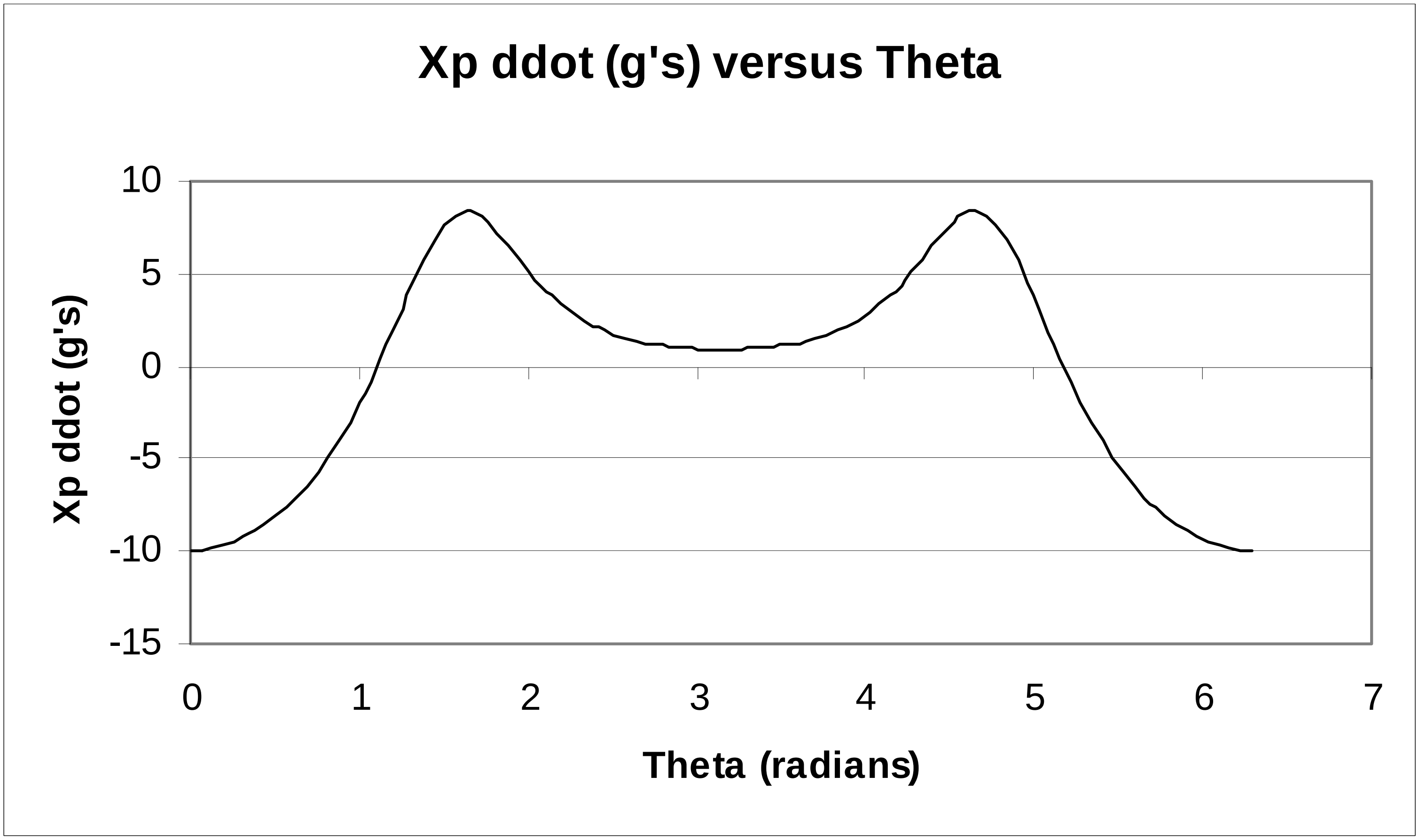
2. For each θ_i value, solve Eq.(4.16) to determine corresponding values for φ_i .

3. Use Eqs.(4.18) with known values for θ_i and φ_i to determine $\dot{\varphi}_i$.

4. Use Eqs.(4.19) with known values for θ_i , φ_i , and $\dot{\varphi}_i$ to determine $\ddot{X}_{Pi}$.

5. Plot $\ddot{X}_{Pi}$ versus θ_i.

Spread-sheet solution for $\ddot{X}_p$ for one θ cycle with $l_1 = 250\,mm, l_2 = 300\,mm, \omega = 14.6\,rad/\sec$.



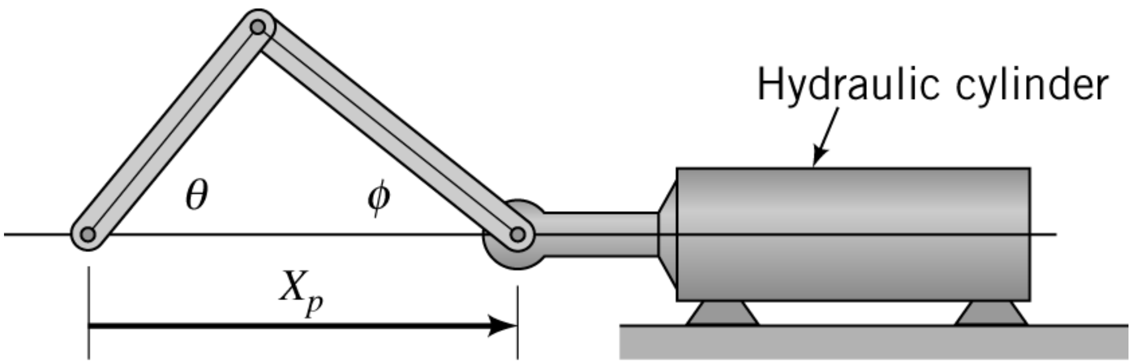


Figure 4.18 Slider crank mechanism with displacement input from a hydraulic cylinder.

For $X_P(t)$ as the input, with (θ and φ), ($\dot{\theta}$ and $\dot{\varphi}$), and ($\ddot{\theta}$ and $\ddot{\varphi}$) as the desired output coordinates. The equations for the coordinates are:

$$l_1\cos\theta + l_2\cos\varphi = X_P \;, \quad l_1\sin\theta - l_2\sin\varphi = 0 \;. \qquad \textbf{(4.20a)}$$

From Eq.(4.20a), the velocity relationships are:

$$l_2\sin\varphi\,\dot{\varphi} + l_1\sin\theta\,\dot{\theta} = -\dot{X}_P \;, \quad l_2\cos\varphi\,\dot{\varphi} - l_1\cos\theta\,\dot{\theta} = 0 \;. \qquad \textbf{(4.20b)}$$

From Eq.(4.20b), the required acceleration component equations are:

$$\begin{aligned} l_2\sin\varphi\,\ddot{\varphi} + l_1\sin\theta\,\ddot{\theta} &= -\ddot{X}_P - l_1\cos\theta\,\dot{\theta}^2 - l_2\cos\varphi\,\dot{\varphi}^2 \\ l_2\cos\varphi\,\ddot{\varphi} - l_1\cos\theta\,\ddot{\theta} &= -l_1\sin\theta\,\dot{\theta}^2 + l_2\sin\varphi\,\dot{\varphi}^2 \;. \end{aligned} \qquad \textbf{(4.20c)}$$

The problem solution is obtained for specified values of $X_P(t)$, $\dot{X}_P(t)$, $\ddot{X}_P(t)$ by proceeding *sequentially* through Eqs.(4.20a), (4.20b), and (4.20c). Note that Eqs.(4.20a) defining θ and φ are nonlinear, while Eqs.(4.20b) for $\dot{\varphi}$ and $\dot{\theta}$, and Eqs.(4.20c) for $\ddot{\varphi}$ and $\ddot{\theta}$ are linear.

The essential first step in developing kinematic equations for planar mechanisms via geometric relationships is drawing a picture of the mechanism in a *general* orientation, yielding equations that can be subsequently differentiated.

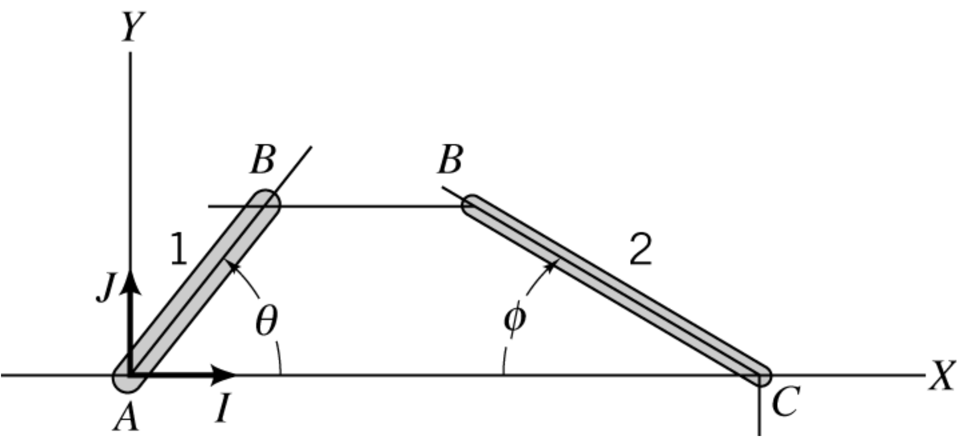


Figure 4.19 Disassembled view of the slider-crank mechanism for vector analysis.

Vector Approach for Velocity and Acceleration Results

Applying the velocity result of Eq.(4.3) separately to links 1 and 2, gives:

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}} \;, \quad \boldsymbol{v_C} = \boldsymbol{v_B} + \boldsymbol{\omega_2} \times \boldsymbol{r_{BC}} \;.$$

Equating the two answers that these equations provide for $\boldsymbol{v_B}$,

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}} = \boldsymbol{v_C} - \boldsymbol{\omega_2} \times \boldsymbol{r_{BC}} \ .$$

Since point A is fixed in the X ,Y system, $\boldsymbol{v_A} = 0$. Similarly, given that point C can only move horizontally, $\boldsymbol{v_C} = \boldsymbol{I}\ \dot{X}_P$. The vector $\boldsymbol{\omega_1}$ is the angular velocity of link 1 with respect to the X,Y system. Using the right-hand rule,

$$\boldsymbol{\omega_1} = \boldsymbol{K}\dot{\theta} \ , \quad \boldsymbol{\omega_2} = -\boldsymbol{K}\dot{\varphi}$$

The position vectors $\boldsymbol{r_{AB}}$ and $\boldsymbol{r_{BC}}$ are defined by

$$\boldsymbol{r_{AB}} = l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\cos\theta) \ , \quad \boldsymbol{r_{BC}} = l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$

Substitution gives

$$0 + \boldsymbol{K}\dot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) = \boldsymbol{I}\dot{X}_p - (-\boldsymbol{K}\dot{\varphi}) \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi) \ .$$

Carrying out the cross products and gathering terms,

$$\boldsymbol{I}: \quad -l_1\dot{\theta}\sin\theta = \dot{X}_p + l_2\sin\varphi\,\dot{\varphi}$$

$$\boldsymbol{J}: \quad l_1\dot{\theta}\cos\theta = l_2\dot{\varphi}\cos\varphi \ .$$

To find the acceleration relationships, applying the second of Eqs.(4.3) to figure 4.17 :

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_{AB} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_{AB})$$

$$\boldsymbol{a}_C = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{BC} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{BC}) \ .$$

Equating the separate definitions for $\boldsymbol{a}_B$ gives

$$\begin{aligned} \boldsymbol{a}_B &= \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_{AB} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_{AB}) \\ &= \boldsymbol{a}_C - \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{BC} - \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{BC}) \ . \end{aligned}$$

Since point A is fixed, $\boldsymbol{a}_A = 0$. Also, since point C is constrained to move in the horizontal plane, $\boldsymbol{a}_C = \boldsymbol{I}\,\ddot{X}_P$. The remaining undefined variables are $\dot{\boldsymbol{\omega}}_1 = \boldsymbol{K}\ddot{\theta}$, the angular acceleration of link 1 with respect to the X, Y system, and $\dot{\boldsymbol{\omega}}_2 = -\boldsymbol{K}\ddot{\varphi}$, the angular acceleration of link 2 with respect to the X, Y system. Substituting gives

$$\begin{aligned} 0 + \boldsymbol{K}\ddot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) + \boldsymbol{K}\dot{\theta} \times [\boldsymbol{K}\dot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)] \\ = \boldsymbol{I}\ddot{X}_P - (-\boldsymbol{K}\ddot{\varphi}) \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi) \\ - (-\boldsymbol{K}\dot{\varphi}) \times [-\boldsymbol{K}\dot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)] \ . \end{aligned}$$

Completing the cross products and algebra gives the following component equations:

$$\boldsymbol{I}: \quad -l_1\ddot{\theta}\sin\theta - l_1\dot{\theta}^2\cos\theta = \ddot{X}_P + l_2\ddot{\varphi}\sin\varphi + l_2\dot{\varphi}^2\cos\varphi$$

$$\boldsymbol{J}: \quad l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta = l_2\ddot{\varphi}\cos\varphi - l_2\dot{\varphi}^2\sin\varphi \quad ,$$

4.5b A Four-Bar-Linkage Example

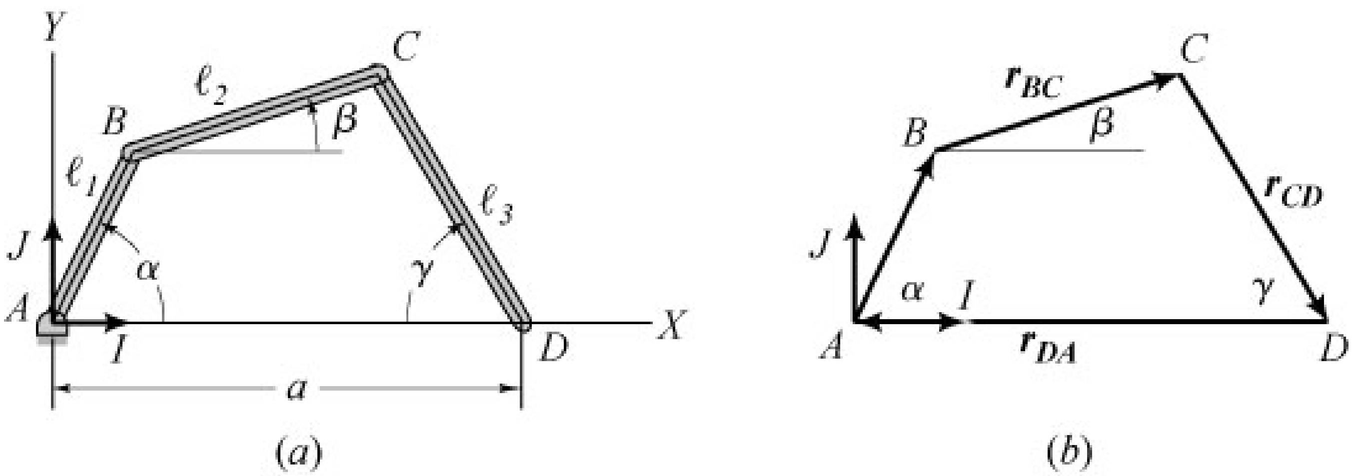


Figure 4.19 (a) Four-bar linkage, (b) Vector diagram for linkage

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Consider the following engineering-analysis task: *For a constant rotation rate* $,\dot{\alpha} = \omega_o$ *determine the angular velocities* $\dot{\beta}$, $\dot{\gamma}$ *and angular accelerations* $\ddot{\beta}$, $\ddot{\gamma}$ *for one rotation of* α.

Geometric Approach

Inspecting figure 4.19a yields:

$$X: \quad l_1\cos\alpha + l_2\cos\beta + l_3\cos\gamma = a$$

$$Y: \quad l_1\sin\alpha + l_2\sin\beta - l_3\sin\gamma = 0 \quad . \qquad \textbf{(4.21)}$$

Figure 4.19b shows a closed-loop vector representation that can be formally used to obtain Eqs.(4.21). The results from figure 4.19b can be stated, $\boldsymbol{r}_{AB} + \boldsymbol{r}_{BC} + \boldsymbol{r}_{CD} + \boldsymbol{r}_{DA} = 0$. Substituting:

$$\boldsymbol{r}_{AB} = l_1(\boldsymbol{I}\cos\alpha + \boldsymbol{J}\sin\alpha) \ , \ \boldsymbol{r}_{BC} = l_2(\boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta)$$
$$\boldsymbol{r}_{CD} = l_3(\boldsymbol{I}\cos\gamma - \boldsymbol{J}\sin\gamma) \ , \ \boldsymbol{r}_{DA} = -\boldsymbol{I}a$$

gives the same result as Eqs.(4.21).

Restating Eqs.(4.21) as:

$$\begin{aligned} l_2\cos\beta + l_3\cos\gamma &= a - l_1\cos\alpha \\ l_2\sin\beta - l_3\sin\gamma &= -l_1\sin\alpha \ . \end{aligned} \qquad \textbf{(4.22)}$$

shows α as the input coordinate and β and γ as output coordinates. Differentiating with respect to time gives:

$$\begin{aligned} -l_2\sin\beta\,\dot{\beta} - l_3\sin\gamma\,\dot{\gamma} &= l_1\sin\alpha\,\dot{\alpha} = l_1\,\omega\sin\alpha \\ l_2\cos\beta\,\dot{\beta} - l_3\cos\gamma\,\dot{\gamma} &= -l_1\cos\alpha\,\dot{\alpha} = -l_1\,\omega\cos\alpha \ . \end{aligned} \qquad \textbf{(4.23)}$$

In matrix format,

$$\begin{bmatrix} \sin\beta & \sin\gamma \\ -\cos\beta & \cos\gamma \end{bmatrix} \begin{Bmatrix} l_2\dot{\beta} \\ l_3\dot{\gamma} \end{Bmatrix} = l_1\,\omega \begin{Bmatrix} -\sin\alpha \\ \cos\alpha \end{Bmatrix} . \qquad \textbf{(4.23)}$$

Using Cramers rule to solve these equations gives

$$l_2\,\dot{\beta} = \frac{-l_1\,\omega\,\sin(\alpha+\gamma)}{\sin(\beta+\gamma)}$$

$$l_3\,\dot{\gamma} = \frac{l_1\,\omega\,\sin(\beta-\alpha)}{\sin(\beta+\gamma)}\ , \tag{4.24}$$

The solution is undefined for $\beta+\gamma=\pi, 0$

Differentiating Eqs.(4.23) gives:

$$-l_2\sin\beta\,\ddot{\beta} - l_3\sin\gamma\,\ddot{\gamma} = l_1\sin\alpha\,\ddot{\alpha} + l_1\cos\alpha\,\dot{\alpha}^2 + l_2\cos\beta\,\dot{\beta}^2 + l_3\cos\gamma\,\dot{\gamma}^2$$

$$l_2\cos\beta\,\ddot{\beta} - l_3\cos\gamma\,\ddot{\gamma} = -l_1\cos\alpha\,\ddot{\alpha} + l_1\sin\alpha\,\dot{\alpha}^2 + l_2\sin\beta\,\dot{\beta}^2 - l_3\sin\gamma\,\dot{\gamma}^2$$

Setting $\ddot{\alpha}$=0, and $\dot{\alpha}=\omega_o$ reduces them to:

$$-l_2\sin\beta\,\ddot{\beta} - l_3\sin\gamma\,\ddot{\gamma} = l_1\cos\alpha\,\omega_o^2 + l_2\cos\beta\,\dot{\beta}^2 + l_3\cos\gamma\,\dot{\gamma}^2$$

$$l_2\cos\beta\,\ddot{\beta} - l_3\cos\gamma\,\ddot{\gamma} = -l_1\sin\alpha\,\omega_o^2 + l_2\sin\beta\,\dot{\beta}^2 - l_3\sin\gamma\,\dot{\gamma}^2\ ,$$

or, in matrix format,

$$\begin{bmatrix} -\sin\beta & -\sin\gamma \\ \cos\beta & -\cos\gamma \end{bmatrix} \begin{Bmatrix} l_2\ddot{\beta} \\ l_3\ddot{\gamma} \end{Bmatrix} =$$

$$\begin{Bmatrix} l_1\omega^2\cos\alpha & + & l_2\dot{\beta}^2\cos\beta & + & l_3\dot{\gamma}^2\cos\gamma \\ -l_1\omega^2\sin\alpha & + & l_2\dot{\beta}^2\sin\beta & - & l_3\dot{\gamma}^2\sin\gamma \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}.$$

Using Cramer's rule, the solution is

$$l_2\ddot{\beta} = \frac{-g_1\cos\gamma + g_2\sin\gamma}{\sin(\beta+\gamma)}$$

$$l_3\ddot{\gamma} = \frac{-g_1\cos\beta - g_2\sin\beta}{\sin(\beta+\gamma)} \,. \qquad \textbf{(4.26)}$$

The solution is undefined for $\beta + \gamma = \pi$.

The engineering-analysis task is accomplished by executing the following sequential steps:

1. Vary α over the range [0, 2π], yielding discrete values α_i.

2. For each α_i value, solve Eq.(4.22) to determine corresponding values for β_i and γ_i.

3. Enter Eqs.(4.23a) with known values for α_i, β_i and γ_i. to determine $\dot{\beta}_i$ and $\dot{\gamma}_i$.

4. Enter Eqs.(4.26a) with known values for α_i, β_i, γ_i, $\dot{\beta}_i$ and $\dot{\gamma}_i$ to determine $\ddot{\beta}_i$ and $\ddot{\gamma}_i$.

5. Plot $\dot{\beta}_i$,$\dot{\gamma}_i$, $\ddot{\beta}_i$ and $\ddot{\gamma}_i$ versus α_I.

Eqs.(4.22) can be solved analytically, starting with the restatement

$$l_2 \cos\beta = (a - l_1 \cos\alpha) - l_3 \cos\gamma = h_1 - l_3 \cos\gamma$$

$$l_2 \sin\beta = -l_1 \sin\alpha + l_3 \sin\gamma = h_2 + l_3 \sin\gamma \ .$$

$h_1 = a - l_1 \cos\alpha$ and $h_2 = -l_1 \sin\alpha$ are defined in terms of α and are known quantities. Squaring both of these equations and adding them together gives

$$l_2^2(\cos^2\beta + \sin^2\beta) = h_1^2 - 2h_1 l_3 \cos\gamma + l_3^2 \cos^2\gamma + h_2^2$$
$$+ 2h_2 l_3 \sin\gamma + l_3^2 \sin^2\gamma$$

$$\therefore \ l_2^2 = h_1^2 + h_2^2 + l_3^2 - 2h_1 l_3 \cos\gamma + 2h_2 l_3 \sin\gamma \, .$$

Rearranging gives

$$2h_1 l_3 \cos\gamma = h_1^2 + h_2^2 + l_3^2 - l_2^2 + 2h_2 l_3 \sin\gamma = 2d + 2h_2 l_3 \sin\gamma$$

$$\therefore\ h_1^2 l_3^2 \cos^2\gamma = h_1^2 l_3^2 (1 - \sin^2\gamma) = d^2 + 2dh_2 l_3 \sin\gamma + h_2^2 l_3^2 \sin^2\gamma\ ,$$

where $2d = h_1^2 + h_2^2 + l_3^2 - l_2^2$. Restating this result gives

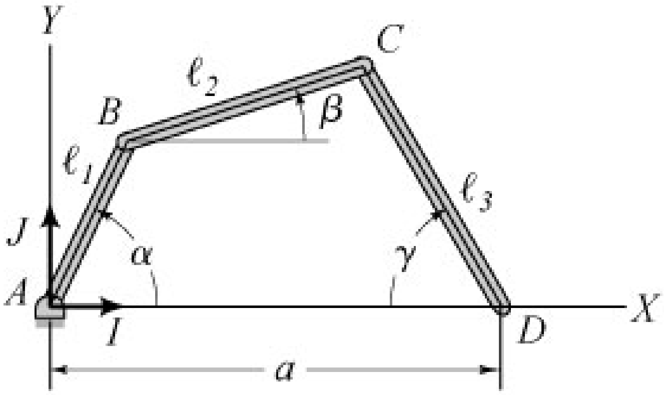
$$\sin^2\gamma + \frac{2dh_2}{l_3(h_1^2 + h_2^2)} \sin\gamma + \frac{(d^2 - h_1^2 l_3^2)}{l_3^2(h_1^2 + h_2^2)} = 0$$

The equation $\sin^2\gamma + B\sin\gamma + C = 0$ has the two roots:

$$\sin\gamma = -\frac{B}{2} + \frac{1}{2}\sqrt{B^2 - 4C}$$

$$\sin\gamma = -\frac{B}{2} - \frac{1}{2}\sqrt{B^2 - 4C}\ . \qquad \textbf{(4.27)}$$

Depending on values for *B* and *C*, this equation can have one real root, two real roots, or two complex roots. Two real roots implies two distinct solutions, and this possibility is illustrated by figure 4.20 below where the same α value gives an orientation that differs from figure 4.19a.



(a)

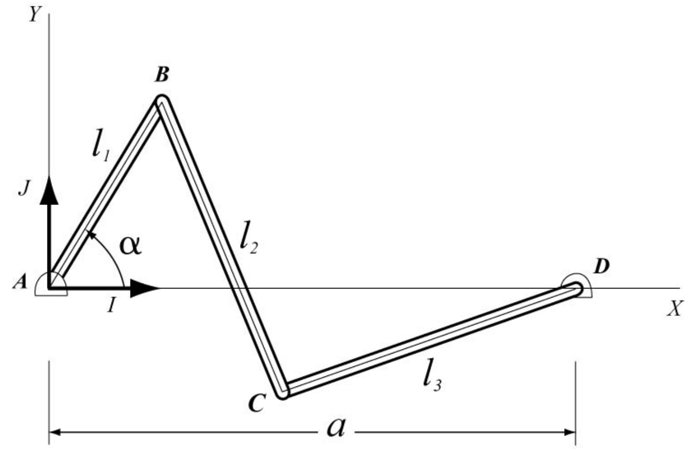


Figure 4.20 Alternate configuration for the linkage of figure 4.19a

The one real-root solution corresponding to $B^2 = 4C$ defines an extreme "locked" position for the mechanism, as illustrated in figure 4.21. Note that this position corresponds to $\beta_1 = -\gamma_1$ netting $\sin(\beta_1 + \gamma_1) = 0$, which also caused the angular velocities and angular accelerations to be undefined in Eqs.(4.23) and (4.25), respectively.

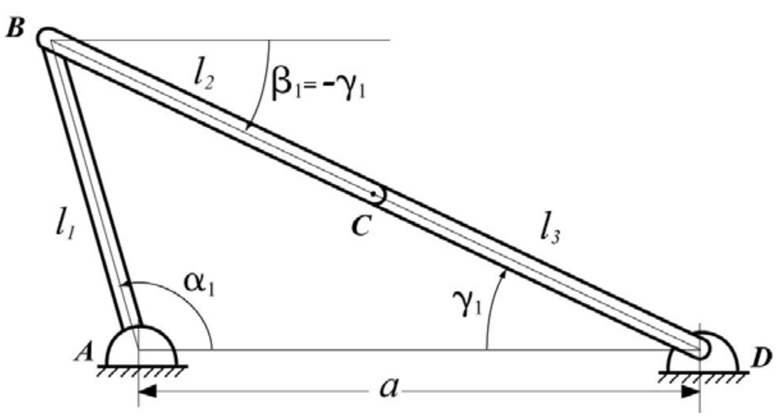


Figure 4.21 Locked position for the linkage of figure 4.19a with $l_1 = l_3 = 1.45\, l_1, a = 2.28\, l_1$

We can solve for the limiting α value in figure 4.21 by substituting $\beta_1 = -\gamma_1$ into Eq.(4.22) to get

$$l_2 \cos\gamma_1 + l_3 \cos\gamma_1 = (l_2 + l_3)\cos\gamma_1 = a - l_1 \cos\alpha_1$$

$$-l_2 \sin\gamma_1 - l_3 \sin\gamma_1 = -(l_2 + l_3)\sin\gamma_1 = -l_1 \sin\alpha_1 \ .$$

Squaring both equations and adding them together gives

$$a^2\, 2 a l_1 \cos\alpha_1 + l_1^2 = (l_2 + l_3)^2$$

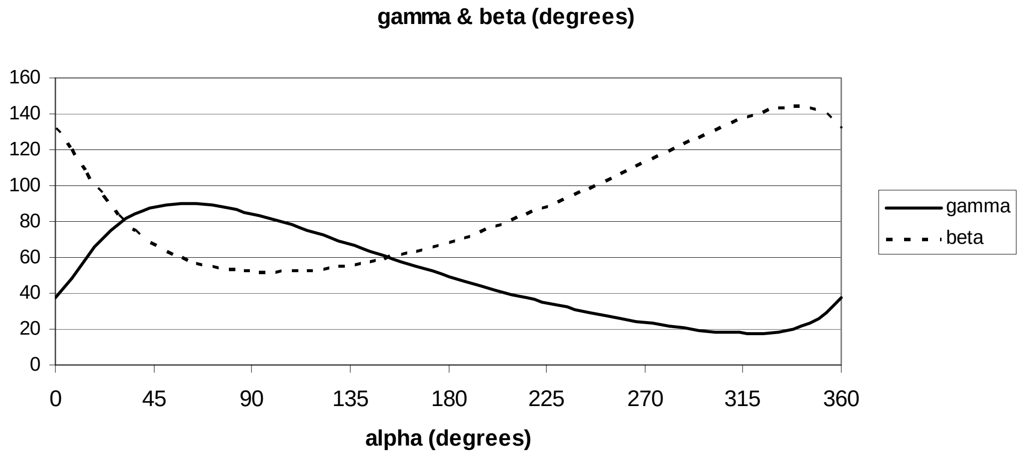
$$\therefore \ \cos\alpha_1 = \frac{a^2 + l_1^2 - (l_2 + l_3)^2}{2 a l_1} \ . \qquad \textbf{(4.28)}$$

If the parameters l_1, l_2, l_3, a are such that a solution exists for α_1, then a locked position can occur. For figure 4.21, the limiting value for α_1 corresponds to

$$\alpha_1 = \pi \ \Rightarrow \ (a + l_1)^2 = (l_2 + l_3)^2 \ \Rightarrow \ a + l_1 = l_2 + l_3 \ .$$

For $l_2 + l_3 > a + l_1$ there is no limiting rotation angle α_1, and the left hand link can rotate freely through 360 degrees.

Figure 4.22 illustrates a solution for $\beta, \gamma, \dot{\beta}, \dot{\gamma}$ with $l_1 = 0.35\,m$; $l_2 = .816\,m$; $l_3 = 1\,m$; $a = 0.6\,m$ and $\omega = 3\,rad/\sec$ for α_i over [0, 2π]. The solution illustrated corresponds to the first solution (positive square root) in Eq.(4.27).



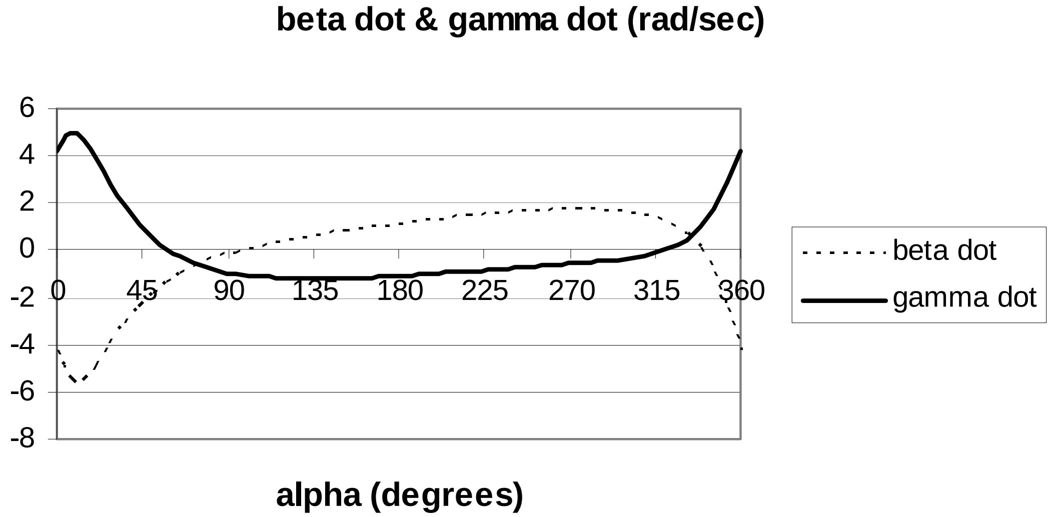


Figure 4.22 Numerical solution for $\beta, \gamma, \dot{\beta}, \dot{\gamma}$ versus α for $l_1 = 0.35\,m$; $l_2 = .816\,m$; $l_3 = 1\,m$; $a = 0.6\,m$

Caution is advisable in using Eq.(4.27) to solve for γ_i to make sure that the solution is in the correct quadrant.

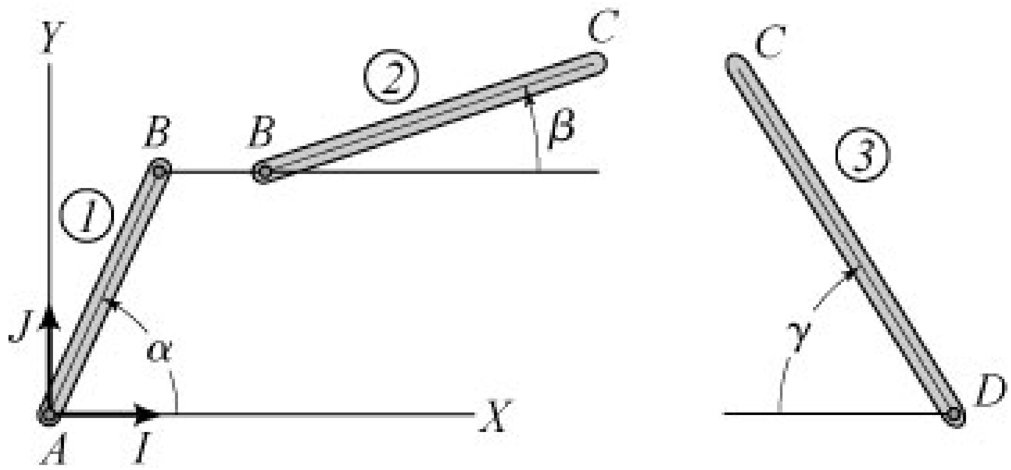


Figure 4.21 Disassembled view of the three-bar linkage of figure 4.19 for vector analysis.

Starting on the left with link 1, and looking from point A to B gives

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \omega_1 \times \boldsymbol{r_{AB}} \quad .$$

Next, for link 3, looking from point D back to point C we can write

$$\boldsymbol{v_C} = \boldsymbol{v_D} + \omega_3 \times \boldsymbol{r_{DC}} \quad .$$

Finally, for link 2, looking from point B to point C gives

$$\boldsymbol{v_C} = \boldsymbol{v_B} + \omega_2 \times \boldsymbol{r_{BC}} \quad .$$

Substituting into this last equation for $\boldsymbol{v_B}$ and $\boldsymbol{v_C}$, and observing that $\boldsymbol{v_A} = \boldsymbol{v_D} = 0$ gives

$$\omega_3 \times \boldsymbol{r_{DC}} = \omega_1 \times \boldsymbol{r_{AB}} + \omega_2 \times \boldsymbol{r_{BC}} \quad .$$

Eqs.(4.22) define the position vectors of this equation. Using the right-hand rule, the angular velocity vectors are defined as $\boldsymbol{\omega_1} = \boldsymbol{K}\dot{\alpha}$, $\boldsymbol{\omega_2} = \boldsymbol{K}\dot{\beta}$, and $\boldsymbol{\omega_3} = -\boldsymbol{K}\dot{\gamma}$. Substituting gives

$$\begin{aligned} &-\boldsymbol{K}\dot{\gamma} \times l_3(-\boldsymbol{I}\cos\gamma + \boldsymbol{J}\sin\gamma) \\ &\quad = \boldsymbol{K}\dot{\alpha} \times l_1(\boldsymbol{I}\cos\alpha + \boldsymbol{J}\sin\alpha) + \boldsymbol{K}\dot{\beta} \times l_2(\boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta) \quad . \end{aligned}$$

Carrying out the cross products gives:

$$\boldsymbol{I} : \quad l_3\dot{\gamma}\sin\gamma = -l_1\dot{\alpha}\sin\alpha - l_2\dot{\beta}\sin\beta$$

$$\boldsymbol{J} : \quad l_3\dot{\gamma}\cos\gamma = l_1\dot{\alpha}\cos\alpha + l_2\dot{\beta}\cos\beta \quad .$$

The acceleration relationships are obtained via the same logic from

$$\begin{aligned} \boldsymbol{a_B} &= \boldsymbol{a_A} + \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} \times \boldsymbol{r_{AB}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}) \\ \boldsymbol{a_C} &= \boldsymbol{a_D} + \dot{\boldsymbol{\omega}}_{\boldsymbol{3}} \times \boldsymbol{r_{DC}} + \boldsymbol{\omega_3} \times (\boldsymbol{\omega_3} \times \boldsymbol{r_{DC}}) \\ \boldsymbol{a_C} &= \boldsymbol{a_B} + \dot{\boldsymbol{\omega}}_{\boldsymbol{2}} \times \boldsymbol{r_{BC}} + \boldsymbol{\omega_2} \times (\boldsymbol{\omega_2} \times \boldsymbol{r_{BC}}) \quad . \end{aligned}$$

Substituting from the first and second equations for $\boldsymbol{a}_{B}$ and $\boldsymbol{a}_{C}$ into the last gives

$$\boldsymbol{a}_{D}+\dot{\boldsymbol{\omega}}_{3}\times\boldsymbol{r}_{DC}+\boldsymbol{\omega}_{3}\times(\boldsymbol{\omega}_{3}\times\boldsymbol{r}_{BC})$$

$$=\boldsymbol{a}_{A}+\dot{\boldsymbol{\omega}}_{1}\times\boldsymbol{r}_{AB}+\boldsymbol{\omega}_{1}\times(\boldsymbol{\omega}_{1}\times\boldsymbol{r}_{AB})+\dot{\boldsymbol{\omega}}_{2}\times\boldsymbol{r}_{BC}+\boldsymbol{\omega}_{2}\times(\boldsymbol{\omega}_{2}\times\boldsymbol{r}_{AC})\ .$$

Noting that $\boldsymbol{a}_{D}$ and $\boldsymbol{a}_{A}$ are zero, and substituting $\dot{\boldsymbol{\omega}}_{1}=\boldsymbol{K}\ddot{\alpha}$, $\dot{\boldsymbol{\omega}}_{2}=\boldsymbol{K}\ddot{\beta}$, and $\dot{\boldsymbol{\omega}}_{3}=-\boldsymbol{K}\ddot{\gamma}$ into this equation gives

$$-\boldsymbol{K}\ddot{\gamma}\times l_{3}(-\boldsymbol{I}\cos\gamma+\boldsymbol{J}\sin\gamma)-\boldsymbol{K}\dot{\gamma}\times[-\boldsymbol{K}\dot{\gamma}\times l_{3}(-\boldsymbol{I}\cos\gamma+\boldsymbol{J}\sin\gamma)]$$

$$=\boldsymbol{K}\ddot{\alpha}\times l_{1}(\boldsymbol{I}\cos\alpha+\boldsymbol{J}\sin\alpha)+\boldsymbol{K}\dot{\alpha}\times[\boldsymbol{K}\dot{\alpha}\times l_{1}(\boldsymbol{I}\cos\alpha+\boldsymbol{J}\sin\alpha)]$$
$$+\boldsymbol{K}\ddot{\beta}\times l_{2}(\boldsymbol{I}\cos\beta+\boldsymbol{J}\sin\beta)+\boldsymbol{K}\dot{\beta}\times[\boldsymbol{K}\dot{\beta}\times l_{2}(\boldsymbol{I}\cos\beta+\boldsymbol{J}\sin\beta)]\ .$$

Carrying out the cross products and gathering terms,

$$\boldsymbol{I}\ :\quad l_{3}\ddot{\gamma}\sin\gamma+l_{3}\dot{\gamma}^{2}\cos\gamma=-l_{1}\ddot{\alpha}\sin\alpha-l_{1}\dot{\alpha}^{2}\cos\beta$$
$$-l_{2}\ddot{\beta}\sin\beta-l_{2}\dot{\beta}^{2}\cos\beta$$

$$\boldsymbol{J}\ :\quad l_{3}\ddot{\gamma}\cos\gamma-l_{3}\dot{\gamma}^{2}\sin\gamma=l_{1}\ddot{\alpha}\cos\alpha-l_{1}\dot{\alpha}^{2}\sin\alpha$$
$$+l_{2}\ddot{\beta}\cos\beta-l_{2}\dot{\beta}^{2}\sin\beta.$$

If general governing equations are required, the geometric relationships of Eq.(4.23) *must be developed.*

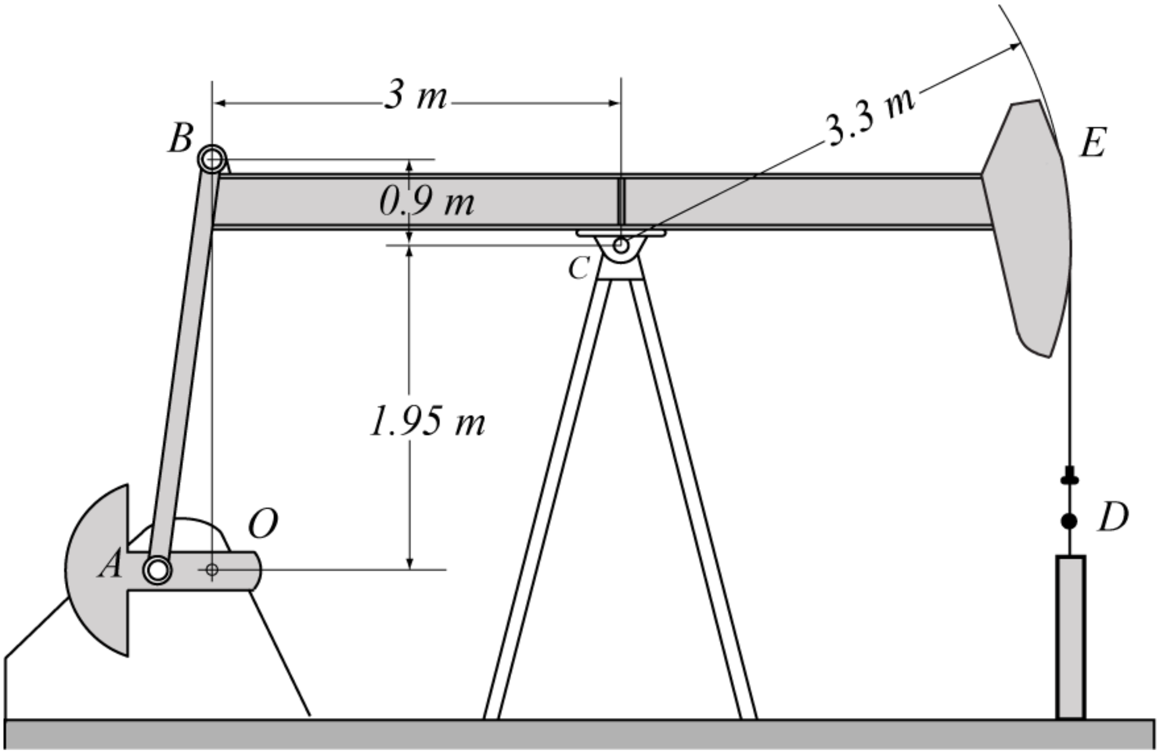


Figure XP5.4a Oil well pumping rig, adapted from Meriam and Kraige (1992)

Example Problem 4.5 Figure XP4.5a illustrates an oil pumping rig that is typically used for shallow oil wells. An electric motor drives the rotating arm *OA* at a constant, clock-wise angular velocity $\omega = 20\, rpm$. A cable attaches the pumping rod at *D* to the end of the rocking arm *BE*. Rotation of the driving link produces a vertical oscillation that drives a positive-displacement pump at the bottom of the well.

Tasks:

a. Draw the rig in a general position and select coordinates to define the bars' general position. State the kinematic constraint equations defining the angular positions of bars *AB* and *BCE* in terms of bar *OA*'s angular position.

b. Outline a solution procedure to determine the orientations of bars *AB* and *BCE* in terms of bar *OA*'s angular position.

c. Derive general expressions for the angular velocities of bars *AB* and *BCE* in terms of bar *OA*'s angular position and angular velocity. Solve for the unknown angular velocities.

d. Derive general expressions for the angular accelerations of bars *AB* and *BCE* in terms of bar *OA*'s angular position, velocity, and acceleration. Solve for the unknown angular accelerations.

e. Derive general expressions for the change in vertical position and vertical acceleration of point *D* as a function of bar *OA*'s angular position.

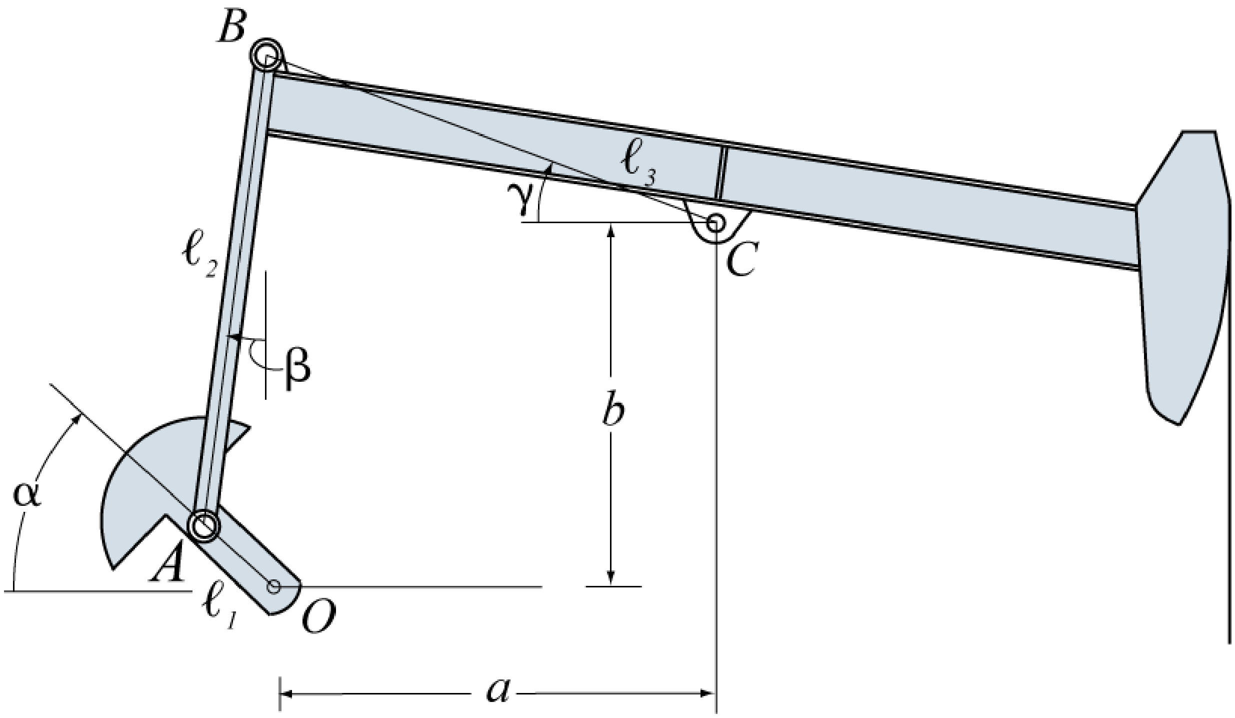


Figure XP5.4b Pumping rig in a general position with coordinates

Solution. The sketch of figure XP4.5b shows the angles α, β, γ defining the angular positions of bars *OA*, *AB* and *BC*, respectively. α is the (known) input variable, while β and γ are the (unknown) output variables. The length l_3 extends from *B* to *C*. Stating the components of the bars in the *X* and *Y* directions gives:

$$-l_1 \cos\alpha + l_2 \sin\beta + l_3 \cos\gamma = a$$
$$\Rightarrow\ l_2 \sin\beta + l_3 \cos\gamma = a + l_1 \cos\alpha = h_1(\alpha)$$

(i)

$$l_1 \sin\alpha + l_2 \cos\beta - l_3 \sin\gamma = b$$
$$\Rightarrow\ l_2 \cos\beta - l_3 \sin\gamma = b - l_1 \sin\alpha = h_2(\alpha),$$

and concludes *Task a*.

As a first step in solving for (β,γ), we state the equations as
$l_2 \sin\beta = h_1 - l_3 \cos\gamma$, $l_2 \cos\beta = h_2 + l_3 \sin\gamma$

Squaring these equations and adding them together gives:

$$l_2^2(\sin^2\beta + \cos^2\beta) = l_2^2$$
$$= h_1^2 - 2h_1 l_3 \cos\gamma + l_3^2 \cos^2\gamma + h_2^2 + 2h_2 l_3 \sin\gamma + l_3^2 \sin^2\gamma$$

$$\therefore\ 2h_1 l_3 \cos\gamma = h_1^2 + h_2^2 + l_3^2 - l_2^2 + 2h_2 l_3 \sin\gamma = 2d + 2h_2 l_3 \sin\gamma\ ,$$

where $2d = h_1^2 + h_2^2 + l_3^2 - l_2^2$. Substituting $\cos\gamma = \sqrt{1 - \sin^2\gamma}$ nets

$$h_1^2 l_3^2(1 - \sin^2\gamma)$$
$$= d^2 + 2dh_2 l_3 \sin\gamma + h_2^2 l_3^2 \sin^2\gamma$$

$$\therefore\ \sin^2\gamma + \frac{2dh_2}{l_3(h_1^2 + h_2^2)}\sin\gamma + \frac{d^2 - h_1^2 l_3^2}{l_3^2(h_1^2 + h_2^2)} = 0\ . \tag{ii}$$

For a specified value of α, solving this quadratic equation gives $\sin\gamma \Rightarrow \gamma = \sin^{-1}\gamma$, and back substitution into Eq.(i) nets β. These steps concludes *Task b*, and figure XP4.5b illustrates the results for the lengths of figure XP4.5a.

Proceeding to *Task c*, we can differentiate Eq.(i) with respect to time to obtain:

$$l_2 \cos\beta\,\dot{\beta} - l_3 \sin\gamma\,\dot{\gamma} = -l_1 \sin\alpha\,\dot{\alpha} = -l_1 \omega \sin\alpha$$

$$-l_2 \sin\beta\,\dot{\beta} - l_3 \cos\gamma\,\dot{\gamma} = -l_1 \cos\alpha\,\dot{\alpha} = -l_1 \omega \cos\alpha .$$

(iii)

In matrix format, these equations become

$$\begin{bmatrix} \cos\beta & -\sin\gamma \\ \sin\beta & \cos\gamma \end{bmatrix} \begin{Bmatrix} l_2 \dot{\beta} \\ l_3 \dot{\gamma} \end{Bmatrix} = l_1 \omega \begin{Bmatrix} -\sin\alpha \\ \cos\alpha \end{Bmatrix} .$$

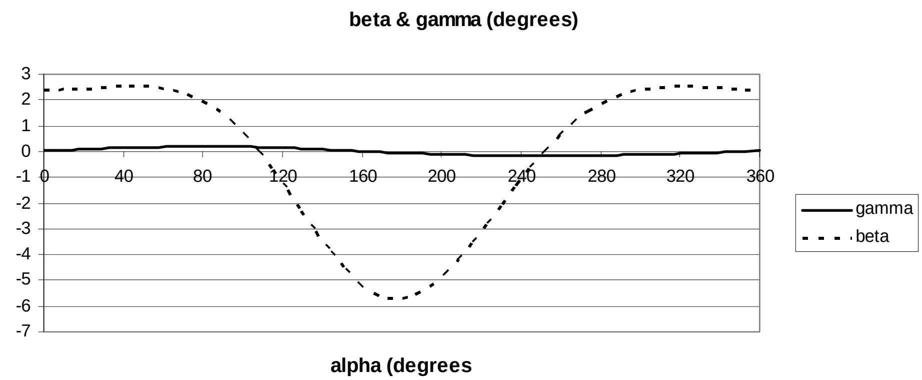
Using Cramer's rule (Appendix A), their solution can be stated:

$$l_2 \dot{\beta} = \frac{l_1 \omega}{\cos\gamma \cos\beta + \sin\gamma \cos\beta} \begin{vmatrix} -\sin\alpha & -\sin\gamma \\ \cos\alpha & \cos\gamma \end{vmatrix} = \frac{l_1 \omega \sin(\gamma - \alpha)}{\cos(\beta - \gamma)}$$

$$l_3 \dot{\gamma} = \frac{l_1 \omega}{\cos\gamma \cos\beta + \sin\gamma \cos\beta} \begin{vmatrix} \cos\beta & -\sin\alpha \\ \sin\beta & \cos\alpha \end{vmatrix} = \frac{l_1 \omega \cos(\alpha - \beta)}{\cos(\beta - \gamma)} ,$$

(iv)

concluding *Task c*. Figure XP 4.5c illustrates $\dot{\gamma}, \dot{\beta}$ versus α



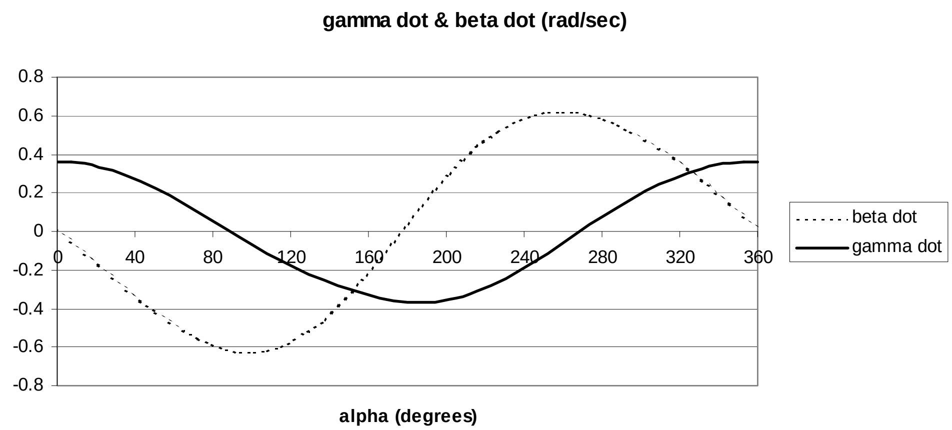


Figure XP 4.5c γ, β and $\dot{\gamma}, \dot{\beta}$ versus α

Moving to *Task d*, we can differentiate Eq.(iii) with respect to time to obtain:

$$\begin{aligned}
&l_2 \cos\beta\,\ddot{\beta} - l_3 \sin\gamma\,\ddot{\gamma} \\
&= -l_1 \omega^2 \cos\alpha + l_2 \sin\beta\,\dot{\beta}^2 + l_3 \cos\gamma\,\dot{\gamma}^2 = g_1
\end{aligned}$$

$$\begin{aligned}
&-l_2 \sin\beta\,\ddot{\beta} - l_3 \cos\gamma\,\ddot{\gamma} \\
&= l_1 \omega^2 \sin\alpha + l_2 \cos\beta\,\dot{\beta}^2 - l_3 \sin\gamma\,\dot{\gamma}^2 = -g_2.
\end{aligned}$$

In matrix format, these equations become

$$\begin{bmatrix} \cos\beta & -\sin\gamma \\ \sin\beta & \cos\gamma \end{bmatrix} \begin{Bmatrix} l_2\ddot{\beta} \\ l_3\ddot{\gamma} \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}.$$

The solution can be stated:

$$l_2\,\ddot{\beta} = \frac{1}{\cos\gamma\cos\beta + \sin\gamma\cos\beta} \begin{vmatrix} g_1 & -\sin\gamma \\ g_2 & \cos\gamma \end{vmatrix} = \frac{g_1\cos\gamma + g_2\sin\gamma}{\cos(\beta-\gamma)}$$

$$l_3\,\ddot{\gamma} = \frac{1}{\cos\gamma\cos\beta + \sin\gamma\cos\beta} \begin{vmatrix} \cos\beta & g_1 \\ \sin\beta & g_2 \end{vmatrix} = \frac{g_2\cos\beta - g_1\sin\beta}{\cos(\beta-\gamma)},$$

(vi)

concluding *Task d*.

In regard to *Task e*, as long as the circular arc at the end of the rocker arm is long enough, the tangent point of the cable with the circular-faced end of the rocker arm will be at a horizontal line running through *C*. Hence, the change in the horizontal position of point *D* is the amount of cable rolled off the arc due to a change in the rocker arm γ, i.e., $\delta Y = -3.3\,\delta\gamma$. Similarly, the vertical acceleration of the sucker rod at *D* is the circumferential acceleration of a point on the arc, i.e, $a_\theta = r\ddot{\theta} \Rightarrow \ddot{Y} = 3.3\,\ddot{\gamma}$. Figure XP4.5 d illustrates $\delta Y, \ddot{Y}$ as a function of alpha. The distance traveled by the pump rod in one cyle is .633 - (-.633) = 1.27 m. The peak positive acceleration is 1.17 g and the minimum is -0.63 g. Note that $|-0.63\,g| < 1\,g$, indicating from a rigid-body viewpoint that the cable will remain in tension during its downward motion.

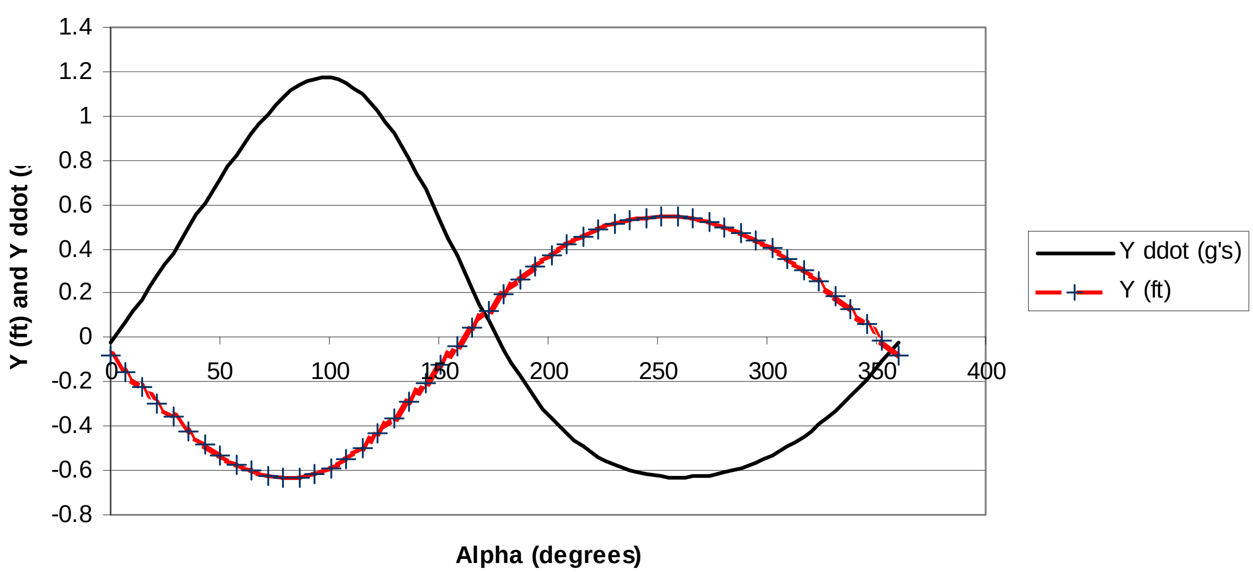


Figure XP4.5d Vertical acceleration and change in position of the pumping rod