Example Problem XP3.3.

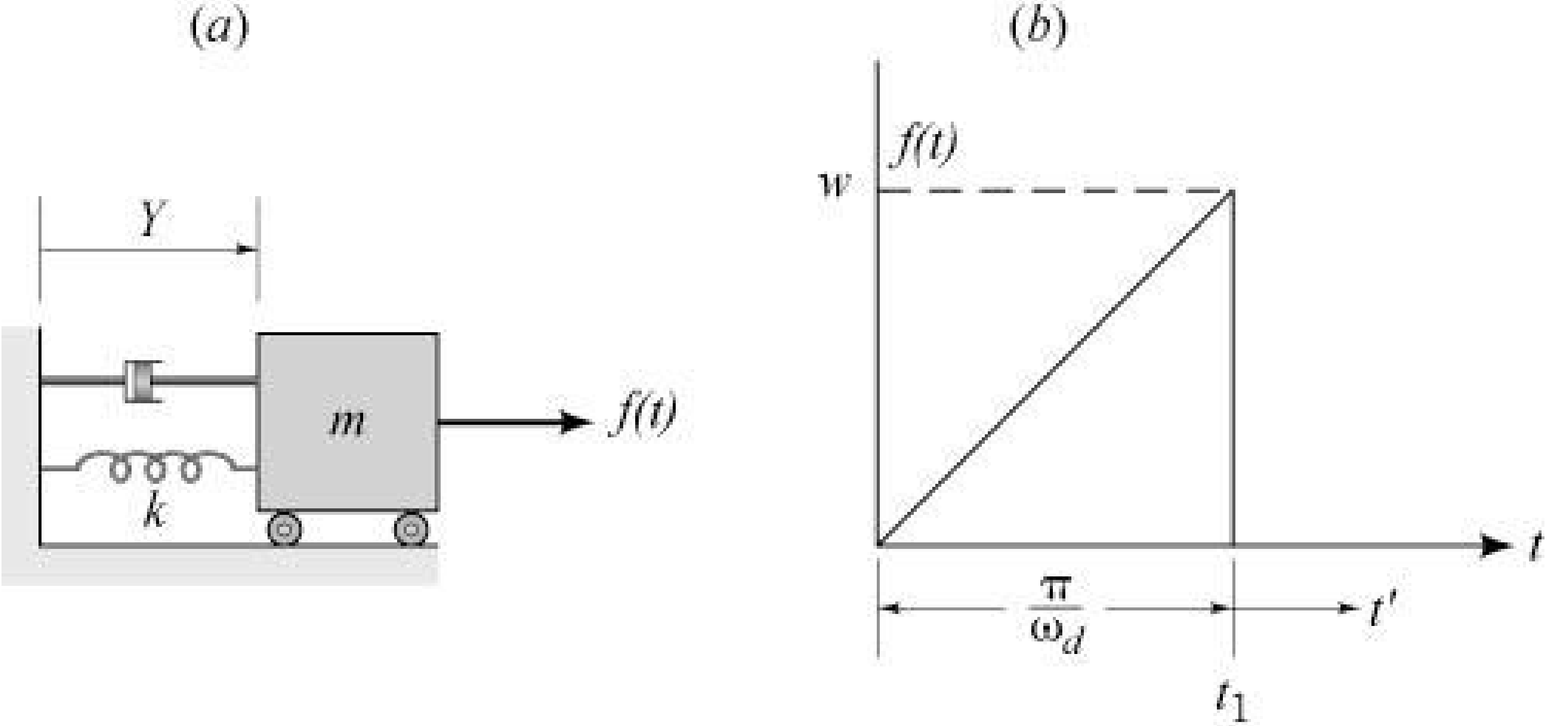


Figure XP3.3 (a). Spring-mass-damper system with an applied force, (b). Applied force definition

Figure XP3.3a illustrates a spring-mass-damper system characterized by the following parameters: ($m = 100\,kg$, $k = 9.87E+04\ N/m$, $c = 3.1416E+02 N\sec/m$), with the forcing function illustrated. The mass system start from rest with the spring undeflected. The force increases linearly until $t = t_1 = \pi/\omega_d = \tau_d/4$ when it reaches a magnitude equal to $w = mg$. For $t \geq t_1$, $f(t) = 0$.

Tasks: *Determine the mass's position and velocity at* $t = t_2 = t_1 + 2\pi/\omega_d$. Plot $Y(t)$ for $0 \leq t \leq t_1$ and $t_1 \leq t \leq t_2$.

Solution. This Example problem has the same stiffness and mass as Example problem 9.1; hence,

$$\omega_n = 31.416 \frac{rad}{\sec} \Rightarrow f_n = 5\,Hz \ , \ \zeta = 0.5 \ , \ \omega_d = 26.97 \frac{rad}{\sec}$$

The equation of motion for $0 \le t \le \pi/\omega_d$ is

$$m\ddot{Y} + c\dot{Y} + kY = ht \ ,$$

$$h = \frac{w\omega_d}{\pi} = (100kg \times 9.81 \frac{m}{\sec^2} \times 26.97 \sec^{-1})/\pi = 8422.\frac{N}{\sec} . \quad \textbf{(i)}$$

For $t \ge \pi/\omega_d$, the equation of motion is $m\ddot{Y} + c\dot{Y} + kY = 0$.
We will use the solution for the first equation for $t = t_1 = \pi/\omega_d$ to determine $Y(t_1)$ and $\dot{Y}(t_1)$, which we will then use as initial conditions in stating the solution to the second equation of motion for $t \ge t_1$.

The applicable homogeneous solution is

$$Y_h = e^{-\zeta\omega_n t} (A \cos\omega_d t + B \sin\omega_d t) \ , \quad \textbf{(ii)}$$

From Table B.2, the particular solution for $\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y = at$, is

$$Y_p = \frac{a}{\omega_n^2}(t - \frac{2\zeta}{\omega_n}) \ ,$$

Hence, the particular solution for $\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = ht/m$ is

$$Y_p = \frac{h}{m\omega_n^2}(t - \frac{2\zeta}{\omega_n}) = \frac{h}{k}(t - \frac{2\zeta}{\omega_n}) \ ,$$

and the complete solution is

$$Y = Y_h + Y_p = e^{-\zeta\omega_n t}\ (A\ \cos\omega_d t\ +\ B\ \sin\omega_d t) + \frac{h}{k}(t - \frac{2\zeta}{\omega_n})\ . \quad \textbf{(iii)}$$

The velocity is

$$\begin{aligned} \dot{Y} = &-\zeta\omega_n e^{-\zeta\omega_n t}\ (A\ \cos\omega_d t\ +\ B\ \sin\omega_d t) \\ &+\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t\ +\ B\cos\omega_d t) + \frac{h}{k}\ . \end{aligned} \quad \textbf{(iv)}$$

Solving for the constants from the initial conditions gives

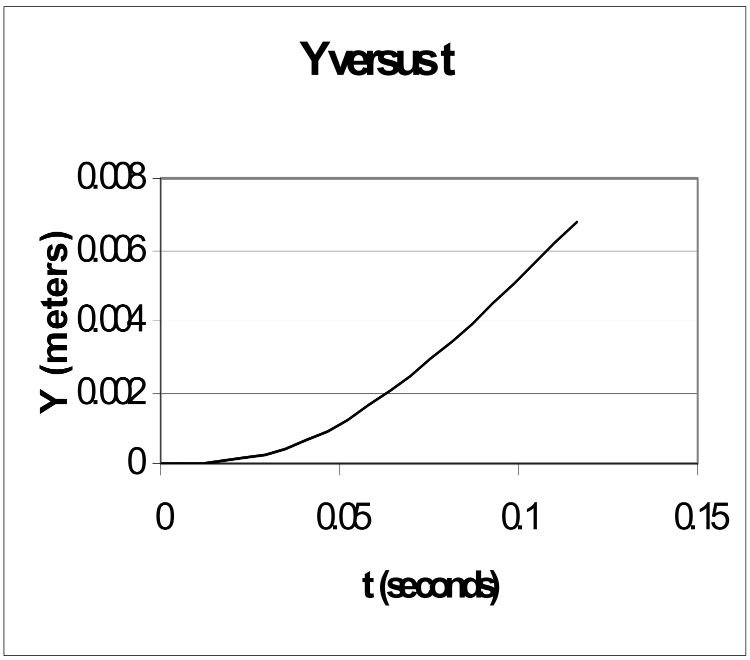
$$Y(0) = 0 = A - \frac{2\zeta h}{k\omega_n} \ \Rightarrow\ A = \frac{2\zeta h}{k\omega_n}$$

$$\dot{Y}(0) = 0 = -\zeta\omega_n A + \omega_d B + \frac{h}{k} \ \Rightarrow\ B = \frac{h(2\zeta^2 - 1)}{k\omega_d}\ ,$$

The complete solution satisfying the initial conditions is

$$Y(t) = \frac{h}{k}\{e^{-\zeta\omega_n t}[\frac{2\zeta}{\omega_n}\cos\omega_d t + \frac{(2\zeta^2-1)}{\omega_d}\sin\omega_d t] + (t - \frac{2\zeta}{\omega_n})\} \; ; \; 0 \le t \le t_1. \quad \textbf{(v)}$$

The complete solution for $t \le 0 \le t_1$ is illustrated below



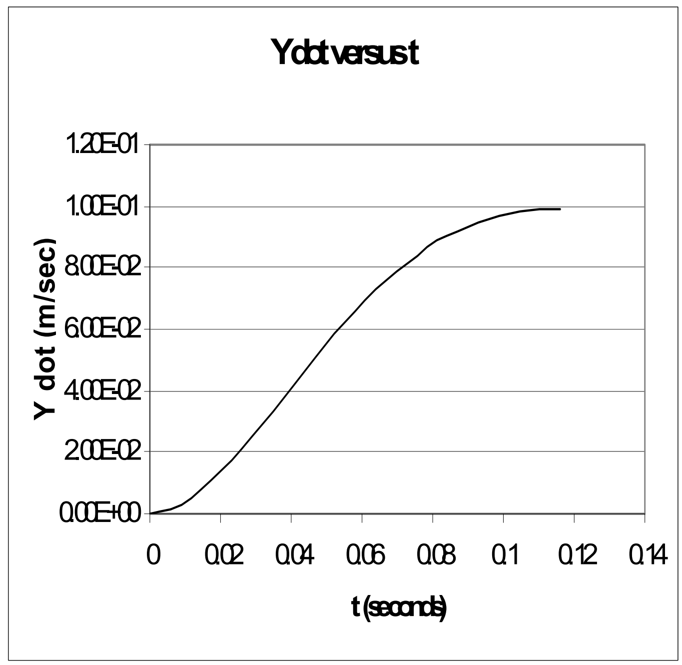


Figure XP3.3 b. Solution for $0 \le t \le t_1$.

The final conditions from figure XP3.3b are $\dot{Y}(t_1) = 9.92E-02\, m/\text{sec}$, and $Y(t_1) = 6.766E-3\, m$ at $t_1 = \pi/\omega_d = \pi/(27.207\, rad/\text{sec}) = 0.1162\,\text{sec}$.

For $t \ge \pi/\omega_d$, the equation of motion $m\ddot{Y} + c\dot{Y} + kY = 0$ has the solution

$$Y = Y_h = e^{-\zeta\omega_n t}\,(A\cos\omega_d t + B\sin\omega_d t)$$

and derivative

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t}\,(A\cos\omega_d t + B\sin\omega_d t) + \omega_d e^{-\zeta\omega_n t}\,(-A\sin\omega_d t + B\cos\omega_d t)\ .$$

We could solve for the unknown constants A and B via $Y(t_1) = 6.766E{-}3\,m$ and $\dot{Y}(t_1) = 9.92E{-}02\,m/\text{sec}$. We are going to take an easier tack by restarting time via the definition $t' = t - t_1$. Since,

$$\frac{DY}{dt'} = \frac{DY}{dt}\,\frac{dt}{dt'} = \frac{DY}{dt}\ ,$$

the equation of motion is unchanged. The solution in terms of t' is

$$Y = e^{-\zeta\omega_n t'}\,(A\cos\omega_d t' + B\sin\omega_d t')$$

with the derivative

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t'}\,(A\cos\omega_d t' + B\sin\omega_d t') + \omega_d e^{-\zeta\omega_n t'}\,(-A\sin\omega_d t' + B\cos\omega_d t')\ .$$

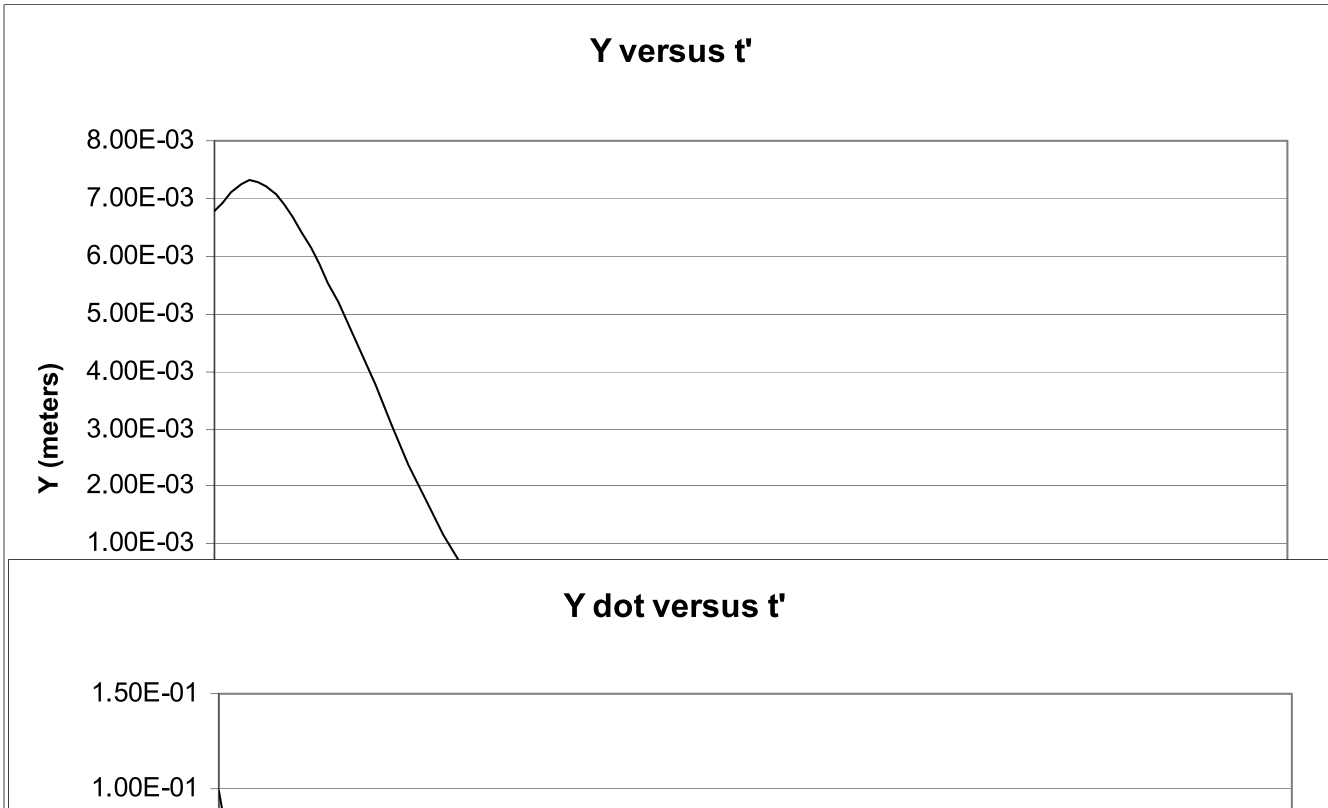
where $t' = t - t_1$, $Y(t' = 0) = Y_0 = Y(t_1) = 6.766E-3\,m$, and $\dot{Y}(t' = 0) = \dot{Y}_0 = \dot{Y}(t_1) = 9.92E-02\,m/\text{sec}$. Again, we are basically restarting time to simplify this solution. Solving for the constants in terms of the initial conditions gives

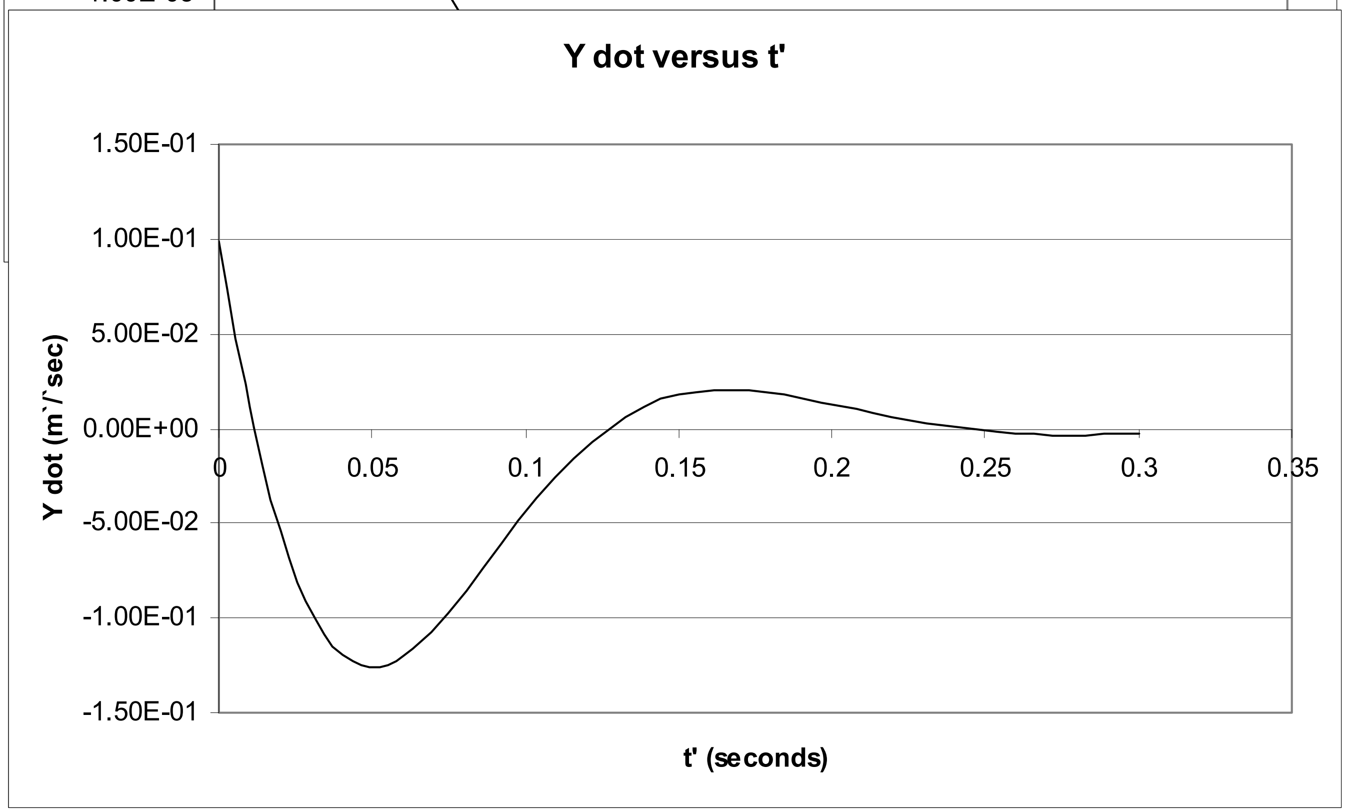
$$Y_0 = A \ , \ \dot{Y}_0 = -\zeta\omega_n A + \omega_d B \ \Rightarrow \ B = (\dot{Y}_0 + \zeta\omega_n Y_0)/\omega_d \ ,$$

and the complete solution in terms of the initial conditions is

$$Y = e^{-\zeta\omega_n t'} \left[Y_0 \cos\omega_d t' + \frac{(\dot{Y}_0 + \zeta\omega_n Y_0)}{\omega_d} \sin\omega_d t' \right]$$

The solution is plotted below


Y versus t'
8.00E-03
7.00E-03
6.00E-03
5.00E-03
4.00E-03
3.00E-03
2.00E-03
1.00E-03
Y (meters)


Y dot versus t'
1.50E-01
1.00E-01
5.00E-02
0.00E+00
-5.00E-02
-1.00E-01
-1.50E-01
Y dot (m`/`sec)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
t' (seconds)

Example Problem Xp3.5

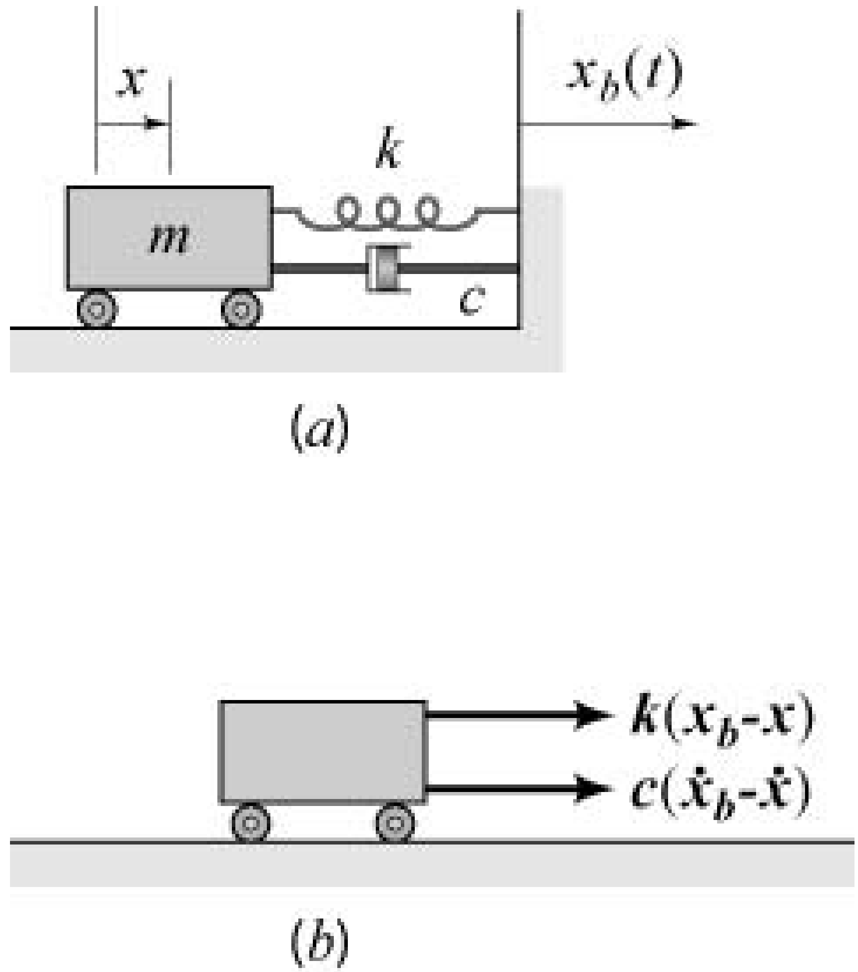


Figure Xp3.5 Movable cart with base excitation. (a). Coordinates, (b). Free-body diagram for $x_b > x, \dot{x}_b > \dot{x}$

Figure XP3.5a illustrates a cart that is connected to a movable base via a spring and a damper. The cart's change of position is defined by x. The movable base's location is defined by x_b.

First, we need to derive the cart's equation of motion, starting with an assumption of the relative positions and velocities. The free-body diagram of XP 3.4a is drawn assuming that $x_b > x > 0$ (placing the spring in tension) and $\dot{x}_b > \dot{x} > 0$ (placing the damper in tension). Applying Newton's 2nd law of motion $\sum \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ nets

$$m\ddot{x} = \sum f_x = k(x_b - x) + c(\dot{x}_b - \dot{x}) \tag{i}$$

$$\therefore\ m\ddot{x} + c\dot{x} + kx = c\dot{x}_b + kx_b$$

In Eq.(i), the spring and damper forces are positive because they are in the $+x$ direction.

The model of Eq.(i) can be used to approximately predict the motion of a small trailer being towed behind a much larger vehicle with the spring and damper used to model the hitch connection between the trailer and the towing vehicle. (Note: To correctly model two vehicles connected by a spring and damper, we need to write $\sum \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for both bodies and account for the influence of the trailer's mass on the towing vehicle's motion. We will consider motion of connected bodies following the next test.)

Assume that the cart and the base are initially motionless with the spring connection undeflected. The base is given the constant acceleration $\ddot{x}_b = g/5$. Complete the following engineering analysis tasks:

a. State the equation of motion.

b. State the homogeneous and particular solutions, and state the complete solution satisfying the initial conditions.
c. For $w = 300\, lbs, k = 776.\, lb/in, c = 24.4\, lb\sec/in$, plot the cart's motion for two periods of damped oscillations.

Stating the equation of motion simply involves proceeding from $\ddot{x}_b = g/5$ to find $\dot{x}_b, x_b$ and plugging them into Eq.(i), via

$$\dot{x}_b = \dot{x}_b(0) + \int_0^t \ddot{x}_b d\tau = \int_0^t \frac{g}{5} d\tau = \frac{gt}{5}$$

$$x_b = x_b(0) + \int_0^t \dot{x}_b d\tau = \frac{gt^2}{10} \ .$$

Substituting these results into Eq.(i) produces

$$m\ddot{x} + c\dot{x} + kx = c\dot{x}_b + kx_b = \frac{g}{5}(ct + k\frac{t^2}{2})$$

The homogeneous solution is

$$x_h = e^{-\zeta\omega_n t} (A \cos\omega_d t + B \sin\omega_d t)$$

We can pick the particular solutions from Table B.2 below as follows

Table B.2. Particular solutions for $\ddot{Y}_p + 2\zeta\omega_n \dot{Y}_p + \omega_n^2 Y_p = u(t)$.

Excitation, $u(t)$	$Y_p(t)$
$h = constant$	h/ω_n^2
$a\,t$	$\frac{a}{\omega_n^2}(t - \frac{2\zeta}{\omega_n})$
bt^2	$\frac{b}{\omega_n^2}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$

$$\frac{gc}{5m}t \Rightarrow x_{p1} = \frac{gc}{5m}\frac{1}{\omega_n^2}(t - \frac{2\zeta}{\omega_n}) = \frac{2\zeta g}{5\omega_n}(t - \frac{2\zeta}{\omega_n})$$

$$\frac{gk}{10m}t^2 \Rightarrow x_{p2} = \frac{gk}{10m}\frac{1}{\omega_n^2}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$$

$$= \frac{g}{10}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)] \ .$$

The complete solution is

$$x = x_h + x_{p1} + x_{p2}$$

$$= e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t)$$
$$+ \frac{2\zeta g}{5\omega_n}(t - \frac{2\zeta}{\omega_n}) + \frac{g}{10}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$$

The constant A is obtained via,

$$x(0) = 0 = A - \frac{4\zeta^2 g}{5\omega_n^2} - \frac{g}{5\omega_n^2}(1 - 4\zeta^2) \Rightarrow A = \frac{g}{5\omega_n^2} \ .$$

To obtain B, first

$$\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)$$
$$+\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t + B\cos\omega_d t)$$
$$+\frac{2\zeta g}{5\omega_n} + \frac{g}{10}(2t - \frac{4\zeta}{\omega_n})\ .$$

Hence,

$$\dot{x}(0) = 0 = -\zeta\omega_n A + \omega_d B + \frac{2\zeta g}{5\omega_n} - \frac{2\zeta g}{5\omega_n} \Rightarrow B = \frac{g\zeta}{5\omega_n^2\sqrt{1-\zeta^2}}\ .$$

The complete solution satisfying the boundary conditions is

$$x = e^{-\zeta\omega_n t}\ [\frac{g}{5\omega_n^2}\ \cos\omega_d t\ + \frac{g\zeta}{5\omega_n^2\sqrt{1-\zeta^2}}\sin\omega_d t]$$
$$+\frac{2\zeta g}{5\omega_n}(t - \frac{2\zeta}{\omega_n}) + \frac{g}{10}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1-4\zeta^2)]\ , \qquad \textbf{(ii)}$$

and Eq.(ii) completes Task b.

Moving towards Task c,
$m = w/g = 300\,lbs/(386.4\,in/\text{sec}^2) = 0.766\,lb/(\text{sec}^2\,in)$.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{776.\,lb/in}{.776\,lb\sec^2/in}} = 31.62\,\frac{rad}{\sec}$$

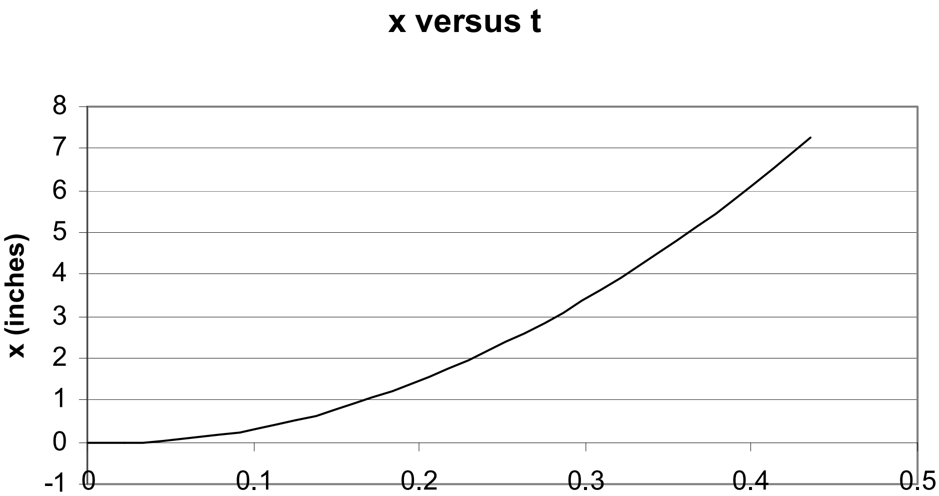
$$\therefore\ f_n = 31.62\frac{rad}{\sec} \times \frac{1\,cycle}{2\pi\,rad} = 5.03\frac{cycles}{\sec} = 5.03\,Hz$$

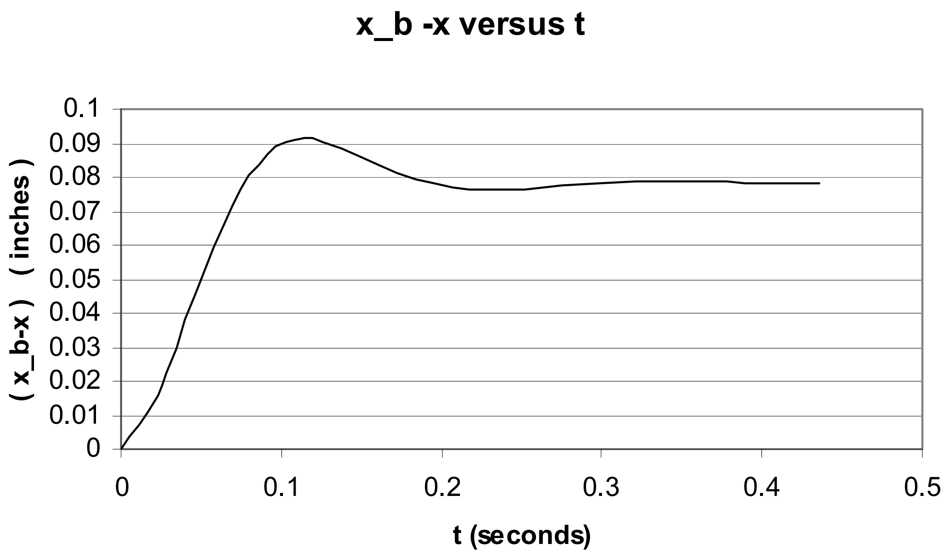
$$\zeta = \frac{c}{2m\omega_n} = 24.4\,\frac{lb\sec}{in} \times \frac{1}{2 \times .776\,snails \times 31.62\sec^{-1}} = 0.497$$

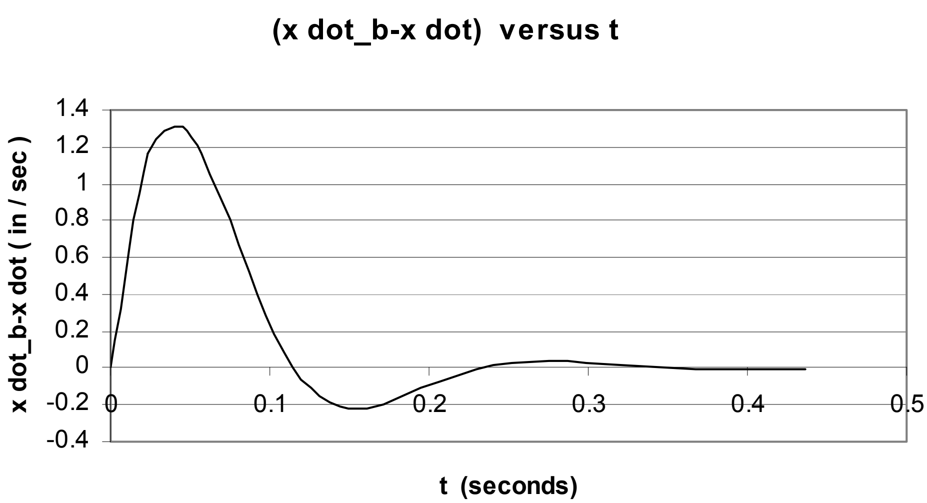
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.62\sqrt{1-0.497^2} = 27.4\,\frac{rad}{\sec}$$

$$\therefore\ f_d = 27.4\,\frac{rad}{\sec} \times \frac{1\,cycle}{2\pi\,rad} = 4.36\,\frac{cycles}{\sec} = 4.36\,Hz\ .$$

Two cycles of damped oscillations will be completed in $2 \times \tau_d$ seconds, where $\tau_d = 1/f_d = 1/4.36\,(cycles/\sec) = 0.229\sec/cycle$. The solutions are shown below.







Discussion. After an initial transient, the relative displacement

$(x_b - x)$ approaches a constant value $(x_b - x) \cong 0.078\,in$, while the relative velocity approaches zero; i.e., $(\dot{x}_b - \dot{x}) \cong 0$. The scaling for $x(t)$ hides the initial transient, but the plot shows quadratic increase with time, which is consistent with its constant acceleration at $g/5$. The spring force $k(x - x_b) \cong (776.lbs/in) \times .078\,in \cong 60.1\,lbs$, while the force required to accelerate the mass at *g/5* is

$$f = m\ddot{x} = 0.776\left(\frac{lb\,\mathrm{sec}}{in^2}\right) \times \frac{386.4}{5} = 60\,lbs \quad .$$

Hence, the asymptotic value for $(x_b - x)$ creates the spring force required to accelerate the mass at $g/5$.

Example XP3.6
Think about an initially motionless cart being "snagged" by a moving vehicle with velocity v_0 through a connection consisting of a parallel spring-damper assembly. The connecting spring is undeflected prior to contact. That circumstance can be modeled by giving the base end of the model in figure XP3.5 the velocity v_0; i.e., $\dot{x}_b = v_0 \Rightarrow x_b = x_b(0) + v_0 t = v_0 t$, and the governing equation of motion is

$$m\ddot{x} + c\dot{x} + kx = cv_0 + kv_0 t \ , \qquad \textbf{(iii)}$$

with initial conditions $x(0) = \dot{x}(0) = 0$. The engineering analysis tasks for this example are:

a. Determine the complete solution that satisfies the initial conditions.

b. For the data set, $m = 10\,kg$, $k = 1580\,N/m$, and $c = 1257\,N/m$, determine $\omega_n, \zeta, \omega_d$.

c. For $v_0 = 20\,km/hr$ produce plots for the mass displacement $x(t)$, relative displacement $x_b(t) - x(t)$, and the mass velocity $\dot{x}(t)$ for two cycles of motion.

Solution. From Table B.2, the particular solutions corresponding to the right-hand terms are:

$$\frac{cv_0}{m} \Rightarrow x_{p1} = \frac{cv_0}{m}\frac{1}{\omega_n^2} = \frac{2\zeta v_0}{\omega_n}$$

$$\frac{kv_0}{m}t \Rightarrow x_{p2} = \frac{kv_0}{m}\frac{1}{\omega_n^2}(t - \frac{2\zeta}{\omega_n}) = v_0(t - \frac{2\zeta}{\omega_n}) \ .$$

The complete solution is

$$x = x_h + x_{p1} + x_{p2} = e^{-\zeta\omega_n t}(A\ \cos\omega_d t\ +\ B\ \sin\omega_d t) + \frac{2\zeta v_0}{\omega_n} + v_0(t - \frac{2\zeta}{\omega_n}).$$

Solving for A ,

$$x(0) = 0 = A + \frac{2\zeta v_0}{\omega_n} - \frac{2\zeta v_0}{\omega_n} \Rightarrow A = 0 \ .$$

Solving for B from

$$\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \omega_d e^{-\zeta\omega_n t}(-A\sin\omega_d t + B\cos\omega_d t) + v_0 \ ,$$

gives

$$\dot{x}(0)=0=-\zeta\omega_n A+\omega_d B+v_0 \;\Rightarrow\; B=-\frac{v_0}{\omega_d}\;.$$

The complete solution satisfying the initial conditions is

$$x=\frac{v_0}{\omega_d}\,e^{-\zeta\omega_n t}\,\sin\omega_d t+\frac{2\zeta v_0}{\omega_n}+v_0\,(t-\frac{2\zeta}{\omega_n})\;, \qquad \textbf{(iv)}$$

which completes Task a.

For Task b,

$$\omega_n=\sqrt{\frac{k}{m}}=\sqrt{\frac{1580}{10}}=12.57\,\frac{rad}{\sec}$$

$$\therefore\; f_n=12.57\frac{rad}{\sec}\times\frac{1\,cycle}{2\pi\,rad}=2\frac{cycles}{\sec}=2\,Hz$$

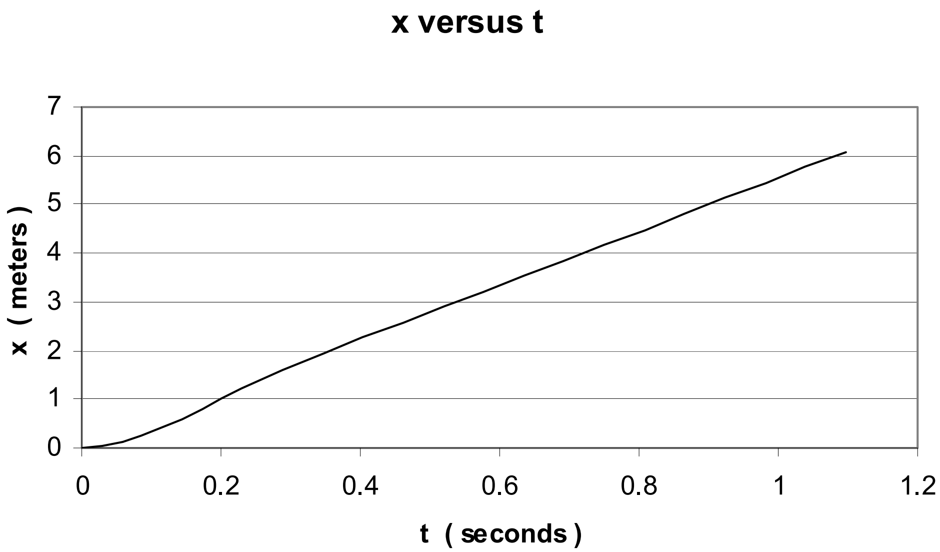
$$\zeta=\frac{c}{2m\omega_n}=1257\,\frac{N\sec}{m}\times\frac{1}{2\times 10\,Kg\times 12.57\sec^{-1}}=0.5$$

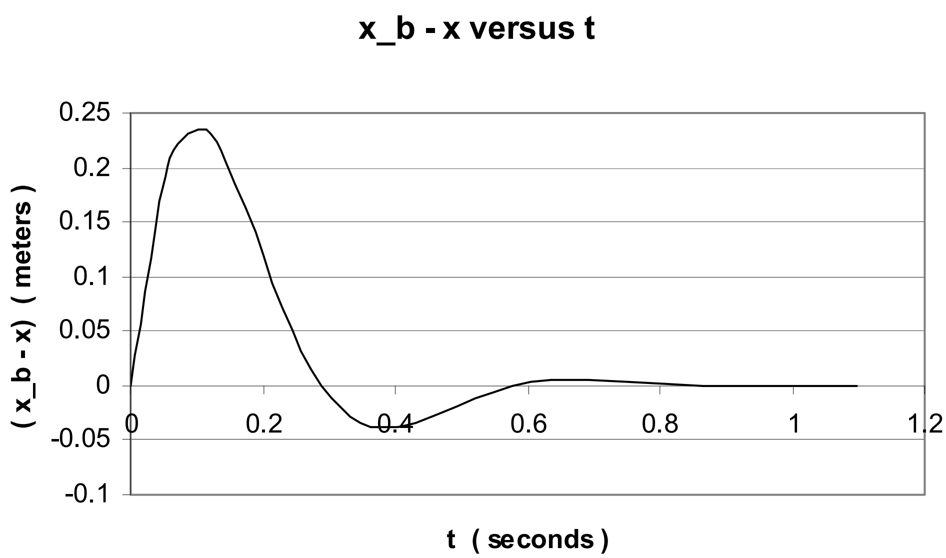
$$\omega_d=\omega_n\sqrt{1-\zeta^2}=12.57\sqrt{1-0.5^2}=10.89\,\frac{rad}{\sec}$$

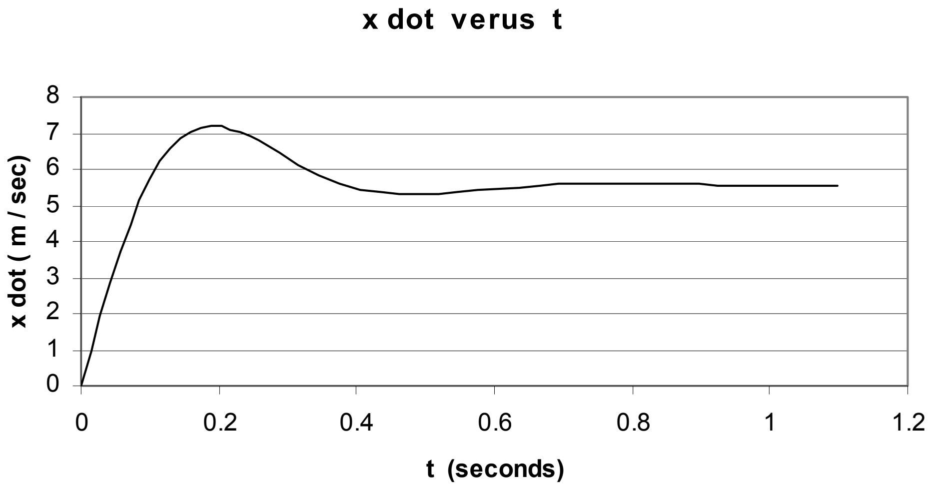
$$\therefore\; f_d=10.89\frac{rad}{\sec}\times\frac{1\,cycle}{2\pi\,rad}=1.73\frac{cycles}{\sec}=1.73\,Hz\;.$$

From the last of these results, the period for a damped oscillation is $\tau_d = 1/f_d = 0.577\,\sec$.

For Task c,
$v_0 = 20\,km/hr \times 1hr/3600\sec \times 1000\,m/km = 5.55\,m/\sec$. The figures below illustrate the solution for two cycles of motion.







Discussion. The first and last figures show the mass rapidly moving towards the towing velocity $v_0 = 5.55\, m/\sec$. The second figure shows the relative position approaching zero. Note that at about 0.3 *seconds*, the towed cart appears to actually passes the towing vehicle when $(x_b - x)$ becomes negative. However, the present analysis does not account for the initial spring length. A the towed vehicle could "crash" into the back of the towing vehicle, but it's not likely. You could plot $(v_0 - \dot{x}_b)$ to find the relative velocity if impact occurs.