LECTURE 11. HARMONIC EXCITATION

Forced Excitation

$$m\ddot{Y} + c\dot{Y} + kY = f_o \sin\omega t \ . \qquad \textbf{(3.32)}$$

Dividing through by the mass m gives

$$\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = (f_o/m)\sin\omega t \ . \qquad \textbf{(3.33)}$$

Seeking a particular solution for the right hand term gives

$$\begin{aligned} Y_p &= C\sin\omega t + D\cos\omega t \\ \dot{Y}_p &= \omega C\cos\omega t - \omega D\sin\omega t \\ \ddot{Y}_p &= -\omega^2 C\sin\omega t - \omega^2 D\cos\omega t. \end{aligned} \qquad \textbf{(3.34)}$$

Substituting this solution into Eq.(3.32) yields

$$(-\omega^2 C\sin\omega t - \omega^2 D\cos\omega t)$$
$$+ 2\zeta\omega_n (C\omega\cos\omega t - \omega D\sin\omega t)$$
$$+\omega_n^2 (C\sin\omega t + D\cos\omega t) = (f_0/m)\sin\omega t \ .$$

Gathering the $\sin\omega t$ and $\cos\omega t$ coefficients gives the following two equations:

$$\sin\omega t:\ -\omega^2 C - 2\zeta\omega_n\omega D + \omega_n^2 C = f_o/m$$
$$\cos\omega t:\ -\omega^2 D + 2\zeta\omega_n\omega C + \omega_n^2 D = 0\ .$$

The matrix statement for the unknowns D and C is

$$\begin{bmatrix} (\omega_n^2 - \omega^2) & -2\zeta\omega_n\omega \\ 2\zeta\omega_n\omega & (\omega_n^2 - \omega^2) \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} f_0/m \\ 0 \end{Bmatrix}\ .$$

Using Cramer's rule for their solution gives:

$$\begin{aligned} C &= \frac{1}{\Delta}\begin{vmatrix} f_o/m & -2\zeta\omega_n\omega \\ 0 & (\omega_n^2 - \omega^2) \end{vmatrix} = \frac{f_o(\omega_n^2 - \omega^2)}{m\Delta} \\ D &= \frac{1}{\Delta}\begin{vmatrix} (\omega_n^2 - \omega^2) & f_o/m \\ 2\zeta\omega_n\omega & 0 \end{vmatrix} = \frac{-f_o\ 2\ \zeta\omega_n\omega}{m\Delta}\ . \end{aligned} \tag{3.35}$$

where Δ is the determinant of the coefficient matrix defined by

$$\Delta = (\omega_n^2 - \omega^2)^2 + 4\ \zeta^2\ \omega_n^2\ \omega^2\ .$$

The solution defined by Eq.(3.34) can be restated

$$
\begin{aligned}
Y_p(t) &= C\ \sin\omega t\ +\ D\ \cos\omega t \\
&= Y_{op}\ \sin(\omega t+\psi) \qquad\qquad (3.36) \\
&= Y_{op}\ (\sin\omega t\ \cos\psi\ +\ \cos\omega t\ \sin\psi)\ \ .
\end{aligned}
$$

where Y_{op} is the amplitude of the solution, and ψ is the phase between the solution $Y_p(t)$ and the input excitation force $f(t) = f_o \sin\omega t$. Figure 3.11 illustrates the phase relation between $f(t)$ and $Y_p(t)$, showing $Y_p(t)$ "leading" $f(t)$ by the phase angle ψ.

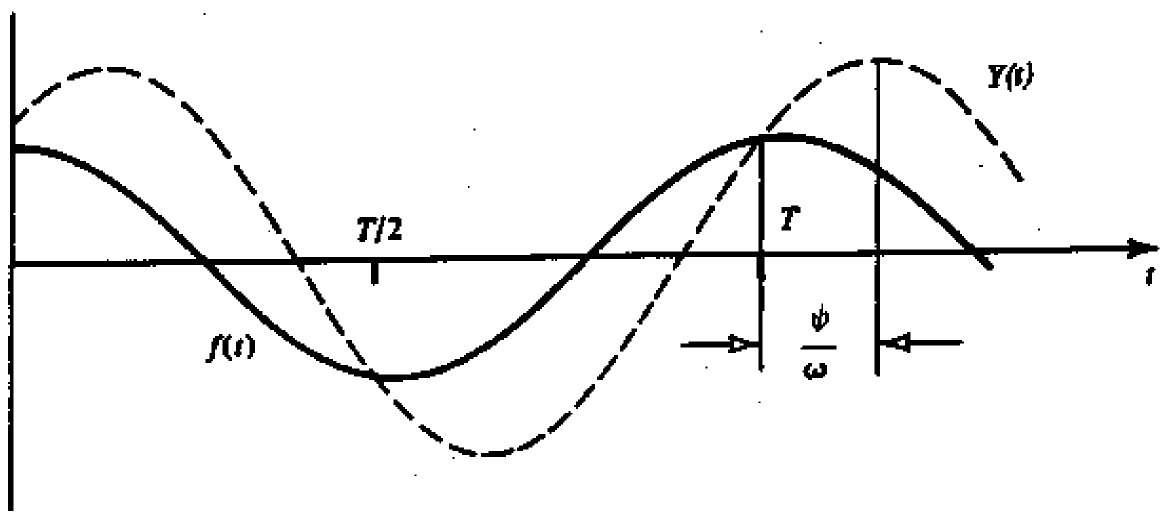


Figure 3.11 Phase relation between the response $Y_P(t)$ and the input harmonic excitation force $f(t) = f_o \sin\omega t$.

The solution for C and D provided by Eq.(3.35) gives:

$$\begin{aligned} Y_{op} &= \sqrt{C^2 + D^2} \\ &= \frac{f_o}{m} \cdot \frac{1}{\Delta} \left[(\omega_n^2 - \omega^2)^2 + 4\,\zeta^2\,\omega_n^2\,\omega^2 \right]^{1/2} \\ &= \frac{f_o}{m} \cdot \frac{1}{\left[(\omega_n^2 - \omega^2)^2 + 4\,\zeta^2\,\omega_n^2\,\omega^2 \right]^{1/2}} \\ &= \frac{f_o}{m\left(\dfrac{k}{m}\right)} \cdot \frac{1}{\left\{ [1 - (\omega/\omega_n)^2]^2 + 4\zeta^2(\omega/\omega_n)^2 \right\}^{1/2}} \,, \end{aligned}$$

where

$$\frac{\omega}{\omega_n} = \textit{Frequency ratio} = r \,, \quad \frac{f_o}{k} = \textit{static deflection}$$

Amplification Factor

$$\frac{Y_{op}}{f_o/k} = \frac{1}{\left\{ [1 - r^2]^2 + 4\zeta^2 r^2 \right\}^{1/2}} = H(r) \,. \tag{3.38}$$

The maximum amplification factor $H(r)$ is found from

$dH(r)/dr = 0$ as

$$r_{\max} = \sqrt{1 - 2\zeta^2} \tag{3.40a}$$

Note that $\omega = \omega_n \Rightarrow r = 1$

$$H(r = 1) = \frac{1}{2\zeta} \doteq q\text{-}factor$$

<u>This is another way to characterize damping.</u>

Eqs.(3.35) and (3.36) define the *phase* as

$$\begin{aligned}\psi(\omega/\omega_n) &= \tan^{-1}\left(\frac{D}{C}\right) = \tan^{-1}\left[\frac{-2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}\right] \\ &= -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right].\end{aligned} \tag{3.39}$$

Complete Solution

$$Y = Y_h + Y_p = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + Y_{op}\sin(\omega t + \psi) \ .$$

The particular solution is frequently referred to as the"steady-state" solution because of this "persisting" nature. It continues indefinitely after the homogeneous solution has disappeared.

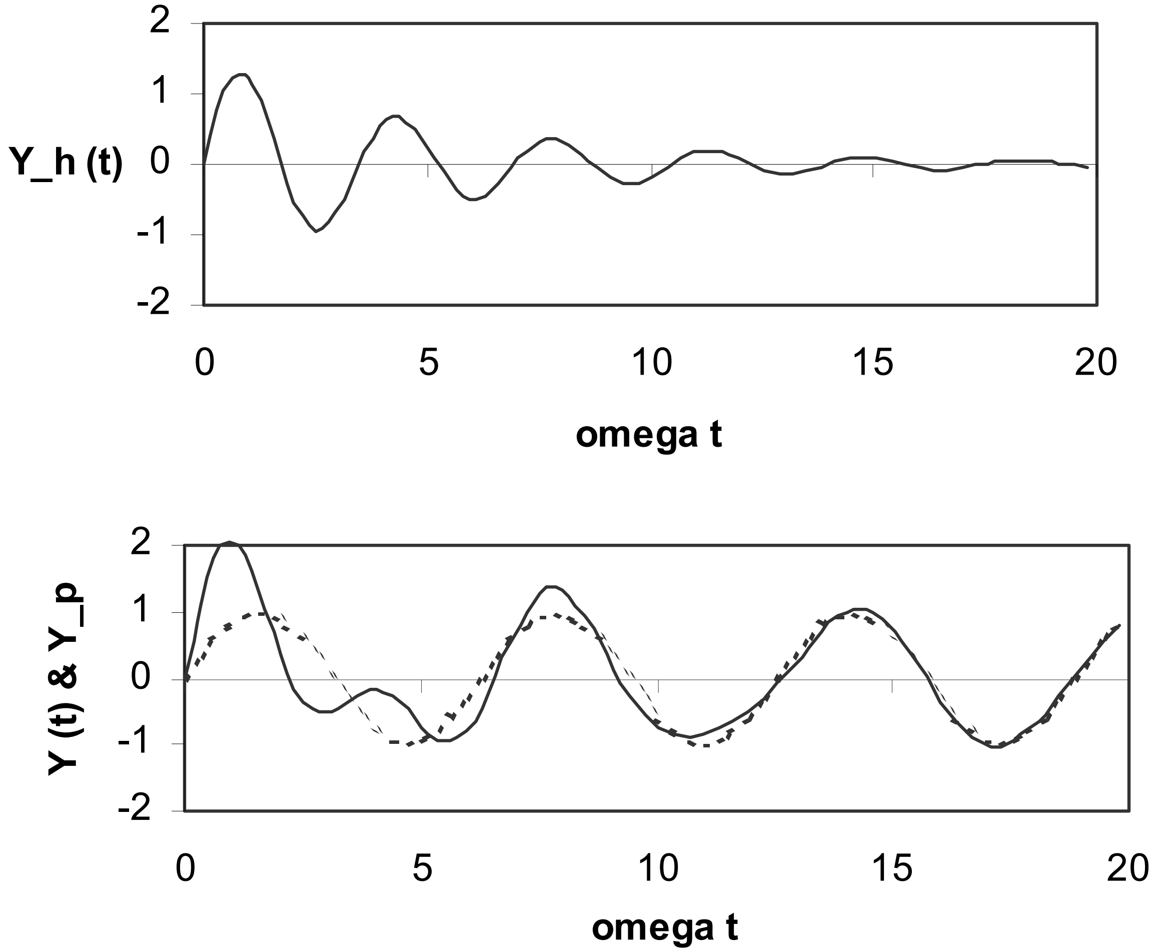


Figure 3.15 (a). Homogeneous solution for $\zeta<1$. (b). Complete $[Y(t) = Y_h(t) + Y_p(t)]$ and steady-state particular solution $Y_p(t)$ for harmoinc excitation $f_o \sin\omega t$

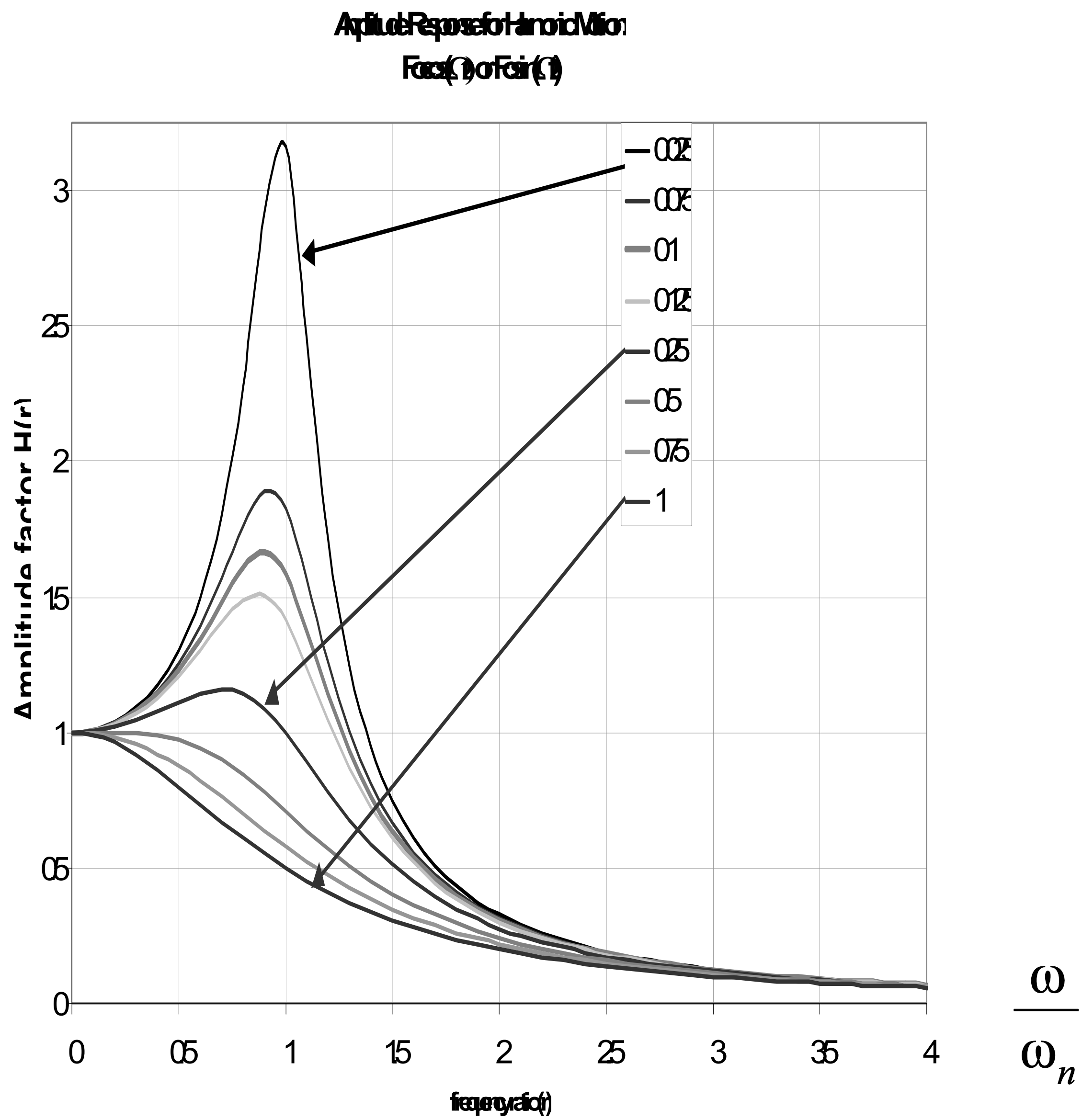


Figure 3.13 Amplification factor for harmonic motion

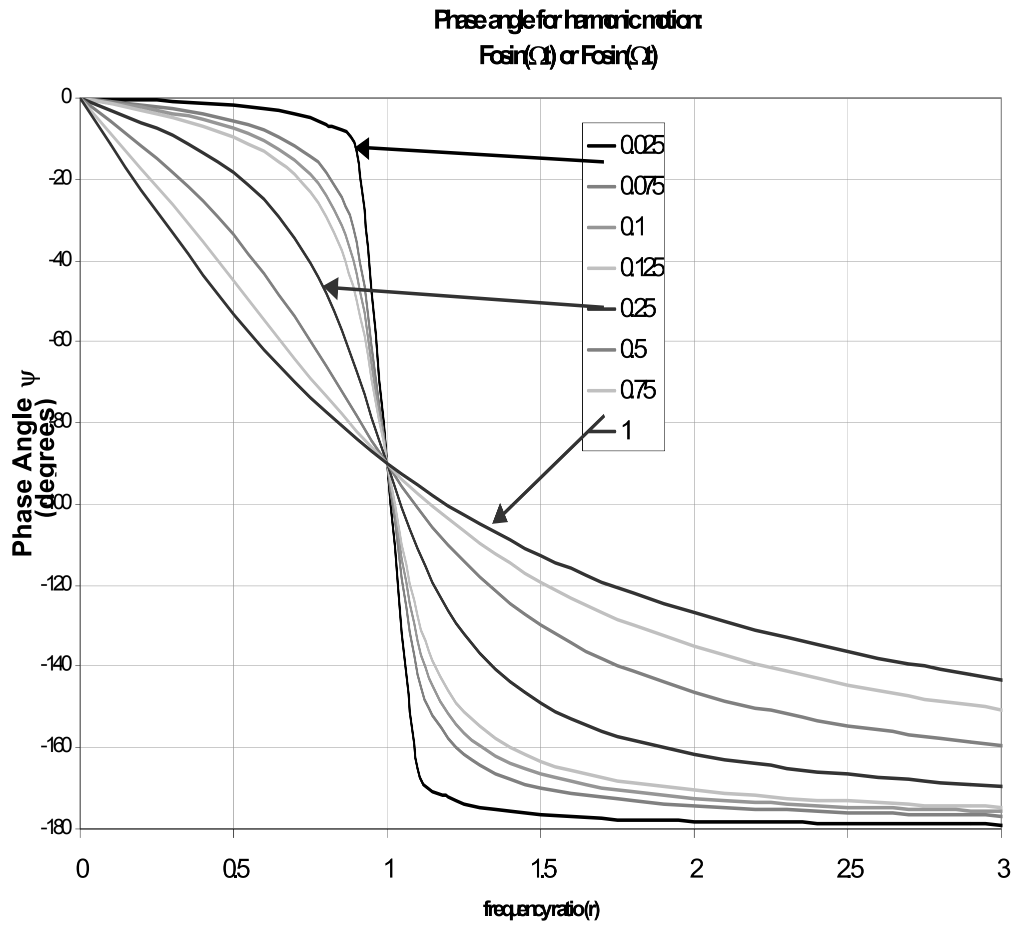


Figure 3.14 Phase angle for harmonic excitation.

Example Problem 3.8 The spring-mass-damper system of figure XP 3.8 is acted on by the external harmonic force $f = f_0 \cos\omega t$, netting the differential equation of motion

$$m\ddot{Y} + c\dot{Y} + kY = f_o \cos\omega t \ . \qquad \textbf{(i)}$$

For the data, $m = 30\ kg$, $k = 12000\ N/m$, $f_o = 200\ N$, *carry out the following engineering-analysis tasks:*

a. For $c = 0$, determine the range of excitation frequencies for which the amplitudes will be less than 76 mm,

b. Determine the damping value that will keep the steady-state response below 76 mm for all excitation frequencies.

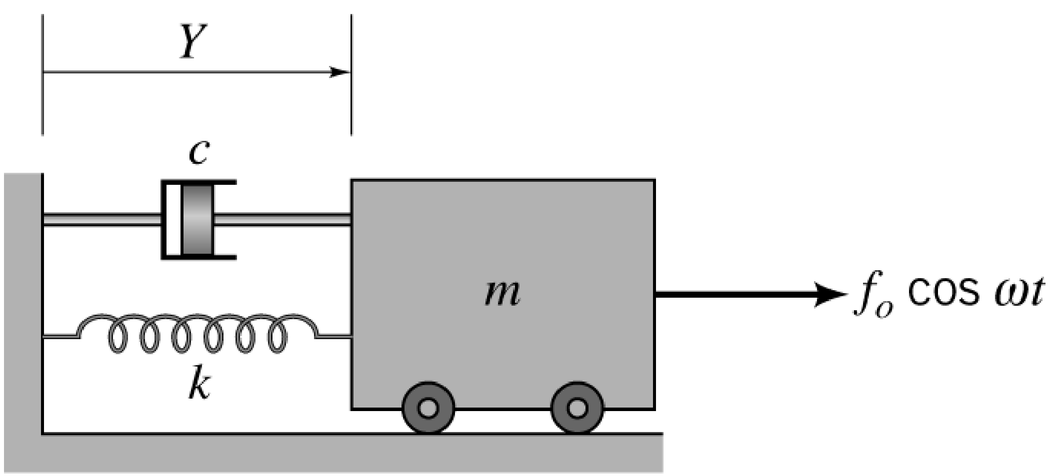


Figure XP3.8 Harmonically excited spring-mass-damper system.

Solution. Note first that Eq.(i) has $f_o \cos\omega t$ as the excitation term, versus $f_o \sin\omega t$ in Eq.(3.32). This change means that the steady-state response is now $Y_p(t) = Y_{op}\cos(\omega t + \psi)$ instead of $Y_p(t) = Y_{op}\sin(\omega t + \psi)$ of Eq.(3.36). The steady-state amplification factor and phase continue to be defined by Eqs.(3.38a) and (3.39), respectively. From the data provided, we can calculate

$$\omega_n = \sqrt{k/m} = \sqrt{12000/30} = 20\, rad/\sec$$

$$\delta_{static} = f_0/k = 200\,N/(12000\,N/m) = 1.667 \times 10^{-2}\, m \ .$$

The amplification factor corresponding to the amplitude $Y_{op} = 76\, mm = 7.6 \times 10^{-2} m$ is $H(r) = 7.6 \times 10^{-2} m/(1.667 \times 10^{-2} m) = 4.56$. Looking back at figure 3.14, this amplification factor would be associated with frequency ratios that are fairly close to $r = 1$. For zero damping, We can use Eq.(3.38a) to solve for the two frequency ratios, via

$$\frac{Y_{op}}{f_o/k} = \frac{1}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}} \Rightarrow 4.56 = \frac{1}{[(1 - r^2)^2]^{1/2}} \ .$$

Restating the last equation gives

$$1 - 2r^2 + r^4 = \frac{1}{4.56^2} = .0481 \ \Rightarrow \ r^4 - 2r^2 + .952 \ = 0 \ .$$

Solving this quadratic equation defines the two frequencies by:

$$r_1^2 = .781 \;\Rightarrow\; r_1 = .883 \;\;,\;\; \omega_1 = r_1 \times \omega_n = .883 \times 20\, rad/\sec$$
$$= 17.7\, rad/\sec$$

$$r_2^2 = 1.10 \;\Rightarrow\; r_2 = 1.05 \;\;,\;\; \omega_2 = r_2 \times \omega_n = 1.05 \times 20\, rad/\sec$$
$$= 21.0\, rad/\sec \;\; .$$

As expected, ω_1 and ω_2 are close to the natural frequency ω_n. The steady-state amplitudes will be less than the specified 76 *mm* , for the frequency ranges, $0 < \omega < \omega_1 = 17.7\, rad/\sec$ and $\omega > \omega_2 = 21.0\, rad/\sec$, and we have competed *Task a*.

Moving to *Task b*, the amplification factor is a maximum at $r = \sqrt{1 - \zeta^2}$, and its maximum value is $H_{\max} = 1/2\zeta\sqrt{1 + 2\zeta^2}$. Hence,

$$H_{\max} = \frac{1}{2\zeta\sqrt{1 + 2\zeta^2}} = 4.56 \;\Rightarrow\; \zeta = .1084 \;\; ,$$

and the required damping to achieve this ζ value is, from Eq.(3.22),

$$c = 2\zeta\omega_n m = 2(.1084)(20.\frac{rad}{\sec})(30\,kg) = 130.0\,N\sec/m \ .$$

Note that the derived units for the newton is $kg\,m/\sec^2$; hence, $kg/\sec \Rightarrow N\sec/m$. Damping coefficients of this value or higher will keep the peak response amplitudes at less than the specified 76 *mm*, and *Task b* is completed.

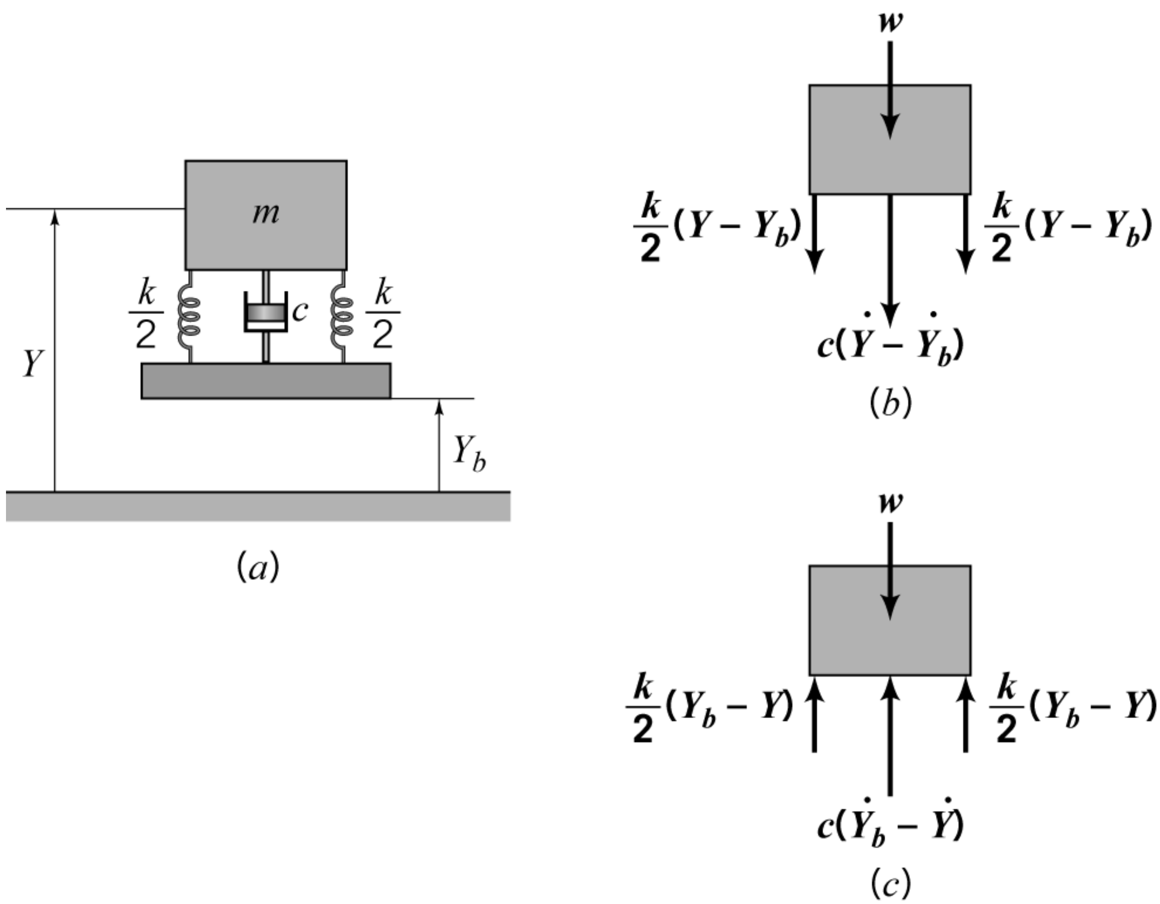


Figure 3.15. (a). Suspended mass and movable base. (b). Free-body diagram for tension in the springs and damper, (c). Free-body diagram for compression in the springs and damper.

Harmonic Base Excitation. Base harmonic base motion can be defined by $Y_b = A\sin\omega t$, $\dot{Y}_b = A\,\omega\cos\omega t$, and $m\ddot{Y} + c\dot{Y} + kY = -w + kY_b + c\dot{Y}_b$ becomes

$$m\ddot{Y} + c\dot{Y} + kY = -w + kA\sin\omega t + cA\omega\cos\omega t \ .$$

Dividing through by m gives

$$\begin{aligned}
&\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y \\
&\quad = -g + A(\omega_n^2\sin\omega t + 2\zeta\omega_n\omega\cos\omega t) \\
&\quad = -g + B\sin(\omega t + \varphi) \\
&\quad = -g + B(\sin\omega t\cos\varphi + \cos\omega t\sin\varphi) \ .
\end{aligned} \tag{3.41}$$

We can solve for B and φ from the last two lines of this equation, obtaining

$$B = A\,\omega_n^2[1 + 4\zeta^2(\omega/\omega_n)^2]^{1/2} \ , \quad \varphi = \tan^{-1}[2\zeta(\frac{\omega}{\omega_n})] \ . \tag{3.42}$$

Looking at Eq.(3.41), our sole interest is the steady-state solution due to the harmonic excitation term $B\sin(\omega t + \varphi)$.

Based on the earlier results of Eq.(3.36), the expected steady-state solution format to Eq.(3.41) is

$$Y_p = Y_{op}\sin(\omega t + \varphi + \psi) \ , \tag{3.43}$$

with ψ defined by Eq.(3.39). This solution is sinusoidal at the input frequency ω, having the same phase lag ψ with respect to the input force excitation $B\sin(\omega t+\varphi)$ as determined earlier for the harmonic force excitation $f(t)=f_o\sin(\omega t)$.

Comparison Eqs.(3.33) and (3.41), shows B replacing f_0/m. Substituting $B=f_0/m$ into Eq.(3.37) gives

$$Y_{op} = \frac{B}{\omega_n^2}\cdot\frac{1}{\left\{[1-(\omega/\omega_n)^2]^2+4\zeta^2(\omega/\omega_n)^2\right\}^{1/2}}$$

$$= A\cdot\frac{[1+4\zeta^2(\omega/\omega_n)^2]^{1/2}}{\left\{[1-(\omega/\omega_n)^2]^2+4\zeta^2(\omega/\omega_n)^2\right\}^{1/2}}\ .$$

Hence, the ratio of the steady-state-response amplitude to the base-excitation amplitude is

$$\frac{Y_{op}}{A} = \frac{[1+4\zeta^2r^2]^{1/2}}{[(1-r^2)^2+4\zeta^2r^2]^{1/2}} = G(r)\ . \tag{3.44}$$

Figure 3.16 illustrates $G(r)$, showing a strong similarity to $H(r)$ of figure 3.13 with the peak amplitudes occurring near $r=\omega/\omega_n=1$. For $\zeta=0$, the two transfer functions coincide. The maximum for $G(r)$ is obtained via $dG(r)/dr=0$, yielding

$$r_{\max} = \frac{1}{2\zeta}\left[(1+8\zeta^2)^{1/2} - 1\right]^{1/2} \quad . \qquad \textbf{(3.46)}$$

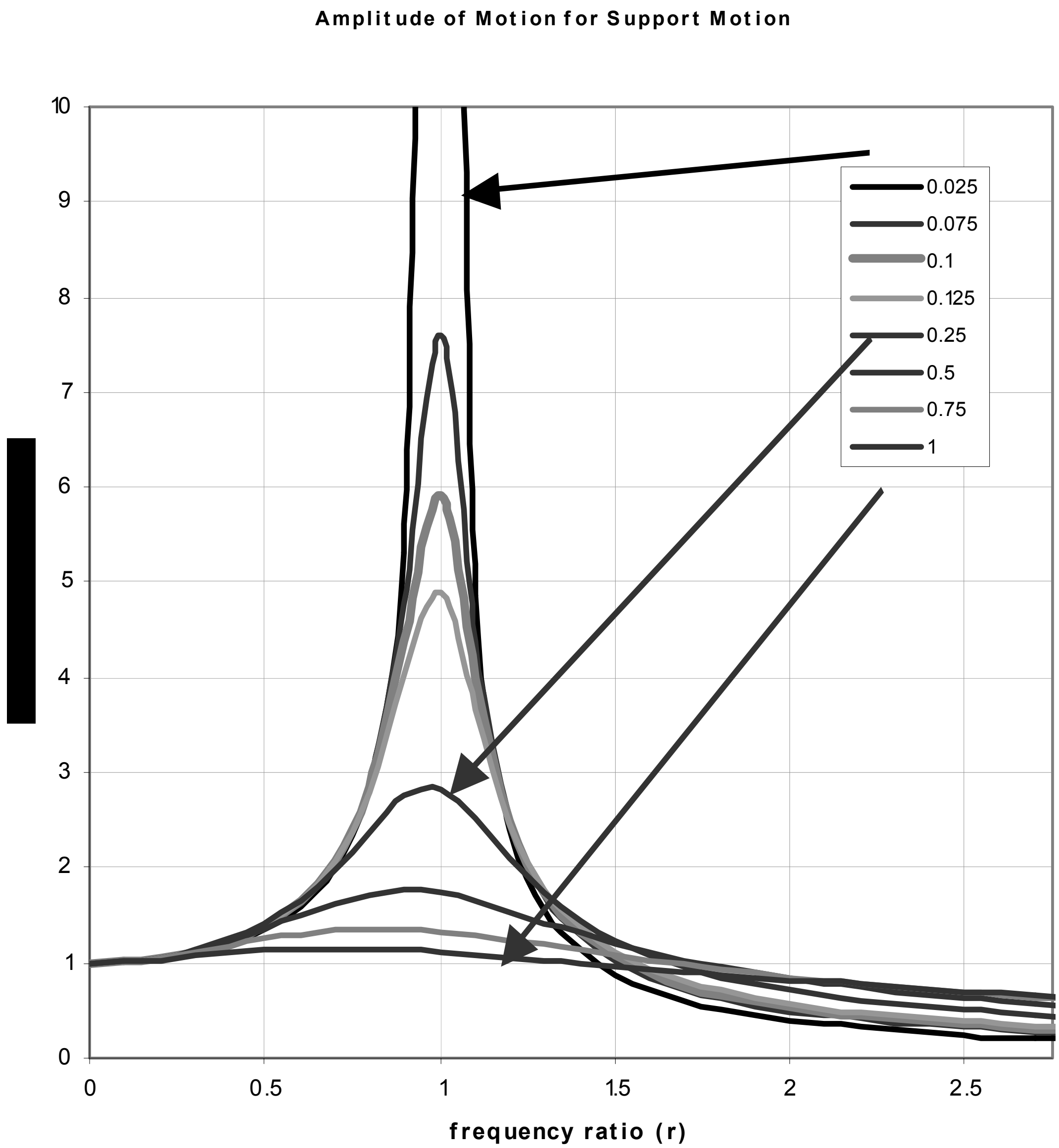


Figure 3.16. Amplification factor $G(r)$ defined by Eq.(3.44).

From the trigonometric identity $\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$, we can use Eqs.(3.39), (3.42), and (3.43), to define the phase between the steady-state solution $Y_p = Y_{op}\sin(\omega t + \varphi + \psi)$ and the base motion excitation $Y_b = A\sin\omega t$ as

$$\varphi + \psi = \tan^{-1}\left(\frac{-2\zeta r^3}{1 + 4\zeta^2 - r^2}\right) \qquad (3.45)$$

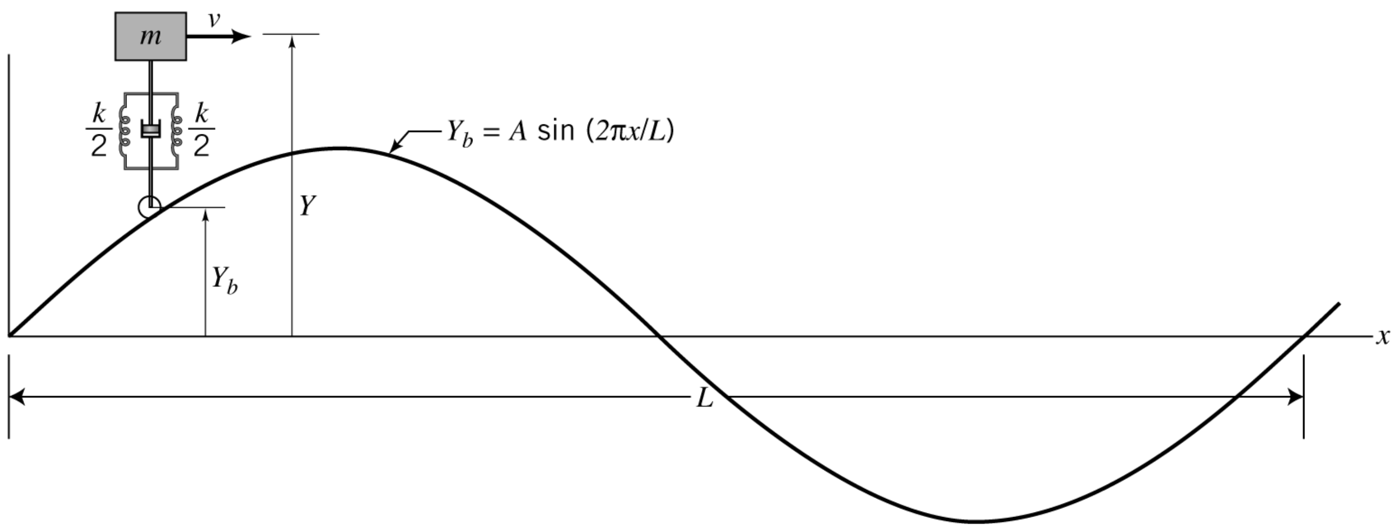


Figure XP3.9. Vehicle moving horizontally along a path defined by $Y_b = A\sin(2\pi x/L)$, $A = .03\,m$, $L = .6\,m$.

The vehicle has a constant velocity $v = \dot{x} = 100\ km/hr$. For convenience, we will assume that it starts at $x = 0$; hence,

$$x = x_o + \dot{x}t = 0 + 100\left(\frac{km}{hr}\right) \times \left(\frac{1\,hr}{3600\,\mathrm{sec}}\right) \times \frac{1000\,m}{1\,km}\,t = 27.8\,t\ m\ ,$$

and, substituting into $Y_b = A\sin(2\pi x/L) = A\sin(2\pi/L)vt$,

$$Y_b(t) = .03\sin\left(\frac{2\times\pi\times 27.8\,t}{.6}\right) = .03\sin(291.\,t)\,m\ . \qquad \text{(ii)}$$

From Eq.(ii), as long as the vehicle's tires remain in contact with the road surface, the vehicle's steady velocity to the right will generate base excitation at the frequency $\omega = 291.\,rad/\sec$, $f = 46.3\ hz$ and amplitude .03 *m*. Driving faster will increase the excitation frequency; driving slower will decrease it.

Tests show that the vehicle's damped natural frequency is 2.62 Hz, and the damping factor is $\zeta = .3$. *Carry out the following engineering-analysis tasks:*

a. Determine the amplitude and phase of vehicle motion for $100\,km/hr$.

b. Determine the speed for which the response is a maximum and determine the response amplitude at this speed.

Solution. For *Task a*, First, the undamped natural frequency is defined by $f_n = f_d/\sqrt{1-\zeta^2} = 2.62/\sqrt{1-.3^2} = 2.5\,Hz$. the frequency ratio is $r = \omega/\omega_n = 46.3/2.5 = 18.5$. Hence, from Eq.(3.44),

$$\frac{Y_{op}}{A} = \frac{[1+4(.3^2)(18.5)^2]^{1/2}}{[\ (1\ -\ 18.5^2)^2\ +\ 4(.3^2)(18.5)^2]^{1/2}]^{1/2}} = .0326 \tag{i}$$

$$\therefore\ Y_{op} = .0326(.03) = 9.77\times 10^{-4}\ m = .977\ mm.$$

From Eq.(3.45), the phase of *m*'s motion with respect to the base excitation is

$$\begin{aligned}\varphi+\psi &= \tan^{-1}[\frac{-2(.3)18.5^3}{1+4(.3^2)-18.5^2}] \\ &= \tan^{-1}(\frac{-3813}{-342}) = -95.1\ degrees\ ,\end{aligned} \tag{ii}$$

which concludes *Task a*.

The response amplitude will be a maximum when $r = r_{\max}$ as defined in Eq.(3.46), i.e.,

$$\begin{aligned} r_{\max} &= \frac{1}{2\zeta}[(1+8\zeta^2)^{1/2}-1]^{1/2} \\ &= \frac{1}{2\times .3}\{[(1+8(.3)^2]^{1/2}-1\}^{1/2} = 0.930\ . \end{aligned}$$

Hence,

$$\omega = \frac{2\pi \times v(m/\sec)}{.6m} = 0.930 \times \omega_n$$

$$= 0.930 \times 2.5 \frac{cycle}{\sec} \times \frac{2\pi\, rad}{1\, cycle} = 14.6 \frac{rad}{\sec}$$

$$\therefore\ v = \frac{.6(14.6)}{2\pi} = 1.40 \frac{m}{\sec} = 1.40 \times \left(\frac{100\, km/hr}{27.8\, m/\sec}\right)$$

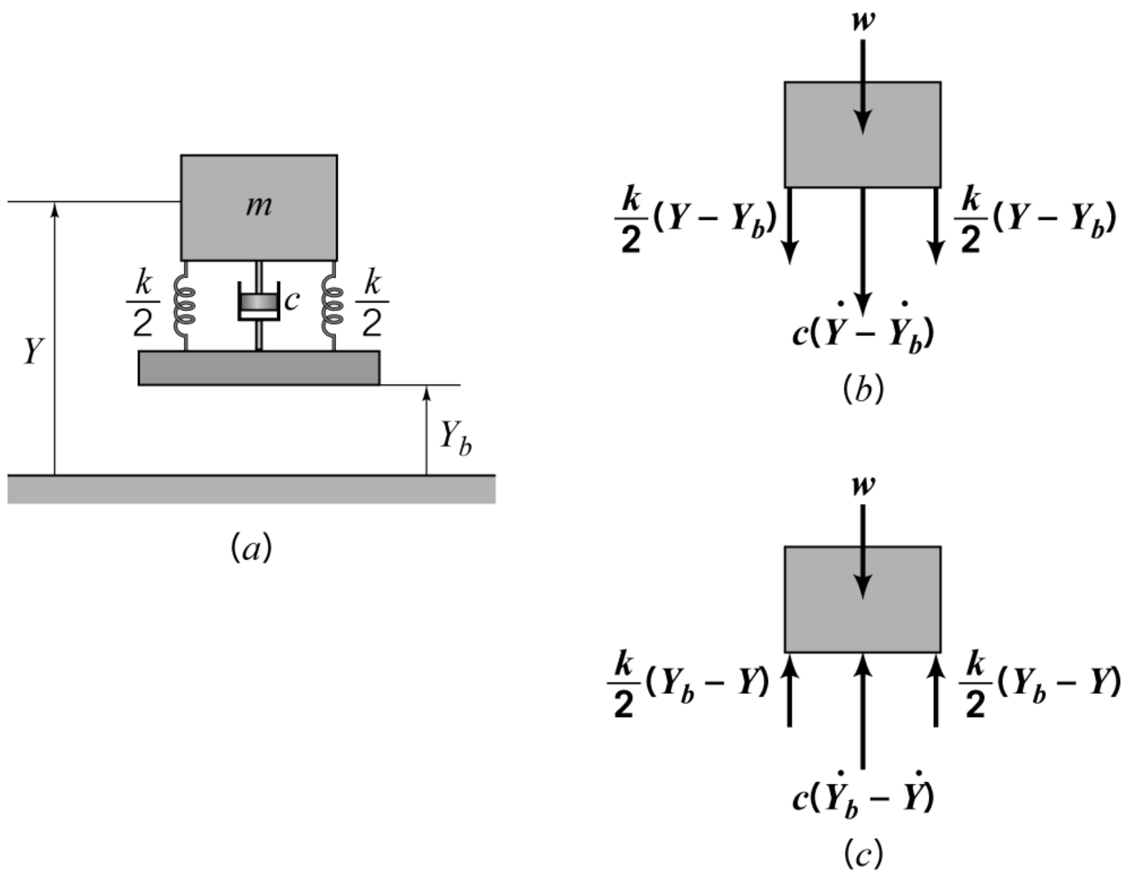
$$= 5.04\ \ km/hr.$$

From Eq.(3.45), at this speed, the steady-state amplification factor is

$$\frac{Y_{op}}{A} = \frac{[1 + 4(.3^2).930^2]^{1/2}}{[\ (1 - 0.930^2)^2 + 4(.3^2).930^2]^{1/2}} = 1.99\ ,$$

$$\therefore\ Y_{op} = .03(1.99) = .060\, m\ .$$

Steady State Amplitude for <u>Relative</u> Deflection with Harmonic Base Excitation



The EOM,

$$m\ddot{Y} + c\dot{Y} + kY = -w + kY_b + c\dot{Y}_b \ .$$

can be written

$$m(\ddot{Y} - \ddot{Y}_B) + c(\dot{Y} - \dot{Y}_B) + k(Y - Y_B) = -w - m\ddot{Y}_B \ \ or$$

$$m\ddot{\delta} + c\dot{\delta} + k\delta = -w - m\ddot{Y}_B \ .$$

Hence, for harmonic base excitation defined by $Y_B = A\sin\omega t$, the

EOM is

$$m\ddot{\delta} + c\dot{\delta} + k\delta = mA\omega^2 \sin\omega t \ . \qquad \textbf{(i)}$$

We dropped the weight term $-w$ in arriving at this equation, which is equivalent to looking at disturbed motion about the equilibrium position. We want a steady-state solution to Eq.(i) of the form, $\delta = \Delta \sin(\omega t + \beta)$. Eq.(i) has the same form as

$$m\,\ddot{Y} + c\,\dot{Y} + k\,Y = f_o \sin\omega t \ . \qquad \textbf{(3.32)}$$

except $mA\omega^2$ has replaced f_0. Hence, by comparison to

$$\frac{Y_{op}}{f_o/k} = \frac{1}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}} = H(r) \ ; \quad r = \omega/\omega_n \ , \qquad \textbf{(3.38)}$$

the steady-state relative amplitude due to harmonic base excitation is

$$\frac{\Delta}{mA\omega^2/k} = \frac{\Delta}{A(\frac{\omega}{\omega_n})^2} = \frac{\Delta}{Ar^2} = H(r) \ ; \quad r = \omega/\omega_n$$

$$\therefore \quad \frac{\Delta}{A} = r^2 H(r) = \frac{r^2}{\left[\,(1 - r^2)^2 + 4\zeta^2 r^2\right]^{1/2}} = J(r) \ .$$

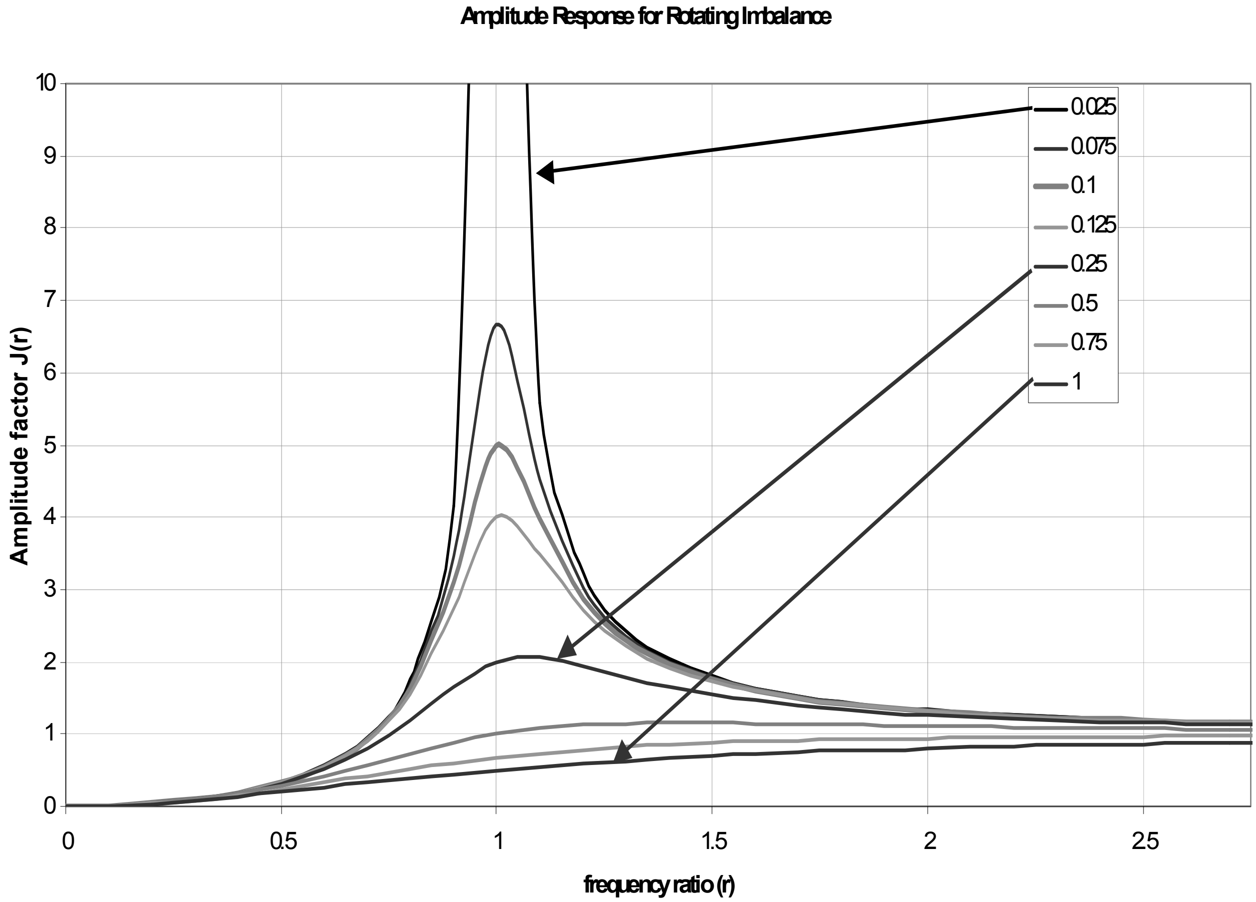


Figure 3.18. $J(r)$ versus frequency ratio $r = \omega / \omega_n$ from Eq.(3.49) for a range of damping ratio values.

Example Problem 3.9 Revisited. Solve for the steady-state <u>relative</u> amplitude at $v = 100\ km/hr$ as

$$\frac{\Delta}{A} = \frac{18.5^2}{[\ (1\ -\ 18.5^2)^2\ +\ 4(.3^2)(18.5)^2\ ]^{1/2}} = 1.002$$

$$\therefore\ \Delta = .03 \times 1.002 = .030\ m = 30.\ mm.$$

This value is much greater than the absolute amplitude of motion

$Y_{op} = 0.977\,mm$ we calculated earlier. This result shows that while the vehicle has small absolute vibration amplitudes, its base is following the ground contour; hence, the relative deflection (across the spring and damper) is approximately equal to the amplitude of the base oscillation.

Example Problem 3.10. An instrument package is to be attached to the housing of a rotating machine. Measurements on the casing show a vibration at 3600 rpm with an acceleration level of .25 *g*. The instrument package has a mass of .5 kg. Tests show that the support bracket to be used in attaching the package to the vibrating structure has a stiffness of 10^5 *N/m*. How much damping is needed to keep the instrument package vibration levels below .5 g?

Solution. The frequency of excitation at 3600 *rpm* converts to $\omega = 3600 rev/\min \times (1 \min/60 \sec) \times (2\pi rad/rev) = 377. rad/\sec$. With harmonic motion, the housing's amplitude of motion is related to its acceleration by $a = .25\,g = -\omega^2 A$; hence, the amplitude corresponding to the housing acceleration levels of .25 *g* is

$$A = .25\,(9.81\,m/\sec^2)/377.^2 = 1.726 \times 10^{-5}\,m = .017\,mm\ \ .$$

Similarly, an 0.5 g acceleration level for the instrument package means its steady-state amplitude is

$$Y_{op} = .5(9.81\,m/\text{sec}^2)/377.^2 = 1.726\times10^{-5}m = .034\,mm \ .$$

Hence, the target amplification factor is $G = 0.034/0.017 = 2$.

The natural frequency of the instrument package is

$$\omega_n = \sqrt{k/m} = \sqrt{10^5 N/.5\,kg} = 447.2\,rad/\text{sec} \ .$$

Hence, the frequency ratio is $r = \omega/\omega_n = 377./477. = .79$.

Plugging $r = .79$ into Eq.(3.44) gives

$$\frac{Y_{op}}{A} = 2 = \frac{[1 + 4\zeta^2 .79^2]^{1/2}}{[\ (1\ -\ .79^2)^2\ +\ 4\zeta^2 .79^2]^{1/2}} \Rightarrow 4 = \frac{1. + 2.496\zeta^2}{.1413 + 2.496\zeta^2}$$

The solution to this equation is $\zeta^2 = .058 \ \Rightarrow \ \zeta = .241$. Hence, the required damping is

$$c = 2\zeta\omega_n m = 2(.241)(447.2\frac{rad}{\text{sec}})(.5\,kg) = 215.5\,N\text{sec}/m \ ,$$

which concludes the engineering-analysis task. Note that the derived units for the Newton are $kg\,m/\text{sec}^2$; hence, kg/sec nets $N\text{sec}/m$. Note that, since only one excitation frequency is involved, we could have simply taken the ratio of the acceleration levels directly to get $G = 2$.