Lecture 5. NEWTON'S LAWS OF MOTION

Newton's Laws of Motion

Law 1. Unless a force is applied to a particle it will either remain at rest or continue to move in a straight line at constant velocity,

Law 2. The acceleration of a particle in an inertial reference frame is proportional to the force acting on the particle, and

Law 3. For every action (force), there is an equal and opposite reaction (force).

Newton's second Law of Motion is a Second-Order Differential Equation

$$\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$$

where $\Sigma \boldsymbol{f}$ is the resultant force acting on the particle, $\ddot{\boldsymbol{r}}$ is the particle's acceleration with respect to an inertial coordinate system, and m is the particle's mass.

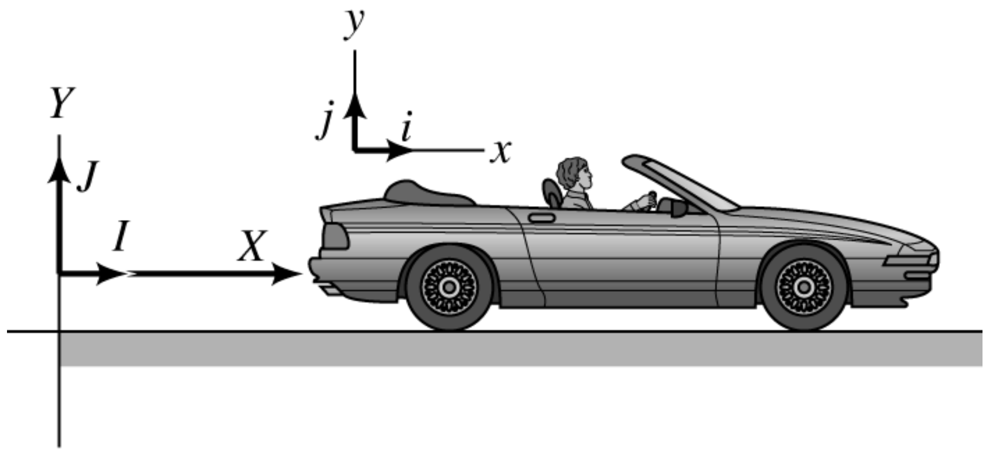


Figure 3.1 A car located in the *X, Y* (inertial) system by ***IX*** and a particle located in the *x, y* system by ***ix***.

A particle in the car can be located by

$$x_p = X + x$$

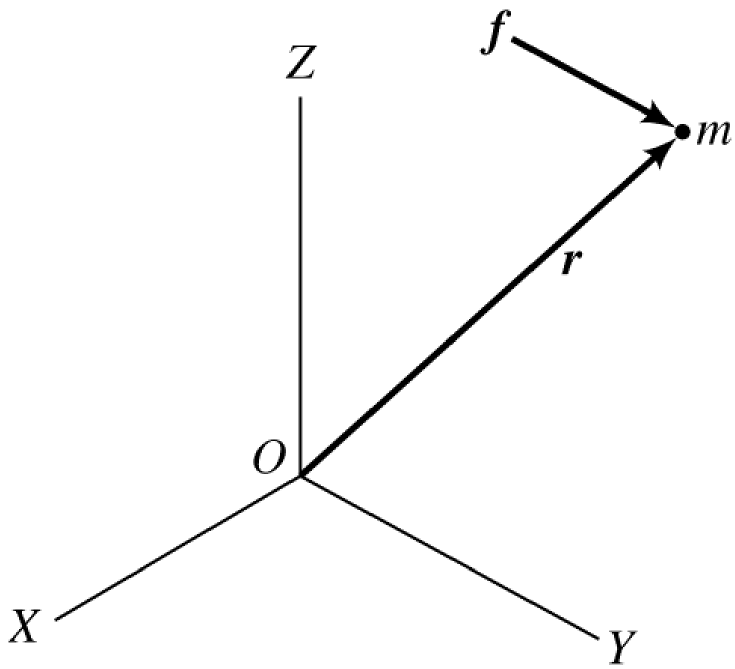
The equation of motion,

$$\Sigma f_X = m\ddot{x}_p = m(\ddot{X} + \ddot{x}),$$

is correct. However,

$$\Sigma f_X \neq m\ddot{x}\ ,$$

because the *x, y* system is not inertial.



X, Y,Z Inertial coordinate system

3-D Version of Newton's 2nd Law of Motion

$$f_X = m\,a_X = m\ddot{X} = m\frac{d}{dX}\left(\frac{\dot{X}^2}{2}\right)$$

$$f_Y = m\,a_Y = m\ddot{Y} = m\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right)$$

$$f_Z = m\,a_Z = m\ddot{Z} = m\frac{d}{dZ}\left(\frac{\dot{Z}^2}{2}\right) \; .$$

Using the Energy-integral substitution

$$\ddot{X} = \frac{d\dot{X}}{dt} = \frac{d\dot{X}}{dX}\frac{dX}{dt} = \dot{X}\frac{d\dot{X}}{dX} = \frac{d}{dX}\left(\frac{\dot{X}^2}{2}\right).$$

Multiplying the equations by dX, dY, dZ , respectively, and adding gives

$$f_X dX + f_Y dY + f_Z dZ = \frac{m}{2} d[(\dot{X}^2) + (\dot{Y}^2) + (\dot{Z}^2)]$$

$$d\,Work = d(\frac{mv^2}{2}) = dT \quad .$$

This "work-energy" equation is an integrated form of Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$. The expressions are fully equivalent and are not independent.

The central task of dynamics is deriving equations of motion for particles and rigid bodies using either Newton's second law of motion or the work-energy equation.

Constant Acceleration: Free-Fall of a Particle Without Drag

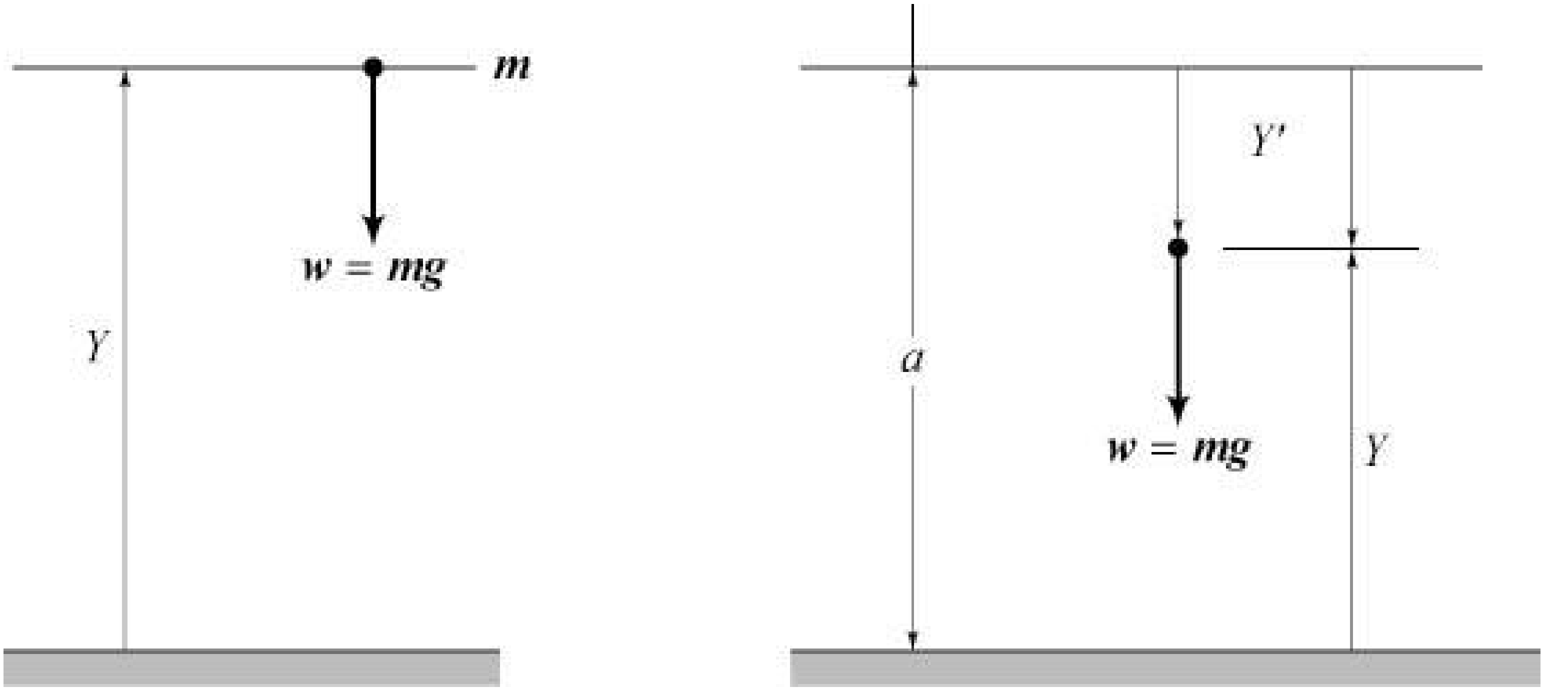


Figure 3.3 Particle acted on by its weight and located by Y (left) and Y' (right)

Equation of Motion using Y

$$\Sigma f_Y = -w = -mg = m\ddot{Y} \Rightarrow \ddot{Y} = -g \; .$$

Let us consider rewriting Newton's law in terms of the new coordinate Y'. Note that $Y' + Y = a \Rightarrow \dot{Y}' + \dot{Y} = 0 \Rightarrow \ddot{Y}' + \ddot{Y} = 0$. From the free-body diagram, the equation of motion using Y' is

$$\Sigma f_{Y'} = w = mg = m\ddot{Y}' \Rightarrow \ddot{Y}' = g \; .$$

This equation conveys the same physical message as the differential equation for Y; namely, the particle has a constant acceleration downwards of g, the acceleration of gravity.

Time Solution for the D. Eq. of motion for *Y*,

$$\frac{d^2 Y}{dt^2} = -g \ .$$

Integrating once with respect to time gives

$$\frac{dY}{dt} = \dot{Y}(t) = \dot{Y}_0 - g\,t \ ,$$

where $\dot{Y}_0 = \dot{Y}(0)$ is the initial (time $t = 0$) velocity. Integrating a second time w.r.t. time gives

$$Y(t) = Y_0 + \dot{Y}_0\,t - \frac{g\,t^2}{2} \ ,$$

where $Y_0 = Y(0)$ is the initial (time $t = 0$) position. Y_0 and $\dot{Y}_0$ are the "initial conditions."

The solution can also be developed more formally via the following steps:

a. Solve the homogeneous equation $\ddot{Y}_h = 0$ (obtained by setting the right-hand side to zero) with the solution as

$$Y_h = A + Bt \ .$$

b. Determine a particular solution to the original equation $d^2Y/dt^2 = -g$ that satisfies the right-hand side. By inspection, the right-hand side is satisfied by the particular solution $Y_p = Ct^2 \Rightarrow \dot{Y}_p = 2Ct \Rightarrow \ddot{Y}_p = 2C$. Substituting this result nets $\ddot{Y}_p = 2C = -g \Rightarrow C = -g/2$, and

$$Y_p = -\frac{gt^2}{2} \ .$$

The complete solution is the sum of the particular and homogeneous solution as

$$Y = Y_h + Y_p = A + Bt - \frac{gt^2}{2} \ .$$

The constants *A* and *B* are solved in terms of the initial conditions starting with

$$Y(0) = Y_0 = A \Rightarrow A = Y_0 \ .$$

Continuing, $\dot{Y} = B - gt$ netting

$$\dot{Y}(0) = \dot{Y}_0 = B \;\Rightarrow\; B = \dot{Y}_0 \ ,$$

and the complete solution — satisfying the initial conditions — is

$$Y(t) \;=\; Y_0 \;+\; \dot{Y}_0\, t \;-\; \frac{g\, t^2}{2} \ ,$$

which duplicates our original results.

Engineering Analysis Task: *If the particle is released from rest* ($\dot{Y}(0) = 0$) *at* $Y(0) = H$, *how fast will it be going when it hits the ground* ($Y = 0$)*?*

Solution a. When the particle hits the ground at time $\bar{t}$,

$$\dot{Y}(\bar{t}) \;=\; \dot{Y}_0 \;-g\,\bar{t} \;=\; -\; g\,\bar{t}$$

$$Y(\bar{t}) \;=\; H \;-\; \frac{g\,\bar{t}^{\,2}}{2} = 0 \ .$$

Solving for $\bar{t}$,

$$\bar{t} = \sqrt{\frac{2\,H}{g}} \ .$$

Solving for $\dot{Y}(\bar{t})$,

$$\dot{Y}(\bar{t}) = -g\sqrt{\frac{2H}{g}} = -\sqrt{2gH} \ .$$

Solution b. Using the energy-integral substitution,

$$\ddot{Y} = \frac{d\dot{Y}}{dt} = \frac{d\dot{Y}}{dY}\frac{dY}{dt} = \dot{Y}\frac{d\dot{Y}}{dY} = \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) \ ,$$

changes the differential equation $m\ddot{Y} = -w$ to

$$m\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = -w = -mg \ .$$

Multiplying through by dY and integrating gives

$$\frac{m\dot{Y}^2}{2} - \frac{m\dot{Y}_0^2}{2} = -mg\int_H^Y dy = mg(H-Y) \ .$$

Since $\dot{Y}_0 = 0$, $\dot{Y}(Y=0)$ is

$$\dot{Y}(Y=0) = \sqrt{2gH}$$

Solution c. There is no energy dissipation; hence, we can work directly from the conservation of mechanical energy equation,

$$T + V = T_0 + V_0,$$

where $T = m\dot{Y}^2/2$ is the kinetic energy. The potential energy of the particle is its weight w times the vertical distance *above* a horizontal datum. Choosing ground as datum gives $V = wY$ and

$$m\frac{\dot{Y}^2}{2} + wY = 0 + wH \;\Rightarrow\; \dot{Y}(Y=0) = \sqrt{2gH}$$

The weight is a conservative force, and in the differential equation is $-w$, pointing in the $-Y$ direction. Strictly speaking, a conservative force is defined as a force that is the negative of its gradient with respect to a potential function V. For this simple example, with $V = wY$

$$-\frac{dV}{dY} = -\frac{d(wY)}{dY} = -w \;.$$

Hence (as noted above) the potential energy function for gravity is the weight times the distance above a datum plane; i.e.,

$$V_g = wY \;.$$

Acceleration as a Function of Displacement: Spring Forces

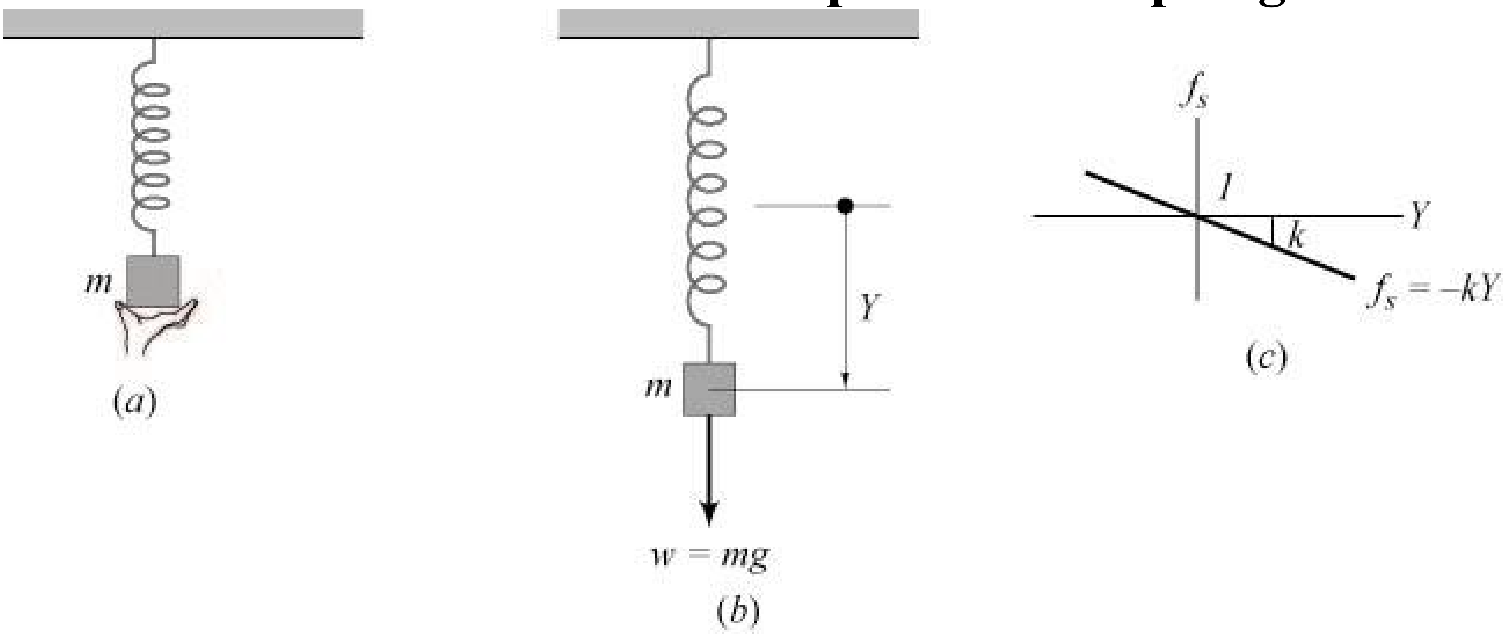


Figure 3.5 (a) Particle being held prior to release, (b) Y coordinate defining m's position below the release point, (c) Spring reaction force $f_s = -kY$

Particle suspended by a spring and acted on by its weight. The spring is undeflected at $Y = 0$; i.e., $Y = 0 \Rightarrow f_s = 0$ (zero spring force). The spring force $f_s = -kY$ has a sign that is the opposite from the displacement Y, and it acts to restore the particle to the position $Y = 0$.

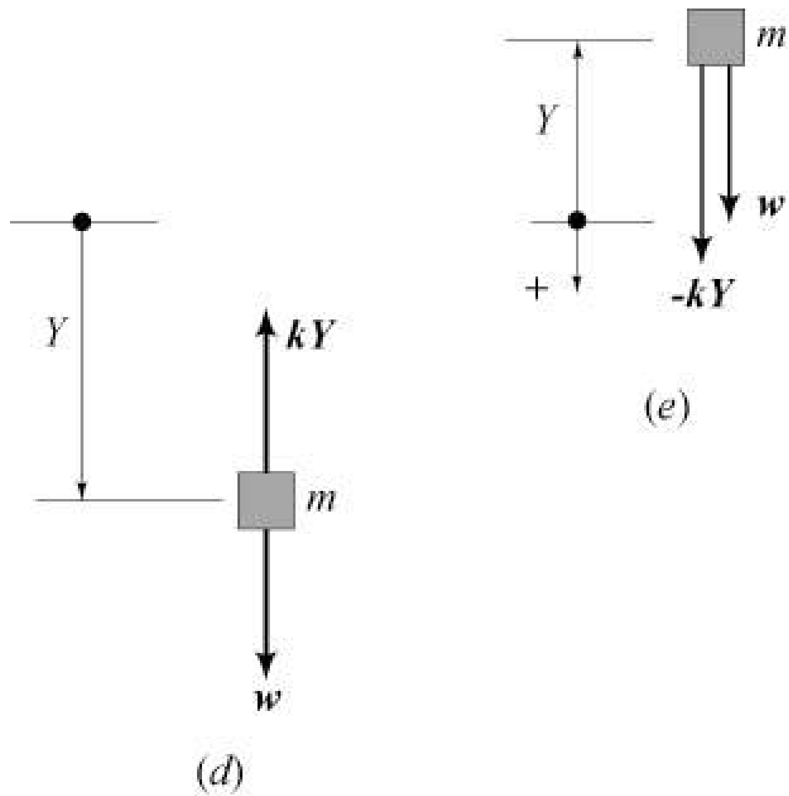


Figure 3.5 (d) Free-body diagram for $Y>0$, (e) Free-body diagram for $Y<0$

From the free-body diagram of figure 3.5d, Newton's second law of motion gives the differential equation of motion,

$$\Sigma f_Y = m\ddot{Y} = w - kY \Rightarrow m\ddot{Y} + kY = w \; . \qquad \textbf{(3.13)}$$

Figure 3.5d-e shows that $f_s = -kY$ for $Y>0$ and $Y<0$

Deriving the Equation of Motion for Motion about Equilibrium

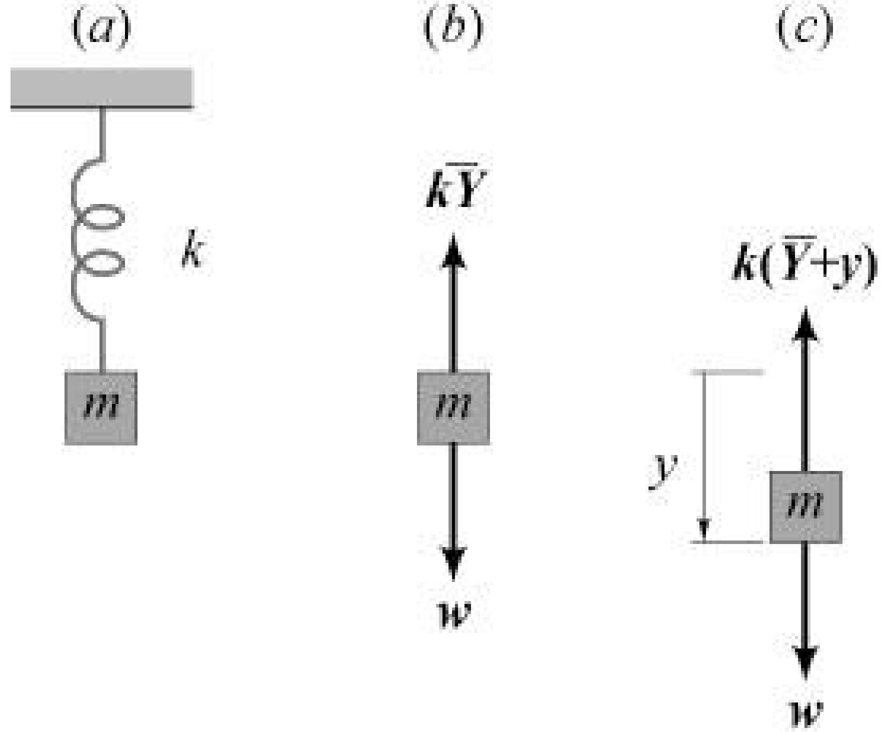


Figure 3.5 (a) Mass *m* in equilibrium, (b) Equilibrium free-body diagram, (c) General-position free-body diagram

Figure 3.5c applies for *m* displaced the distance *y* below the equilibrium point. The additional spring displacement generates the spring reaction force $f_s = k(\bar{Y} + y)$, and yields the following equation of motion.

$$m\ddot{y} = \sum f_Y = w - k(\bar{Y} + y) = w - k\bar{Y} - ky = -ky$$

$$\therefore \; m\ddot{y} + ky = 0 \; . \qquad \textbf{(3.14)}$$

This result holds for a *linear* spring and shows that w is eliminated, leaving only the perturbed spring force ky.

MORE EQUILIBRIUM, 1 MASS - 2 SPRINGS

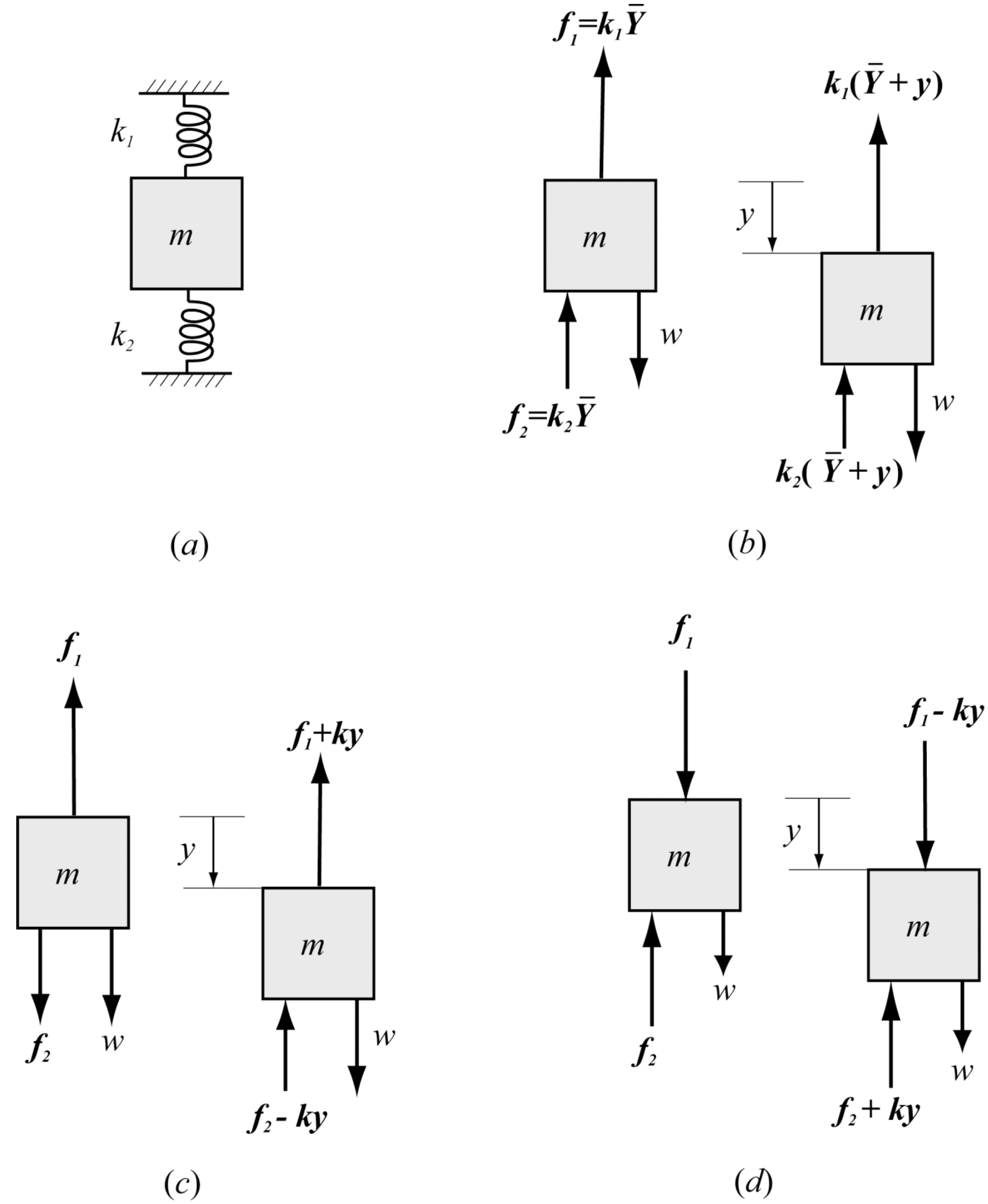


Figure 3.6 (a) Equilibrium, (b) Equilibrium with spring 1 in tension and spring 2 in compression, (c) Equilibrium with both springs in tension, (d) Equilibrium with both springs in compression

Spring 1 in tension, spring 2 in compression
Figure 3.6b, equilibrium starting from undeflected springs:
$w = f_1 + f_2 = (k_1 + k_2)\bar{y}$

$$m\ddot{y} = \sum f_y = w - k_1(\bar{y} + y) - k_2(\bar{y} + y) = w - (k_1 + k_2)\bar{y} + (k_1 + k_2)y$$
$$= 0 - (k_1 + k_2)y$$
$$\therefore\ m\ddot{y} + (k_1 + k_2)y = 0\ .$$

Figure 3.6c, Equilibrium with both springs in tension, $w = f_1 - f_2$

$$m\ddot{y} = \sum f_y = w - (f_1 + k_1 y) + (f_2 - k_2 y) = (w - f_1 + f_2) - (k_1 + k_2)y$$
$$= 0 - (k_1 + k_2)y$$

$$\therefore\ m\ddot{y} + (k_1 + k_2)y = 0\ .$$

Figure 3.6c, Equilibrium with both springs in compression, $w = f_2 - f_1$

$$m\ddot{y} = \sum f_y = w + (f_1 - k_1 y) - (f_2 + k_2 y) = (w + f_1 - f_2) + (k_1 + k_2)y$$
$$= 0 - (k_1 + k_2)y$$
$$\therefore\ m\ddot{y} + (k_1 + k_2)y = 0\ .$$

CONCLUDE: For linear springs, the weight drops out of the equation of motion.

Natural Frequency Definition and calculation

Dividing $m\ddot{Y} + kY = w$ by m gives

$$\ddot{Y} + \omega_n^2 \, Y = g \; ,$$

where

$$\omega_n = \sqrt{\frac{k}{m}} \; ,$$

and ω_n is the undamped natural frequency.

Students frequently have trouble in getting the dimensions correct in calculating the undamped natural frequency. Starting with the ft-lb-sec system, *k* has the units lb/ft. The mass has the derived units of slugs. From

$$w = mg \;\Rightarrow\; m(slugs) = w(lbs)/g(ft/\sec^2) = \frac{w}{g}\frac{(lb-\sec^2)}{ft} ,$$

The dimensions for ω_n are

$$\omega_n = \sqrt{\frac{k(\frac{lb}{ft})}{\frac{m(lb-\sec^2)}{ft}}} = \sqrt{\frac{k}{m}\frac{rad}{\sec}} = \sqrt{\frac{k}{m}}\;\sec^{-1} ,$$

Starting with the SI system using m-kg-sec, the units for k are N/m. We can use $w = mg$ to convert Newtons into $kg-m/\sec^2$. Alternatively, the units for kg from $m = w/g$ are $N-\sec^2/m$, and

$$\omega_n = \sqrt{\frac{k(\frac{N}{meter})}{\frac{m(N-\sec^2)}{meter}}} = \sqrt{\frac{k}{m}}\ \sec^{-1},$$

Note that the <u>correct</u> units for ω_n are rad/sec not cycles/sec. The undamped natural frequency can be given in terms of cycles/sec as

$$f_n = \omega_n \frac{rad}{\sec} \times \frac{1cycle}{2\pi rad} = \frac{\omega_n}{2\pi}\frac{cycles}{\sec} = \frac{\omega_n}{2\pi} Hertz$$

Time Solution From Initial Conditions.
The homogeneous differential equation corresponding to $\ddot{Y} + \omega_n^2 Y = g$ is

$$\ddot{Y}_h + \omega_n^2 Y_h = 0 .$$

Substituting the guessed $Y_h = A \cos pt$ yields

$$(-p^2 + \omega_n^2) A \cos pt = 0 \Rightarrow p = \omega_n$$

Guessing $Y_h = A \sin pt$ yields the same result; hence, the general solution is solution

$$Y_h = A \cos \omega_n t + B \sin \omega_n t \; .$$

The particular solution is

$$Y_p = \frac{g}{\omega_n^2} = \frac{w}{k} \; .$$

Note that this is also the static solution; i.e., $m\ddot{Y} + kY = w \Rightarrow Y_{static} = w/k$. The complete solution is

$$Y = Y_h + Y_p = A \cos \omega_n t + B \sin \omega_n t + \frac{w}{k} \; .$$

For the initial conditions $\dot{Y}(0) = 0, Y(0) = Y_0 = 0$, the constant A *is* obtained as

$$Y(0) = Y_0 = 0 = A + \frac{w}{k} \Rightarrow A = -\frac{w}{k} \; .$$

From

$$\dot{Y} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t ,$$

one obtains

$$\dot{Y}(0) = \dot{Y}_0 = 0 = B\omega_n \Rightarrow B = 0 ,$$

and the complete solution is

$$Y = \frac{w}{k} (1 - \cos \omega_n t) . \qquad \textbf{(3.17)}$$

For arbitrary initial conditions $(Y_0, \dot{Y}_0)$, the solution is

$$Y(t) = Y_0 \cos\omega_n t + \frac{\dot{Y}_0}{\omega_n} \sin\omega_n t + \frac{w}{k}(1 - \cos\omega_n t)$$

Note that the maximum displacement defined by Eq.(3.17) occurs for $\omega_n t = \pi \Rightarrow \cos\omega_n t = -1$, and is defined by

$$Y_{\max} = \frac{w}{k}[1-(-1)] = 2\frac{w}{k} = 2 \times Y_{static} .$$

Sometimes, engineers use 2 as a design factor of safety to account for dynamic loading verus static loading.

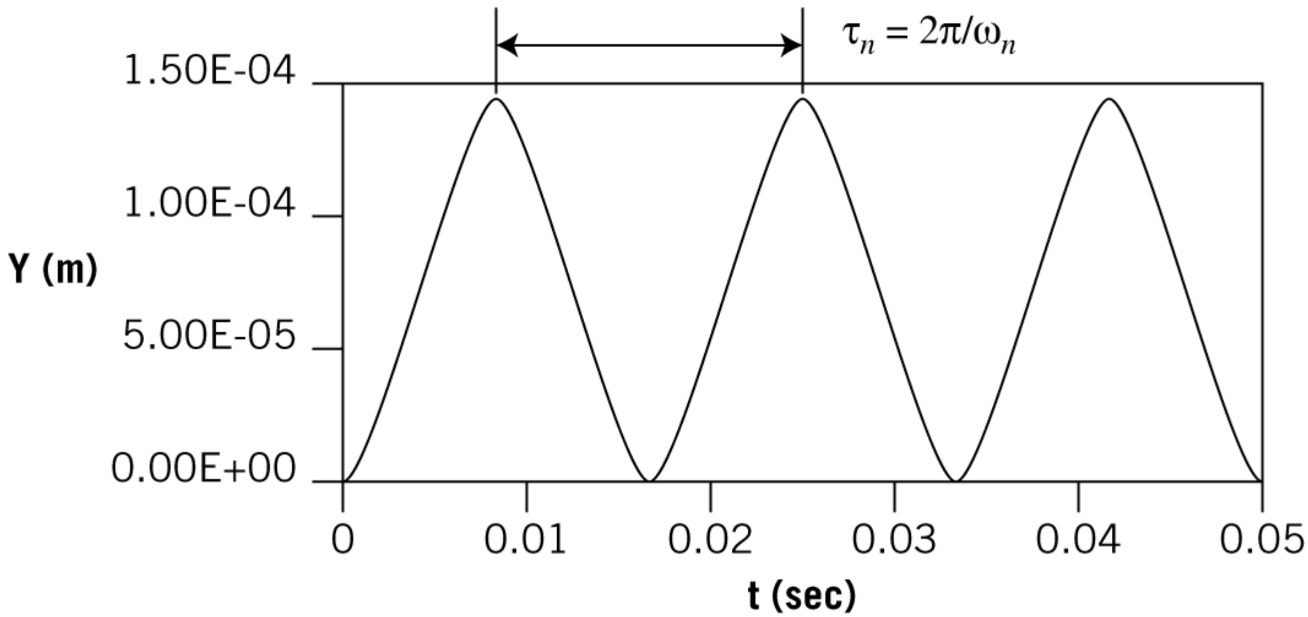


Figure 3.6 Solution $Y(t)$ from Eq.(3.17) with $m = 40\,kg$, $k = 5.685 \times 10^6 N/m$, yielding $\omega_n = 377.rad/\sec \Rightarrow f_n = 60\,Hz$ and $\tau_n = 2\pi/\omega_n = .01667\sec$.

The period for undamped motion is

$$\tau_n = \frac{2\pi}{\omega_n}(\sec)$$

Note that

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi}(\sec^{-1})$$

Energy-Integral Substitution

Substituting,

$$\ddot{Y} = \frac{d\dot{Y}}{dt} = \frac{d\dot{Y}}{dY}\frac{dY}{dt} = \dot{Y}\frac{d\dot{Y}}{dY} = \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right),$$

into the differential equation of motion gives

$$m\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = w - kY .$$

Multiplying through by dY and integrating gives

$$\frac{m\dot{Y}^2}{2} - \frac{m\dot{Y}_0^2}{2} = w\,(Y - Y_0) - \left(\frac{kY^2}{2} - \frac{kY_0^2}{2}\right) .$$

Rearranging gives,

$$\frac{m\dot{Y}^2}{2} - wY + \frac{kY^2}{2} = \frac{m\dot{Y}_0^2}{2} - w\,Y_0 + \frac{kY_0^2}{2},$$

which is the physical statement

$$T + V = T_0 + V_0,$$

where

$$V = V_g + V_s \ , \ V_g = -wY \ , \quad V_s = k\frac{Y^2}{2} \ .$$

The gravity potential-energy function is negative because the coordinate *Y* defines the body's distance *below* the datum.

The potential energy of a linear spring is $V_s = k\delta^2/2$ where δ is *the change in length of the spring from its undeflected position.* Note

$$f_s = -\frac{dV_s}{dY} = -\frac{d}{dY}(k\frac{Y^2}{2}) = -kY \ .$$

Hence, the spring force is the negative derivative of the potential-energy function.

Units

With the notable exception of the United States of America, all engineers use the SI system of units involving the meter, newton, and kilogram, respectively, for length, force, and mass. The metric system, which preceded the SI system, was legalized for commerce by an act of the United States Congress in 1866. The act of 1866 reads in part[1],

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[1] Mechtly E.A. (1969), *The International System of Units*, NASA SP-7012

> It shall be lawful throughout the United States of America to employ the weights and measures of the metric system; and no contract or dealing or pleading in any court, shall be deemed invalid or liable to objection because the weights or measures referred to therein are weights or measures of the metric system.

None the less, in the 21^{st} century, USA engineers, manufacturers, and the general public continue to use the foot and pound as standard units for length and force. Both the SI and US systems use the second as a unit of tim. The US Customary system of units began in England, and continues to be referred to in the United States as the "English System" of units. However, Great Britain adopted and has used the SI system for many years.

Both the USA and SI systems use the same standard symbols for exponents of 10 base units. A partial list of these symbols is provided in Table 1.1 below. Only the "m = -3" (mm = millimeter = 10^{-3}m) and "k = +3" (km = kilometer = 10^{3} m) exponent symbols are used to any great extent in this book.

Table 1.1 Standard Symbols for exponents of 10.

Factor by which unit is multiplied	Prefix	Symbol
10^6	mega	M
10^3	kilo	k
10^2	hecto	h
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ

Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ ties the units of time, length, mass, and force together. In a vacuum, the vertical motion of a mass *m* is defined by

$$w = m\ddot{Y} = mg \quad , \tag{1.3}$$

where *Y* is pointed directly downwards, *w* is the weight force due to gravity, and *g* is the acceleration of gravity. At sea level, the standard acceleration of gravity[2] is

2

$$g = 9.81\, m/\sec^2 = 9810.\, mm/\sec^2$$
$$= 32.2\, ft/\sec^2 = 386.\, in/\sec^2 \quad . \qquad \textbf{(1.4)}$$

We start our discussion of the connection between force and mass with the SI system, since it tends to be more rational (not based on the length of a man's foot or stride). The kilogram (mass) and meter (length) are fundamental units in the SI system, and the Newton (force) is a derived unit. The formal definition of a Newton is ,"that force which gives to a mass of 1 kilogram an acceleration of 1 meter per second per second." From Newton's second law as expressed in Eq.(1.3), 9.81 newtons would be required to accelerate 1 kilogram at the constant acceleration rate of $g = 9.81\, m/\sec^2$, i.e.,

$$9.81\, N = 1\, kg \times 9.81\, m/\sec^2 = 9.81\, kg\; m/\sec^2$$

Hence, the newton has derived dimensions of $kg\; m/\sec^2$.

From Eq.(1.3), changing the length unit to the millimeter (mm) while retaining the kg as the mass unit gives $mg = m(kg) \times 9810\, mm/\sec^2$, which would imply a thousand fold increase in the weight force; however, 1 newton is still

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From the universal law of gravitation provided by Eq.(1.1), the acceleration of gravity varies with altitude; however, the standard value for g in Eq.(1.4) is used for most engineering analysis.

required to accelerate 1 kg at $g = 9.81\, m/\sec^2 = 9810\, mm/\sec^2$, and

$$m(kg) \times 9810\, mm/\sec^2 = w(10^{-3} N) = w(mN) \ .$$

Hence, for a kg-mm-second system of units, the derived force unit is
10^{-3} newton = 1 mN (1 milli newton).

Another view of units and dimensions is provided by the undamped-natural-frequency definition $\omega_n = \sqrt{K/M}$ of a mass M supported by a linear spring with spring coefficient K . Perturbing the mass from its equilibrium position causes harmonic motion at the frequency ω_n, and ω_n's dimension is $radians/second$, or $\sec^{-1}$ (since the radian is dimensionless) . Using kg-meter-second system for length, mass, and time, the dimensions for ω_n^2 follow from

$$\omega_n^2 = \frac{K(N/m)}{M(kg)} = K\left(\frac{kg\, m}{\sec^2} \times \frac{1}{m}\right) \times \frac{1}{M(kg)} = \frac{K}{M}(\sec^{-2}) \ , \quad \textbf{(1.5)}$$

confirming the expected dimensions.

Shifting to mm for the length unit while continuing to use the newton as the force unit would change the dimensions of the stiffness coefficient K to N/mm and reduce K by a factor of 1000. Specifically, the force required to deflect the spring 1 mm

should be smaller by a factor of 1000 than the force required to displace the same spring
1 m = 1000 mm. However, substituting *K* with dimensions of N/mm into Eq.(1.5), while retaining *M* in kg would cause an decrease in the undamped natural frequency by a factor of $\sqrt{1000}$. Obviously, changing the units should not change the undamped natural frequency; hence, this proposed dimensional set is wrong. The correct answer follows from using *mN* as the derived unit for force. This choice gives the dimensions of *mN* */mm* for *K*, and leaves both *K* and ω_n unchanged. To confirm that *K* is unchanged (numerically) by this choice of units, suppose $K = 1000\,N/m = 1\,N/mm = 1000\,mN/mm$. The reaction force produced by the deflection $\delta = 1\,mm = 10^{-3}\,m$ is

$$f_s = K\delta = 1000\frac{N}{m} \times 10^{-3}\,m = 1\,N$$

$$= 1\,\frac{N}{mm} \times 1mm = 1\,N$$

$$= 1000\frac{mN}{mm} \times 1\,mm = 10^3\,mN = 1\,N\ ,$$

confirming that *mN* is the appropriate derived force unit for a *kg-mm-sec* unit system.

Table 1.2 SI base and derived units

Base mass unit	Base time unit	Base length unit	Derived force unit	Derived force unit dimensions
kilogram (kg)	second (sec)	meter (m)	Newton (N)	kg m / sec^2
kilogram (kg)	second (sec)	millimeter (mm)	milli Newton (mN)	kg mm / sec^2

Shifting to the US Customary unit system, the unit of pounds for force is the base unit, and the mass dimensions are to be derived. Applying Eq.(1.3) to this situation gives

$$w = mg \;\Rightarrow\; m = \frac{w(lb)}{32.2 ft/\sec^2} = \frac{w}{32.2}\,\frac{lb\sec^2}{ft} \;.$$

The derived mass unit has dimensions of $lb\sec^2/ft$ and is called a “slug.” When acted on by a resultant force of 1 lb, a mass of one slug will accelerate at $1 ft/\sec^2$. Alternatively, under standard conditions, a mass of one slug weighs 32.2 lbs.

If the inch-pound-second unit system is used for displacement, force, and time, respectively, Eq.(1.3) gives

$$w = mg \;\Rightarrow\; m = \frac{w(lb)}{386.\, in/\sec^2} = \frac{w}{386.}\,\frac{lb\sec^2}{in} \;,$$

and the mass has derived dimensions of $lb\sec^2/in$. Within the author's 1960's aerospace employer, a mass weighing one pound with the derived units of $lb\sec^2/in$ was called a "snail". To the author's knowledge, there is no commonly accepted name for this mass, so *snail* will be used in this discussion. When acted upon by a resultant 1 lb force, a mass of 1 snail will accelerate at 386. in/sec^2, and under standard conditions, a snail weighs 386. lbs.

Returning to the undamped natural frequency discussion, from Eq.(1.5) for a pound-ft-sec system,

$$\omega_n^2 = \frac{K(lb/ft)}{M(\frac{lb\sec^2}{ft})} = \frac{K}{M}(\sec^{-2}) \ .$$

Switching to the inch-pound-second unit system gives

$$\omega_n^2 = \frac{K(lb/in)}{M(\frac{lb\sec^2}{in})} = \frac{K}{M}(\sec^{-2}) \ .$$

Table 1.3 US Customary base and derived units

Base force unit	Base time unit	Base length unit	Derived mass unit	Derived mass unit dimensions
pound (lb)	second (sec)	foot (ft)	slug	lb sec^2/ ft
pound (lb)	second (sec)	inch (in)	snail	lb sec^2/ in

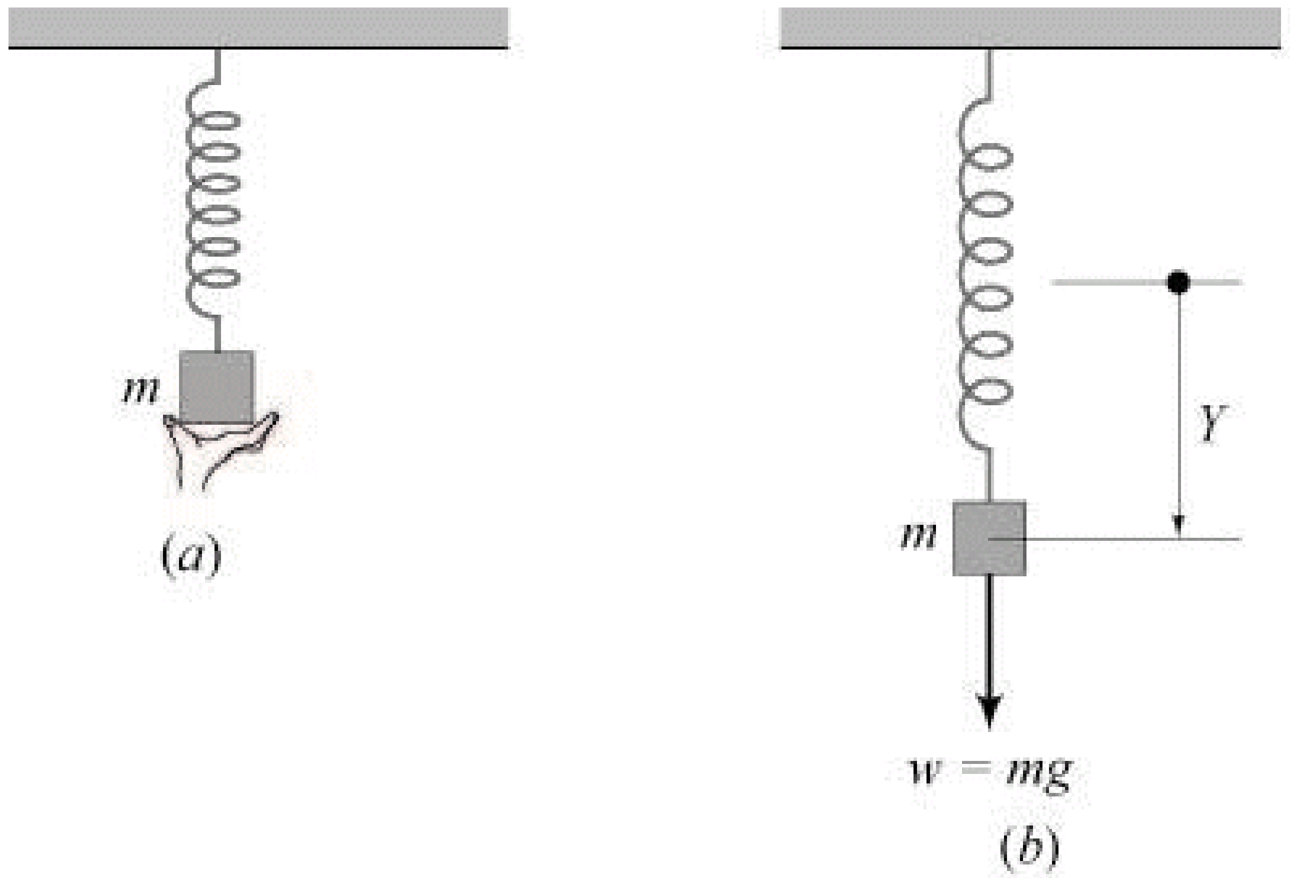
Many American students and engineers use the pound mass (lbm) unit in thermodynamics, fluid mechanics, and heat transfer. A one pound mass weighs one pound under standard conditions. However, the lbm unit makes no sense in dynamics. Inserting $m = 1\,lbm$ into $w = mg$ with $g = 32.2\,ft/\sec^2$ would require a new force unit equal to 32.2 lbs, called the *Poundal*. *Stated briefly, unless you are prepared to use the poundal force unit, the pound-mass unit should never be used in dynamics.*

From one viewpoint, the US system makes more sense than the SI system in that a US-system scale states your weight (the force of gravity) in pounds, a force unit. A scale in SI units reports your weight in kilos (kilograms), the SI mass unit, rather than in Newtons, the SI force unit. Useful (and exact) conversion factors between the SI and USA systems are: 1 lbm = .4535 923 7* kg, 1 in = .0254* m, 1 ft = .3048* m, 1 lb = 4.448 221 615 260 5* N. The * in these definitions denotes internationally-agreed-upon *exact* conversion factors.

Conversions between SI and US Customary unit systems should be checked carefully. An article in the 4 October 1999 issue of *Aviation Week and Space Technology* states," Engineers have discovered that use of English instead of metric units in a navigation software table contributed to, if not caused, the loss of Mars Climate Orbiter during orbit injection on Sept. 23." This press report covers a highly visible and public failure; however, less spectacular mistakes are regularly made in unit conversions.

Lecture 6. MORE VIBRATIONS

Deriving the Equation of Motion, Starting From an Energy Equation



Assume that the spring is undeflected for $Y=0$ and the gravity potential-energy datum is also at $Y=0$. Starting with,

$$T+V=T_0+V_0,$$

we can state

$$\frac{m\,\dot{Y}^2}{2} - w\,Y + \frac{k\,Y^2}{2} \;=\; \frac{m\,\dot{Y}_0^2}{2} - w\;\;Y_0 + \frac{k\,Y_0^2}{2}\;\;.$$

The negative sign applies for the gravity potential energy because Y is below the datum. Differentiating with respect to Y gives

$$\frac{d}{dY}(\frac{m\,\dot{Y}^2}{2}) - w + \frac{d}{dY}(\frac{k\,Y^2}{2}) = \;0\;\Rightarrow\quad m\,\ddot{Y} = w - k\,Y\;.$$

In many cases, the differential equation of motion is obtained more easily from an energy equation than from Newton's 2nd law.

Equilibrium Conditions. Equilibrium for the particle governed by the mass-spring differential equation,

$$\Sigma\, f_Y \;=\; m\,\ddot{Y} \;=\; w \;-\; k\,Y\;,$$

occurs for $\ddot{Y}=0$, and defines the equilibrium position

$$0 = w - k\,\overline{Y} \;\Rightarrow\quad \overline{Y} = \frac{w}{k}\;\;.$$

We looked at motion about the equilibrium position by defining $Y = \bar{Y} + y \Rightarrow \ddot{Y} = \ddot{y}$. Substituting these results into the differential equation of motion gives the following *perturbed* differential equation of motion

$$m\ddot{y} + k(\bar{Y} + y) = w \Rightarrow m\ddot{y} + ky = 0 ,$$

since $k\bar{Y} = w$. This equation has the particular solution $Y_p = 0$ and the complete solution

$$Y = Y_h + Y_p = A \cos\omega_n t + B \sin\omega_n t + 0 .$$

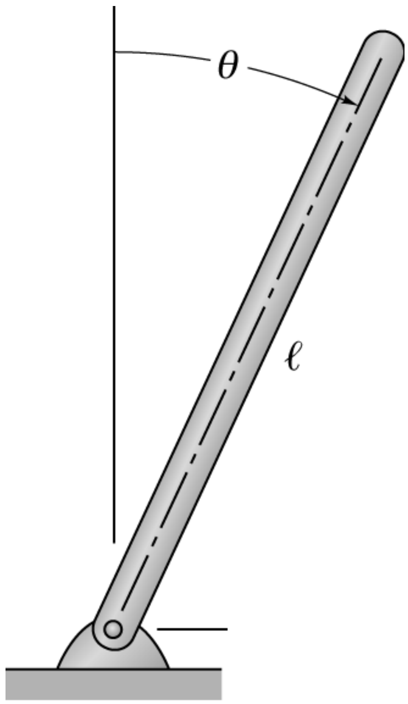
The motion is stable, oscillating about the equilibrium position at the undamped natural frequency $\omega_n = \sqrt{k/m}$; hence, $Y = \bar{Y} = w/k$ defines a *stable* equilibrium position.

For small motion about an unstable static equilibrium position, the perturbed differential equation of motion will have a "negative stiffness coefficient,"such as

$$m\ddot{y} - ky = 0 ,$$

and an unstable time solution

$$Y = A \cosh\omega_n t + B \sinh\omega_n t$$



Inverted compound pendulum with an unstable static equilibrium position about $\theta = 0$.

For small motion about the equilibrium position $\theta = 0$, the inverted compound pendulum has the differential equation of motion

$$\ddot{\theta} - \frac{3g}{2l}\theta = 0 .$$

External time-varying force. Adding the external time-varying force $f(t) = f_0 \sin \omega t$ to the harmonic oscillator (mass-spring) system yields the differential equation

$$m\ddot{Y} = w - kY + f_0 \sin \omega t .$$

The energy-integral substitution,

$$\ddot{Y} = \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) ,$$

gives

$$m \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = w - kY + f_0 \sin \omega t ,$$

and integration gives

$$\left(\frac{m\dot{Y}^2}{2} - wY + \frac{kY^2}{2}\right) - \left(\frac{m\dot{Y}_0^2}{2} - w Y_0 + \frac{kY_0^2}{2}\right) = \int_{Y_0}^{Y} f_0 \sin \omega t \, dy = \int_0^t f_0 \sin \omega \tau \times \dot{y}(\tau) \, d\tau .$$

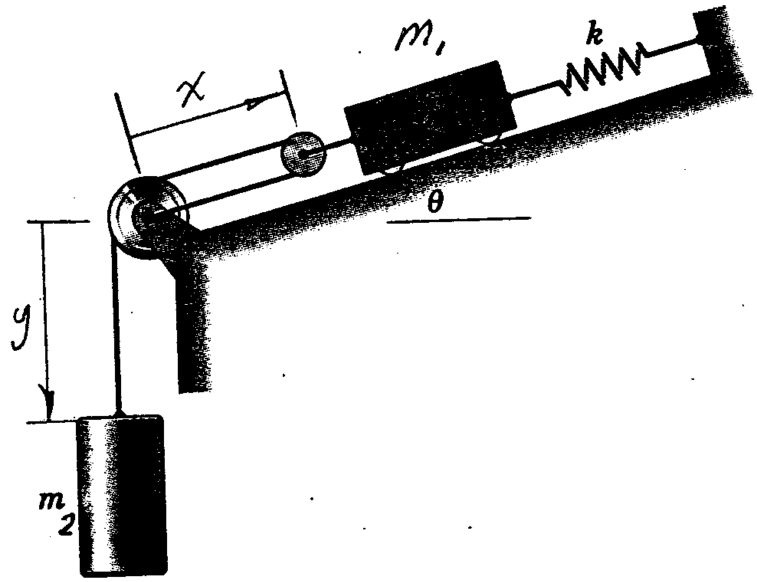
This equation is a specific example of the general equation

$$\Delta(T+V) = Work_{nonconservative} \ .$$

An external time-varying force is a nonconservative force and produces nonconservative work. The right-hand side integral cannot be evaluated unless the solution $Y(t)$ for the differential equation is known.

Hence, for nonconservative forces that are functions of time, neither the energy-integral substitution nor the work-energy equation is useful in solving the differential equation of motion. The substitution is normally helpful in solving the differential equation when the acceleration can be expressed as a function of displacement only.

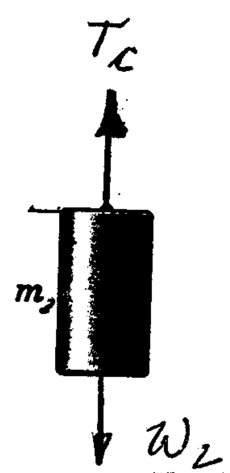
Example 6.1

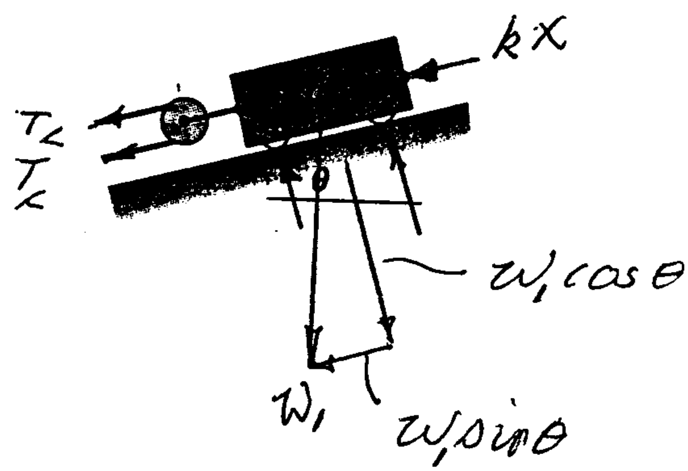


2 bodies, one degree of freedom
⇒ Kinematic constraint equations

$$l_c = cord\ length = 2x + y + a \;\Rightarrow\; \dot{y} = -2\dot{x} \;\Rightarrow\; \ddot{y} = -2\ddot{x}\ ,$$

where a is the amount of cord wrapped around the pulley.
Free-body diagrams





E

quations of Motion

$$\Sigma f_x = -2T_c - w_1 \sin\theta - kx = m_1 \ddot{x}$$
$$\Sigma f_y = w_2 - T_c = m\ddot{y} \quad \Rightarrow \quad T_c = w_2 - m_2 \ddot{y}.$$

Substituting for T_c, the cord tension, and using the kinematic constraint to eliminate $\ddot{y}$ gives

$$-2(w_2 - m_2 \ddot{y}) - w_1 \sin\theta - kx = m_1 \ddot{x}$$

$$-2[w_2 - m_2(-2\ddot{x})] - w_1 \sin\theta - kx = m_1 \ddot{x}$$

$$\therefore \ (m_1 + 4m_2)\ddot{x} + kx = -2w_2 - w_1 \sin\theta \quad \Rightarrow \quad \omega_n^2 = k/(m_1 + 4m_2).$$

This equation of motion can be stated

$$m_{eq}\ddot{x} + k_{eq}x = f_{eq} \ .$$

Note that both entries in $m_{eq} = m_1 + 4m_2$ are positive. If either were negative, the answer would be wrong. We don't have negative masses in dynamics, and ***a negative contribution in this type of coefficient always indicates a mistake in developing the equation of motion.***

Strategy for Deriving and Verifying the Equation of Motion from $\sum f_X = m\ddot{X}$:

1. Select Coordinates and draw them on your figure. Your choice for coordinates will establish the + and - signs for displacements, velocities, accelerations and forces. Check to see if there is a relation between your coordinates. If there is a relationship, write out the corresponding kinematic constraint equation(s).

2. Draw free-body diagrams corresponding to positive displacements and velocities for your bodies. In the present example a positive *x* displacement produced a compression force in the spring acting in the -*x* direction.

3. Use $\sum \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to state the equations of motion with + and - forces defined by the + and - signs of your coordinates.

4. Use the kinematic constraint equation(s) to eliminate excess variables to produce an equation of motion.

5. If your equation has the form $m_{eq}\ddot{x} + k_{eq}x = f\ (t)$, check to see that the individual contributions for m_{eq} and k_{eq} are positive.

Think about the degrees of freedom for your system. A one degree-of-freedom (1DOF) system needs one coordinate to define all of the bodies' positions. A 2DOF system needs two coordinate to define all of the bodies' positions, etc. Example 6.1 has two coordinates but only one degree of freedom.

Equation of Motion for motion about equilibrium

Equilibrium is defined by $\sum f_x = 0 \Rightarrow \ddot{x} = 0$ and implies

$$\bar{x} = -(2w_2 + w_1 \sin\theta)/k \ .$$

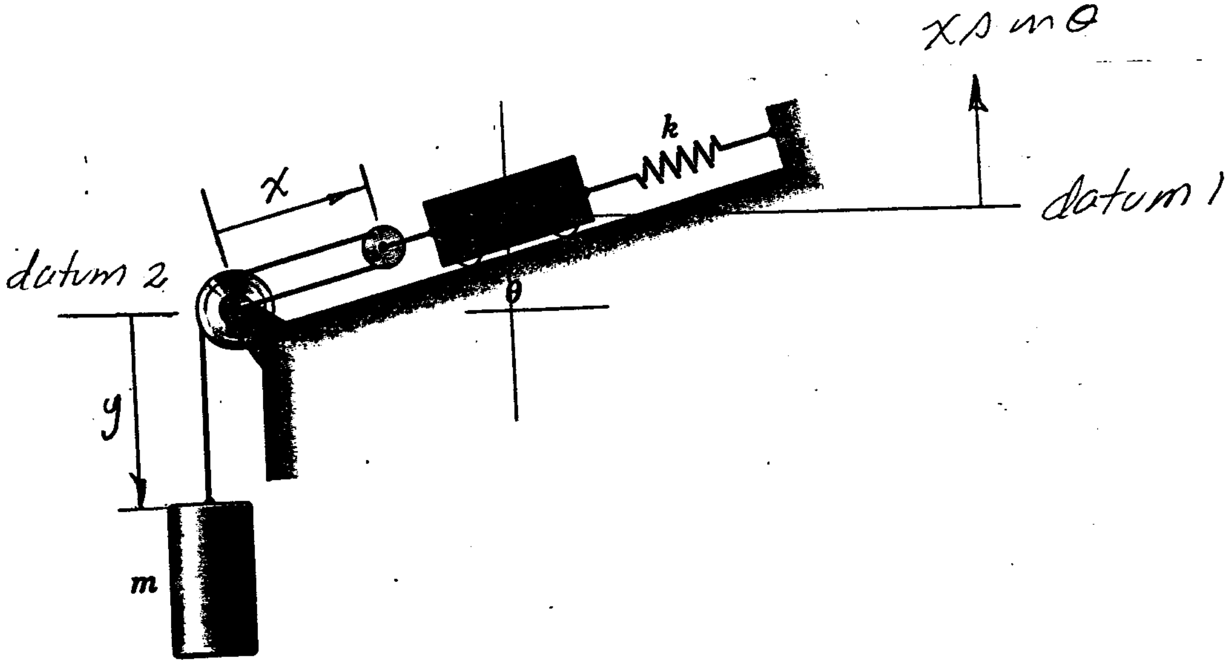
For motion about the equilibrium position defined by $x = \bar{x} + \delta x \Rightarrow \ddot{x} = \delta\ddot{x}$,

$$(m_1 + 4m_2)\delta\ddot{x} + k(\delta x + \bar{x}) = -2w_2 - w_1 \sin\theta$$

$$\therefore \ (m_1 + 4m_2)\delta\ddot{x} + k\delta x = 0 \ .$$

ENERGY APPROACH

Data for Potential Energy Functions



Conservation-of-Energy Equation: $T + V = T_0 + V_0$

$$m_2 \frac{\dot{y}^2}{2} + m_1 \frac{\dot{x}^2}{2} - w_2 y + w_1 x \sin\theta + k\frac{x^2}{2} = T_0 + V_0 \ , or$$

$$m_2 \frac{(-2\dot{x})^2}{2} + m_1 \frac{\dot{x}^2}{2} - w_2 (l_c - 2x) + w_1 x \sin\theta + k\frac{x^2}{2} = T_0 + V_0 , or$$

$$(m_1 + 4m_2)\frac{\dot{x}^2}{2} - w_2 (l_c - 2x) + w_1 x \sin\theta + k\frac{x^2}{2} = T_0 + V_0 \ .$$

Differentiating w.r.t. x

$$(m_1 + 4m_2)\frac{d}{dx}(\frac{\dot{x}^2}{2}) + 2w_2 + w_1 \sin\theta + kx = 0$$

$$\therefore \; (m_1 + 4m_2)\ddot{x} + kx = -2w_2 - w_1 \sin\theta$$

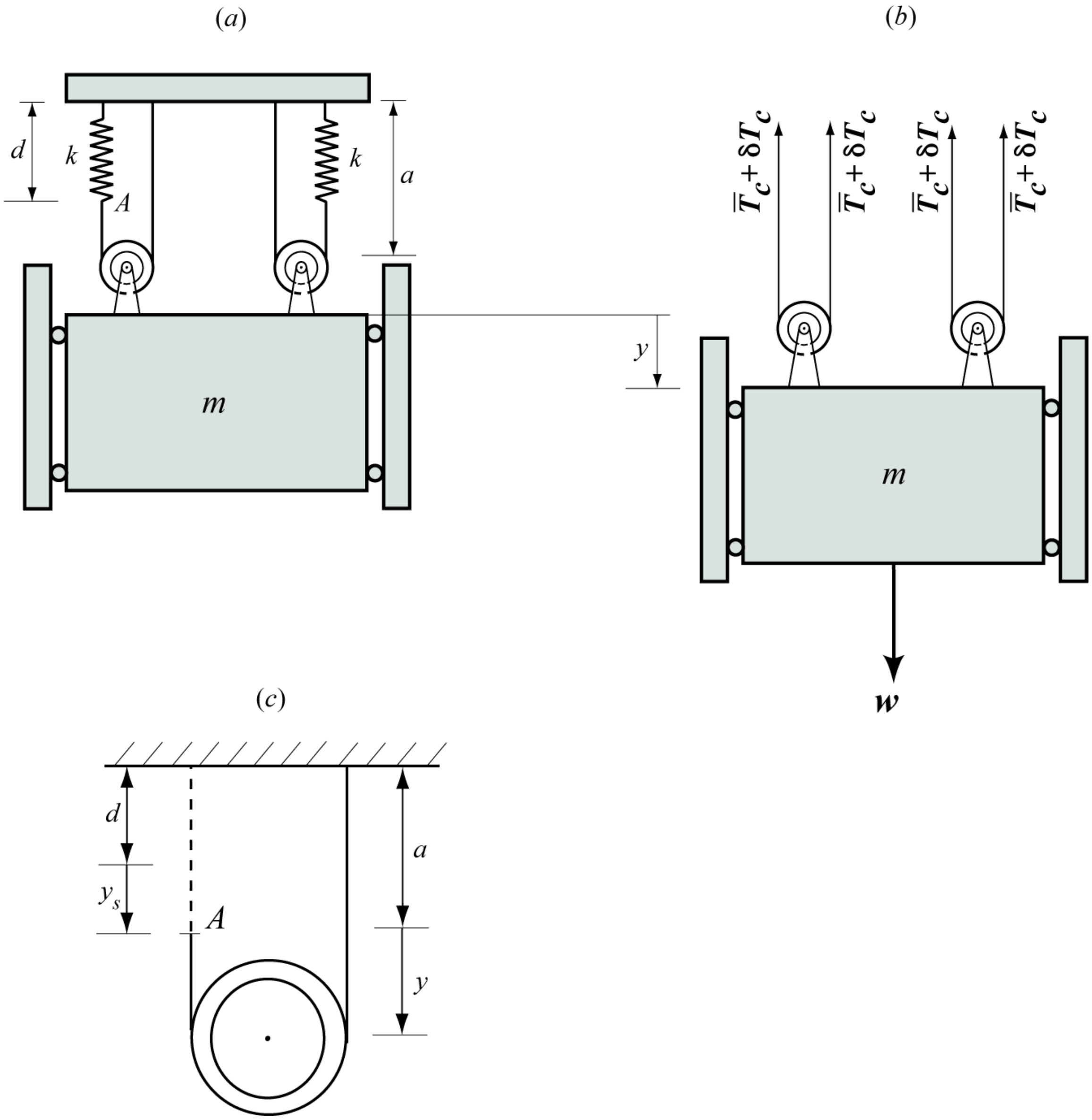


Figure XP3.1 (a) Mass in equilibrium, (b) Free-body diagram, (c) Kinematic constraint relation

Example Problem 3.1 Figure XP3.1 illustrates a mass m that is *in equilibrium* and supported by two spring-pulley combinations. **Tasks:** *a.* Draw a free-body diagram and derive the equations of motion, and *b.* For w=100 lbs, and $k = 200\, lb/in$. Determine the natural frequency.

For equilibrium,

$$\Sigma f_Y = w - 4\bar{T}_c$$

Applying $\Sigma f_y = m\ddot{y}$ to the free-body diagram gives

$$m\ddot{y} = \Sigma f_y = w - 4T_c = w - 4\bar{T}_c - 4\delta T_c = -4\delta T_c \ . \qquad \textbf{(i)}$$

The reaction forces are defined as $\delta T_c = k y_s$ where y_s is the change in length of the spring due to the displacement y. We need a relationship between y and y_s.

The cord wrapped around a pulley is inextensible with length l_c. Point *A* in figure XP3.1a locates the end of the cord. From this figure, $l_c = a + (a - d) = 2a - d$. Figure XP3.1c shows the displaced position and provides the following relationship,

$$\begin{aligned} l_c &= (a + y) + [(a + y) - (d + y_s)] = (2a - d) + 2y - y_s \\ &= l_c + 2y - y_s \end{aligned} \qquad \textbf{(ii)}$$

$$\therefore \ y_s = 2y \ .$$

Applying these results to Eq.(i) gives

$$n\ddot{y} = -4\delta T_c = -4ky_s = = -4k_s(2y) = -8ky \Rightarrow m\ddot{y} + 8ky = 0 \qquad \textbf{(iii)}$$

This step concludes *Task a.* Eq.(i) has the form

$m\ddot{y} + k_{eq} y = 0$ where $k_{eq} = 8k$ is the "equivalent stiffness."
Dividing through by m puts the equation into the form
$\ddot{y} + \omega_n^2 y = 0$; hence,

$$\omega_n^2 = \frac{k_{eq}}{m} = \frac{8 \times 100\, lb/in}{[100\, lb/(386.\, in/\sec^2)]} = 3088\left(\frac{rad}{\sec}\right)^2$$

$$\therefore \quad \omega_n = 55.6 \frac{rad}{\sec}$$

and

$$f_n = 55.6 \frac{rad}{\sec} \times \left(\frac{1\, cycle}{2\pi\, rad}\right) = 8.84 \frac{cycle}{\sec} = 8.84\, Hz$$

Note the conversion from weight to mass via $m = w/g$ where for the inch-pound-second system, $g = 386.\, in/\sec^2$.

$$Period = \tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = .113 \sec \quad .$$

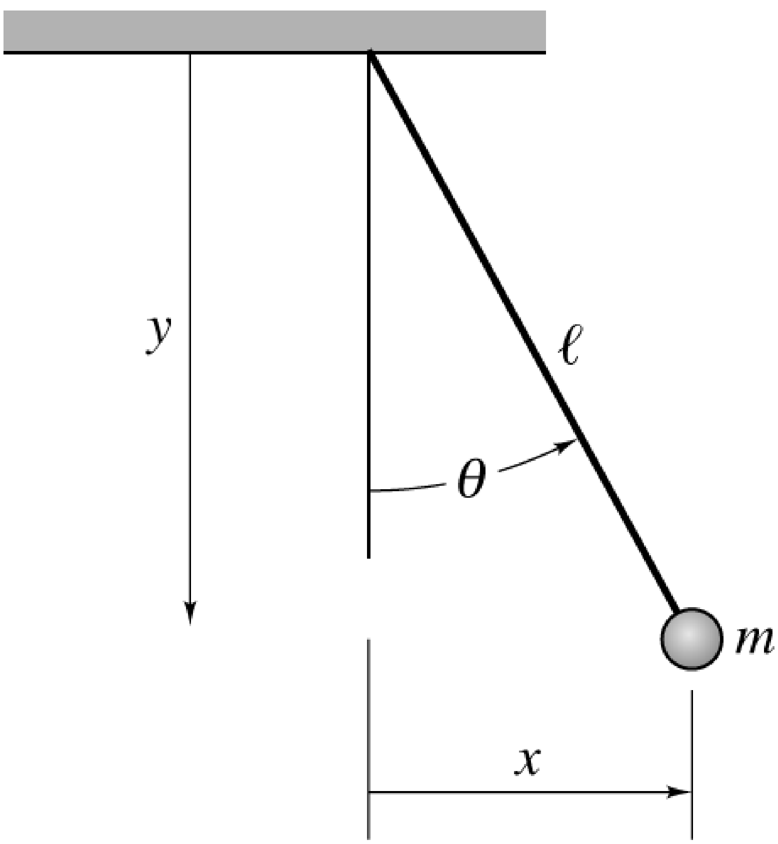
Deriving Equation of Motion From Conservation of Energy

$$T+V=T_0+V_0 \quad \Rightarrow \quad m\frac{\dot{y}^2}{2}+2\left[\frac{k}{2}(2y)^2\right]=Constant$$

$$m\frac{\dot{y}^2}{2}+4y^2=Constant$$

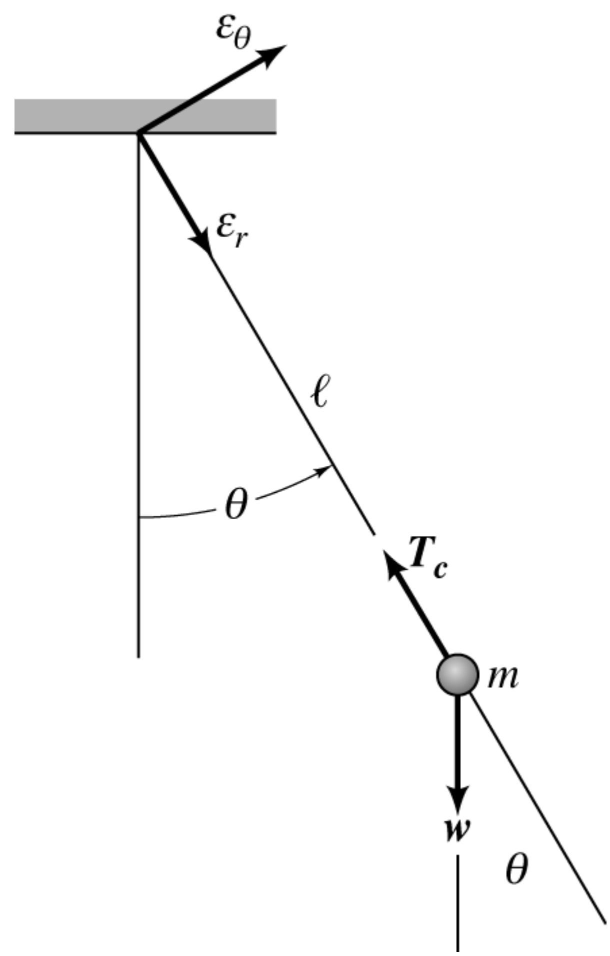
Differentiating w.r.t. *y* gives

$$m\frac{d}{dy}\left(\frac{\dot{y}^2}{2}\right)+8ky=0 \;\Rightarrow\; m\ddot{y}+8ky=0 \;\; .$$



The Simple Pendulum, Linearization of nonlinear Differential Equations for small motion about an equilibrium

Pendulum
Free-Body Diagram



The pendulum cord is inextensible.

Application of $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ in polar coordinates:

$$\begin{aligned}\Sigma f_r &= -T_c + w\cos\theta = m(\ddot{l} - ml\dot{\theta}^2) = -ml\dot{\theta}^2 \\ \Sigma f_\theta &= -w\sin\theta = m(l\ddot{\theta} + 2\dot{l}\dot{\theta}) = ml\ddot{\theta}\end{aligned} \qquad (3.82)$$

Nonlinear Equation of Motion

$$ml\ddot{\theta} = -w\sin\theta \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0 . \qquad (3.83)$$

Equilibrium is defined by $\ddot{\theta} = 0 \Rightarrow \sin\overline{\theta} = 0 \Rightarrow \overline{\theta} = 0 \ or\ \pi$

Expanding the nonlinear $\sin\theta$ term in a Taylor series about $\theta = 0$ gives

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \ldots \tag{3.84}$$

Discarding second-order and higher terms in θ gives the linearized D.Eq.

$$\ddot{\theta} + \frac{g}{l}\theta \cong 0 \quad \Rightarrow \quad \ddot{\theta} + \omega_n^2\theta = 0 \ . \tag{3.85}$$

where

$$\omega_n = \sqrt{g/l} \ , \tag{3.86}$$

Linearization validity:

$$\begin{gathered} \sin 15^\circ = \sin(.2618\,rad) = .2588 \\ \theta = .2618 \\ \theta^3/6 = .002990 \\ \theta^5/120 = .000010 \ . \end{gathered} \tag{3.87}$$

Linearized model is reasonable for $\theta \le 15^\circ$.

Pendulum Differential Equation of Motion From Conservation of Energy

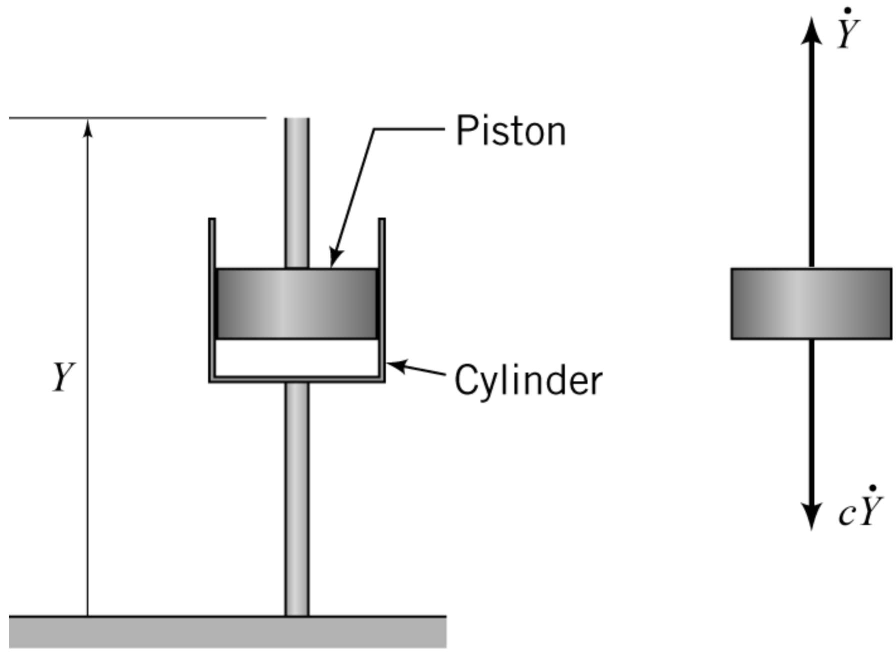
$$T + V = T_0 + V_0 \quad \Rightarrow \; m\frac{(l\dot{\theta})^2}{2} - wl\cos\theta = constant \;\; .$$

Differentiating w.r.t. θ gives

$$ml^2\frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) + wl\sin\theta = 0 \quad \Rightarrow \; \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \;\; .$$

ENERGY DISSIPATION—Viscous Damping

Automatic door-closers and shock absorbers for automobiles provide common examples of energy dissipation that is deliberately introduced into mechanical systems to limit peak response of motion.



(a) (b)

As illustrated above, a thin layer of fluid lies between the piston and cylinder. As the piston is moved, shear flow is produced in the fluid which develops a resistance force that is proportional to the velocity of the piston relative to the cylinder. The flow between the piston and the cylinder is laminar, versus turbulent flow which occurs commonly in pipe flow and will be covered in your fluid mechanics course. The piston free-body diagram shows a <u>reaction force</u> that is proportional to the piston's velocity and acting in a direction that is <u>opposite</u> to the piston's velocity

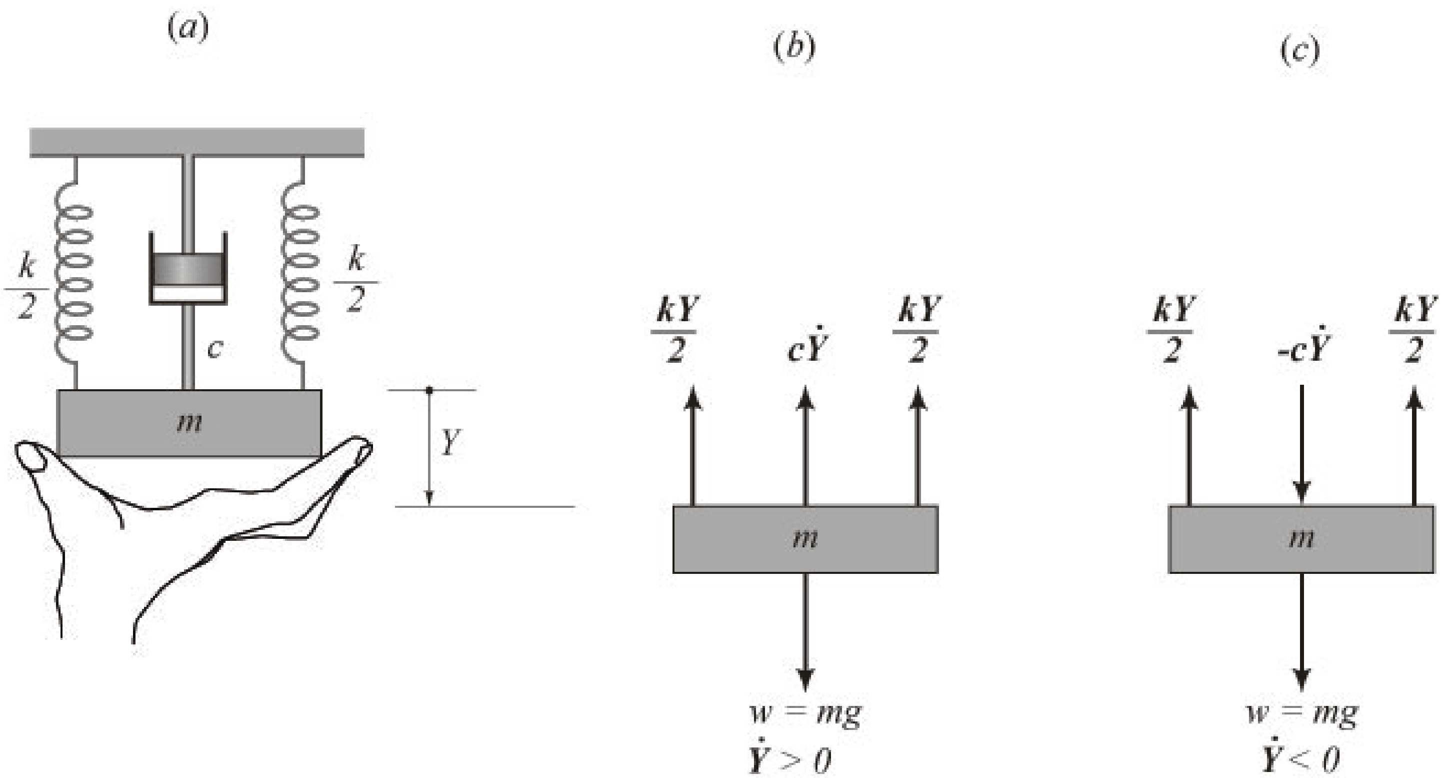


Figure 3.10 (a) Body acted on by its weight and supported by two springs and a viscous damper, (b) Free-body diagram for $Y>0, \dot{Y}>0$, (c) Free-body for $Y>0, \dot{Y}<0$

Figure 3.10a shows a body of mass *m* that is supported by two equal-stiffness linear springs with stiffness coefficients $k/2$ and a viscous damper with damping coefficient c. Figure 3.10b provides the free-body diagram for $Y>0, \dot{Y}>0$. The springs are undeflected at $Y=0$. Applying Newton's equation of motion to the free-body diagram gives:

$$m\ddot{Y} = \Sigma F_Y = w - \frac{kY}{2} - c\dot{Y} - \frac{kY}{2} ,$$

or

$$m\ddot{Y} + c\dot{Y} + kY = w \; . \qquad \textbf{(3.21)}$$

Dividing through by *m* gives

$$\ddot{Y} + 2\,\zeta\,\omega_n\,\dot{Y} + \omega_n^2\,Y = g \; ,$$

where

$$\omega_n = \sqrt{k/m} \quad , \quad 2\,\zeta\,\omega_n = c/m \Rightarrow \zeta = \frac{c}{2\sqrt{km}} .$$

ζ is called the linear damping factor.

Transient Solution due to Initial Conditions and Weight.

Homogeneous differential Equation

$$\ddot{Y}_h + 2\,\zeta\,\omega_n\,\dot{Y}_h + \omega_n^2\,Y_h = 0 \qquad (3.22)$$

Assumed solution: $Y_h = Ae^{st}$ yields:

$$(s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2)\,A e^{st} = 0 \; .$$

Since $A \neq 0$, and $e^{st} \neq 0$

$$s^2 + 2 \zeta \omega_n s + \omega_n^2 = 0 . \qquad \textbf{(3.23)}$$

<u>This is the characteristic equation</u>. For $\zeta = 1$, the mass is critically damped and does not oscillate. For $\zeta<1$, the roots are

$$\begin{aligned} s &= - \zeta \omega_n \pm j \omega_n \sqrt{1-\zeta^2} \\ &= - \zeta \omega_n \pm j \omega_d . \end{aligned} \qquad \textbf{(3.24)}$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ is called the *damped natural frequency*. The two roots defined by Eq.(3.24) are

$$s_1 = -\zeta\omega_n + \boldsymbol{j}\omega_d , \; s_2 = -\zeta\omega_n - \boldsymbol{j}\omega_d .$$

The homogeneous solution looks like

$$\begin{aligned} Y_h &= A_1 e^{s_1 t} + A_2 e^{s_2 t} \\ &= A_1 e^{(-\zeta \omega_n + j \omega_d) t} + A_2 e^{(-\zeta \omega_n - j \omega_d) t} , \end{aligned} \qquad \textbf{(3.25)}$$

where $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are *complex* coefficients. Substituting the identities

$$e^{j\omega_d t} = \cos\omega_d t + j \sin\omega_d t , \quad e^{-j\omega_d t} = \cos\omega_d t - j \sin\omega_d t$$

into Eq.(3.25) yields a final homogeneous solution of the form

$$Y_h = e^{-\zeta \omega_n t} (A \cos\omega_d t + B \sin\omega_d t) \ . \qquad \textbf{(3.26)}$$

where A and B are *real* constants. The complete solution is

$$Y = Y_h + Y_P = e^{-\zeta \omega_n t} (A \cos\omega_d t + B \sin\omega_d t) + \frac{w}{k} \ . \qquad \textbf{(3.27)}$$

For I.C.'s $Y(0) = \dot{Y}(0) = 0$, starting with Eq.(3.27) yields

$$Y(0) = 0 = A + \frac{w}{k} \Rightarrow A = -\frac{w}{k} \ .$$

To evaluate B, we differentiate Eq.(3.27), obtaining

$$\dot{Y} = e^{-\zeta \omega_n t} \omega_d (-A \sin\omega_d t + B \cos\omega_d t)$$

$$-\zeta \omega_n e^{-\zeta \omega_n t} (A \cos\omega_d t + B \sin\omega_d t) \ .$$

Evaluating this expression at $t = 0$ gives

$$\dot{Y}(0) = 0 = B \omega_d - \zeta \omega_n A \ .$$

Hence,

$$B = \frac{\zeta \omega_n A}{\omega_d} = \frac{\zeta}{\sqrt{1-\zeta^2}} A = -\frac{w \zeta/k}{\sqrt{1-\zeta^2}} \ .$$

The complete solution is now given as

$$Y = \frac{w}{k}\left[1 - e^{-\zeta\,\omega_n\,t}\left(\cos\,\omega_d\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\sin\,\omega_d\,t\right)\right] . \quad \textbf{(3.28)}$$

Figure 3.10 illustrates this solution for the data

$$m = 1\ kg\ ,\quad k = 3{,}948.\ N/m\ ,\quad c = 12.57\ N\,\text{sec}/m\ ,$$

$$\omega_n = 62.8\ rd/\text{sec} \Rightarrow f_n = 10\ cycle/\text{sec} = 10\ Hz$$
$$\zeta = .10\ ,\quad \omega_d = 62.2\ rd/\text{sec}\ . \quad \textbf{(3.29)}$$

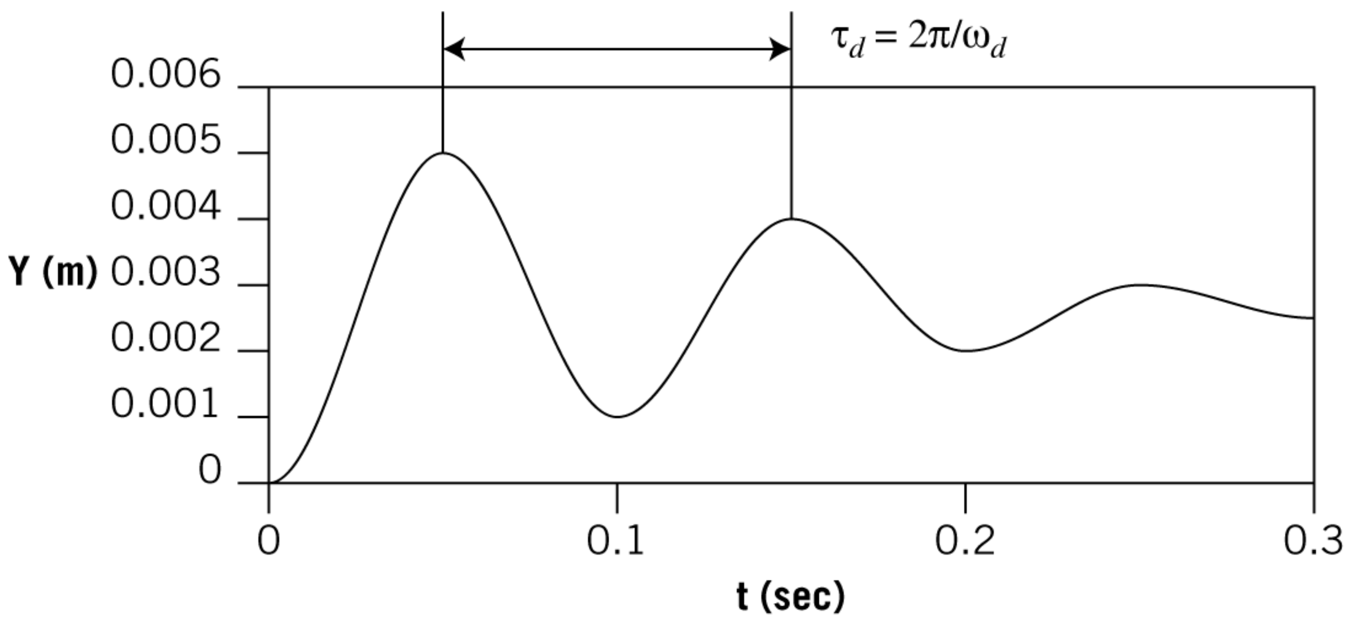


Figure 3.10 Time solution of Eq.(3.28) with the data of Eq.(3.29).

Substituting the "energy-integral" substitution $\ddot{Y} = d(\dot{Y}^2/2)/dY$ into Eq.(3.21)

$$m \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = -kY + w - c\dot{Y} .$$

We can still multiply through by dY, but we can not execute the integral,

$$\int_{Y_0}^{Y} c\,\dot{Y}\,dY = \int_{t_0}^{t} c\,\dot{Y}\,\frac{dY}{dt}\,dt = \int_{t_0}^{t} c\,\dot{Y}^2 dt$$

since $\dot{Y}$ is generally a function of t, not Y. You need the *solution* $Y(t)$ to evaluate this integral and determine how much energy is dissipated.

Incorrect Sign for Damping.

Suppose you get the sign wrong on the damping force, netting,

$$m\ddot{Y} - c\dot{Y} + kY = w .$$

Dividing through by *m gives*

$$\ddot{Y} - 2\,\zeta\,\omega_n\,\dot{Y} + \omega_n^2\,Y = g .$$

the general solution is now

$$Y = e^{\zeta\omega_n t} \, (A \, \cos\omega_d t \, + \, B \, \sin\omega_d t) \, + \, \frac{w}{k} \; .$$

Instead of decaying exponentially with time, the solution grows exponentially with time.

UNITS — Damping Coefficients

The damping force is defined by $f_{damping} = -c\dot{x}$; hence, the appropriate dimensions for the damping coefficient c in SI units is

$$c = \frac{f(N)}{\dot{x}(m/\sec)} = c(\frac{N\sec}{m}) \; .$$

In the US system with *lb-ft-sec* units,

$$c = \frac{f(lb)}{\dot{x}(ft/\sec)} = c(\frac{lb\sec}{ft}) \; .$$

Similarly, in the US system with *lb-in-sec* units,

$$c = \frac{f(lb)}{\dot{x}(in/\sec)} = c(\frac{lb\sec}{in}) \; .$$

The damping factor ζ is always dimensionless, defined by,

$$2\zeta\omega_n = \frac{c}{M} \Rightarrow \zeta = \frac{c}{2M\omega_n} \ ,$$

where M is the mass. In the SI system, we have

$$\zeta = c(\frac{N\sec}{m}) \times \frac{1}{2M(kg)} \times \frac{1}{\omega_n(\sec^{-1})} = \frac{c}{2M\omega_n}(\frac{N\sec^2}{m\,kg}) \ .$$

However, recall that the derived units for the newton are $kg\ \ m/\sec^2$; hence, ζ is dimensionless.

Using the *ft-lb-sec* US standard system of units,

$$\zeta = c(\frac{lb\sec}{ft}) \times \frac{1}{2M(slug)} \times \frac{1}{\omega_n(\sec^{-1})} = \frac{c}{2M\omega_n}(\frac{lb\sec^2}{ft\,slug}) \ .$$

However, the derived units for the slug is $lb\sec^2/ft$, so this result is also dimensionless.

Finally, Using the *in-lb-sec* US standard system of units,

$$\zeta = c(\frac{lb\sec}{in}) \times \frac{1}{2M(snail)} \times \frac{1}{\omega_n(\sec^{-1})} = \frac{c}{2M\omega_n}(\frac{lb\sec^2}{in\,snail}) \ .$$

However, the derived units for the snail is $lb\sec^2/in$, so this result is also dimensionless.

“Logarithmic Decrement” or simply “log dec” δ to Characterize Damping

The log dec can be determined directly from an experimentally-measured transient response. From Eq.(3.26), the motion about the equilibrium position can be stated

$$Y_h = e^{-\zeta\omega_n t} D \cos(\omega_d t - \varphi) \ ,$$

where $D = (A^2 + B^2)^{-1/2}$, and $\varphi = \tan^{-1}(-B/A)$. Peaks in the response curves occur when $\cos(\omega_d t - \varphi) = 1$, at time intervals equal to the damped period $\tau_d = 2\pi/\omega_d$. Hence, the ratio of two successive peaks would be

$$\frac{Y_1}{Y_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d} \ ,$$

and the log dec δ is defined as

$$\delta = \ln\left(\frac{Y_1}{Y_2}\right) = \zeta\omega_n\tau_d = \zeta\omega_n \frac{2\pi}{\omega_n(1-\zeta^2)^{1/2}} = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}} \ . \qquad \textbf{(3.30)}$$

The log dec can also be defined in terms of the ratio of a peak to the *n*th successive peaks as:

$$\delta = \frac{1}{(n-1)} \ln(\frac{Y_1}{Y_n}) = \frac{1}{(n-1)} \ln[\frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + (n-1)\tau_d)}}]$$
$$= \zeta\omega_n \tau_d = \frac{2\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad . \qquad \textbf{(3.31)}$$

Note in Eq.(3.30) that δ becomes unbounded as $\zeta \Rightarrow 1$. In many dynamic systems, the damping factor is small, $\zeta << 1$, and $\delta \cong 2\pi\zeta$.

In stability calculations for systems with unstable eigenvalues, negative log dec's are regularly stated.

Solving for the damping factor ζ in terms of δ from Eq.(3.30) gives

$$\zeta = \frac{\delta}{[(2\pi)^2 + \delta^2]^{1/2}} \quad . \qquad \textbf{(3.31)}$$

The effect of damping in the model $m\ddot{x} + c\dot{x} + kx = 0$ can be characterized in terms of the damping factor ζ and the log-dec δ.

Example Problem 3.4

Figure XP3.4 illustrates a transient response result for a mass that has been disturbed from its equilibrium position. The first peak occurs at $t = 1.6\,\text{sec}$ with an amplitude of 0.018 m. The fourth peak occurs at $t = 10.6\,\text{sec}$ with an amplitude of .003 m.

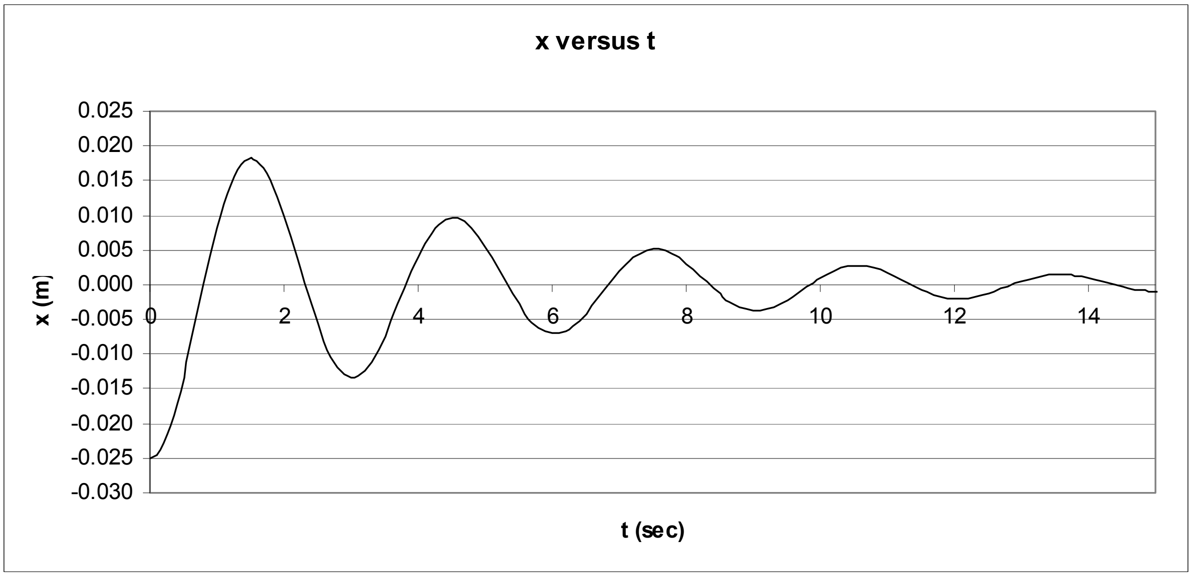


Figure XP3.4 Transient response

Tasks. Determine the log dec and the damping factor. Also, what is the damped natural frequency?

Solution Applying Eq.(3.31) gives

$$\delta = \frac{1}{n-1}\ln\left(\frac{Y_1}{Y_n}\right) = \frac{1}{3}\ln\left(\frac{Y_1}{Y_4}\right) = \frac{1}{3}\ln\left(\frac{.018}{.003}\right) = 0.597 \qquad \textbf{(3.31)}$$

for the log dec. From Table 3.1, the damping factor is

$$\zeta = \frac{\delta}{[(2\pi)^2 + \delta^2]^{1/2}} = \frac{0.597}{\sqrt{4\pi^2 + .597^2}} = 0.095$$

The damped period for the system is obtained as
$3\tau_d = 10.6 - 1.6 = 9\,\text{sec}$. Hence,
$\tau_d = 3\,\text{sec} \Rightarrow \omega_d = 2\pi/\tau_d = 2.09\, rad/\text{sec}$.

Percent of Critical Damping
Recall that $\zeta = 1 \Rightarrow$ critical damping A "percent of critical damping" is also used to specify the amount of available damping. For example, $\zeta = 0.1$ implies 10% of critical damping.

Table 3.1 demonstrates how to proceed from one damping Characterization to another

Table 3.1 Relationship between damping characterizations

Output Column	ζ	δ	% of critical damping	q factor
ζ	1	$\delta/[(2\pi)^2+\delta^2]^{1/2}$	% of critical damping/ 100	$1/(2\,q)$
δ	$2\pi\zeta/(1-\zeta^2)^{1/2}$	1	find ζ first	find ζ first
% of critical damping	$100\times\zeta$	find ζ first	1	$50\,/\,q$
q factor	$1\,/\,(2\,\zeta)$	find ζ first	50/ % of critical damping	1

Important Concept and Knowledge Questions

How is the reaction force due to viscous damping defined?

In the differential equation of motion for a damped, spring-mass system, what is the correct sign for *c , the linear damping coefficient?*

What is implied by a negative damping coefficient *c?*

What are *c*'s dimensions in the inch-pound-second, foot-pound-second, newton-kilogram-second, newton-millimeter-second systems?

What is the damped natural frequency?

What is the damping factor ζ ?

How is the critical damping factor defined, and what does critical damping imply about free motion?

What is the log dec?

Lecture 8. TRANSIENT SOLUTIONS 1, FORCED RESPONSE AND INITIAL CONDITIONS

Introduction

Differential equations in dynamics normally arise from Newton's second law of motion, $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$. Accordingly, in dynamics, systems of coupled second order differential equations are the norm. Occasionally, the equation of motion for a particle or rigid body has the form, $\ddot{y} = d^2y/dt^2 = g(y)$, and the energy-integral substitution,

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{y}}{dy} = \frac{d}{dy}\left(\frac{\dot{y}^2}{2}\right) , \qquad \textbf{(B.1)}$$

reduces the second-order equation, with time *t* as the independent variable, to the first-order differential equation,

$$\frac{d}{dy}(\frac{\dot{y}^2}{2}) = g(y) , \qquad \textbf{(B.2)}$$

with displacement *y* as the independent variable and $\dot{y}^2/2$ as the dependent variable. Systems of first order equations arise in heat transfer, RC circuits, population studies, etc.

Undamped Spring-Mass Model

The undamped spring-mass system with weight acting as a forcing function can be stated

$$\ddot{Y} + \omega_n^2 Y = w/m = g \ . \qquad \textbf{(B.8)}$$

The homogeneous differential equation is obtained by setting the right-hand side to zero, netting in this case $\ddot{Y}_h + \omega_n^2 Y_h = 0$. In solving any linear second-order differential equation, we use the following steps:

(i). Solve the homogeneous equation for $Y_h(t)$, which will include two arbitrary constants *A* and *B*,

(ii). Solve for the particular solution $Y_p(t)$ that satisfies the right-hand side of the equation,

(iii). Form the complete solution $Y(t) = Y_h(t) + Y_p(t)$, and

(iv). Use the complete solution to solve for the two unknown constants *A* and *B* that satisfy the problem's initial conditions.

The formal solution to the homogeneous differential equation, $\ddot{Y}_h + \omega_n^2 Y_h = 0$, is obtained by assuming a solution of the form $Y_h = A e^{st} \Rightarrow \ddot{Y}_h = s^2 A e^{st}$. We have previously developed the homogeneous solution as

$$Y_h = A \cos \omega_n t + B \sin \omega_n t \ , \qquad \textbf{(B.9)}$$

where $\omega_n = \sqrt{k/m}$.

Any constant-coefficient linear ordinary differential equation can be solved by Laplace transforms including the particular solution. However, most particular solutions can be obtained by an inspection of the right-hand side terms and the differential equation itself. For Eq.(B.8), the right hand side is constant, and a guessed constant solution of the form $Y_p = c \Rightarrow \dot{Y}_p = \ddot{Y}_p = 0$ yields

$$(0 + c\omega_n^2) = g \Rightarrow c = \frac{g}{\omega_n^2} = \frac{w}{m\omega_n^2} \Rightarrow Y_p = \frac{w}{k} = Y_s \ ,$$

where $Y_s = w/k$ is the static deflection due to the weight *w*. Note that this particular solution is linearly proportional to the excitation on the right-hand side of Eq.(B.8).

The complete solution to $\ddot{Y} + \omega_n^2 Y = w/m = g$ is

$$Y = Y_h + Y_p = A\cos\omega_n t + B\sin\omega_n t + \frac{w}{k} \ . \qquad \textbf{(B.10)}$$

Assuming that the initial conditions are $Y(0) = Y_0$, $\dot{Y}(0) = \dot{Y}_0$, we can first solve for the constant A via

$$Y_0 = A + \frac{g}{\omega_n^2} \Rightarrow A = Y_0 - \frac{w}{k} \ .$$

Similarly, $\dot{Y} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t$, nets

$$\dot{Y}_0 = B\omega_n \Rightarrow B = \dot{Y}_0/\omega_n \ ,$$

and the complete solution (satisfying the specified initial conditions) is

$$Y = Y_0\cos\omega_n t + \frac{w}{k}(1 - \cos\omega_n t) + \frac{\dot{Y}_0}{\omega_n}\sin\omega_n t \ . \qquad \textbf{(B.11)}$$

Suppose the spring-mass system is acted on by an external force that increases linearly with time, netting the differential equation of motion,

$$\ddot{Y} + \omega_n^2 \, Y = a t \ . \qquad \textbf{(B.12)}$$

This equation has the same homogeneous differential equation and solution; however, a new particular solution is required. By inspection, a solution of the form $Y_p = D t \ \Rightarrow \ \dot{Y}_p = D \ \Rightarrow \ \ddot{Y}_p = 0$ will work. Substituting this guessed solution into Eq.(B.12) produces

$$(0 + \omega_n^2 D t) = a t \ \Rightarrow \ D = \frac{a}{\omega_n^2} \ , \ Y_p = \frac{a t}{\omega_n^2} \ .$$

Eq.(B.12)'s complete solution is now

$$Y = A \cos \omega_n t + B \sin \omega_n t + \frac{a t}{\omega_n^2} \ . \qquad \textbf{(B.13)}$$

Table B.1 provides three particular solutions.

Table B.1. Particular solutions for $\ddot{Y}_p + \omega_n^2 Y_p = u(t)$.

Excitation, $u(t)$	$Y_p(t)$
$h = constant$	h/ω_n^2
$a\,t$	at/ω_n^2
bt^2	$-\dfrac{2b}{\omega_n^4} + \dfrac{bt^2}{\omega_n^2}$.

Consider the following version of Eq.(B.8)

$$\ddot{Y} + \omega_n^2 \, Y = c + a t + b t^2 \ .$$

Since this equation is linear, from Table B.1 and the homogeneous solution defined by Eq.(B.9), from superposition, the complete solution is

$$Y = A \cos\omega_n t + B \sin\omega_n t + \frac{c}{\omega_n^2} + \frac{a t}{\omega_n^2} + \left(- \frac{2 b}{\omega_n^4} + \frac{b t^2}{\omega_n^2} \right) \ .$$

Spring-Mass-Damper Model

The equation of motion for a spring-mass-damper system acted upon by weight can be stated

$$\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = \frac{w}{m} = g . \qquad \textbf{(B.18)}$$

The homogeneous solution is obtained (again) by assuming a solution of the form $Y_h = A e^{st} \Rightarrow \dot{Y} = s A e^{st} \Rightarrow \ddot{Y}_h = s^2 A e^{st}$. Substituting into the homogeneous differential equation nets

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) A e^{st} = 0 \Rightarrow (s^2 + 2\zeta\omega_n s + \omega_n^2) = 0 \ . \qquad \textbf{(B.19)}$$

This result holds, since neither A nor e^{st} is zero.

The following three solution possibilities exist for Eq.(B.19):

(i). $\zeta = 1$, critically damped motion,

(ii). $\zeta > 1$, over-damped motion, and

(iii). $\zeta < 1$, under-damped motion.

For $\zeta > 1$, the roots to the characteristic Eq.(B.19) are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \Rightarrow s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} < 0$$
$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} < 0 \ . \qquad \textbf{(B.20)}$$

Note that two real, negative roots are obtained, netting the homogeneous solution,

$$Y_h = A_1 e^{-|s_1|t} + A_2 e^{-|s_2|t} \ , \qquad \textbf{(B.21)}$$

where A_1, A_2 are real constants. The overdamped solution is the sum of two exponentially decaying terms.

For $\zeta = 1$ (critical damping), the characteristic Eq.(B.19) produces the single root $s = -\omega_n$, and single solution, $Y_h = A e^{-\omega_n t}$. This homogeneous solution (containing only one constant) is not adequate to satisfy two initial conditions (position and velocity). For less than obvious reasons, the complete homogeneous solution is

$$Y_h = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} \ , \qquad \textbf{(B.22)}$$

where A_1, A_2 are real constants. The second term in this solution also satisfies the differential equation as can be confirmed by substituting for Y_h . The critically-damped solution is interesting from a mathematical viewpoint as a limiting condition, but has minimal direct engineering value.

For $\zeta < 1$, the homogeneous equation, $\ddot{Y}_h + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y_h = 0$ has the solution form

$$Y_h = e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t) \ . \qquad \textbf{(B.25)}$$

where A, B are *real* constants, and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

The particular solution for $\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = w/m = g$ is $Y_p = g/\omega_n^2 = w/k$, and the complete solution is

$$Y = e^{-\zeta \omega_n t} (A \cos\omega_d t + B \sin\omega_d t) + \frac{w}{k} \qquad \textbf{(B.26)}$$

For the initial conditions, $Y(0) = Y_0$, Eq.(B.26) yields

$$Y(0) = Y_0 = A + \frac{w}{k} \;\Rightarrow\; A = Y_0 - \frac{w}{k} \; . \tag{B.27}$$

Differentiating Eq.(B.26) gives

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t} (A \cos\omega_d t + B \sin\omega_d t)$$
$$+ e^{-\zeta\omega_n t} (-\omega_d A \sin\omega_d t + \omega_d B \cos\omega_d t) \; .$$

Hence, the initial condition, $\dot{Y}(0) = \dot{Y}_0$, defines B via,

$$\dot{Y}_0 = -\zeta\omega_n A + \omega_d B$$

$$\therefore \; B = \frac{\dot{Y}_0}{\omega_d} + \frac{\zeta\omega_n A}{\omega_d} = \frac{\dot{Y}_0}{\omega_d} + \frac{\zeta}{\sqrt{1-\omega_n^2}} (Y_0 - \frac{w}{k}) \; , \tag{B.26}$$

and the complete solution is

$$Y = e^{-\zeta\omega_n t} \{ (Y_0 - \frac{w}{k}) \cos\omega_d t$$
$$+ [\frac{\dot{Y}_0}{\omega_d} + \frac{\zeta}{\sqrt{1-\omega_n^2}} (Y_0 - \frac{w}{k})] \sin\omega_d t \} + \frac{w}{k} \; .$$

Table B.2 provides three particular solutions for the spring-mass-damper system.

Table B.2. Particular solutions for $\ddot{Y}_p + 2\zeta\omega_n \dot{Y}_p + \omega_n^2 Y_p = u(t)$.

Excitation, $u(t)$	$Y_p(t)$
c = constant	c/ω_n^2
$a\,t$	$\frac{a}{\omega_n^2}(t - \frac{2\zeta}{\omega_n})$
bt^2	$\frac{b}{\omega_n^2}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$

As with the undamped equation, the following basic steps are taken to produce a complete solution for $\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y = u(t)$:

a. The homogeneous solution, $Y_h(t) = e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t)$, is developed for $\ddot{Y}_h + 2\zeta\omega_n\dot{Y}_h + \omega_n^2 Y_h = 0$ and involves the two constants A and B.

b. The particular solution $Y_p(t)$ is developed to satisfy the right hand side of the equation $\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y = u(t)$.

c. The complete solution, $Y(t) = Y_h(t) + Y_p(t)$, is formed, and the arbitrary constants A and B are determined from the complete solution such that the complete solution satisfies the initial conditions.

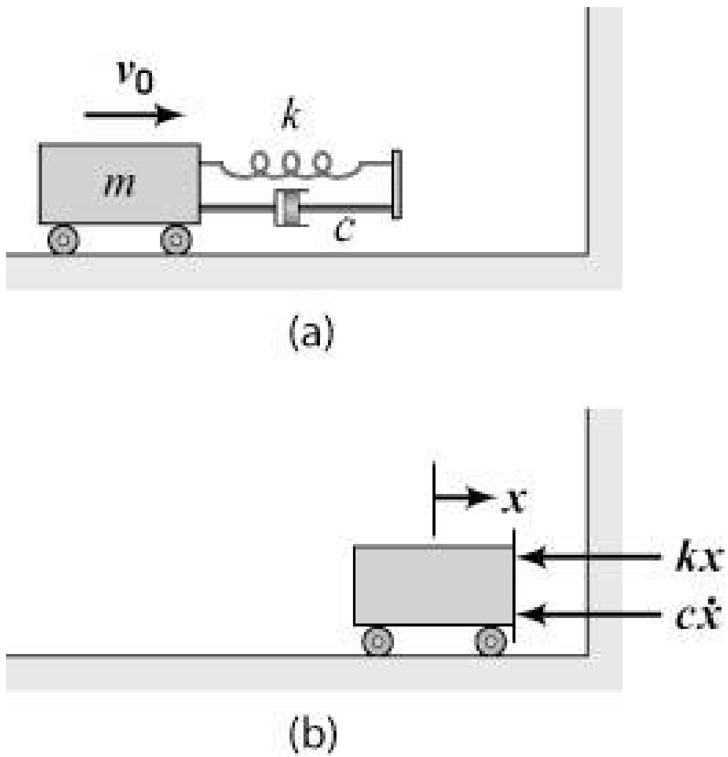


Figure XP3.2 (a). Cart approaching collision, (b). Free-body diagram during contact

Figure XP3.2a illustrates a cart that is rolling along at the speed v_0. It has a collision-absorption unit consisting of a spring and damper connected to a rigid plate of negligible mass. The engineering tasks for this example are listed below:

a. Select a coordinate and derive the equation of motion.

b. For $m = 100\,kg$, $k = 9.87E+04\;N/m$, $c = 3.1416E+03\,N\sec/m$, $v_0 = 10\,m/\sec$ determine the solution for motion while the cart remains in contact with the wall.

c. For arbitrary *k* and *m,* illustrate how a range of damping constants c producing $0 \le \zeta \le 1$ will reduce the stopping time and peak amplitude.

Solution: Figure XP3.2b shows the coordinate choice for x. Contact occurs for $x \ge 0$. The free-body diagram in figure XP3.2c corresponds to $x > 0$ and $\dot{x} > 0$, requiring compression in the spring and damper. Applying Newton's laws gives

$$m\ddot{x} = \sum \boldsymbol{f} = -kx - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0 \ ,$$

with the initial conditions, $x(0) = 0, \dot{x}(0) = v_0$. The spring and damper forces are negative in this equation because they are acting in the -*x* direction. There is no forcing function on the right-hand side of the equation; hence, the homogeneous solution of Eq.(B.25),

$$x = x_h + x_p = e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t) + 0, \qquad \textbf{(i)}$$

is the complete solution, and the velocity solution is

$$\begin{aligned} \dot{x} &= -\zeta\omega_n e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t) \\ &+ e^{-\zeta\omega_n t}\omega_d(-A\sin\omega_d t + B \cos\omega_d t) \ . \end{aligned} \qquad \textbf{(ii)}$$

Applying the initial conditions gives

$$x(0) = 0 = A \ , \ \dot{x}(0) = v_0 = B\omega_d \Rightarrow B = \frac{v_0}{\omega_d} \ .$$

Substituting back into (i) and (ii) gives

$$x = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$$

$$\dot{x} = -\frac{\zeta v_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_d t + v_0 e^{-\zeta\omega_n t} \cos\omega_d t \ , \qquad \textbf{(iii)}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

The cart loses contact with the wall when $x(T_f) = 0$. T_f is defined from the first of Eqs.(iii) by $\sin(\omega_d T_f) = 0 \Rightarrow \omega_d T_f = \pi$ [1]. The problem parameters net

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1

The equation $\sin(\omega_d T_f) = 0$ has an infinite number of solutions defined by: $\omega_d T_f = n\pi \ ; n = 0,1,2,..$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.87E{+}4\,N/m}{100\,kg}} = 31.416\,\frac{rad}{\sec} \;\Rightarrow\; f_n = \frac{\omega_n}{2\pi} = 5\,\frac{cy}{\sec} = 5\,Hz$$

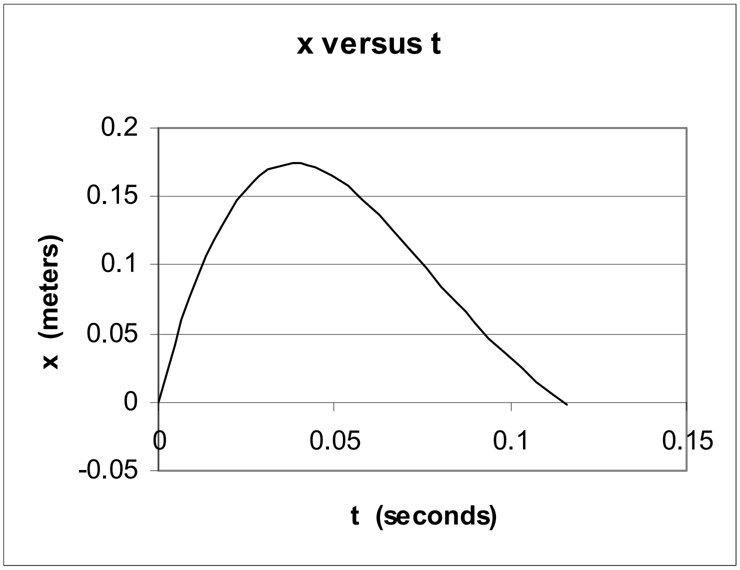
$$2\zeta\omega_n = \frac{c}{m} \;\Rightarrow\; \zeta = \frac{3.1416E{+}3\,Nsec/m}{2\times 31.146\sec^{-1}\times 100\,kg} = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.416\sqrt{1-0.25} = 26.97\,\frac{rad}{\sec}$$

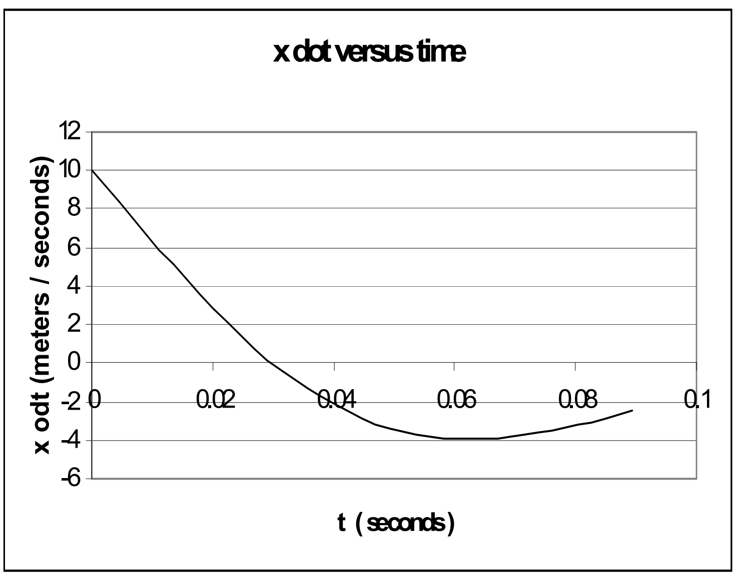
$$f_d = \frac{\omega_d}{2\pi} = 4.29Hz \;\Rightarrow\; \tau_d = \frac{1}{f_d} = 0.233\sec \;\; ; \;\; T_f = \pi/\omega_d = .116\sec.$$

Plots for x and x(t) are provided below for $0 \le t \le T_f$.

Figure XP3.2 a. Position and b. Velocity solution following contact



a. Position



b. Velocity

Moving to *Task c*, the cart's forward motion stops, and the peak deflection occurs at the time T^* defined by $\dot{x}(T^*) = 0$. From the second of Eq.(iii), this "stopping time" T^* is defined by

$$\dot{x}(T^*) = 0 = -\frac{\zeta v_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T^*} \sin\omega_d T^* + v_0 e^{-\zeta\omega_n T^*} \cos\omega_d T^*,$$

$$\therefore \tan(\omega_d T^*) = \frac{\sqrt{1-\zeta^2}}{\zeta} \Rightarrow \omega_d T^* = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{iv}$$

The maximum value for T^* occurs at $\zeta = 0$ and is defined by

$$\omega_d T^*_{\max} = \omega_n T^*_{\max} = \pi/2 \;;$$

i.e., one fourth of the natural period. From Eq.(iv), $\sin(\omega_d T^*) = \sqrt{1-\zeta^2}$. The peak amplitude is defined by substituting this result into the first of Eq.(iii), netting

$$x(T^*) = x_{\max} = \frac{v_0}{\omega_d} e^{-\zeta\omega_n T^*} \sqrt{1-\zeta^2} = \frac{v_0}{\omega_n} e^{-\zeta\omega_n T^*} . \tag{v}$$

For $\zeta = 0$, the peak deflection is

$$x(T^*)\big|_{\zeta=0} = \frac{v_0}{\omega_n}$$

Figure XP3.2c illustrates the stopping-time ratio

$$\frac{T^*}{T^*_{\max}} = \frac{2}{\pi} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

versus ζ. Increasing the damping ratio ζ from 0 to 1 reduces the stopping time by 32% .

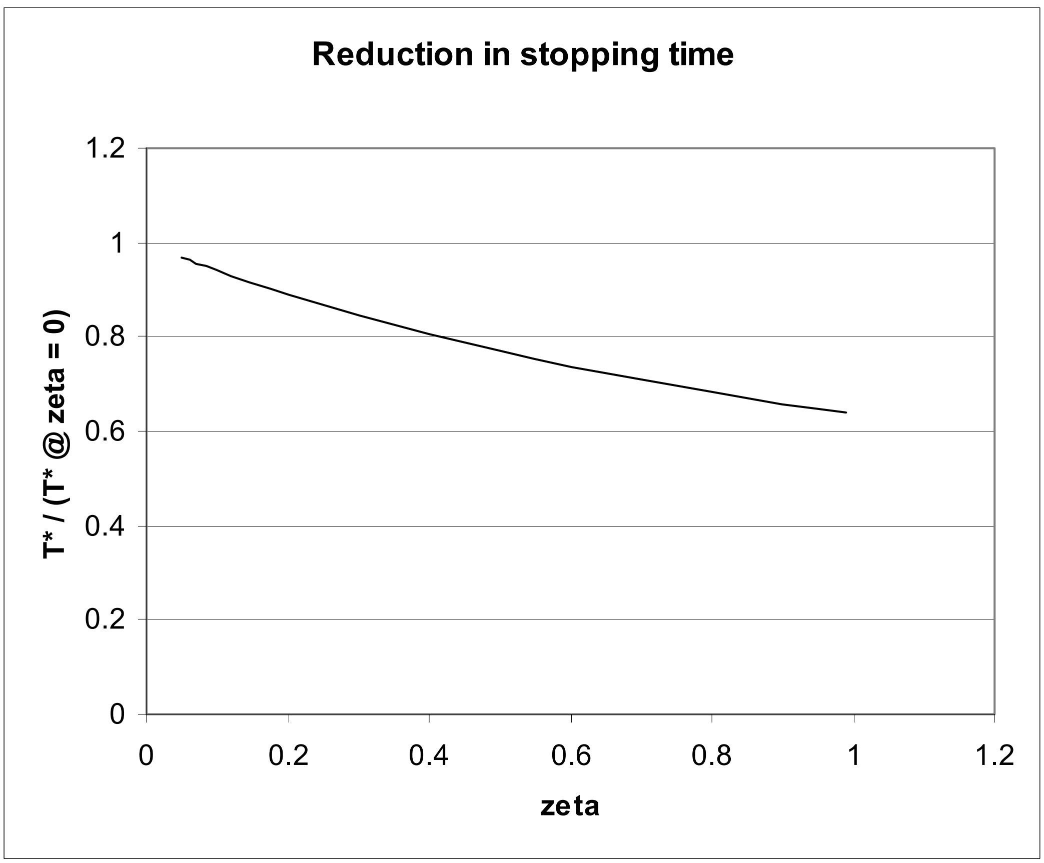


Figure XP3.2c Ratio of stopping time for $0<\zeta<1$.

Figure XPL10.d illustrates the peak deflection ratio

$$\frac{x_{\max}}{x_{\max}(\zeta = 0)} = e^{-\zeta \omega_n T*}$$

versus ζ. Increasing the damping ratio to $\zeta \cong 1$ decreases the peak amplitude by 56%.

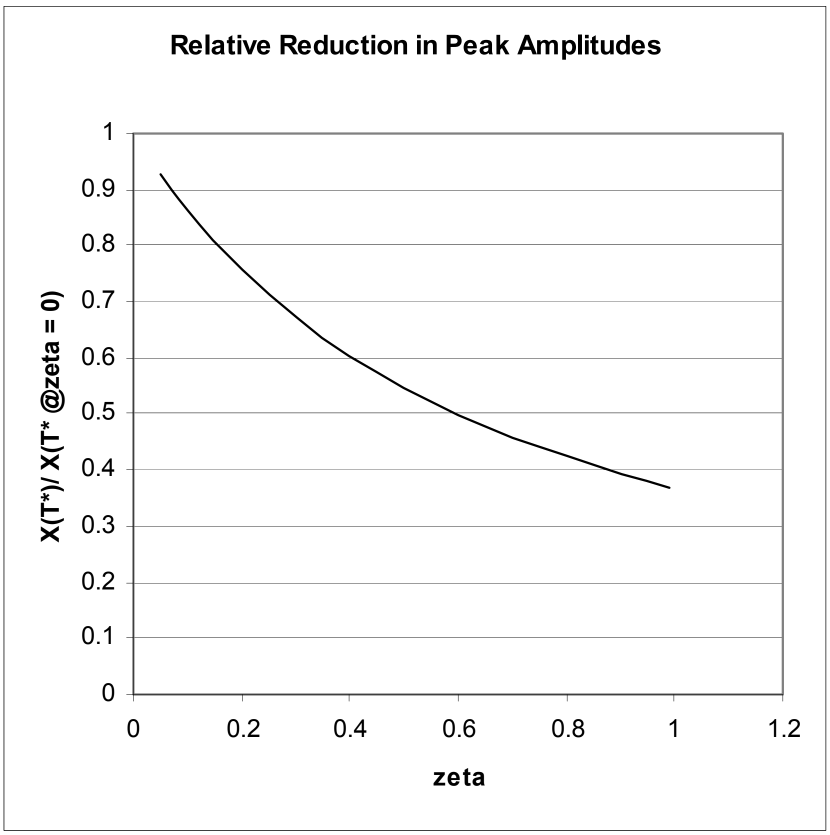


Figure XP3.2d. Ratio of maximum deflection for $0<\zeta<1$ to the $\zeta = 0$ value.

We could increase the damping constant c such that $\zeta \geq 1$; however, the solution provided by Eq.(i) is no longer valid. For $\zeta > 1$, the over-damped solution of Eq.(B.21) applies. For $\zeta = 1$, the critically damped solution of Eq.(B.22) applies.

Lecture 9. More Transient Solutions, Base Excitation

Example Problem L9.1

Consider the model $m\ddot{Y}+c\dot{Y}+kY=f(t)$ with initial conditions $Y(0)=\dot{Y}(0)=0$. The parameters *k*, *m,* and *c* are defined by $k=9.87E+04\,N/m$, $m=100\,Kg$, and $c=314.16\,N\sec/m$. These data produce:

$$\omega_n=\sqrt{\frac{k}{m}}=\sqrt{\frac{9.87E+4\,N/m}{100\,kg}}=31.416\,\frac{rad}{\sec}$$

$$\therefore\ f_n=\frac{\omega_n}{2\pi}=5\,\frac{cy}{\sec}=5\,Hz$$

$$2\zeta\omega_n=\frac{c}{m}\ \Rightarrow\ \zeta=\frac{314.16\,Nsec/m}{2\times 31.146\sec^{-1}\times 100\,kg}=0.05$$

$$\omega_d=\omega_n\sqrt{1-\zeta^2}=31.416\sqrt{1-0.0025}=31.38\,\frac{rad}{\sec}$$

$$f_d=\frac{\omega_n}{2\pi}=5\,\frac{cyc}{\sec}=4.993\,Hz\ ,\ \tau_d=\frac{1}{f_d}=.200\sec\ .$$

The force $f(t)$ is defined by the half-sine wave pulse illustrated below, and defined by

$$f(t) = f_0 \sin\omega_n t \ , \ 0 \le t \le t_1 = \pi/\omega_n$$

$$f(t) = 0 \ , \ t \ge t_1 \ .$$

where $f_0 = w = m \times g = 100\,kg \times 9.81\ m/\sec^2 = 981\,N$.

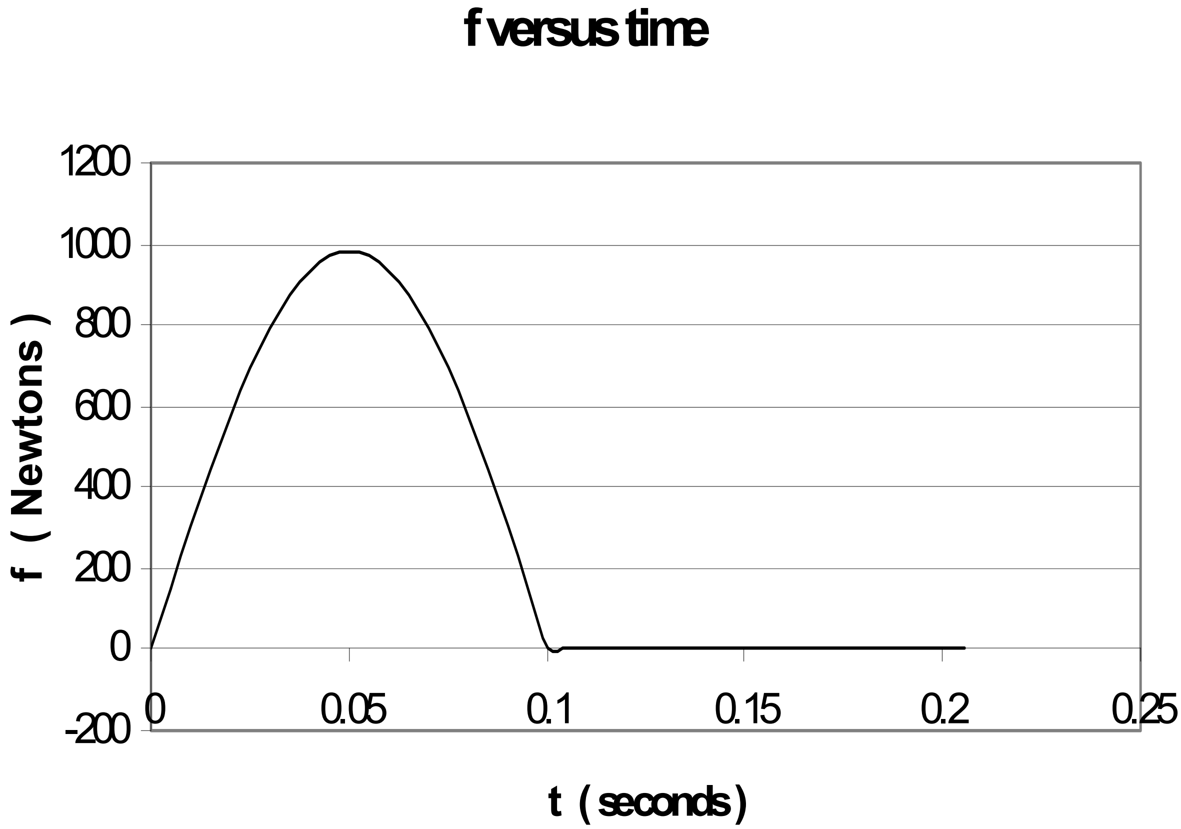


Figure XP9.1a Force excitation $f(t) = 981\ \sin\omega_n t\ (Newtons)$, $0 \le t \le t_1 = \pi/\omega_n$; $f(t) = 0,\ t \ge t_1$.

Complete the following engineering analysis tasks:

a. Determine the solution $Y(t)$ for $0 \le t \le t_1 = \pi/\omega_n$.

b. Determine the solution $Y(t)$ for $t \ge t_1$.

c. Plot the solution for $0 \le t \le 3\tau_d$

Solution. In lecture 11, we will develop the particular solution for $m\ddot{Y} + c\dot{Y} + kY = f_0 \sin\omega t$ as $Y_p = Y_{op}\sin(\omega t + \psi)$, where

$$Y_{op} = \frac{f_o}{k}\frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}} \ , \ r = \omega/\omega_n \ ,$$

$$\psi(r) = -\tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \ .$$

For $\omega = \omega_n \Rightarrow r = 1$, $Y_{op} = (f_0/k)(1/2\zeta) = f_0/(2k\zeta)$, $\psi = -\pi/2$, and the particular solution is

$$Y_p = \frac{f_0}{2k\zeta}\sin(\omega_n t - \frac{\pi}{2}) = -\frac{f_0}{2k\zeta}\cos(\omega_n t)$$

The complete solution is

$$Y = Y_h + Y_p = e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)$$

$$-\frac{f_0}{2k\zeta}\cos\omega_n t\ .$$

A is solved via

$$Y(0) = 0 = A - \frac{f_0}{2k\zeta}\ \Rightarrow\ A = \frac{f_0}{2k\zeta}$$

To determine $B,$ we start with

$$\begin{aligned}\dot{Y} = &-\zeta\omega_n e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)\\ &+\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t + B\cos\omega_d t)\\ &+\omega_n\frac{f_0}{2k\zeta}\sin\omega_n t\ .\end{aligned}$$

Evaluating,

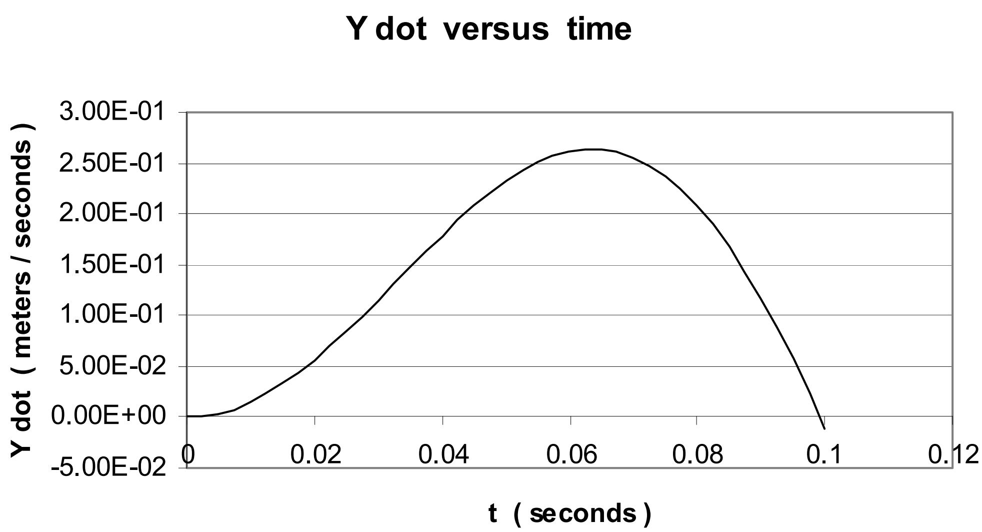
$$\dot{Y}(0) = 0 = -\zeta\omega_n A + B\omega_d\ \Rightarrow\ B = \frac{\zeta A}{\sqrt{1-\zeta^2}} = \frac{f_0}{2k}\frac{1}{\sqrt{1-\zeta^2}}$$

The complete solution satisfying the initial conditions is

$$Y = \frac{f_0}{2k\zeta}[e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t) - \cos\omega_n t] \ ,$$

concluding Task a. Starting on Task b, from the data provided: The solution for $0 \le t \le t_1 = \pi/\omega_n = 0.100\,\sec$ is presented below. At $t = t_1$, the values for the position and velocity are

$$Y(t_1) = .0145\,m, and\ \dot{Y}(t_1) = -.0105\ m/\sec$$



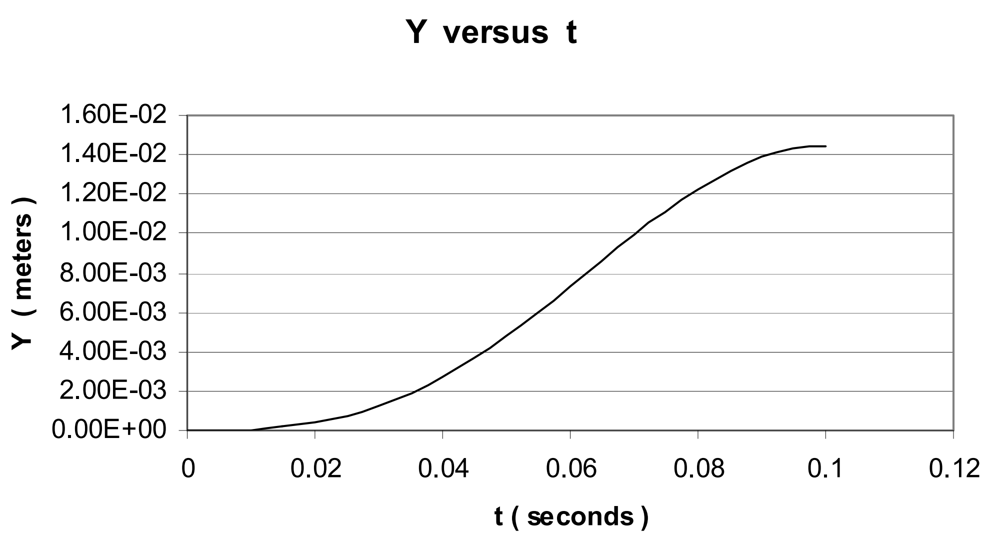


Figure XPL9.1b Solution for $Y(t)$ and $\dot{Y}(t)$ for $0 \le t \le t_1 = \pi/\omega_d$.

For $t \ge t_1$, the model reverts to $m\ddot{Y} + c\dot{Y} + kY = 0$, and the solution is now

$$Y = Y_h = e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)$$

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)$$
$$+\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t + B\cos\omega_d t).$$

We can solve for *A* and *B* via the $t = t_1$ results $Y(t_1) = .0145\,m$, $\dot{Y}(t_1) = -.0105\ m/\sec$ from:

$$Y(t_1) = .0145 = e^{-\zeta\omega_n t_1}\ (A\cos\omega_d t_1 + B\sin\omega_d t_1)$$

$$\dot{Y}(t_1) = -.0105 = -\zeta\omega_n e^{-\zeta\omega_n t_1}\ (A\cos\omega_d t_1 + B\sin\omega_d t_1)$$
$$+\omega_d e^{-\zeta\omega_n t_1}\ (-A\sin\omega_d t_1 + B\cos\omega_d t_1).$$

Two equations for the two unknowns *A* and *B*.

We are going to use $t' = t - t_1$, and "restart " time for $t \geq t_1 \Rightarrow t' \geq 0$. We previously developed the solution in terms of arbitrary initial conditions as

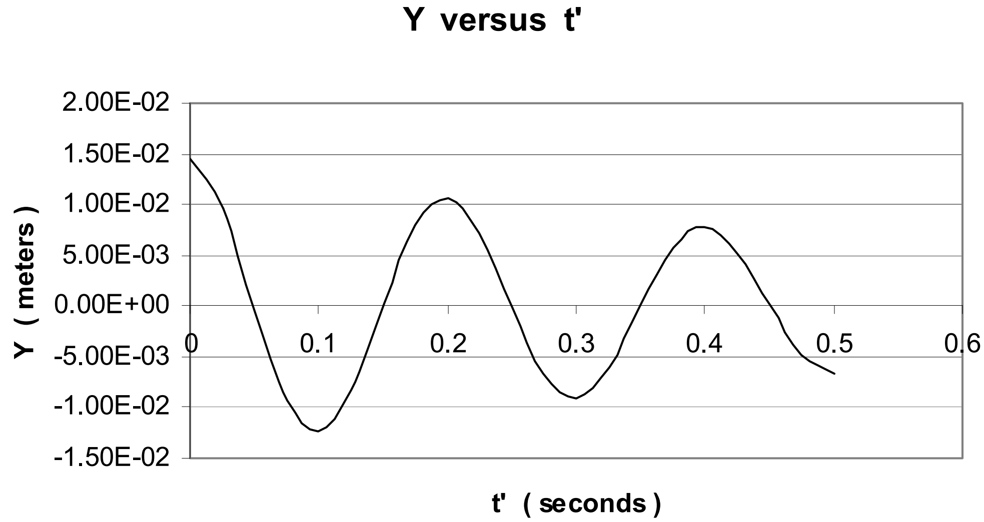
$$Y = e^{-\zeta\omega_n t'}\ [Y_0 \cos\omega_d t' + \frac{(\dot{Y}_0 + \zeta\omega_n Y_0)}{\omega_d} \sin\omega_d t']\ .$$

The initial conditions for this solution are:

$$Y(t'=0) = Y(t=t_1) = Y_0 = = .0145\, m$$

$$\dot{Y}(t'=0) = \dot{Y}(t=t_1) = \dot{Y}_0 = -.0105\, m/\sec\ .$$

The solution for $0 \leq t' \leq (3\tau_d - t_1),\ t_1 \leq t \leq 3\tau_d$ is presented below.



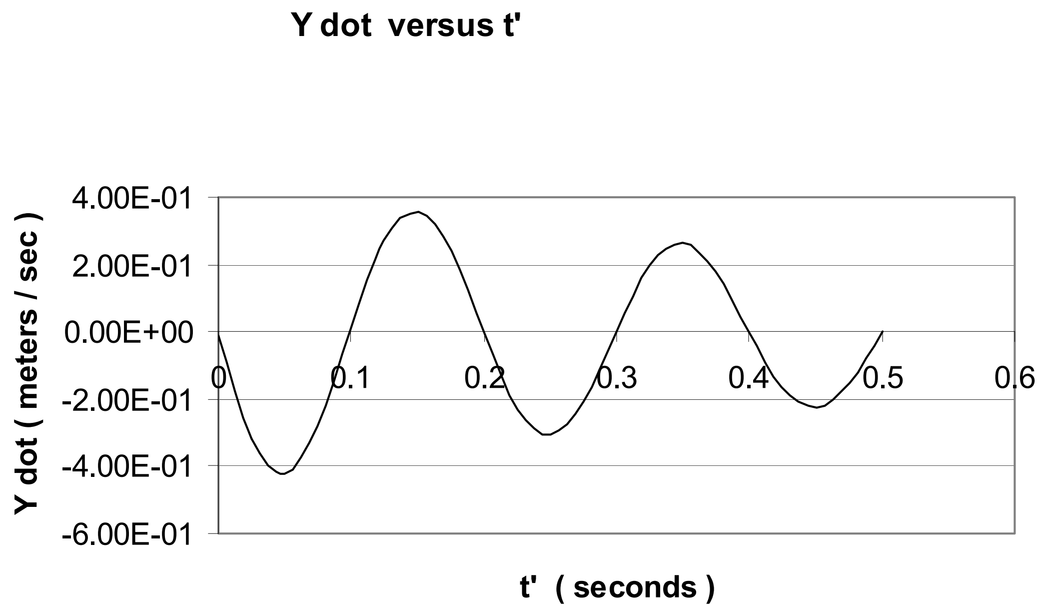


Figure XPL9.1c $Y(t')$ and $\dot{Y}(t')$ for $0 \le t' \le (3\tau_d - t_1)$.

Base Excitation

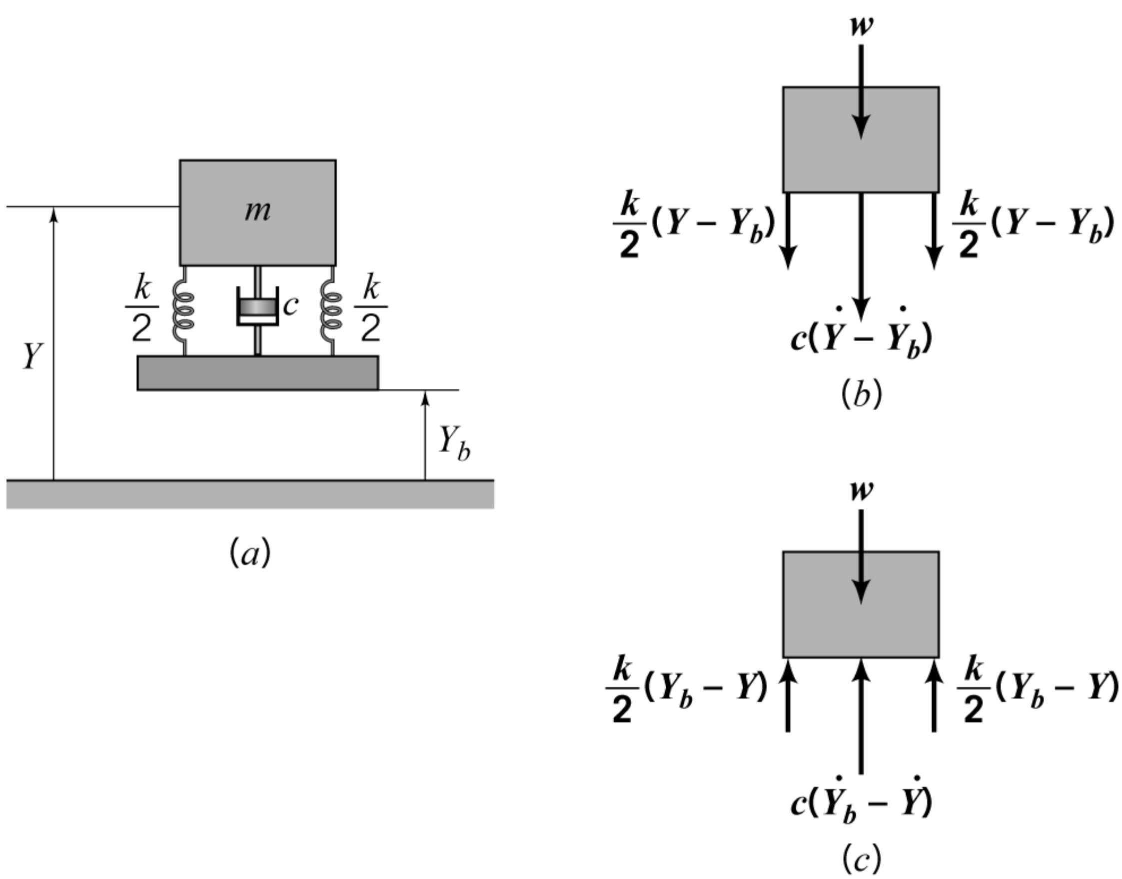


Figure 3.15.
(a). Suspended mass and movable base. (b). Free-body diagram for tension in the springs and damper, (c). Free-body diagram for compression in the springs and damper.

For free-body diagram b, assume that the base and the mass m have positive displacements $(Y > 0 \,, Y_b > 0)$ and velocities $(\dot{Y} > 0 \,, \dot{Y}_b > 0)$.

For zero base motion $(Y_b = 0)$, the springs are undeflected when $Y = 0$.

Assuming that the *m*'s displacement is greater than the base displacement $(Y > Y_b)$ means that the springs are in tension, and

the spring forces are defined by $f_s = (k/2)(Y - Y_b)$.

Assuming that the *m*'s velocity is greater than the base velocity $(\dot{Y} > \dot{Y}_b)$, the damper is also in tension, and the damping force is defined by $f_d = c(\dot{Y} - \dot{Y}_b)$.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the free-body diagram of figure 3.15b gives the differential equation of motion

$$m\ddot{Y} = \Sigma \boldsymbol{f_Y} = -w - 2 \times \frac{k}{2}(Y - Y_b) - c(\dot{Y} - \dot{Y}_b) \quad ,or$$

$$m\ddot{Y} + c\dot{Y} + kY = -w + c\dot{Y}_b + kY_b \ .$$

For free-body diagram c , the base displacement is greater than the mass displacement $(Y_b > Y)$. With this assumed relative motion, the springs are in compression, and the spring forces are defined by $f_s = (k/2)(Y_b - Y)$. Similarly, assuming that the base velocity is greater than *m*'s $(\dot{Y}_b > \dot{Y})$ causes the damper to also be in compression, and the damper force is defined by $f_d = c(\dot{Y}_b - \dot{Y})$. The free-body diagram of figure 3.15C is consistent with this assumed motion and leads to the differential equation of motion

$$m\ddot{Y} = \Sigma f_Y = -w + c(\dot{Y}_b - \dot{Y}) + k(Y_b - Y) \text{ ,or}$$

$$m\ddot{Y} + c\dot{Y} + kY = -w + kY_b + c\dot{Y}_b \text{ ,}$$

The short lesson from these developments is that the same governing equation should result for any assumed motion, since the governing equation applies for any position and velocity. Note that the following procedural steps were taken in arriving at the equation of motion:

a. The nature of the motion was assumed e.g.,$(Y_b > Y)$,

b. The spring or damper force was stated in a manner that was consistent with the assumed motion, e.g., $f_s = k/2 \times (Y_b - Y)$,

c. The free-body diagram was drawn in a manner that was consistent with the assumed motion and its resultant spring and damper forces., i.e., in tension or compression, and

d. Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ was applied to the free-body diagram to obtain the equation of motion.

Note: We have looked at only one arrangement for base excitation. Your homework has several examples for which the same procedure works, but the final equations are different.

Lecture 10. Transient Solutions 3 — More base Excitation

Example Problem XP3.3.

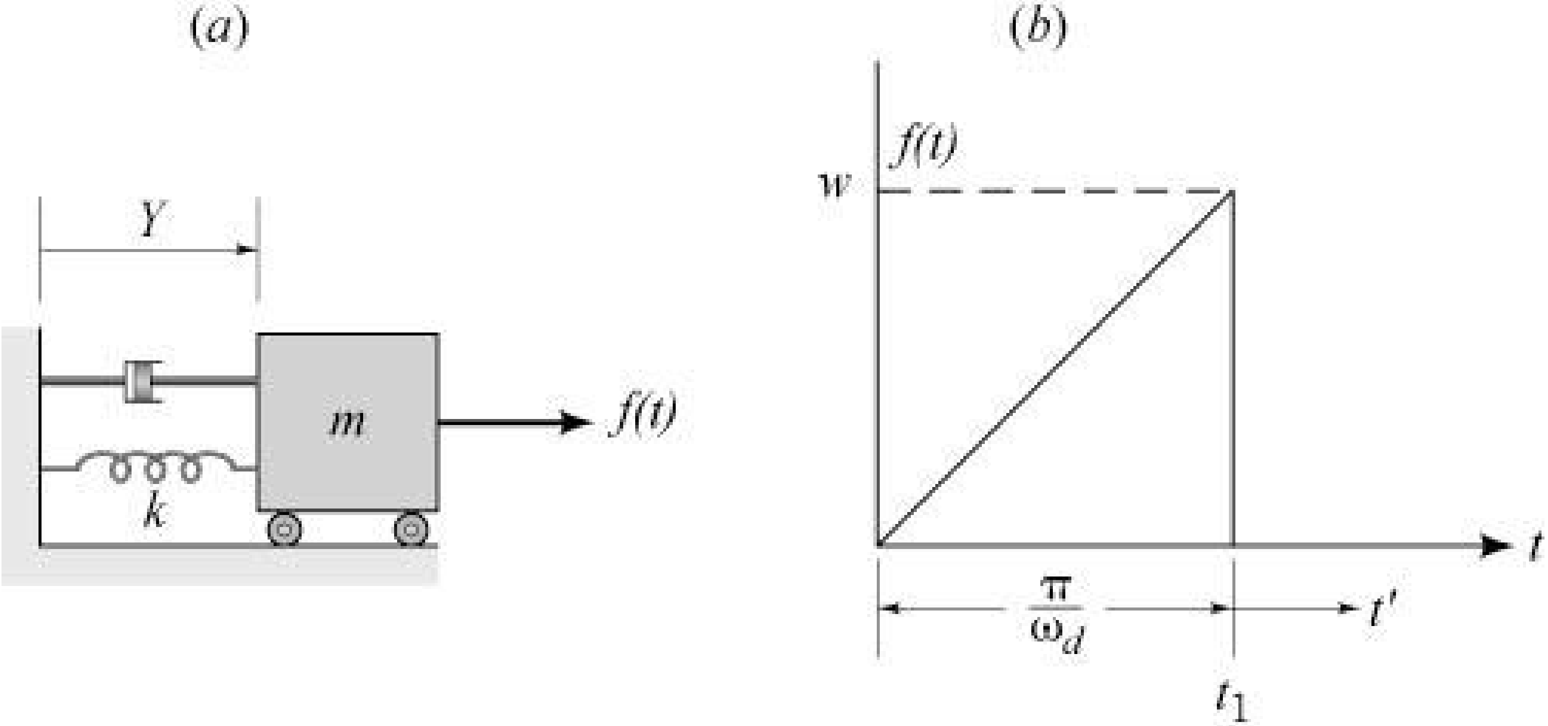


Figure XP3.3 (a). Spring-mass-damper system with an applied force, (b). Applied force definition

Figure XP3.3a illustrates a spring-mass-damper system characterized by the following parameters: ($m = 100\,kg$, $k = 9.87E+04\ N/m$, $c = 3.1416E+02 N\sec/m$), with the forcing function illustrated. The mass system start from rest with the spring undeflected. The force increases linearly until $t = t_1 = \pi/\omega_d = \tau_d/4$ when it reaches a magnitude equal to $w = mg$. For $t \geq t_1$, $f(t) = 0$.

Tasks: *Determine the mass's position and velocity at* $t = t_2 = t_1 + 2\pi/\omega_d$. Plot $Y(t)$ for $0 \leq t \leq t_1$ and $t_1 \leq t \leq t_2$.

Solution. This Example problem has the same stiffness and mass as Example problem 9.1; hence,

$$\omega_n = 31.416 \frac{rad}{\sec} \Rightarrow f_n = 5\,Hz \ , \ \zeta = 0.5 \ , \ \omega_d = 26.97 \frac{rad}{\sec}$$

The equation of motion for $0 \le t \le \pi/\omega_d$ is

$$m\ddot{Y} + c\dot{Y} + kY = ht \ ,$$

$$h = \frac{w\omega_d}{\pi} = (100kg \times 9.81 \frac{m}{\sec^2} \times 26.97 \sec^{-1})/\pi = 8422.\frac{N}{\sec} . \qquad \textbf{(i)}$$

For $t \ge \pi/\omega_d$, the equation of motion is $m\ddot{Y} + c\dot{Y} + kY = 0$.
We will use the solution for the first equation for $t = t_1 = \pi/\omega_d$ to determine $Y(t_1)$ and $\dot{Y}(t_1)$, which we will then use as initial conditions in stating the solution to the second equation of motion for $t \ge t_1$.

The applicable homogeneous solution is

$$Y_h = e^{-\zeta\omega_n t} (A \cos\omega_d t + B \sin\omega_d t) \ , \qquad \textbf{(ii)}$$

From Table B.2, the particular solution for $\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y = at$, is

$$Y_p = \frac{a}{\omega_n^2}(t - \frac{2\zeta}{\omega_n}) \ ,$$

Hence, the particular solution for $\ddot{Y}+2\zeta\omega_n\dot{Y}+\omega_n^2 Y=ht/m$ is

$$Y_p=\frac{h}{m\omega_n^2}(t-\frac{2\zeta}{\omega_n})=\frac{h}{k}(t-\frac{2\zeta}{\omega_n}) \ ,$$

and the complete solution is

$$Y=Y_h+Y_p=e^{-\zeta\omega_n t}\ (A\ \cos\omega_d t\ +\ B\ \sin\omega_d t)+\frac{h}{k}(t-\frac{2\zeta}{\omega_n})\ . \quad \textbf{(iii)}$$

The velocity is

$$\begin{aligned}\dot{Y}= & -\zeta\omega_n e^{-\zeta\omega_n t}\ (A\ \cos\omega_d t\ +\ B\ \sin\omega_d t) \\ & +\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t\ +\ B\cos\omega_d t)+\frac{h}{k}\ .\end{aligned} \quad \textbf{(iv)}$$

Solving for the constants from the initial conditions gives

$$Y(0)=0=A-\frac{2\zeta h}{k\omega_n}\ \Rightarrow\ A=\frac{2\zeta h}{k\omega_n}$$

$$\dot{Y}(0)=0=-\zeta\omega_n A+\omega_d B+\frac{h}{k}\ \Rightarrow\ B=\frac{h(2\zeta^2-1)}{k\omega_d}\ ,$$

The complete solution satisfying the initial conditions is

$$Y(t) = \frac{h}{k}\{e^{-\zeta\omega_n t}[\frac{2\zeta}{\omega_n}\cos\omega_d t + \frac{(2\zeta^2-1)}{\omega_d}\sin\omega_d t] + (t - \frac{2\zeta}{\omega_n})\} \; ; \; 0 \le t \le t_1. \qquad \textbf{(v)}$$

The complete solution for $t \le 0 \le t_1$ is illustrated below

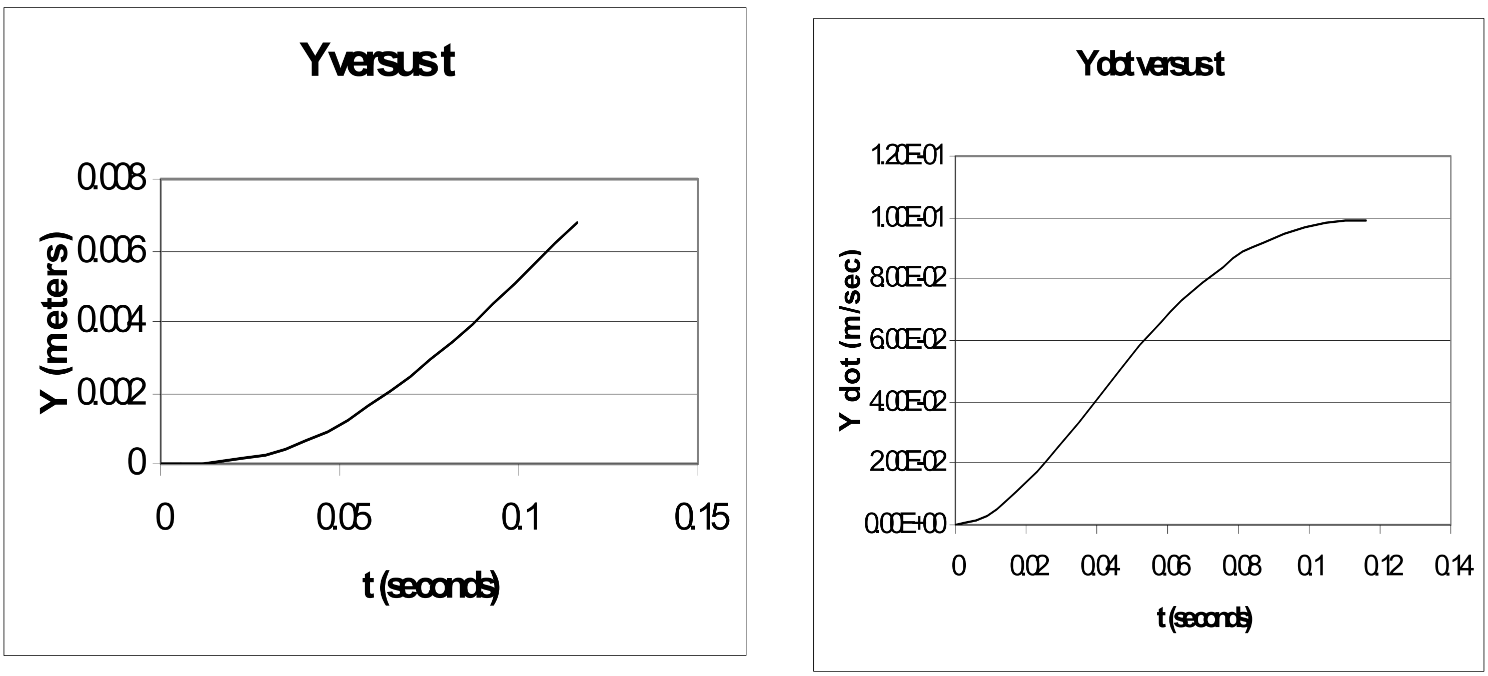


Figure XP3.3 b. Solution for $0 \le t \le t_1$.

The final conditions from figure XP3.3b are $\dot{Y}(t_1) = 9.92E-02\, m/\sec$, and $Y(t_1) = 6.766E-3\, m$ at $t_1 = \pi/\omega_d = \pi/(27.207\, rad/\sec) = 0.1162\sec$.

For $t \ge \pi/\omega_d$, the equation of motion $m\ddot{Y} + c\dot{Y} + kY = 0$ has the solution

$$Y = Y_h = e^{-\zeta\omega_n t}\ (A\ \cos\omega_d t\ +\ B\ \sin\omega_d t)$$

and derivative

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t}\ (A\ \cos\omega_d t\ +\ B\ \sin\omega_d t)$$
$$+\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t\ +\ B\cos\omega_d t)\ .$$

We could solve for the unknown constants A and B via $Y(t_1) = 6.766E{-}3\,m$ and $\dot{Y}(t_1) = 9.92E{-}02\,m/\mathrm{sec}$. We are going to take an easier tack by restarting time via the definition $t' = t - t_1$. Since,

$$\frac{DY}{dt'} = \frac{DY}{dt}\frac{dt}{dt'} = \frac{DY}{dt}\ ,$$

the equation of motion is unchanged. The solution in terms of t' is

$$Y = e^{-\zeta\omega_n t'}\ (A\ \cos\omega_d t'\ +\ B\ \sin\omega_d t')$$

with the derivative

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t'}\ (A\ \cos\omega_d t' + B\sin\omega_d t')$$
$$+\omega_d e^{-\zeta\omega_n t'}\ (-A\sin\omega_d t' + B\cos\omega_d t')\ .$$

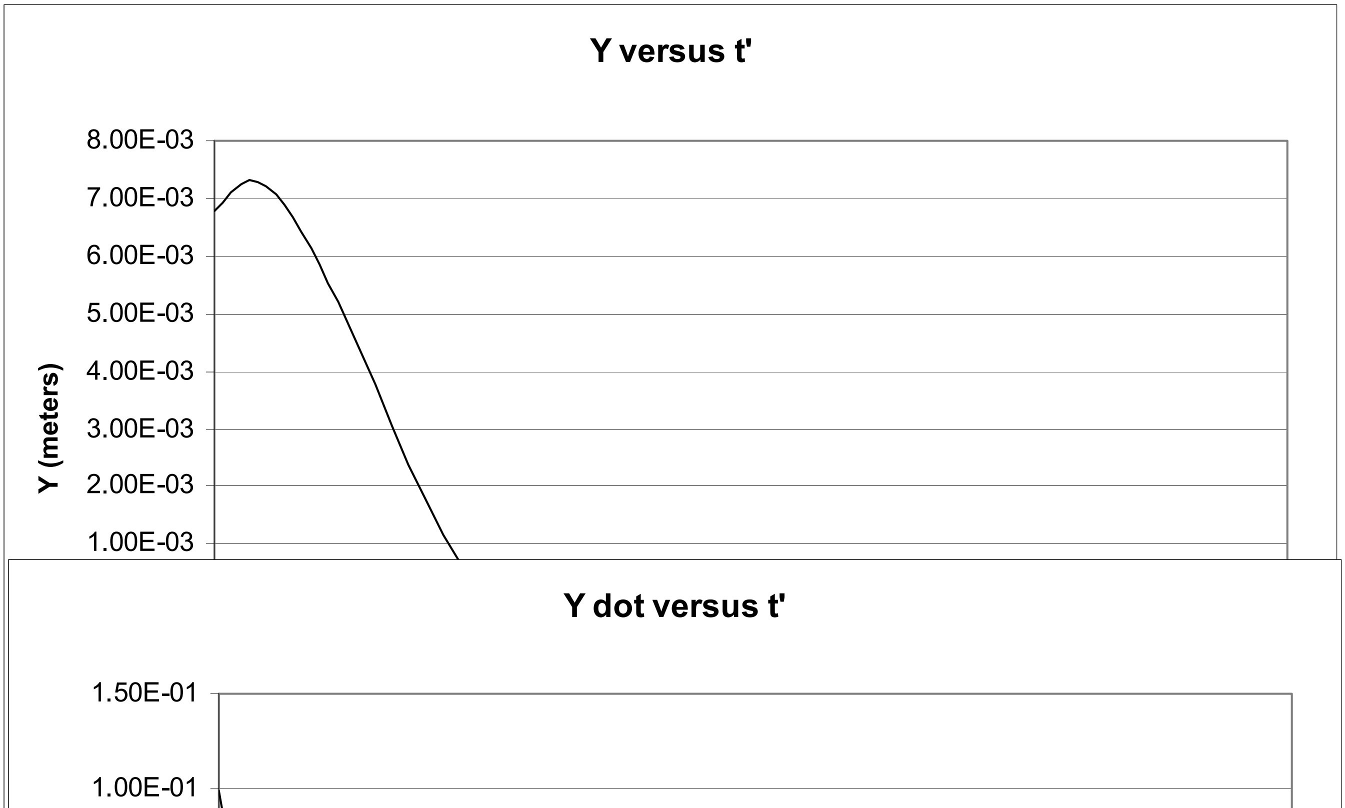
where $t' = t - t_1$, $Y(t'=0) = Y_0 = Y(t_1) = 6.766E-3\, m$, and $\dot{Y}(t'=0) = \dot{Y}_0 = \dot{Y}(t_1) = 9.92E-02\, m/\mathrm{sec}$. Again, we are basically restarting time to simplify this solution. Solving for the constants in terms of the initial conditions gives

$$Y_0 = A \ , \ \dot{Y}_0 = -\zeta\omega_n A + \omega_d B \ \Rightarrow \ B = (\dot{Y}_0 + \zeta\omega_n Y_0)/\omega_d \ ,$$

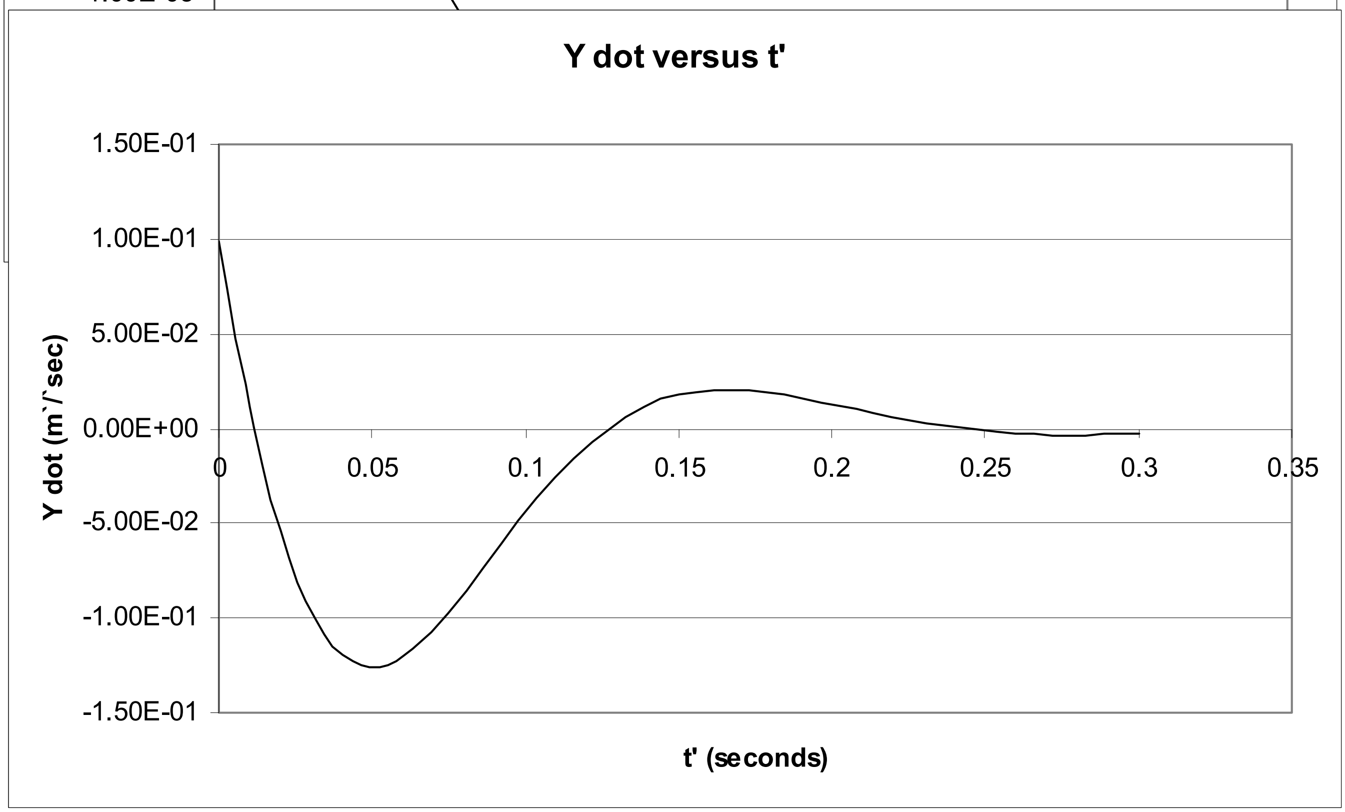
and the complete solution in terms of the initial conditions is

$$Y = e^{-\zeta\omega_n t'} \left[Y_0 \cos\omega_d t' + \frac{(\dot{Y}_0 + \zeta\omega_n Y_0)}{\omega_d} \sin\omega_d t' \right]$$

The solution is plotted below



Y versus t'
8.00E-03
7.00E-03
6.00E-03
5.00E-03
4.00E-03
3.00E-03
2.00E-03
1.00E-03
Y (meters)



Y dot versus t'
1.50E-01
1.00E-01
5.00E-02
0.00E+00
-5.00E-02
-1.00E-01
-1.50E-01
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Y dot (m`/`sec)
t' (seconds)

Example Problem Xp3.5

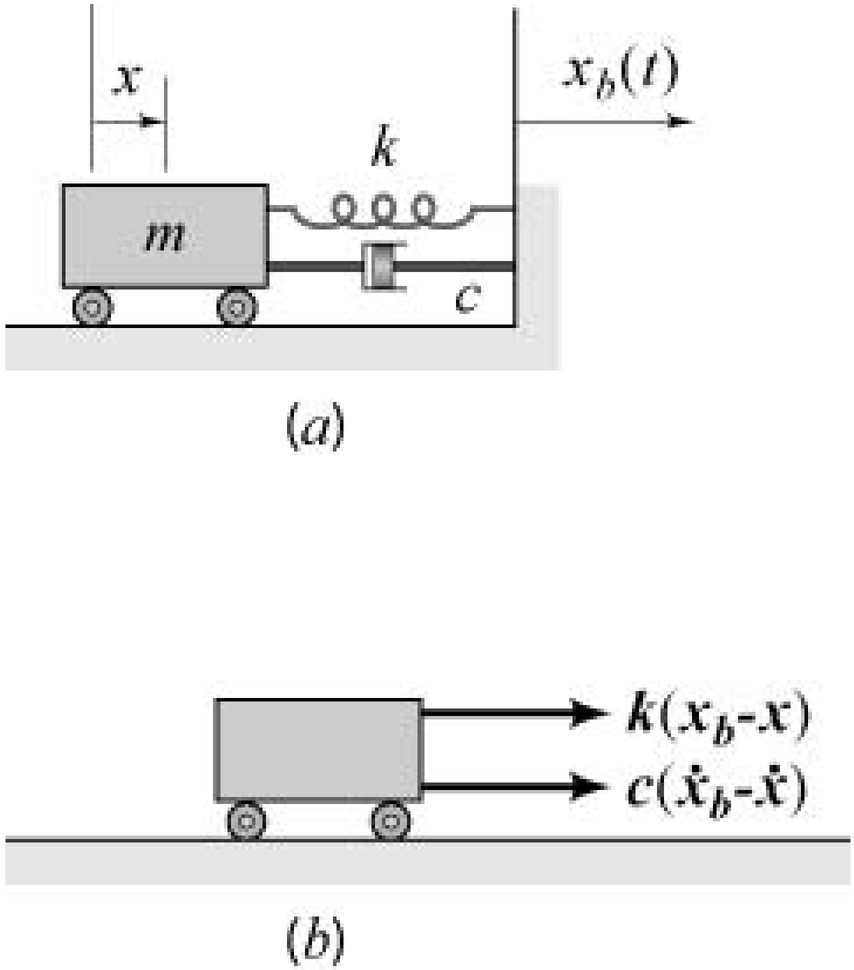


Figure Xp3.5 Movable cart with base excitation. (a). Coordinates, (b). Free-body diagram for $x_b>x, \dot{x}_b>\dot{x}$

Figure XP3.5a illustrates a cart that is connected to a movable base via a spring and a damper. The cart's change of position is defined by *x*. The movable base's location is defined by x_b.

First, we need to derive the cart's equation of motion, starting with an assumption of the relative positions and velocities. The

free-body diagram of XP 3.4a is drawn assuming that $x_b > x > 0$ (placing the spring in tension) and $\dot{x}_b > \dot{x} > 0$
(placing the damper in tension). Applying Newton's 2nd law of motion $\sum \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ nets

$$m\ddot{x} = \sum f_x = k(x_b - x) + c(\dot{x}_b - x) \qquad \textbf{(i)}$$

$$\therefore \; m\ddot{x} + c\dot{x} + kx = c\dot{x}_b + kx_b$$

In Eq.(i), the spring and damper forces are positive because they are in the $+x$ direction.

The model of Eq.(i) can be used to approximately predict the motion of a small trailer being towed behind a much larger vehicle with the spring and damper used to model the hitch connection between the trailer and the towing vehicle. (Note: To correctly model two vehicles connected by a spring and damper, we need to write $\sum \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for both bodies and account for the influence of the trailer's mass on the towing vehicle's motion. We will consider motion of connected bodies following the next test.)

Assume that the cart and the base are initially motionless with the spring connection undeflected. The base is given the constant acceleration $\ddot{x}_b = g/5$. Complete the following engineering analysis tasks:

a. State the equation of motion.

b. State the homogeneous and particular solutions, and state the complete solution satisfying the initial conditions.
c. For $w = 300\, lbs, k = 776.\, lb/in, c = 24.4\, lb\sec/in$, plot the cart's motion for two periods of damped oscillations.

Stating the equation of motion simply involves proceeding from $\ddot{x}_b = g/5$ to find $\dot{x}_b, x_b$ and plugging them into Eq.(i), via

$$\dot{x}_b = \dot{x}_b(0) + \int_0^t \ddot{x}_b d\tau = \int_0^t \frac{g}{5} d\tau = \frac{gt}{5}$$

$$x_b = x_b(0) + \int_0^t \dot{x}_b d\tau = \frac{gt^2}{10} \quad .$$

Substituting these results into Eq.(i) produces

$$m\ddot{x} + c\dot{x} + kx = c\dot{x}_b + kx_b = \frac{g}{5}(ct + k\frac{t^2}{2})$$

The homogeneous solution is

$$x_h = e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t)$$

We can pick the particular solutions from Table B.2 below as follows

Table B.2. Particular solutions for $\ddot{Y}_p + 2\zeta\omega_n \dot{Y}_p + \omega_n^2 Y_p = u(t)$.

Excitation, $u(t)$	$Y_p(t)$
$h = constant$	h/ω_n^2
$a\,t$	$\frac{a}{\omega_n^2}(t - \frac{2\zeta}{\omega_n})$
bt^2	$\frac{b}{\omega_n^2}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$

$$\frac{gc}{5m}t \Rightarrow x_{p1} = \frac{gc}{5m}\frac{1}{\omega_n^2}(t - \frac{2\zeta}{\omega_n}) = \frac{2\zeta g}{5\omega_n}(t - \frac{2\zeta}{\omega_n})$$

$$\frac{gk}{10m}t^2 \Rightarrow x_{p2} = \frac{gk}{10m}\frac{1}{\omega_n^2}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$$

$$= \frac{g}{10}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)] \ .$$

The complete solution is

$$x = x_h + x_{p1} + x_{p2}$$

$$= e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t)$$
$$+ \frac{2\zeta g}{5\omega_n}(t - \frac{2\zeta}{\omega_n}) + \frac{g}{10}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$$

The constant A is obtained via,

$$x(0) = 0 = A - \frac{4\zeta^2 g}{5\omega_n^2} - \frac{g}{5\omega_n^2}(1 - 4\zeta^2) \Rightarrow A = \frac{g}{5\omega_n^2} \ .$$

To obtain B, first

$$\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}\,(A\cos\omega_d t + B\sin\omega_d t)$$
$$+\,\omega_d e^{-\zeta\omega_n t}\,(-A\sin\omega_d t + B\cos\omega_d t)$$
$$+\frac{2\zeta g}{5\omega_n} + \frac{g}{10}(2t - \frac{4\zeta}{\omega_n})\ .$$

Hence,

$$\dot{x}(0) = 0 = -\zeta\omega_n A + \omega_d B + \frac{2\zeta g}{5\omega_n} - \frac{2\zeta g}{5\omega_n} \Rightarrow B = \frac{g\zeta}{5\omega_n^2\sqrt{1-\zeta^2}}\ .$$

The complete solution satisfying the boundary conditions is

$$x = e^{-\zeta\omega_n t}\,[\frac{g}{5\omega_n^2}\cos\omega_d t + \frac{g\zeta}{5\omega_n^2\sqrt{1-\zeta^2}}\sin\omega_d t]$$
$$+\frac{2\zeta g}{5\omega_n}(t - \frac{2\zeta}{\omega_n}) + \frac{g}{10}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1-4\zeta^2)]\ , \qquad \textbf{(ii)}$$

and Eq.(ii) completes Task b.

Moving towards Task c,
$m = w/g = 300\,lbs/(386.4\,in/\text{sec}^2) = 0.766\,lb/(\text{sec}^2\,in)$.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{776.\,lb/in}{.776\,lb\sec^2/in}} = 31.62\,\frac{rad}{\sec}$$

$$\therefore\ f_n = 31.62\frac{rad}{\sec}\times\frac{1\,cycle}{2\pi\,rad} = 5.03\frac{cycles}{\sec} = 5.03\,Hz$$

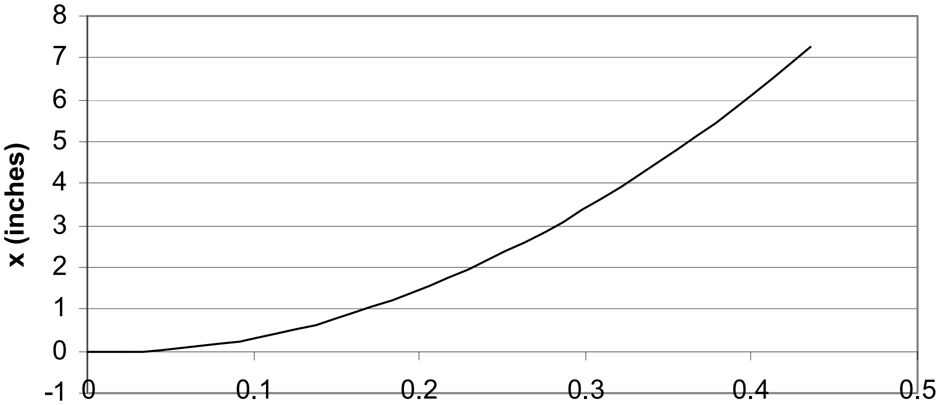
$$\zeta = \frac{c}{2m\omega_n} = 24.4\,\frac{lb\sec}{in}\times\frac{1}{2\times .776\,snails\times 31.62\sec^{-1}} = 0.497$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.62\sqrt{1-0.497^2} = 27.4\,\frac{rad}{\sec}$$

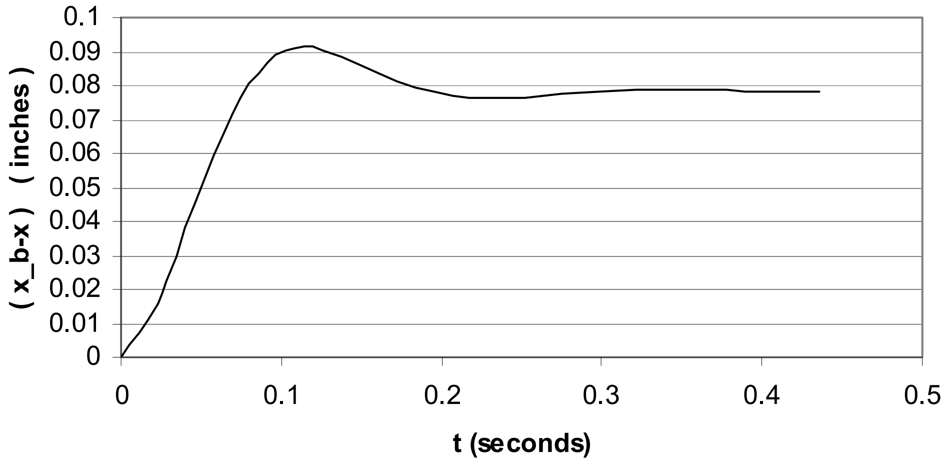
$$\therefore\ f_d = 27.4\,\frac{rad}{\sec}\times\frac{1\,cycle}{2\pi\,rad} = 4.36\,\frac{cycles}{\sec} = 4.36\,Hz\ .$$

Two cycles of damped oscillations will be completed in $2\times\tau_d$ seconds, where $\tau_d = 1/f_d = 1/4.36\,(cycles/\sec) = 0.229\sec/cycle$. The solutions are shown below.

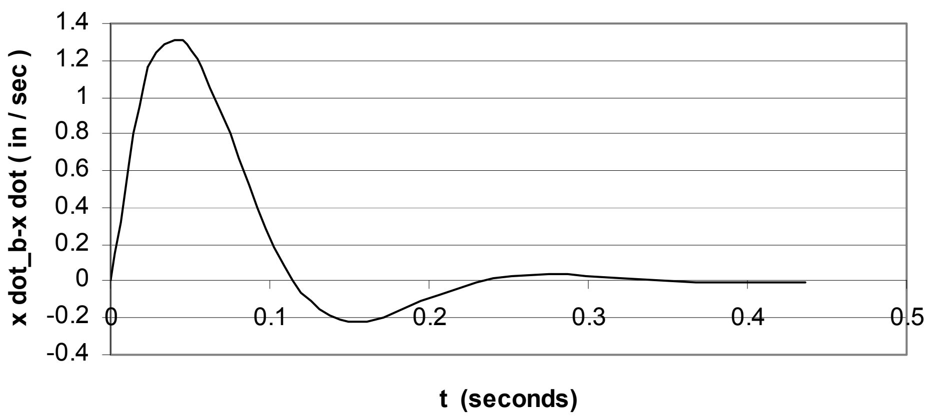
x versus t



x_b -x versus t



(x dot_b-x dot) versus t



Discussion. After an initial transient, the relative displacement

$(x_b - x)$ approaches a constant value $(x_b - x) \cong 0.078\,in$, while the relative velocity approaches zero; i.e., $(\dot{x}_b - \dot{x}) \cong 0$. The scaling for $x(t)$ hides the initial transient, but the plot shows quadratic increase with time, which is consistent with its constant acceleration at $g/5$. The spring force $k(x - x_b) \cong (776.lbs/in) \times .078\,in \cong 60.1\,lbs$, while the force required to accelerate the mass at $g/5$ is

$$f = m\ddot{x} = 0.776\left(\frac{lb\sec}{in^2}\right) \times \frac{386.4}{5} = 60\,lbs \quad .$$

Hence, the asymptotic value for $(x_b - x)$ creates the spring force required to accelerate the mass at $g/5$.

Example XP3.6

Think about an initially motionless cart being "snagged" by a moving vehicle with velocity v_0 through a connection consisting of a parallel spring-damper assembly. The connecting spring is undeflected prior to contact. That circumstance can be modeled by giving the base end of the model in figure XP3.5 the velocity v_0; i.e., $\dot{x}_b = v_0 \Rightarrow x_b = x_b(0) + v_0 t = v_0 t$, and the governing equation of motion is

$$m\ddot{x} + c\dot{x} + kx = cv_0 + kv_0 t \ , \qquad \textbf{(iii)}$$

with initial conditions $x(0) = \dot{x}(0) = 0$. The engineering analysis tasks for this example are:

a. Determine the complete solution that satisfies the initial conditions.

b. For the data set, $m = 10\,kg$, $k = 1580\,N/m$, and $c = 1257\,N/m$, determine $\omega_n, \zeta, \omega_d$.

c. For $v_0 = 20\,km/hr$ produce plots for the mass displacement $x(t)$, relative displacement $x_b(t) - x(t)$, and the mass velocity $\dot{x}(t)$ for two cycles of motion.

Solution. From Table B.2, the particular solutions corresponding to the right-hand terms are:

$$\frac{cv_0}{m} \Rightarrow x_{p1} = \frac{cv_0}{m}\frac{1}{\omega_n^2} = \frac{2\zeta v_0}{\omega_n}$$

$$\frac{kv_0}{m}t \Rightarrow x_{p2} = \frac{kv_0}{m}\frac{1}{\omega_n^2}\left(t - \frac{2\zeta}{\omega_n}\right) = v_0\left(t - \frac{2\zeta}{\omega_n}\right) .$$

The complete solution is

$$x = x_h + x_{p1} + x_{p2} = e^{-\zeta\omega_n t}\left(A \cos\omega_d t + B \sin\omega_d t\right) + \frac{2\zeta v_0}{\omega_n} + v_0\left(t - \frac{2\zeta}{\omega_n}\right).$$

Solving for A ,

$$x(0) = 0 = A + \frac{2\zeta v_0}{\omega_n} - \frac{2\zeta v_0}{\omega_n} \Rightarrow A = 0 .$$

Solving for B from

$$\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + \omega_d e^{-\zeta\omega_n t}\left(-A\sin\omega_d t + B\cos\omega_d t\right) + v_0 ,$$

gives

$$\dot{x}(0) = 0 = -\zeta\omega_n A + \omega_d B + v_0 \;\Rightarrow\; B = -\frac{v_0}{\omega_d} \;.$$

The complete solution satisfying the initial conditions is

$$x = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t + \frac{2\zeta v_0}{\omega_n} + v_0 \left(t - \frac{2\zeta}{\omega_n}\right) \;, \qquad \textbf{(iv)}$$

which completes Task a.

For Task b,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1580}{10}} = 12.57 \frac{rad}{\sec}$$

$$\therefore\; f_n = 12.57 \frac{rad}{\sec} \times \frac{1\,cycle}{2\pi\,rad} = 2 \frac{cycles}{\sec} = 2\,Hz$$

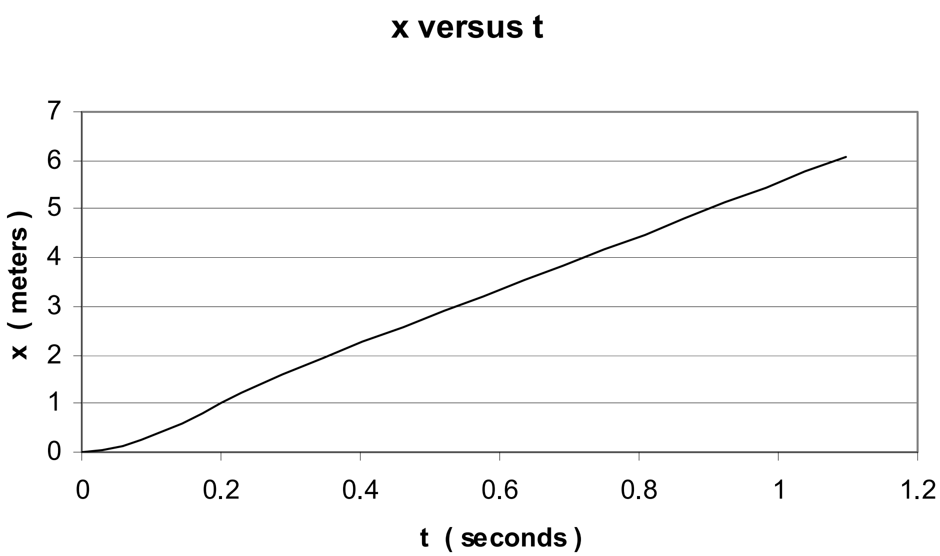
$$\zeta = \frac{c}{2m\omega_n} = 1257 \frac{N\sec}{m} \times \frac{1}{2 \times 10\,Kg \times 12.57 \sec^{-1}} = 0.5$$

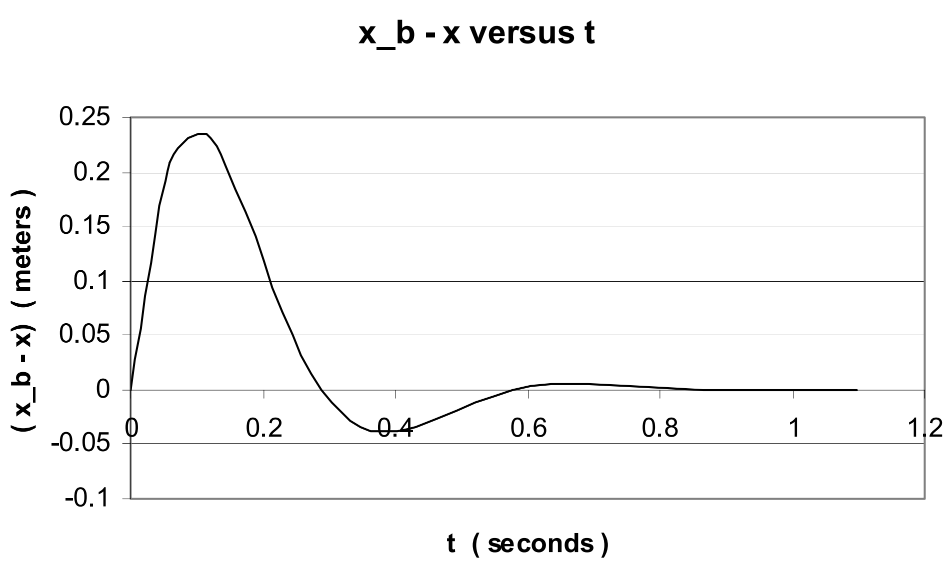
$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 12.57 \sqrt{1 - 0.5^2} = 10.89 \frac{rad}{\sec}$$

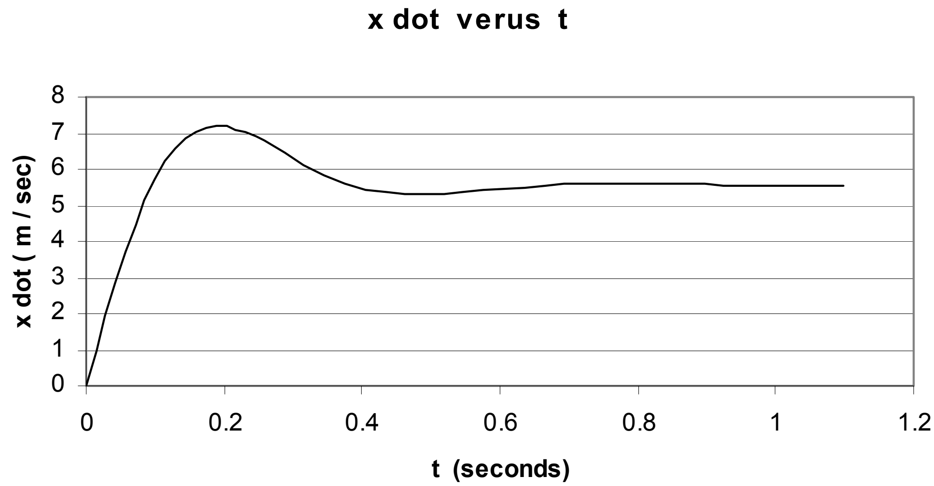
$$\therefore\; f_d = 10.89 \frac{rad}{\sec} \times \frac{1\,cycle}{2\pi\,rad} = 1.73 \frac{cycles}{\sec} = 1.73\,Hz \;.$$

From the last of these results, the period for a damped oscillation is $\tau_d = 1/f_d = 0.577\,\sec$.

For Task c,
$v_0 = 20\,km/hr \times 1\,hr/3600\,\sec \times 1000\,m/km = 5.55\,m/\sec$. The figures below illustrate the solution for two cycles of motion.







Discussion. The first and last figures show the mass rapidly moving towards the towing velocity $v_0 = 5.55\, m/\sec$. The second figure shows the relative position approaching zero. Note that at about 0.3 *seconds*, the towed cart appears to actually passes the towing vehicle when $(x_b - x)$ becomes negative. However, the present analysis does not account for the initial spring length. A the towed vehicle could "crash" into the back of the towing vehicle, but it's not likely. You could plot $(v_0 - \dot{x}_b)$ to find the relative velocity if impact occurs.

LECTURE 11. HARMONIC EXCITATION

Forced Excitation

$$m\ddot{Y} + c\dot{Y} + kY = f_o \sin\omega t \ . \qquad \textbf{(3.32)}$$

Dividing through by the mass m gives

$$\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = (f_o/m)\sin\omega t \ . \qquad \textbf{(3.33)}$$

Seeking a particular solution for the right hand term gives

$$Y_p = C\sin\omega t + D\cos\omega t$$

$$\dot{Y}_p = \omega C\cos\omega t - \omega D\sin\omega t \qquad \textbf{(3.34)}$$

$$\ddot{Y}_p = -\omega^2 C\sin\omega t - \omega^2 D\cos\omega t.$$

Substituting this solution into Eq.(3.32) yields

$$(-\omega^2 C\sin\omega t - \omega^2 D\cos\omega t)$$

$$+ 2\zeta\omega_n(C\omega\cos\omega t - \omega D\sin\omega t)$$

$$+\omega_n^2 (C\sin\omega t + D\cos\omega t) = (f_0/m)\sin\omega t \ .$$

Gathering the $\sin\omega t$ and $\cos\omega t$ coefficients gives the following two equations:

$$\sin\omega t:\ -\omega^2 C - 2\zeta\omega_n\omega D + \omega_n^2 C = f_o/m$$
$$\cos\omega t:\ -\omega^2 D + 2\zeta\omega_n\omega C + \omega_n^2 D = 0\ .$$

The matrix statement for the unknowns D and C is

$$\begin{bmatrix} (\omega_n^2 - \omega^2) & -2\zeta\omega_n\omega \\ 2\zeta\omega_n\omega & (\omega_n^2 - \omega^2) \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} f_0/m \\ 0 \end{Bmatrix}\ .$$

Using Cramer's rule for their solution gives:

$$C = \frac{1}{\Delta}\begin{vmatrix} f_o/m & -2\zeta\omega_n\omega \\ 0 & (\omega_n^2 - \omega^2) \end{vmatrix} = \frac{f_o(\omega_n^2 - \omega^2)}{m\Delta}$$

$$D = \frac{1}{\Delta}\begin{vmatrix} (\omega_n^2 - \omega^2) & f_o/m \\ 2\zeta\omega_n\omega & 0 \end{vmatrix} = \frac{-f_o\ 2\ \zeta\omega_n\omega}{m\Delta}\ . \tag{3.35}$$

where Δ is the determinant of the coefficient matrix defined by

$$\Delta = (\omega_n^2 - \omega^2)^2 + 4\ \zeta^2\ \omega_n^2\ \omega^2\ .$$

The solution defined by Eq.(3.34) can be restated

$$Y_p(t) = C\sin\omega t + D\cos\omega t$$

$$= Y_{op}\sin(\omega t + \psi) \qquad \mathbf{(3.36)}$$

$$= Y_{op}\,(\sin\omega t\cos\psi + \cos\omega t\sin\psi)\ .$$

where Y_{op} is the amplitude of the solution, and ψ is the phase between the solution $Y_p(t)$ and the input excitation force $f(t) = f_o\sin\omega t$. Figure 3.11 illustrates the phase relation between $f(t)$ and $Y_p(t)$, showing $Y_p(t)$ "leading" $f(t)$ by the phase angle ψ.

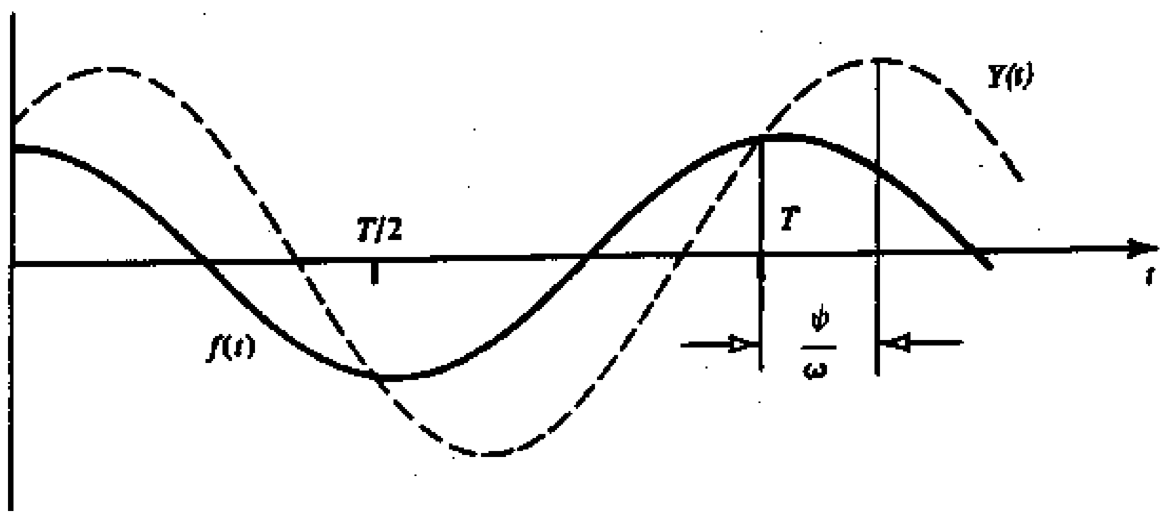


Figure 3.11 Phase relation between the response $Y_P(t)$ and the input harmonic excitation force $f(t) = f_o\sin\omega t$.

The solution for C and D provided by Eq.(3.35) gives:

$$\begin{aligned} Y_{op} &= \sqrt{C^2 + D^2} \\ &= \frac{f_o}{m} \cdot \frac{1}{\Delta} \left[(\omega_n^2 - \omega^2)^2 + 4\,\zeta^2\,\omega_n^2\,\omega^2 \right]^{1/2} \\ &= \frac{f_o}{m} \cdot \frac{1}{\left[(\omega_n^2 - \omega^2)^2 + 4\,\zeta^2\,\omega_n^2\,\omega^2 \right]^{1/2}} \\ &= \frac{f_o}{m\left(\dfrac{k}{m}\right)} \cdot \frac{1}{\left\{ [1 - (\omega/\omega_n)^2]^2 + 4\zeta^2(\omega/\omega_n)^2 \right\}^{1/2}} \,, \end{aligned}$$

where

$$\frac{\omega}{\omega_n} = \textit{Frequency ratio} = r \;, \quad \frac{f_o}{k} = \textit{static deflection}$$

Amplification Factor

$$\frac{Y_{op}}{f_o/k} = \frac{1}{\left\{ [1 - r^2]^2 + 4\zeta^2 r^2 \right\}^{1/2}} = H(r) \;. \qquad (3.38)$$

The maximum amplification factor $H(r)$ is found from

$dH(r)/dr = 0$ as

$$r_{\max} = \sqrt{1 - 2\zeta^2} \qquad \textbf{(3.40a)}$$

Note that $\omega = \omega_n \Rightarrow r = 1$

$$H(r = 1) = \frac{1}{2\zeta} \doteq q\text{-}factor$$

<u>This is another way to characterize damping.</u>

Eqs.(3.35) and (3.36) define the *phase* as

$$\psi(\omega/\omega_n) = \tan^{-1}(\frac{D}{C}) = \tan^{-1}\left[\frac{-2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}\right]$$

$$= -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right]. \qquad (3.39)$$

Complete Solution

$$Y = Y_h + Y_p = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + Y_{op}\sin(\omega t + \psi) .$$

The particular solution is frequently referred to as the"steady-state" solution because of this "persisting" nature. It continues indefinitely after the homogeneous solution has disappeared.

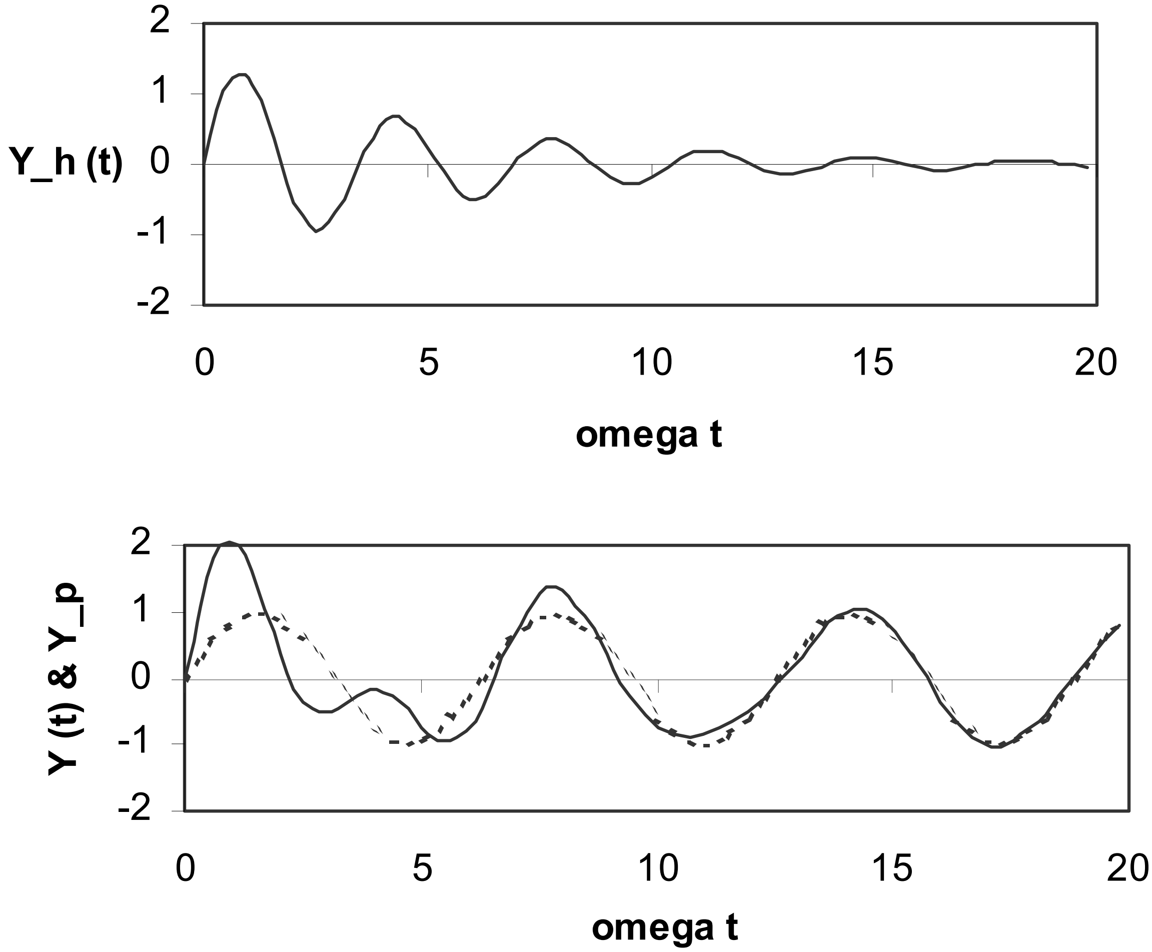


Figure 3.15 (a). Homogeneous solution for $\zeta<1$.
(b). Complete $[Y(t) = Y_h(t) + Y_p(t)]$ and steady-state particular solution $Y_p(t)$ for harmoinc excitation $f_o \sin\omega t$

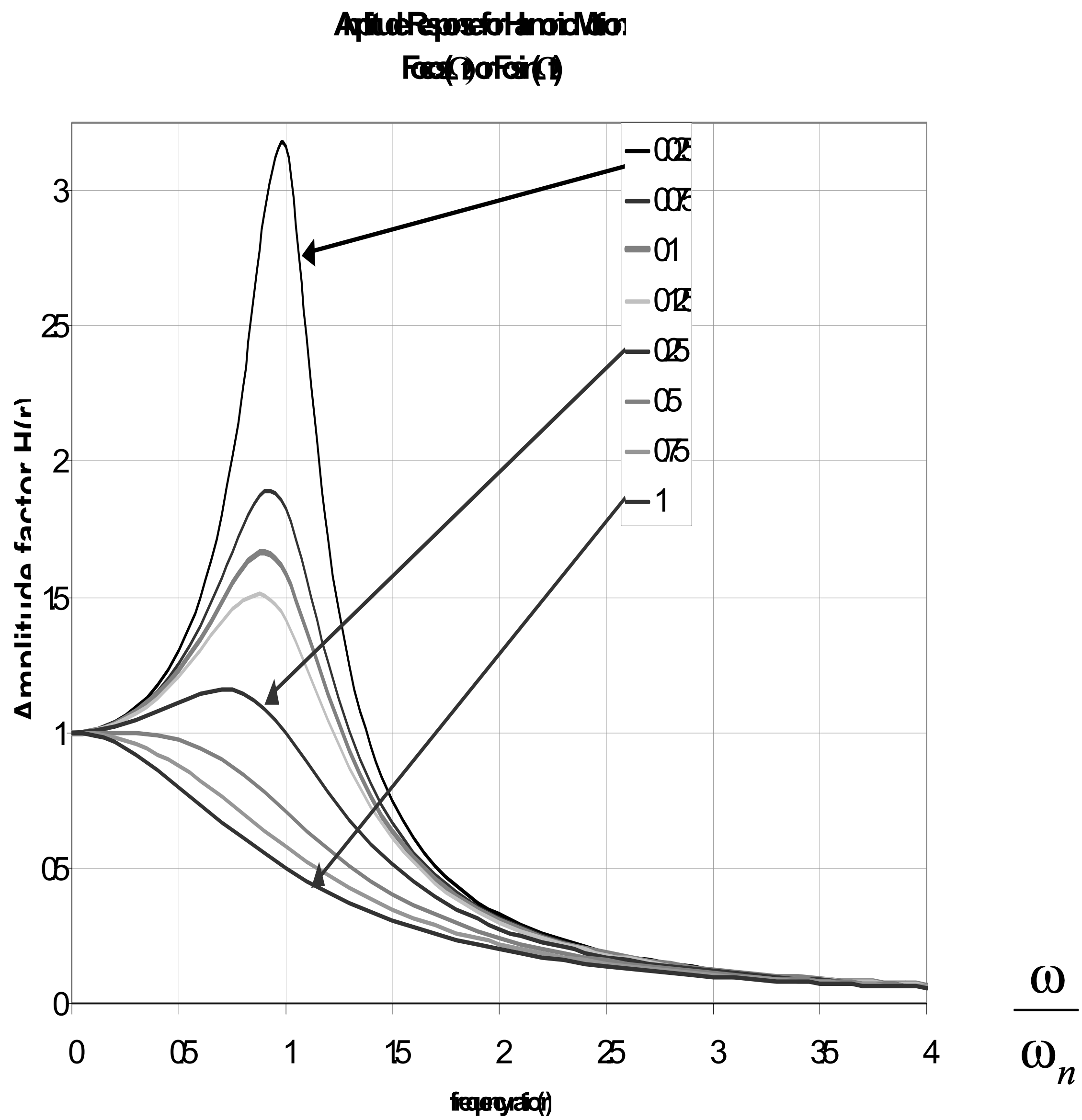


Figure 3.13 Amplification factor for harmonic motion

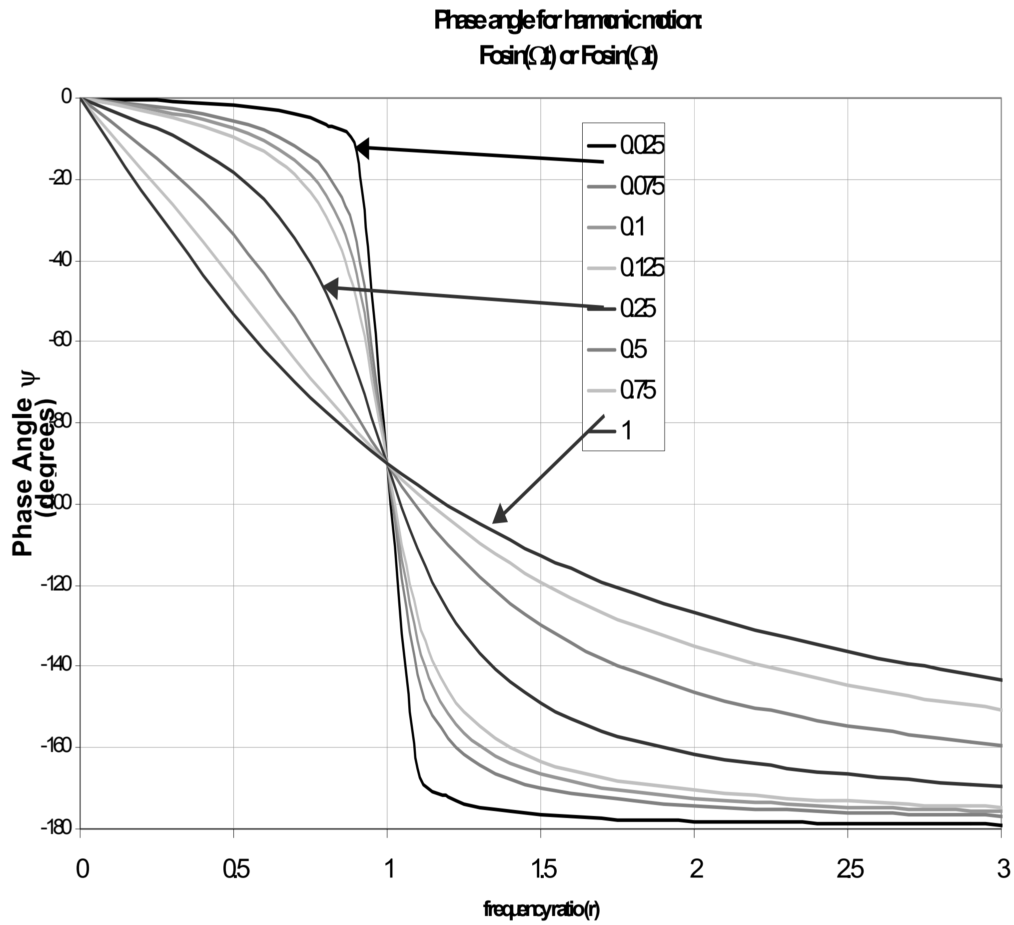


Figure 3.14 Phase angle for harmonic excitation.

Example Problem 3.8 The spring-mass-damper system of figure XP 3.8 is acted on by the external harmonic force $f = f_0 \cos\omega t$, netting the differential equation of motion

$$m\ddot{Y} + c\dot{Y} + kY = f_o \cos\omega t \ . \qquad \textbf{(i)}$$

For the data, $m = 30\ kg$, $k = 12000\ N/m$, $f_o = 200\ N$, *carry out the following engineering-analysis tasks:*

a. For $c = 0$, determine the range of excitation frequencies for which the amplitudes will be less than 76 mm,

b. Determine the damping value that will keep the steady-state response below 76 mm for all excitation frequencies.

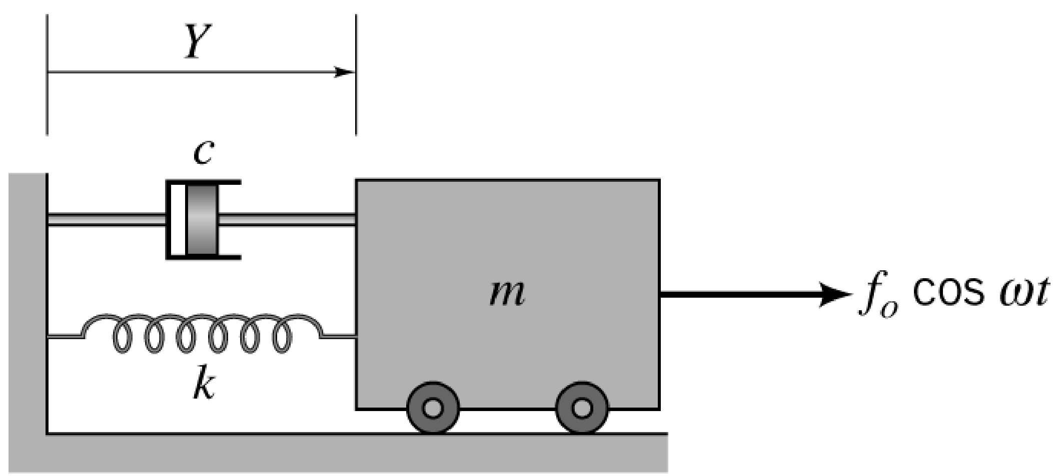


Figure XP3.8 Harmonically excited spring-mass-damper system.

Solution. Note first that Eq.(i) has $f_o \cos\omega t$ as the excitation term, versus $f_o \sin\omega t$ in Eq.(3.32). This change means that the steady-state response is now $Y_p(t) = Y_{op}\cos(\omega t + \psi)$ instead of $Y_p(t) = Y_{op}\sin(\omega t + \psi)$ of Eq.(3.36). The steady-state amplification factor and phase continue to be defined by Eqs.(3.38a) and (3.39), respectively. From the data provided, we can calculate

$$\omega_n = \sqrt{k/m} = \sqrt{12000/30} = 20\, rad/\sec$$

$$\delta_{static} = f_0/k = 200\, N/(12000\, N/m) = 1.667\times 10^{-2}\, m \ .$$

The amplification factor corresponding to the amplitude $Y_{op} = 76\, mm = 7.6\times 10^{-2} m$ is
$H(r) = 7.6\times 10^{-2} m/(1.667\times 10^{-2} m) = 4.56$. Looking back at figure 3.14, this amplification factor would be associated with frequency ratios that are fairly close to $r = 1$. For zero damping, We can use Eq.(3.38a) to solve for the two frequency ratios, via

$$\frac{Y_{op}}{f_o/k} = \frac{1}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}} \Rightarrow 4.56 = \frac{1}{[(1 - r^2)^2]^{1/2}} \ .$$

Restating the last equation gives

$$1 - 2r^2 + r^4 = \frac{1}{4.56^2} = .0481 \Rightarrow r^4 - 2r^2 + .952 = 0 \ .$$

Solving this quadratic equation defines the two frequencies by:

$$r_1^2 = .781 \;\Rightarrow\; r_1 = .883 \;,\;\; \omega_1 = r_1 \times \omega_n = .883 \times 20\, rad/\sec$$
$$= 17.7\, rad/\sec$$

$$r_2^2 = 1.10 \;\Rightarrow\; r_2 = 1.05 \;,\;\; \omega_2 = r_2 \times \omega_n = 1.05 \times 20\, rad/\sec$$
$$= 21.0\, rad/\sec \;\; .$$

As expected, ω_1 and ω_2 are close to the natural frequency ω_n. The steady-state amplitudes will be less than the specified 76 *mm* , for the frequency ranges, $0 < \omega < \omega_1 = 17.7\, rad/\sec$ and $\omega > \omega_2 = 21.0\, rad/\sec$, and we have competed *Task a*.

Moving to *Task b*, the amplification factor is a maximum at $r = \sqrt{1 - \zeta^2}$, and its maximum value is $H_{\max} = 1/2\zeta\sqrt{1 + 2\zeta^2}$. Hence,

$$H_{\max} = \frac{1}{2\zeta\sqrt{1 + 2\zeta^2}} = 4.56 \;\Rightarrow\; \zeta = .1084 \;\; ,$$

and the required damping to achieve this ζ value is, from Eq.(3.22),

$$c = 2\zeta\omega_n m = 2(.1084)(20.\frac{rad}{\sec})(30\,kg) = 130.0\,N\sec/m \ .$$

Note that the derived units for the newton is $kg\,m/\sec^2$; hence, $kg/\sec \Rightarrow N\sec/m$. Damping coefficients of this value or higher will keep the peak response amplitudes at less than the specified 76 *mm*, and *Task b* is completed.

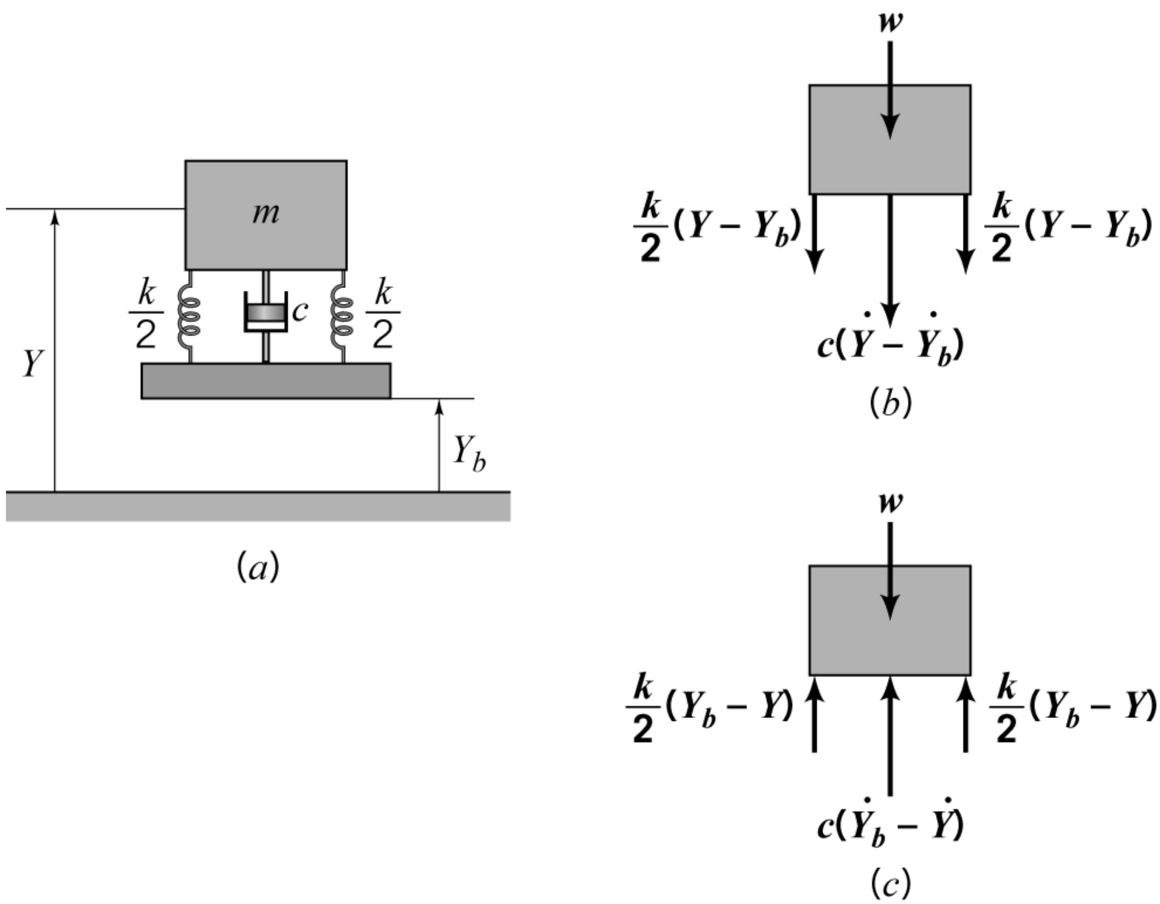


Figure 3.15. (a). Suspended mass and movable base. (b). Free-body diagram for tension in the springs and damper, (c). Free-body diagram for compression in the springs and damper.

Harmonic Base Excitation. Base harmonic base motion can be defined by $Y_b = A\sin\omega t$, $\dot{Y}_b = A\,\omega\cos\omega t$, and $m\ddot{Y} + c\dot{Y} + kY = -w + kY_b + c\dot{Y}_b$ becomes

$$m\ddot{Y} + c\dot{Y} + kY = -w + kA\sin\omega t + cA\omega\cos\omega t \ .$$

Dividing through by *m* gives

$$\begin{aligned}\ddot{Y} + 2\zeta\omega_n\dot{Y} + \omega_n^2 Y \\ &= -g + A(\omega_n^2\sin\omega t + 2\zeta\omega_n\omega\cos\omega t) \\ &= -g + B\sin(\omega t + \varphi) \\ &= -g + B(\sin\omega t\cos\varphi + \cos\omega t\sin\varphi) \ .\end{aligned} \tag{3.41}$$

We can solve for *B* and φ from the last two lines of this equation, obtaining

$$B = A\,\omega_n^2[1 + 4\zeta^2(\omega/\omega_n)^2]^{1/2} \ , \quad \varphi = \tan^{-1}[2\zeta(\frac{\omega}{\omega_n})] \ . \tag{3.42}$$

Looking at Eq.(3.41), our sole interest is the steady-state solution due to the harmonic excitation term $B\sin(\omega t + \varphi)$.

Based on the earlier results of Eq.(3.36), the expected steady-state solution format to Eq.(3.41) is

$$Y_p = Y_{op}\sin(\omega t + \varphi + \psi) \ , \tag{3.43}$$

with ψ defined by Eq.(3.39). This solution is sinusoidal at the input frequency ω, having the same phase lag ψ with respect to the input force excitation $B\sin(\omega t + \varphi)$ as determined earlier for the harmonic force excitation $f(t) = f_o \sin(\omega t)$.

Comparison Eqs.(3.33) and (3.41), shows B replacing f_0/m. Substituting $B = f_0/m$ into Eq.(3.37) gives

$$Y_{op} = \frac{B}{\omega_n^2} \cdot \frac{1}{\left\{ [1 - (\omega/\omega_n)^2]^2 + 4\zeta^2(\omega/\omega_n)^2 \right\}^{1/2}}$$

$$= A \cdot \frac{[1 + 4\zeta^2(\omega/\omega_n)^2]^{1/2}}{\left\{ [1 - (\omega/\omega_n)^2]^2 + 4\zeta^2(\omega/\omega_n)^2 \right\}^{1/2}} .$$

Hence, the ratio of the steady-state-response amplitude to the base-excitation amplitude is

$$\frac{Y_{op}}{A} = \frac{[1 + 4\zeta^2 r^2]^{1/2}}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}} = G(r) . \qquad (3.44)$$

Figure 3.16 illustrates $G(r)$, showing a strong similarity to $H(r)$ of figure 3.13 with the peak amplitudes occurring near $r = \omega/\omega_n = 1$. For $\zeta = 0$, the two transfer functions coincide. The maximum for $G(r)$ is obtained via $dG(r)/dr = 0$, yielding

$$r_{max} = \frac{1}{2\zeta}[(1+8\zeta^2)^{1/2} - 1]^{1/2} \quad . \qquad \textbf{(3.46)}$$

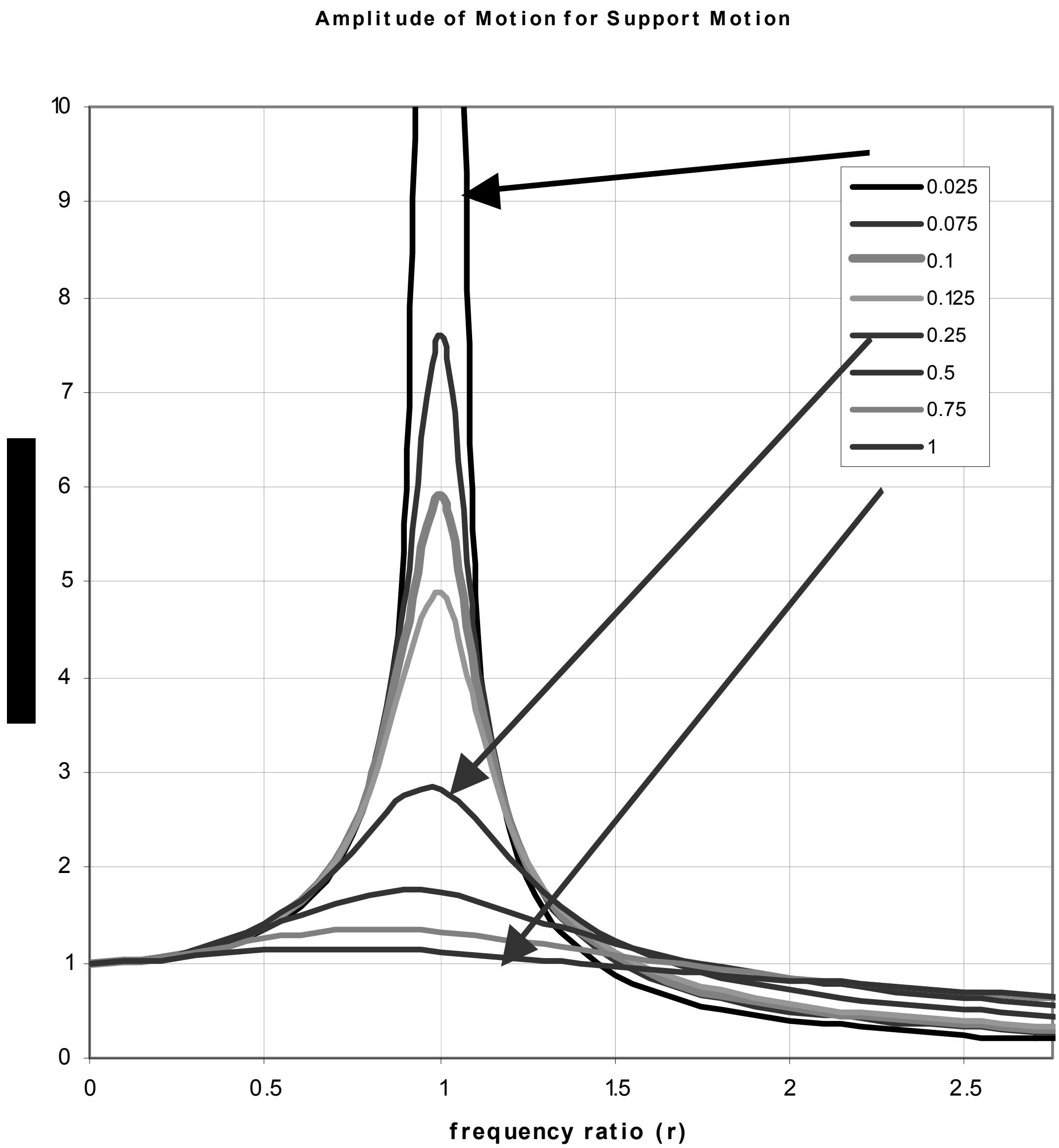


Figure 3.16. Amplification factor $G(r)$ defined by Eq.(3.44).

From the trigonometric identity $\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$, we can use Eqs.(3.39), (3.42), and (3.43), to define the phase between the steady-state solution $Y_p = Y_{op} \sin(\omega t + \varphi + \psi)$ and the base motion excitation $Y_b = A \sin \omega t$ as

$$\varphi + \psi = \tan^{-1}\left(\frac{-2\zeta r^3}{1 + 4\zeta^2 - r^2}\right) \qquad (3.45)$$

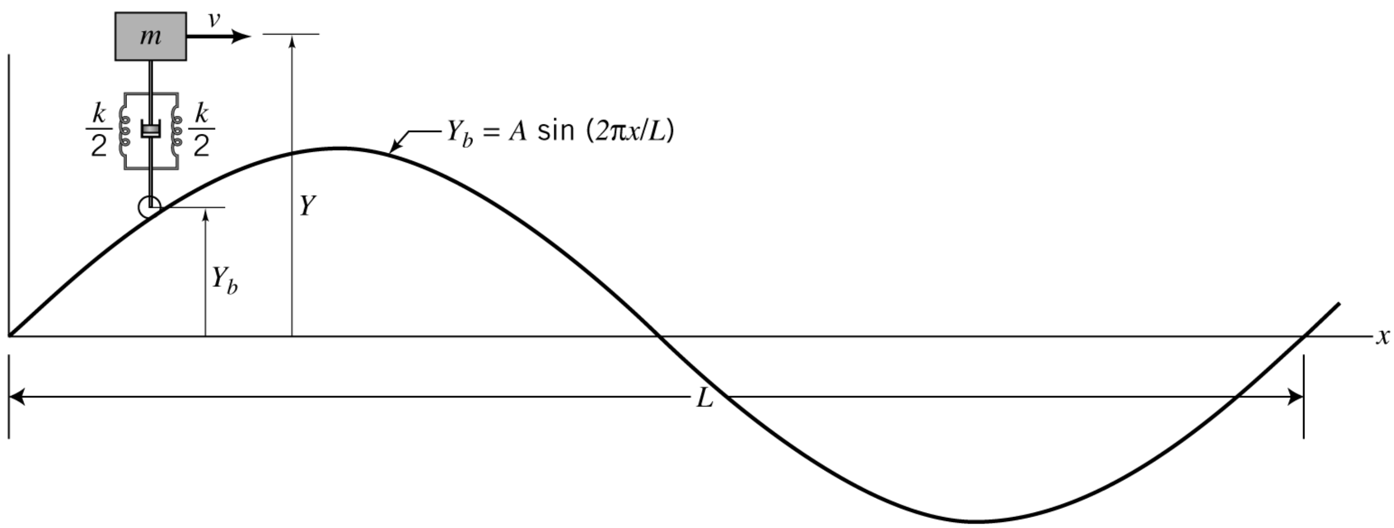


Figure XP3.9. Vehicle moving horizontally along a path defined by $Y_b = A \sin(2\pi x/L)$, $A = .03\,m$, $L = .6\,m$.

The vehicle has a constant velocity $v = \dot{x} = 100\ km/hr$. For convenience, we will assume that it starts at $x = 0$; hence,

$$x = x_o + \dot{x}t = 0 + 100\left(\frac{km}{hr}\right) \times \left(\frac{1\,hr}{3600\,\text{sec}}\right) \times \frac{1000\,m}{1\,km} t = 27.8\,t\ m\ ,$$

and, substituting into $Y_b = A \sin(2\pi x/L) = A \sin(2\pi/L)vt$,

$$Y_b(t) = .03\sin\left(\frac{2\times\pi\times 27.8\,t}{.6}\right) = .03\sin(291.\,t)\,m\ . \qquad \text{(ii)}$$

From Eq.(ii), as long as the vehicle's tires remain in contact with the road surface, the vehicle's steady velocity to the right will generate base excitation at the frequency $\omega = 291.\,rad/\sec$, $f = 46.3\ hz$ and amplitude .03 *m*. Driving faster will increase the excitation frequency; driving slower will decrease it.

Tests show that the vehicle's damped natural frequency is 2.62 Hz, and the damping factor is $\zeta = .3$. *Carry out the following engineering-analysis tasks:*

a. Determine the amplitude and phase of vehicle motion for $100\,km/hr$.

b. Determine the speed for which the response is a maximum and determine the response amplitude at this speed.

Solution. For *Task a*, First, the undamped natural frequency is defined by $f_n = f_d/\sqrt{1-\zeta^2} = 2.62/\sqrt{1-.3^2} = 2.5\,Hz$. the frequency ratio is $r = \omega/\omega_n = 46.3/2.5 = 18.5$. Hence, from Eq.(3.44),

$$\frac{Y_{op}}{A} = \frac{[1+4(.3^2)(18.5)^2]^{1/2}}{[\ (1\ -\ 18.5^2)^2\ +\ 4(.3^2)(18.5)^2]^{1/2}]^{1/2}} = .0326 \qquad \text{(i)}$$

$$\therefore\ Y_{op} = .0326(.03) = 9.77\times 10^{-4}\ m = .977\ mm.$$

From Eq.(3.45), the phase of *m*'s motion with respect to the base excitation is

$$\varphi+\psi = \tan^{-1}[\frac{-2(.3)18.5^3}{1+4(.3^2)-18.5^2}]$$

$$=\tan^{-1}(\frac{-3813}{-342}) = -95.1\ degrees\ , \qquad \text{(ii)}$$

which concludes *Task a*.

The response amplitude will be a maximum when $r = r_{\max}$ as defined in Eq.(3.46), i.e.,

$$r_{\max} = \frac{1}{2\zeta}[(1+8\zeta^2)^{1/2}-1]^{1/2}$$

$$= \frac{1}{2\times .3}\{[(1+8(.3)^2]^{1/2}-1\}^{1/2} = 0.930\ .$$

Hence,

$$\omega = \frac{2\pi \times v(m/\sec)}{.6m} = 0.930 \times \omega_n$$

$$= 0.930 \times 2.5 \frac{cycle}{\sec} \times \frac{2\pi\, rad}{1\, cycle} = 14.6 \frac{rad}{\sec}$$

$$\therefore \; v = \frac{.6(14.6)}{2\pi} = 1.40 \frac{m}{\sec} = 1.40 \times (\frac{100\, km/hr}{27.8\, m/\sec})$$

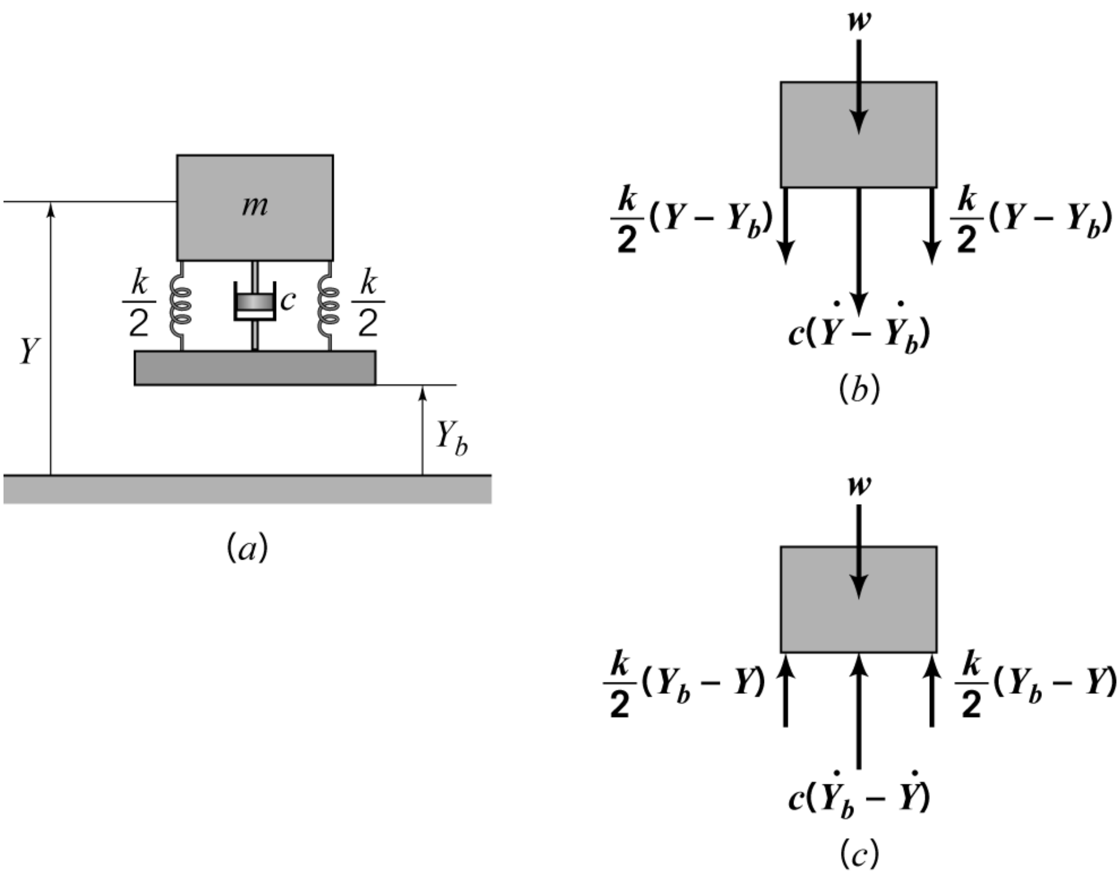
$$= 5.04 \;\; km/hr.$$

From Eq.(3.45), at this speed, the steady-state amplification factor is

$$\frac{Y_{op}}{A} = \frac{[1 + 4(.3^2).930^2]^{1/2}}{[\;(1 - 0.930^2)^2 + 4(.3^2).930^2]^{1/2}} = 1.99 \;,$$

$$\therefore \; Y_{op} = .03(1.99) = .060\, m \;.$$

Steady State Amplitude for <u>Relative</u> Deflection with Harmonic Base Excitation



The EOM,

$$m\ddot{Y} + c\dot{Y} + kY = -w + kY_b + c\dot{Y}_b \ .$$

can be written

$$m(\ddot{Y} - \ddot{Y}_B) + c(\dot{Y} - \dot{Y}_B) + k(Y - Y_B) = -w - m\ddot{Y}_B \ \ or$$

$$m\ddot{\delta} + c\dot{\delta} + k\delta = -w - m\ddot{Y}_B \ .$$

Hence, for harmonic base excitation defined by $Y_B = A\sin\omega t$, the

EOM is

$$m\ddot{\delta} + c\dot{\delta} + k\delta = mA\omega^2 \sin\omega t \ . \qquad \textbf{(i)}$$

We dropped the weight term $-w$ in arriving at this equation, which is equivalent to looking at disturbed motion about the equilibrium position. We want a steady-state solution to Eq.(i) of the form, $\delta = \Delta \sin(\omega t + \beta)$. Eq.(i) has the same form as

$$m\ddot{Y} + c\dot{Y} + kY = f_o \sin\omega t \ . \qquad \textbf{(3.32)}$$

except $mA\omega^2$ has replaced f_0. Hence, by comparison to

$$\frac{Y_{op}}{f_o/k} = \frac{1}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}} = H(r) \ ; \quad r = \omega/\omega_n \ , \qquad \textbf{(3.38)}$$

the steady-state relative amplitude due to harmonic base excitation is

$$\frac{\Delta}{mA\omega^2/k} = \frac{\Delta}{A(\frac{\omega}{\omega_n})^2} = \frac{\Delta}{Ar^2} = H(r) \ ; \quad r = \omega/\omega_n$$

$$\therefore \quad \frac{\Delta}{A} = r^2 H(r) = \frac{r^2}{\left[(1 - r^2)^2 + 4\zeta^2 r^2\right]^{1/2}} = J(r) \ .$$

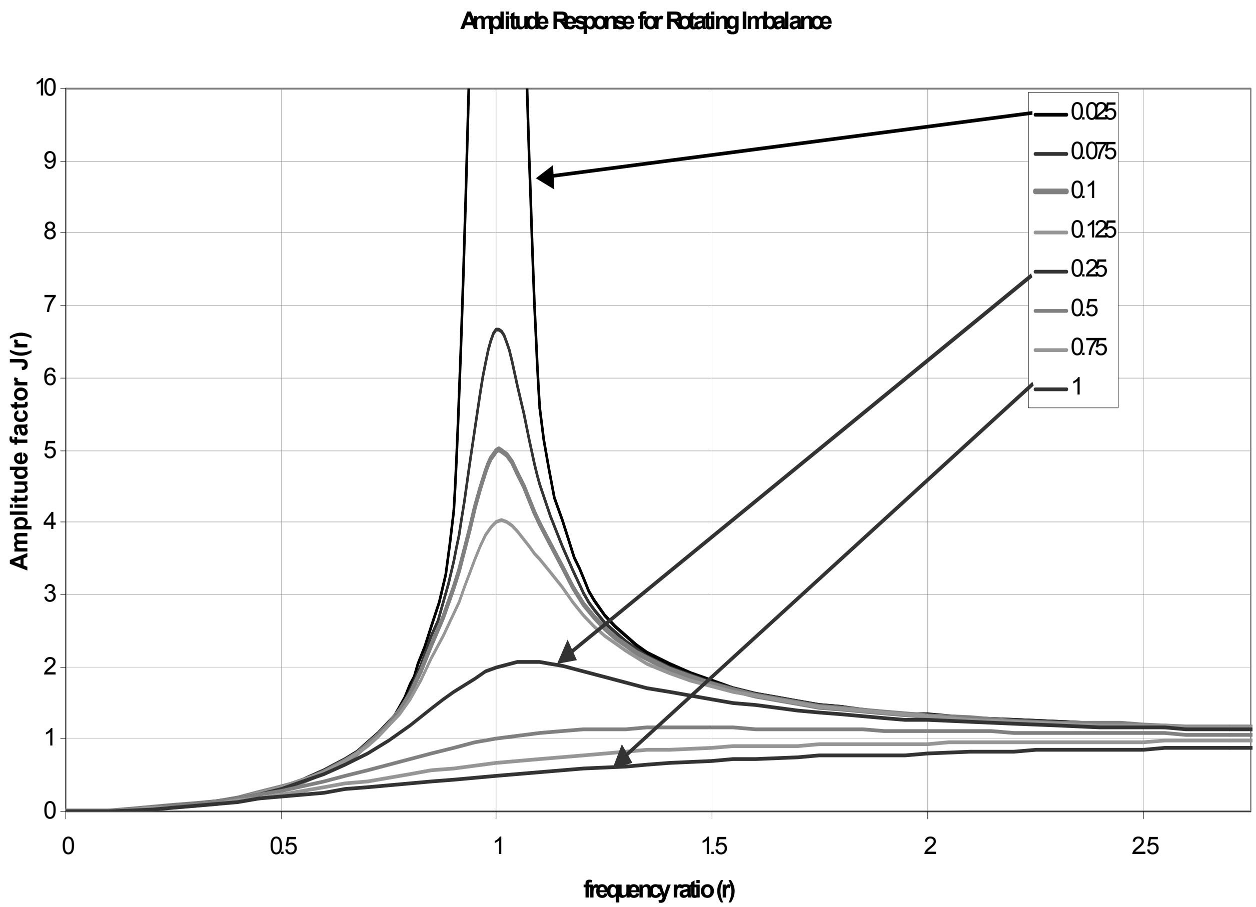


Figure 3.18. $J(r)$ versus frequency ratio $r = \omega/\omega_n$ from Eq.(3.49) for a range of damping ratio values.

Example Problem 3.9 Revisited. Solve for the steady-state relative amplitude at $v = 100\ km/hr$ as

$$\frac{\Delta}{A} = \frac{18.5^2}{[\ (1\ -\ 18.5^2)^2\ +\ 4(.3^2)(18.5)^2\ ]^{1/2}} = 1.002$$

$$\therefore\ \Delta = .03 \times 1.002 = .030\ m = 30.\ mm.$$

This value is much greater than the absolute amplitude of motion

$Y_{op} = 0.977\,mm$ we calculated earlier. This result shows that while the vehicle has small absolute vibration amplitudes, its base is following the ground contour; hence, the relative deflection (across the spring and damper) is approximately equal to the amplitude of the base oscillation.

Example Problem 3.10. An instrument package is to be attached to the housing of a rotating machine. Measurements on the casing show a vibration at 3600 rpm with an acceleration level of .25 *g*. The instrument package has a mass of .5 kg. Tests show that the support bracket to be used in attaching the package to the vibrating structure has a stiffness of 10^5 *N/m*. How much damping is needed to keep the instrument package vibration levels below .5 g?

Solution. The frequency of excitation at 3600 *rpm* converts to $\omega = 3600 rev/\min \times (1\min/60\sec) \times (2\pi rad/rev) = 377. rad/\sec$. With harmonic motion, the housing's amplitude of motion is related to its acceleration by $a = .25g = -\omega^2 A$; hence, the amplitude corresponding to the housing acceleration levels of .25 *g* is

$$A = .25(9.81\,m/\sec^2)/377.^2 = 1.726 \times 10^{-5}\,m = .017\,mm \ .$$

Similarly, an 0.5 g acceleration level for the instrument package means its steady-state amplitude is

$$Y_{op} = .5(9.81\, m/\sec^2)/377.^2 = 1.726 \times 10^{-5} m = .034\, mm \ .$$

Hence, the target amplification factor is $G = 0.034/0.017 = 2$.

The natural frequency of the instrument package is

$$\omega_n = \sqrt{k/m} = \sqrt{10^5 N/.5\, kg} = 447.2\, rad/\sec \ .$$

Hence, the frequency ratio is $r = \omega/\omega_n = 377./477. = .79$.

Plugging $r = .79$ into Eq.(3.44) gives

$$\frac{Y_{op}}{A} = 2 = \frac{[1 + 4\zeta^2 .79^2]^{1/2}}{[\ (1\ -\ .79^2)^2\ +\ 4\zeta^2 .79^2]^{1/2}} \Rightarrow 4 = \frac{1. + 2.496\zeta^2}{.1413 + 2.496\zeta^2}$$

The solution to this equation is $\zeta^2 = .058 \ \Rightarrow \ \zeta = .241$. Hence, the required damping is

$$c = 2\zeta\omega_n m = 2(.241)(447.2\frac{rad}{\sec})(.5\, kg) = 215.5\, N\sec/m \ ,$$

which concludes the engineering-analysis task. Note that the derived units for the Newton are $kg\,m/\sec^2$; hence, $kg/\sec$ nets $N\sec/m$. Note that, since only one excitation frequency is involved, we could have simply taken the ratio of the acceleration levels directly to get $G = 2$.

LECTURE 12. STEADY-STATE RESPONSE DUE TO ROTATING IMBALANCE

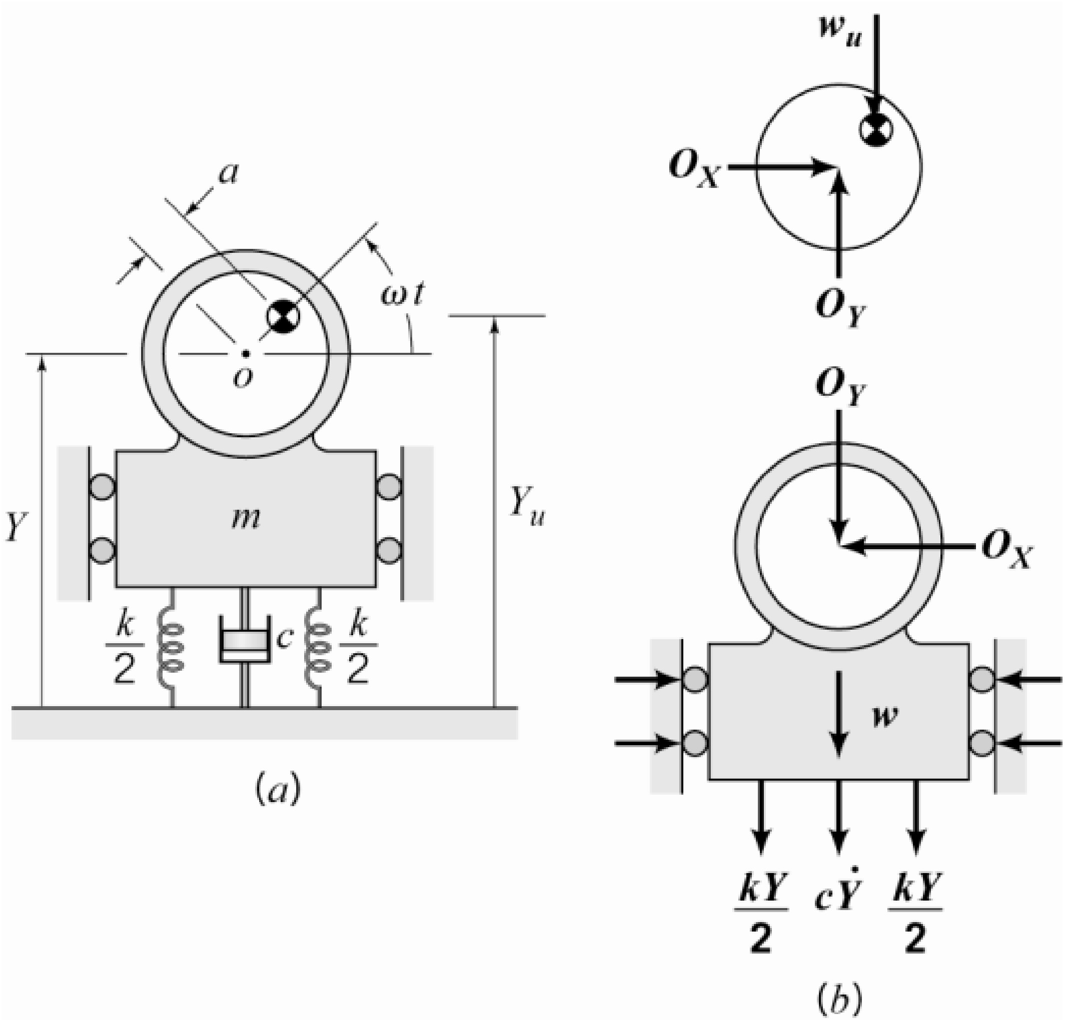


Figure 3.18 (a) Imbalanced motor with mass m_u supported by a housing mass m, (b) Free-body diagram for $Y>0,\ \dot{Y}>0$

The product $m_u \boldsymbol{a}$ is called the "imbalance vector." Y defines m's vertical position with respect to ground. From figure 3.17A, m_u's vertical position with respect to ground is

$$Y_u = Y + a\sin\omega t \;\Rightarrow\; \ddot{Y}_u = \ddot{Y} - a\,\omega^2 \sin\omega t \ . \qquad (3.46)$$

The free-body diagram of figure 3.17B for the support mass applies for upwards motion of the support housing that causes tension in the support $(Y > 0)$ springs and the support damper $(\dot{Y} > 0)$. The internal reaction force (acting at the motor's bearings) between the motor and the support mass is defined by the components (O_X, O_Y).

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the individual masses of figure 3.17B gives:

$$\Sigma f_Y = -w_u + O_Y = m_u \ddot{Y}_u = m_u(\ddot{Y} - a\omega^2 \sin\omega t)$$

$$\Sigma f_Y = -w - O_Y - kY - c\dot{Y} = m\ddot{Y} \ ,$$

where Eq.(3.46) has been used to eliminate $\ddot{Y}_u$. Adding these equations eliminates the vertical reaction force O_Y and gives the single equation of motion,

$$(m + m_u)\ddot{Y} + c\dot{Y} + kY = -(w + w_u) + m_u a\omega^2 \sin\omega t \ . \qquad (3.47)$$

This equation resembles Eq.(3.32),

$$m\ddot{Y} + c\dot{Y} + kY = f_o \sin\omega t \ . \qquad \mathbf{(3.32)}$$

except $(m + m_u)$ and $m_u a\omega^2$ have replaced m and f_0, respectively. Dividing Eq.(3.47) by $(m + m_u)$ gives

$$\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = -g + \frac{m_u a\omega^2}{M} \sin\omega t \ , \qquad (3.48a)$$

where,

$$M = (m + m_u)\ , \quad 2\zeta\omega_n = \frac{c}{M}\ , \quad \omega_n^2 = \frac{k}{M}\ . \qquad (3.48b)$$

We want the steady-state solution to Eq.(3.48a) due to the rotating-imbalance excitation term $[(m_u a\omega^2)/M]\ \sin(\omega t)$ and are not interested in either the homogeneous solution due to initial conditions or the particular solution due to weight. Eq.(3.48a)

$$Y_{op} = \frac{f_o}{m} \cdot \frac{1}{\left[\ (\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\ \right]^{1/2}}\ , \qquad (3.37)$$

has the same form as Eq.(3.32) except $m_u a\omega^2/M$ has replaced f_0/m. Hence, by comparison to Eq.(3.37),

the steady-state response amplitude due to rotating imbalance is

$$Y_{op} = \frac{m_u a\omega^2}{M} \cdot \frac{1}{\left[\ (\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\ \right]^{1/2}}$$

$$= \frac{m_u a\omega^2}{M\omega_n^2} \cdot \frac{1}{\left\{\ [1 - (\omega/\omega_n)^2]^2 + 4\zeta^2(\omega/\omega_n)^2\right\}^{1/2}}\ ,$$

and the amplification factor due to the rotating imbalance is

$$Y_{op}/a\left(\frac{m_u}{M}\right) = \frac{r^2}{\left[(1 - r^2)^2 + 4\zeta^2 r^2\right]^{1/2}} = J(r) . \qquad (3.49)$$

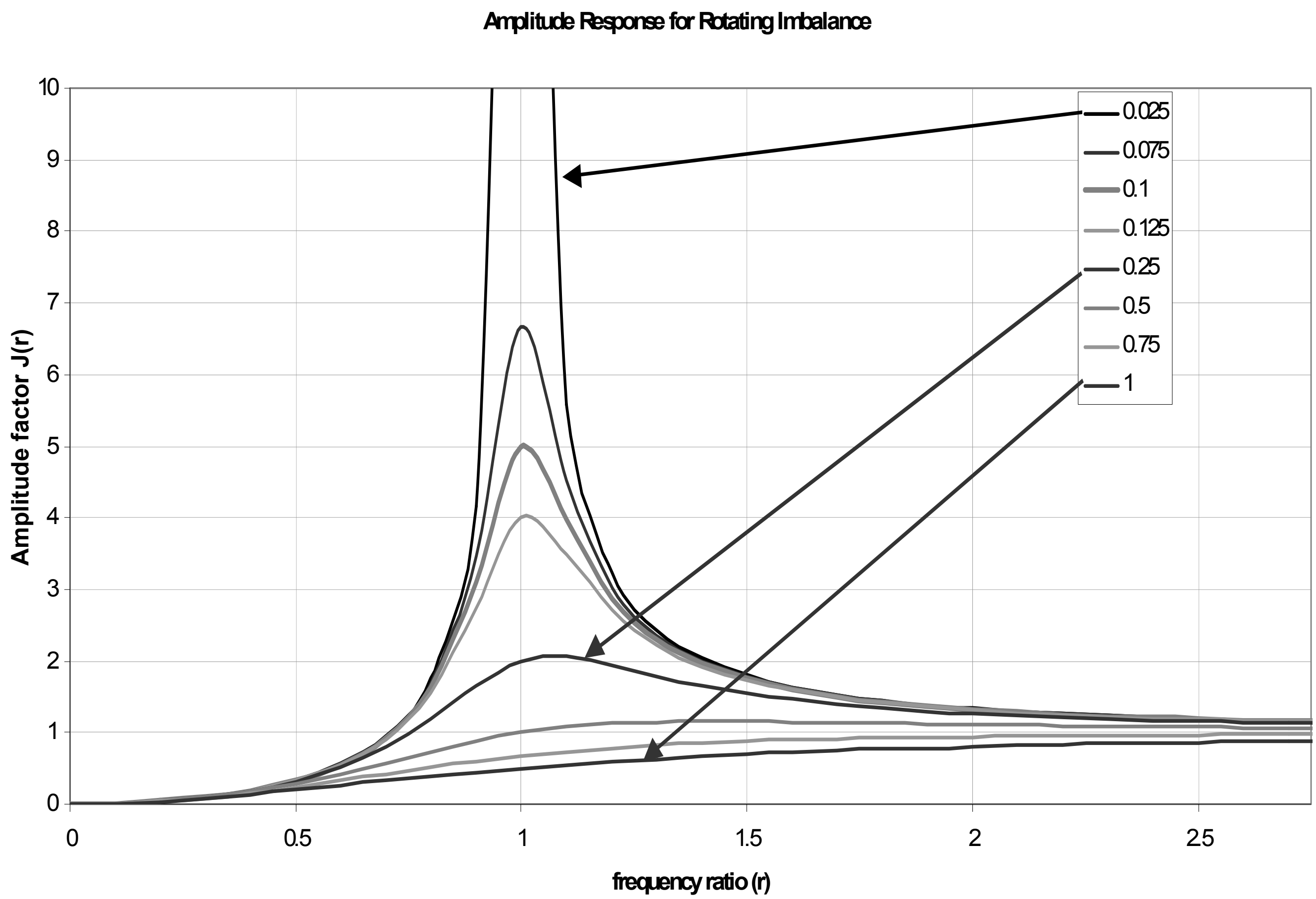


Figure 3.18. $J(r)$ versus frequency ratio $r = \omega/\omega_n$ from Eq.(3.49) for a range of damping ratio values.

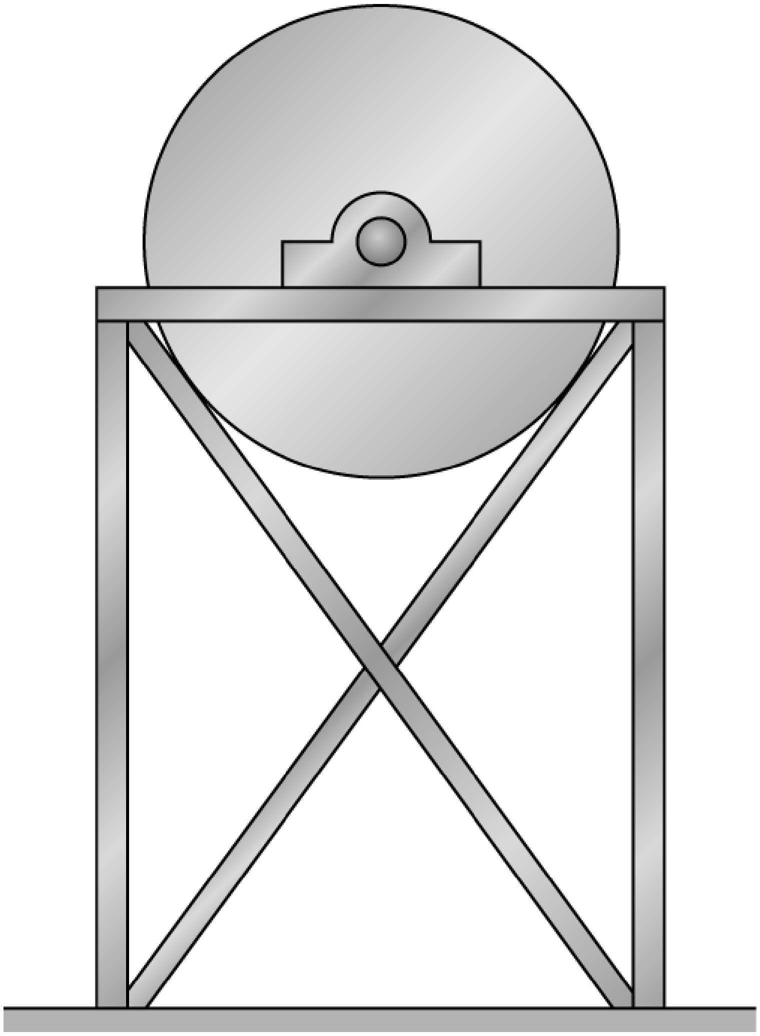


Figure XP3.4 Industrial blower supported by a welded-frame structure.

The support structure is much stiffer in the vertical direction than in the horizontal; hence, the model of Eq.(3.48a) , without the weight, holds for horizontal motion. This unit runs at 500 rpm, and has been running at “high” vibration levels. A “rap” test has been performed by mounting an accelerometer to the fan base, hitting the support structure below the fan with a large hammer, and recording the output of the accelerometer. This test shows a natural frequency of 10.4 Hz = 625 rpm with very little damping ($\zeta \cong .005$). The rotating mass of the fan is 114 kg. The total weight of the fan (including the rotor) and its base plate was stated to weigh 1784 N according to the manufacturer. Answer the following questions:

a. Assuming that the vibration level on the fan is to be less than $0.1g$, how well should the fan be balanced; i.e., to what value should *a* be reduced?

b. Assuming that the support structure could be stiffened laterally by approximately 50%, how well should the fan be balanced?

Solution. In terms of the steady-state amplitude of the housing, the housing acceleration magnitude is $|a_p| = \omega^2 Y_{op} = 0.1g$, where $g = 9.81\,m^2/\sec$. Hence, at

$$\omega = 500(\frac{rev}{\min}) \times (\frac{1\min}{60\sec}) \times (\frac{2\pi\,rad}{1\,rev}) = 52.36\,\frac{rad}{\sec}\ ,$$

the housing-amplitude specification is:

$$Y_{op} \le 0.1 \times 9.81/52.36^2 = 3.578 \times 10^{-4} m = .358\,mm\ .$$

Applying the notation of Eq.(3.49) gives

$$M = W/g = 1784./9.81 = 181.9\,kg\ ;\quad m_u = 114\,kg\ .$$

and $r = \omega/\omega_n = 500/625 = 0.8$. Applying Eq.(3.49) gives

$$Y_{op}/a(\frac{m_u}{M}) = \frac{0.64}{[(1-0.64)^2+4(.005)^2(.64)]^{1/2}} = 1.77 \ .$$

Hence, the imbalance vector magnitude a should be no more than

$$a = Y_{op}/(\frac{m_u}{M})/1.77 = .358\,mm/(114/182.)/1.77 = .323\,mm \ ,$$

which concludes *Task a*. Balancing and maintaining the rotor such that $a \le .323\,mm(.013\,in)$ for a 114 kg fan rotor is not easy.

Moving to Task b, and assuming that the structural stiffening does not appreciably increase the mass of the fan assembly, increasing the lateral stiffness by 50% would change the natural frequency to

$$\omega_n(new) = \sqrt{\frac{1.5\times k}{M}} = \sqrt{1.5}\times\omega_n(old) = 1.225\times 625\,rpm = 765\,rpm \ .$$

With a stiffened housing, $r_{new} = \omega/\omega_n = 500/765. = 0.653$, $r_{new}^2 = .427$. Since $2\zeta\omega_n = c/M \Rightarrow \zeta = c/2M\omega_n = c/2\sqrt{kM}$,

$$\zeta_{old} = \frac{c}{2\sqrt{M}}\times\frac{1}{\sqrt{k_{old}}} \ , \ \zeta_{new} = \frac{c}{2\sqrt{M}}\times\frac{1}{\sqrt{k_{new}}} \ .$$

Hence,

$$\zeta(new) = \zeta(old)\sqrt{\frac{k(old)}{k(new)}} = \frac{\zeta(old)}{\sqrt{1.5}} = \frac{.005}{1.225} = .004 \ .$$

Further,

$$Y_{op}/a(\frac{m_u}{M}) = \frac{0.427}{[(1-0.427)^2 + 4(.004)^2(.427)]^{1/2}} = .745 \ ,$$

and

$$a = Y_{op}/(m_u/M)/.745 = .358\,mm/(114/182.)/.745$$
$$= .767\,mm \ \ (.030\,in) \ .$$

Hence, by elevating the system natural frequency, the imbalance-vector magnitude can be $.767/.323 = 2.4$ times greater without exceeding the vibration limit. Stated differently, the fan can tolerate a much higher imbalance when its operating speed ω is further away from resonance.

Typically, appreciable damping is very difficult to introduce into this type of system; moreover, increasing the damping factor to $\zeta = .05$ in *Task a* reduces only slightly the required value for *a* to meet the housing-acceleration level specification. Damping would be more effective for $r = \omega/\omega_n \cong 1$. Of course, the vibration amplitudes would also be much higher.

Table 3.2 Forced-excitation results where $r = \omega/\omega_n$.

Application	Transfer Function	Amplitude Ratio	r for maximum response
Forced Response	$\frac{Y_{op}}{f_o/k} = H(r)$	$\frac{1}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}}$	$r_{max} = \sqrt{1 - 2\zeta^2}$
Base Excitation	$\frac{Y_{op}}{A} = G(r)$	$\frac{[1 + 4\zeta^2 r^2]^{1/2}}{[(1 - r^2)^2 + 4\zeta^2 r^2]^{1/2}}$	$r_{max} = \frac{1}{2\zeta}\sqrt{(1 + 8\zeta^2)^{1/2} - 1}$
Relative Deflection with Base Excitation	$\frac{\Delta}{A} = J(r)$	$\frac{r^2}{\left[(1 - r^2)^2 + 4\zeta^2 r^2\right]^{1/2}}$	$r_{max} = \frac{1}{\sqrt{1 - 2\zeta^2}}$
Rotating Imbalance	$Y_{op}/a(\frac{m_u}{M}) = J(r)$	$\frac{r^2}{\left[(1 - r^2)^2 + 4\zeta^2 r^2\right]^{1/2}}$	$r_{max} = \frac{1}{\sqrt{1 - 2\zeta^2}}$