Lecture 5. NEWTON'S LAWS OF MOTION

Newton's Laws of Motion

Law 1. Unless a force is applied to a particle it will either remain at rest or continue to move in a straight line at constant velocity,

Law 2. The acceleration of a particle in an inertial reference frame is proportional to the force acting on the particle, and

Law 3. For every action (force), there is an equal and opposite reaction (force).

Newton's second Law of Motion is a Second-Order Differential Equation

$$\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$$

where $\Sigma \boldsymbol{f}$ is the resultant force acting on the particle, $\ddot{\boldsymbol{r}}$ is the particle's acceleration with respect to an inertial coordinate system, and m is the particle's mass.

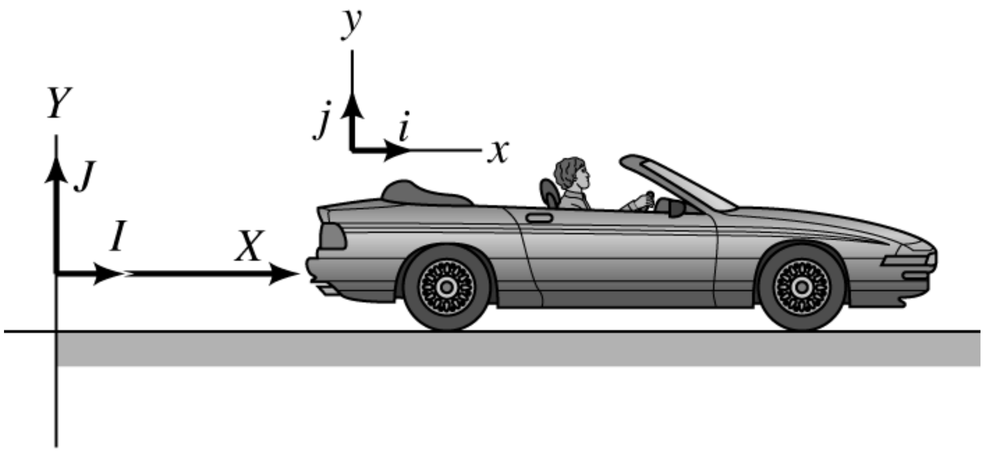


Figure 3.1 A car located in the *X, Y* (inertial) system by ***I****X* and a particle located in the *x, y* system by ***i****x*.

A particle in the car can be located by

$$x_p = X + x$$

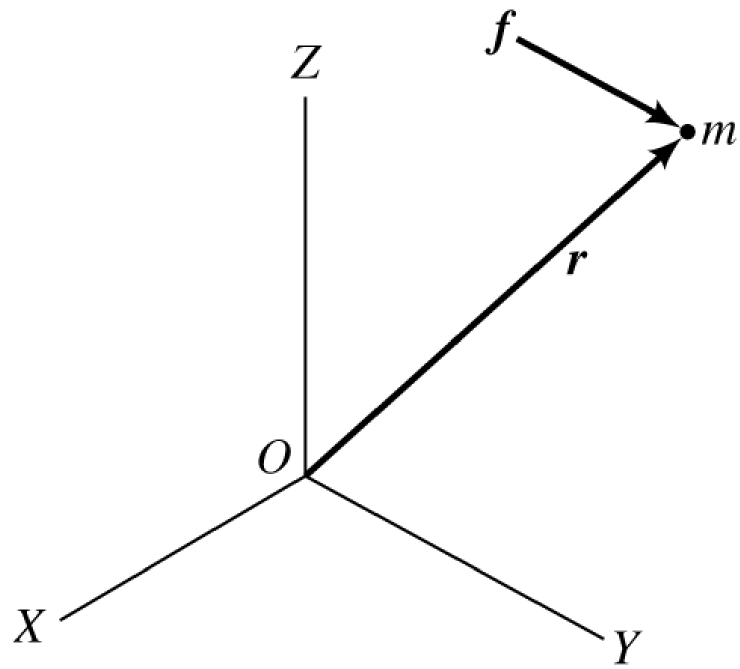
The equation of motion,

$$\Sigma f_X = m\ddot{x}_p = m(\ddot{X} + \ddot{x}),$$

is correct. However,

$$\Sigma f_X \neq m\ddot{x} \ ,$$

because the *x, y* system is not inertial.



X, Y, Z Inertial coordinate system

3-D Version of Newton's 2nd Law of Motion

$$f_X = m\,a_X = m\ddot{X} = m\frac{d}{dX}\left(\frac{\dot{X}^2}{2}\right)$$

$$f_Y = m\,a_Y = m\ddot{Y} = m\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right)$$

$$f_Z = m\,a_Z = m\ddot{Z} = m\frac{d}{dZ}\left(\frac{\dot{Z}^2}{2}\right) \; .$$

Using the Energy-integral substitution

$$\ddot{X} = \frac{d\dot{X}}{dt} = \frac{d\dot{X}}{dX}\frac{dX}{dt} = \dot{X}\frac{d\dot{X}}{dX} = \frac{d}{dX}\left(\frac{\dot{X}^2}{2}\right).$$

Multiplying the equations by dX, dY, dZ , respectively, and adding gives

$$f_X dX + f_Y dY + f_Z dZ = \frac{m}{2} d[(\dot{X}^2) + (\dot{Y}^2) + (\dot{Z}^2)]$$

$$d\,Work = d(\frac{m v^2}{2}) = d\,T \quad .$$

This "work-energy" equation is an integrated form of Newton's second law of motion $\Sigma \boldsymbol{f} = m \ddot{\boldsymbol{r}}$. The expressions are fully equivalent and are not independent.

The central task of dynamics is deriving equations of motion for particles and rigid bodies using either Newton's second law of motion or the work-energy equation.

Constant Acceleration: Free-Fall of a Particle Without Drag

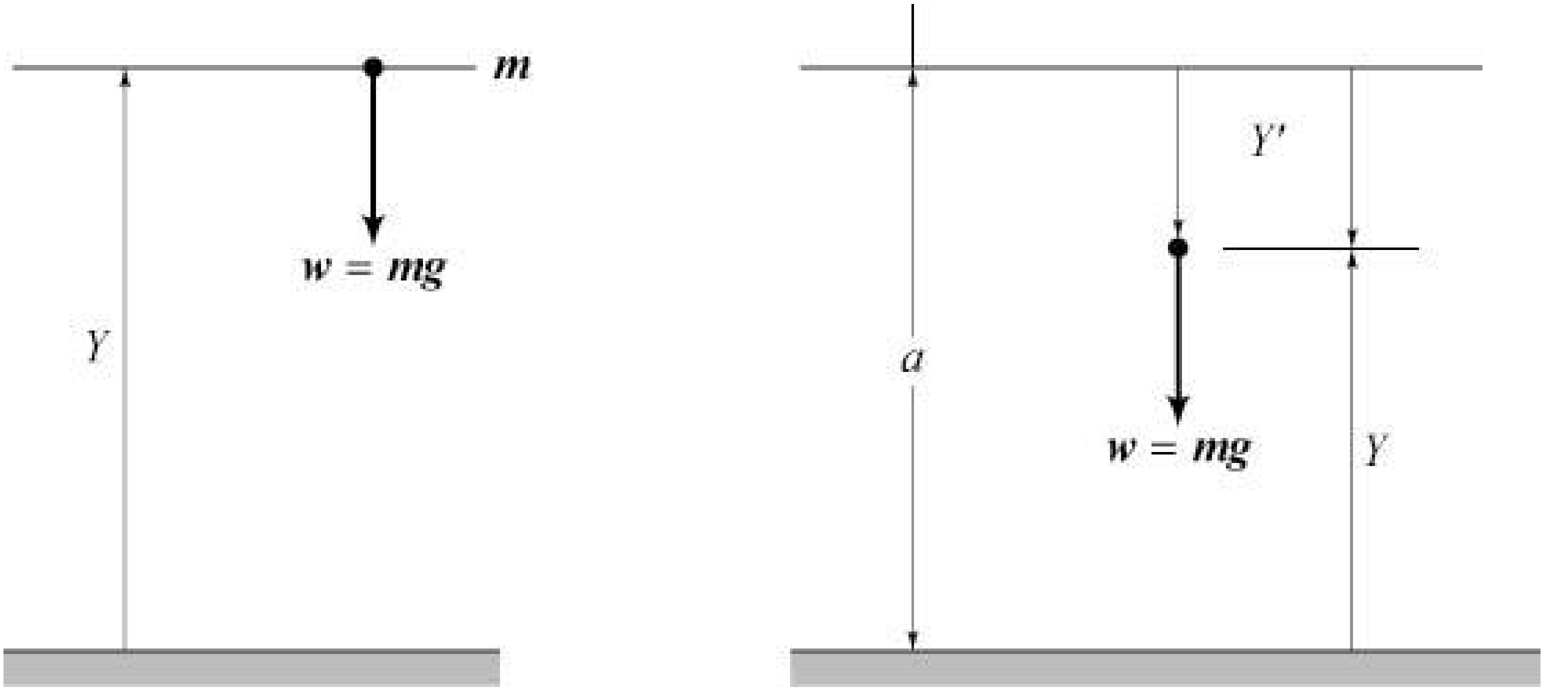


Figure 3.3 Particle acted on by its weight and located by Y (left) and $Y^{/}$(right)

Equation of Motion using Y

$$\Sigma f_Y = -w = -mg = m\ddot{Y} \Rightarrow \ddot{Y} = -g\ .$$

Let us consider rewriting Newton's law in terms of the new coordinate $Y^{/}$. Note that $Y^{/} + Y = a \Rightarrow \dot{Y}^{/} + \dot{Y} = 0 \Rightarrow \ddot{Y}^{/} + \ddot{Y} = 0$. From the free-body diagram, the equation of motion using $Y^{/}$ is

$$\Sigma f_{Y^{/}} = w = mg = m\ddot{Y}^{/} \Rightarrow \ddot{Y}^{/} = g\ .$$

This equation conveys the same physical message as the differential equation for Y; namely, the particle has a constant acceleration downwards of g, the acceleration of gravity.

Time Solution for the D. Eq. of motion for *Y*,

$$\frac{d^2 Y}{dt^2} = -g \ .$$

Integrating once with respect to time gives

$$\frac{dY}{dt} = \dot{Y}(t) = \dot{Y}_0 - g\,t \ ,$$

where $\dot{Y}_0 = \dot{Y}(0)$ is the initial (time $t = 0$) velocity. Integrating a second time w.r.t. time gives

$$Y(t) = Y_0 + \dot{Y}_0\,t - \frac{g\,t^2}{2} \ ,$$

where $Y_0 = Y(0)$ is the initial (time $t = 0$) position. Y_0 and $\dot{Y}_0$ are the "initial conditions."

The solution can also be developed more formally via the following steps:

a. Solve the homogeneous equation $\ddot{Y}_h = 0$ (obtained by setting the right-hand side to zero) with the solution as

$$Y_h = A + Bt \ .$$

b. Determine a particular solution to the original equation $d^2Y/dt^2 = -g$ that satisfies the right-hand side. By inspection, the right-hand side is satisfied by the particular solution $Y_p = Ct^2 \Rightarrow \dot{Y}_p = 2Ct \Rightarrow \ddot{Y}_p = 2C$. Substituting this result nets $\ddot{Y}_p = 2C = -g \Rightarrow C = -g/2$, and

$$Y_p = -\frac{gt^2}{2} \ .$$

The complete solution is the sum of the particular and homogeneous solution as

$$Y = Y_h + Y_p = A + Bt - \frac{gt^2}{2} \ .$$

The constants *A* and *B* are solved in terms of the initial conditions starting with

$$Y(0) = Y_0 = A \Rightarrow A = Y_0 \ .$$

Continuing, $\dot{Y} = B - gt$ netting

$$\dot{Y}(0) = \dot{Y}_0 = B \Rightarrow B = \dot{Y}_0 \ ,$$

and the complete solution — satisfying the initial conditions — is

$$Y(t) = Y_0 + \dot{Y}_0 t - \frac{g t^2}{2} \ ,$$

which duplicates our original results.

Engineering Analysis Task: *If the particle is released from rest* ($\dot{Y}(0) = 0$) *at* $Y(0) = H$, *how fast will it be going when it hits the ground* ($Y = 0$)*?*

Solution a. When the particle hits the ground at time $\bar{t}$,

$$\dot{Y}(\bar{t}) = \dot{Y}_0 - g\bar{t} = -g\bar{t}$$

$$Y(\bar{t}) = H - \frac{g\bar{t}^2}{2} = 0 \ .$$

Solving for $\bar{t}$,

$$\bar{t} = \sqrt{\frac{2H}{g}} \ .$$

Solving for $\dot{Y}(\bar{t})$,

$$\dot{Y}(\bar{t}) = -g\sqrt{\frac{2H}{g}} = -\sqrt{2gH} .$$

Solution b. Using the energy-integral substitution,

$$\ddot{Y} = \frac{d\dot{Y}}{dt} = \frac{d\dot{Y}}{dY}\frac{dY}{dt} = \dot{Y}\frac{d\dot{Y}}{dY} = \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) ,$$

changes the differential equation $m\ddot{Y} = -w$ to

$$m\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = -w = -mg .$$

Multiplying through by dY and integrating gives

$$\frac{m\dot{Y}^2}{2} - \frac{m\dot{Y}_0^2}{2} = -mg\int_H^Y dy = mg(H-Y) .$$

Since $\dot{Y}_0 = 0$, $\dot{Y}(Y=0)$ is

$$\dot{Y}(Y=0) = \sqrt{2gH}$$

Solution c. There is no energy dissipation; hence, we can work directly from the conservation of mechanical energy equation,

$$T + V = T_0 + V_0,$$

where $T = m\dot{Y}^2/2$ is the kinetic energy. The potential energy of the particle is its weight w times the vertical distance *above* a horizontal datum. Choosing ground as datum gives $V = wY$ and

$$m\frac{\dot{Y}^2}{2} + wY = 0 + wH \;\Rightarrow\; \dot{Y}(Y=0) = \sqrt{2gH}$$

The weight is a conservative force, and in the differential equation is $-w$, pointing in the $-Y$ direction. Strictly speaking, a conservative force is defined as a force that is the negative of its gradient with respect to a potential function V. For this simple example, with $V = wY$

$$-\frac{dV}{dY} = -\frac{d(wY)}{dY} = -w \;.$$

Hence (as noted above) the potential energy function for gravity is the weight times the distance above a datum plane; i.e.,

$$V_g = wY \;.$$

Acceleration as a Function of Displacement: Spring Forces

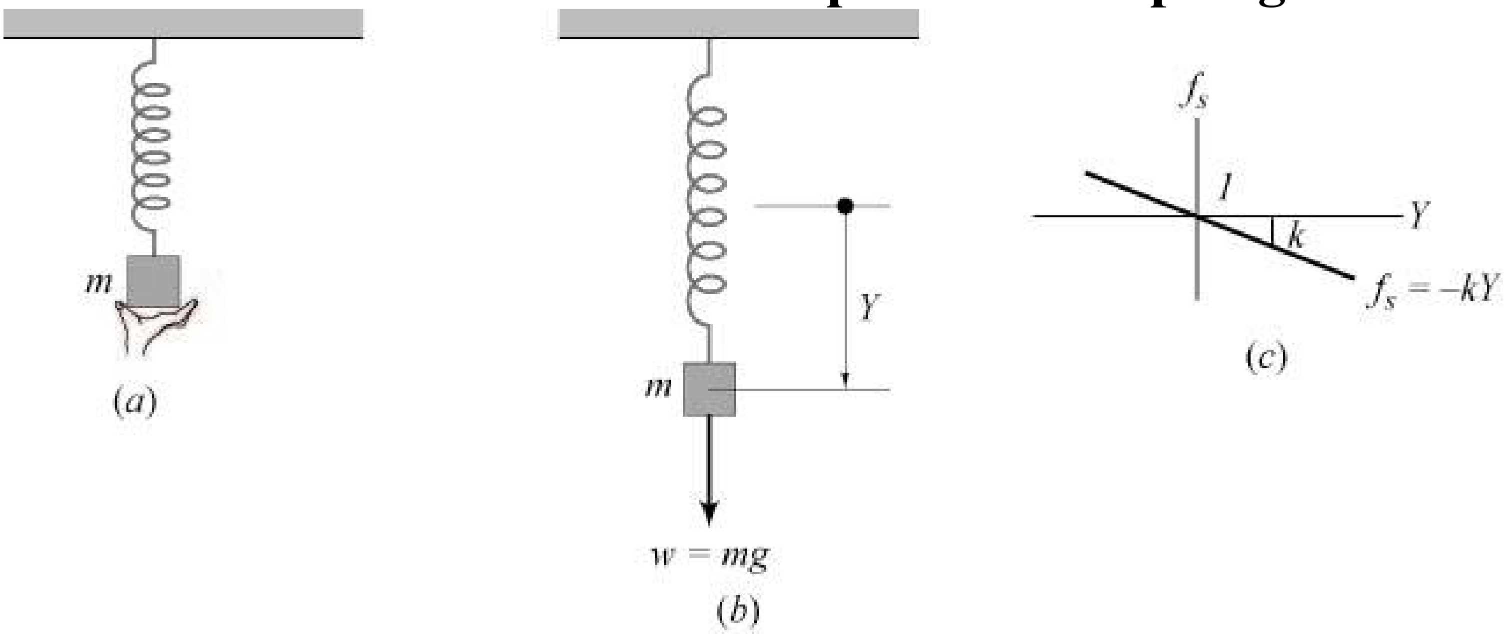


Figure 3.5 (a) Particle being held prior to release, (b) Y coordinate defining *m*'s position below the release point, (c) Spring reaction force $f_s = -kY$

Particle suspended by a spring and acted on by its weight. The spring is undeflected at $Y = 0$; i.e., $Y = 0 \Rightarrow f_s = 0$ (zero spring force). The spring force $f_s = -kY$ has a sign that is the opposite from the displacement Y, and it acts to restore the particle to the position $Y = 0$.

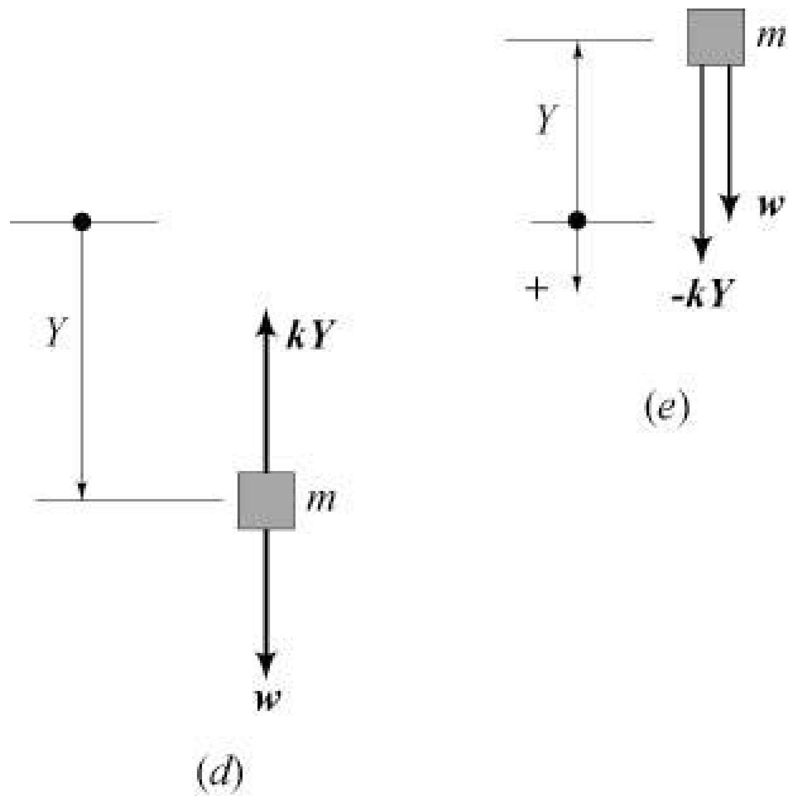


Figure 3.5 (d) Free-body diagram for $Y > 0$, (e) Free-body diagram for $Y < 0$

From the free-body diagram of figure 3.5d, Newton's second law of motion gives the differential equation of motion,

$$\Sigma\, f_Y = m\,\ddot{Y} = w - k\,Y \Rightarrow \quad m\,\ddot{Y} + k\,Y = w \ . \qquad \textbf{(3.13)}$$

Figure 3.5d-e shows that $f_s = -k\,Y$ for $Y > 0$ and $Y < 0$

Deriving the Equation of Motion for Motion about Equilibrium

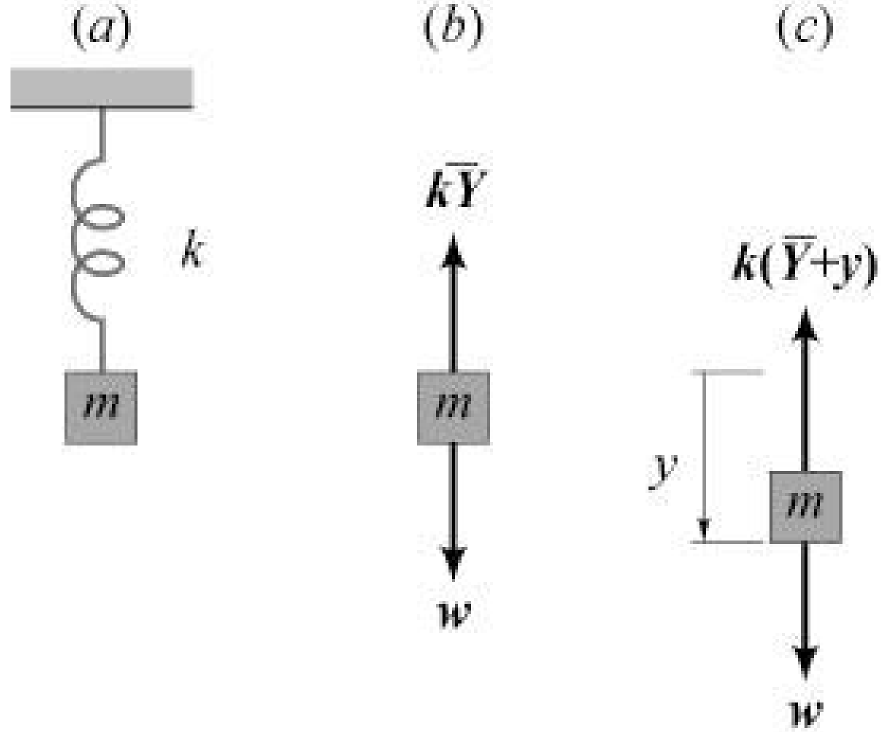


Figure 3.5 (a) Mass m in equilibrium, (b) Equilibrium free-body diagram, (c) General-position free-body diagram

Figure 3.5c applies for m displaced the distance y below the equilibrium point. The additional spring displacement generates the spring reaction force $f_s = k(\bar{Y} + y)$, and yields the following equation of motion.

$$m\ddot{y} = \sum f_Y = w - k(\bar{Y} + y) = w - k\bar{Y} - ky = -ky$$
$$\therefore\ m\ddot{y} + ky = 0\ . \qquad \textbf{(3.14)}$$

This result holds for a *linear* spring and shows that w is eliminated, leaving only the perturbed spring force ky.

MORE EQUILIBRIUM, 1 MASS - 2 SPRINGS

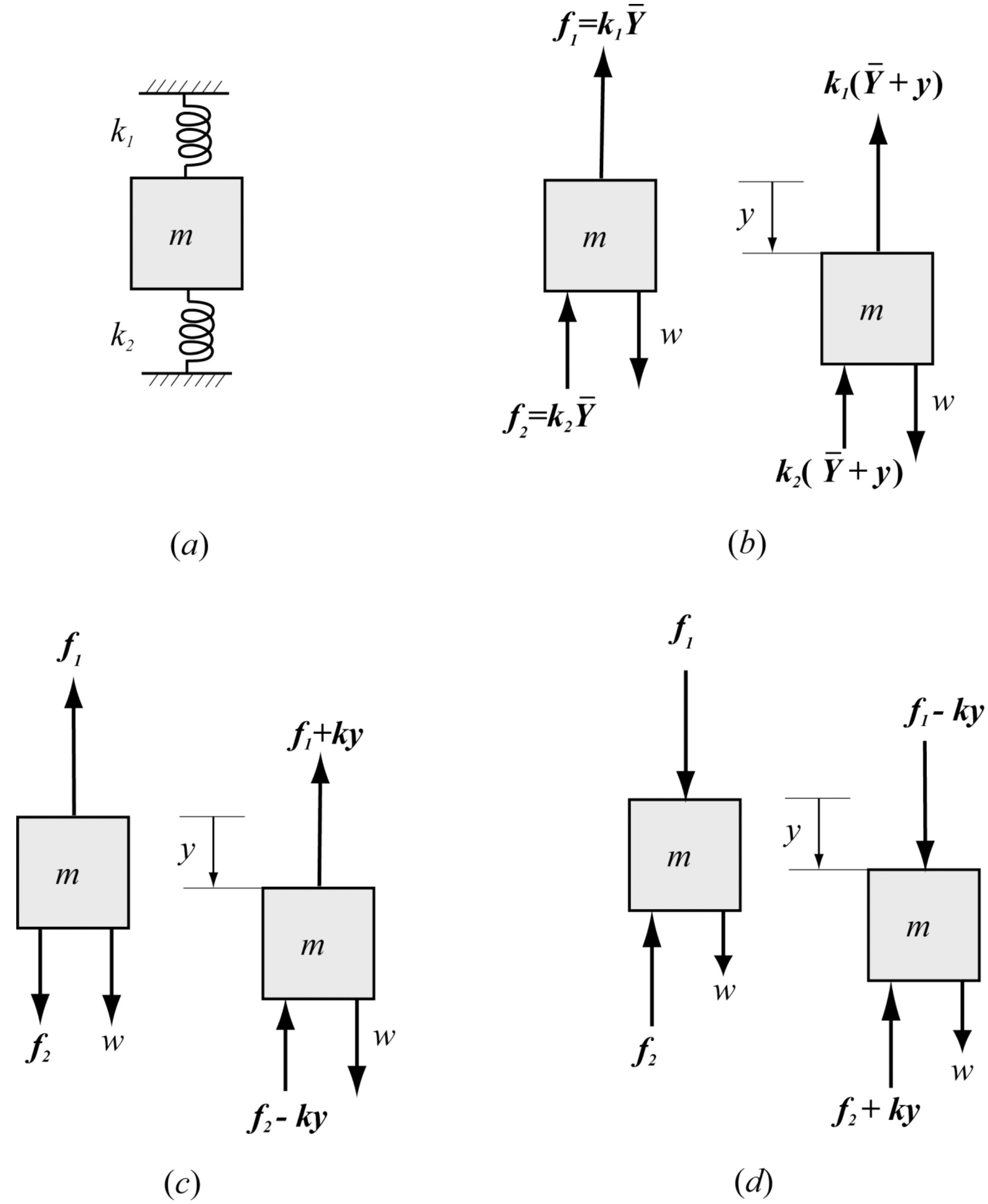


Figure 3.6 (a) Equilibrium, (b) Equilibrium with spring 1 in tension and spring 2 in compression, (c) Equilibrium with both springs in tension, (d) Equilibrium with both springs in compression

Spring 1 in tension, spring 2 in compression
Figure 3.6b, equilibrium starting from undeflected springs:
$w = f_1 + f_2 = (k_1 + k_2)\bar{y}$

$$m\ddot{y} = \sum f_y = w - k_1(\bar{y} + y) - k_2(\bar{y} + y) = w - (k_1 + k_2)\bar{y} + (k_1 + k_2)y$$
$$= 0 - (k_1 + k_2)y$$
$$\therefore \ m\ddot{y} + (k_1 + k_2)y = 0 \ .$$

Figure 3.6c, Equilibrium with both springs in tension, $w = f_1 - f_2$

$$m\ddot{y} = \sum f_y = w - (f_1 + k_1 y) + (f_2 - k_2 y) = (w - f_1 + f_2) - (k_1 + k_2)y$$
$$= 0 - (k_1 + k_2)y$$

$$\therefore \ m\ddot{y} + (k_1 + k_2)y = 0 \ .$$

Figure 3.6c, Equilibrium with both springs in compression, $w = f_2 - f_1$

$$m\ddot{y} = \sum f_y = w + (f_1 - k_1 y) - (f_2 + k_2 y) = (w + f_1 - f_2) + (k_1 + k_2)y$$
$$= 0 - (k_1 + k_2)y$$
$$\therefore \ m\ddot{y} + (k_1 + k_2)y = 0 \ .$$

CONCLUDE: For linear springs, the weight drops out of the equation of motion.

Natural Frequency Definition and calculation

Dividing $m\ddot{Y} + kY = w$ by m gives

$$\ddot{Y} + \omega_n^2 Y = g ,$$

where

$$\omega_n = \sqrt{\frac{k}{m}} ,$$

and ω_n is the undamped natural frequency.

Students frequently have trouble in getting the dimensions correct in calculating the undamped natural frequency. Starting with the ft-lb-sec system, *k* has the units lb/ft. The mass has the derived units of slugs. From

$$w = mg \Rightarrow m(slugs) = w(lbs)/g(ft/\sec^2) = \frac{w}{g}\frac{(lb - \sec^2)}{ft},$$

The dimensions for ω_n are

$$\omega_n = \sqrt{\frac{k(\frac{lb}{ft})}{\frac{m(lb-\sec^2)}{ft}}} = \sqrt{\frac{k}{m}\frac{rad}{\sec}} = \sqrt{\frac{k}{m}} \sec^{-1},$$

Starting with the SI system using m-kg-sec, the units for k are N/m. We can use $w = mg$ to convert Newtons into $kg - m/\sec^2$. Alternatively, the units for kg from $m = w/g$ are $N - \sec^2/m$, and

$$\omega_n = \sqrt{\frac{k(\frac{N}{meter})}{\frac{m(N-\sec^2)}{meter}}} = \sqrt{\frac{k}{m}}\ \sec^{-1},$$

Note that the <u>correct</u> units for ω_n are rad/sec not cycles/sec. The undamped natural frequency can be given in terms of cycles/sec as

$$f_n = \omega_n \frac{rad}{\sec} \times \frac{1 cycle}{2\pi rad} = \frac{\omega_n}{2\pi}\frac{cycles}{\sec} = \frac{\omega_n}{2\pi} Hertz$$

Time Solution From Initial Conditions.
The homogeneous differential equation corresponding to $\ddot{Y} + \omega_n^2 Y = g$ is

$$\ddot{Y}_h + \omega_n^2 Y_h = 0 .$$

Substituting the guessed $Y_h = A \cos pt$ yields

$$(-p^2 + \omega_n^2) A \cos pt = 0 \Rightarrow p = \omega_n$$

Guessing $Y_h = A \sin pt$ yields the same result; hence, the general solution is solution

$$Y_h = A \cos \omega_n t + B \sin \omega_n t .$$

The particular solution is

$$Y_p = \frac{g}{\omega_n^2} = \frac{w}{k} .$$

Note that this is also the static solution; i.e., $m \ddot{Y} + k Y = w \Rightarrow Y_{static} = w/k$. The complete solution is

$$Y = Y_h + Y_p = A \cos \omega_n t + B \sin \omega_n t + \frac{w}{k} .$$

For the initial conditions $\dot{Y}(0) = 0, Y(0) = Y_0 = 0$, the constant A *is* obtained as

$$Y(0) = Y_0 = 0 = A + \frac{w}{k} \Rightarrow A = -\frac{w}{k} .$$

From

$$\dot{Y} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t ,$$

one obtains

$$\dot{Y}(0) = \dot{Y}_0 = 0 = B\omega_n \Rightarrow B = 0 ,$$

and the complete solution is

$$Y = \frac{w}{k}(1 - \cos\omega_n t) . \qquad \textbf{(3.17)}$$

For arbitrary initial conditions $(Y_0, \dot{Y}_0)$, the solution is

$$Y(t) = Y_0 \cos\omega_n t + \frac{\dot{Y}_0}{\omega_n}\sin\omega_n t + \frac{w}{k}(1 - \cos\omega_n t)$$

Note that the maximum displacement defined by Eq.(3.17) occurs for $\omega_n t = \pi \Rightarrow \cos\omega_n t = -1$, and is defined by

$$Y_{\max} = \frac{w}{k}[1 - (-1)] = 2\frac{w}{k} = 2 \times Y_{static} .$$

Sometimes, engineers use 2 as a design factor of safety to account for dynamic loading verus static loading.

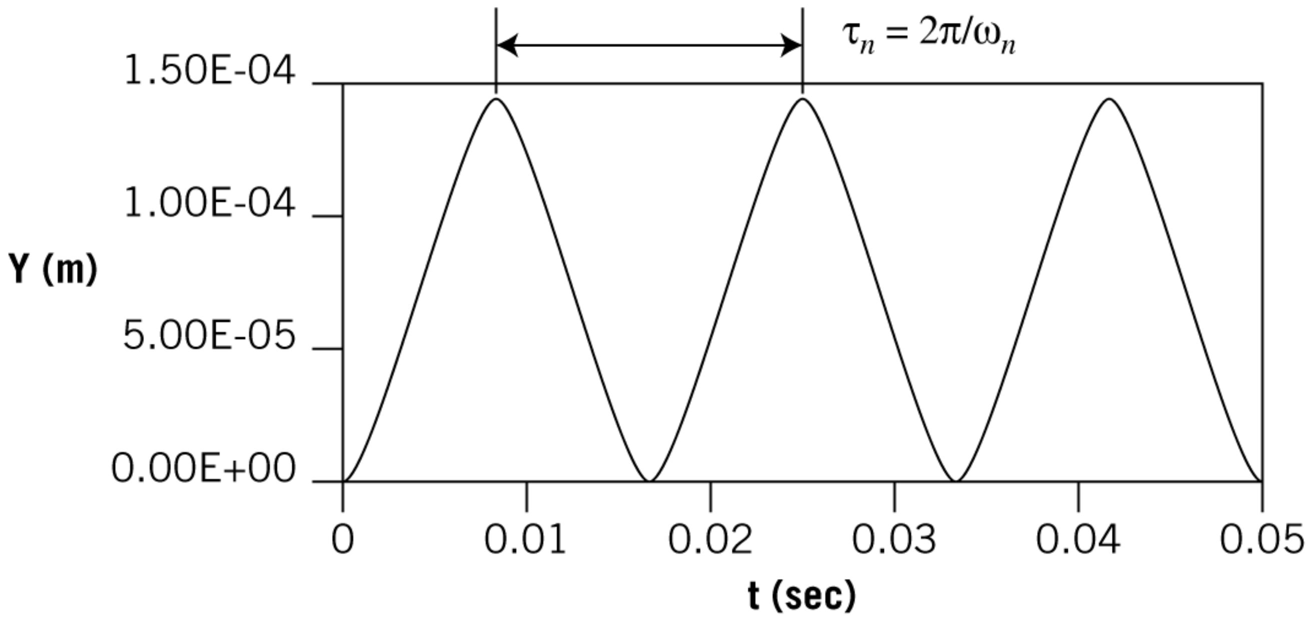


Figure 3.6 Solution $Y(t)$ from Eq.(3.17) with $m = 40\,kg$, $k = 5.685 \times 10^6 N/m$, yielding $\omega_n = 377.rad/\sec \Rightarrow f_n = 60\,Hz$ and $\tau_n = 2\pi/\omega_n = .01667\,\sec$.

The period for undamped motion is

$$\tau_n = \frac{2\pi}{\omega_n}(\sec)$$

Note that

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi}(\sec^{-1})$$

Energy-Integral Substitution

Substituting,

$$\ddot{Y} = \frac{d\dot{Y}}{dt} = \frac{d\dot{Y}}{dY}\frac{dY}{dt} = \dot{Y}\frac{d\dot{Y}}{dY} = \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right),$$

into the differential equation of motion gives

$$m\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = w - kY .$$

Multiplying through by dY and integrating gives

$$\frac{m\dot{Y}^2}{2} - \frac{m\dot{Y}_0^2}{2} = w\,(Y - Y_0) - \left(\frac{kY^2}{2} - \frac{kY_0^2}{2}\right) .$$

Rearranging gives,

$$\frac{m\dot{Y}^2}{2} - wY + \frac{kY^2}{2} = \frac{m\dot{Y}_0^2}{2} - w\,Y_0 + \frac{kY_0^2}{2} ,$$

which is the physical statement

$$T + V = T_0 + V_0,$$

where

$$V = V_g + V_s \ , \ V_g = -wY \ , \quad V_s = k\frac{Y^2}{2} \ .$$

The gravity potential-energy function is negative because the coordinate Y defines the body's distance *below* the datum.

The potential energy of a linear spring is $V_s = k\delta^2/2$ where δ is *the change in length of the spring from its undeflected position.* Note

$$f_s = -\frac{dV_s}{dY} = -\frac{d}{dY}(k\frac{Y^2}{2}) = -kY \ .$$

Hence, the spring force is the negative derivative of the potential-energy function.

Units

With the notable exception of the United States of America, all engineers use the SI system of units involving the meter, newton, and kilogram, respectively, for length, force, and mass. The metric system, which preceded the SI system, was legalized for commerce by an act of the United States Congress in 1866. The act of 1866 reads in part[1],

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[1] Mechtly E.A. (1969), *The International System of Units*, NASA SP-7012

> It shall be lawful throughout the United States of America to employ the weights and measures of the metric system; and no contract or dealing or pleading in any court, shall be deemed invalid or liable to objection because the weights or measures referred to therein are weights or measures of the metric system.

None the less, in the 21st century, USA engineers, manufacturers, and the general public continue to use the foot and pound as standard units for length and force. Both the SI and US systems use the second as a unit of tim. The US Customary system of units began in England, and continues to be referred to in the United States as the "English System" of units. However, Great Britain adopted and has used the SI system for many years.

Both the USA and SI systems use the same standard symbols for exponents of 10 base units. A partial list of these symbols is provided in Table 1.1 below. Only the "m = -3" (mm = millimeter = 10^{-3}m) and "k = +3" (km = kilometer = 10^{3} m) exponent symbols are used to any great extent in this book.

Table 1.1 Standard Symbols for exponents of 10.

Factor by which unit is multiplied	Prefix	Symbol
10^6	mega	M
10^3	kilo	k
10^2	hecto	h
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ

Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ ties the units of time, length, mass, and force together. In a vacuum, the vertical motion of a mass *m* is defined by

$$w = m\ddot{Y} = mg \quad , \tag{1.3}$$

where *Y* is pointed directly downwards, *w* is the weight force due to gravity, and *g* is the acceleration of gravity. At sea level, the standard acceleration of gravity[2] is

2

$$g = 9.81\, m/\sec^2 = 9810.\, mm/\sec^2$$
$$= 32.2\, ft/\sec^2 = 386.\, in/\sec^2 \quad . \qquad \textbf{(1.4)}$$

We start our discussion of the connection between force and mass with the SI system, since it tends to be more rational (not based on the length of a man's foot or stride). The kilogram (mass) and meter (length) are fundamental units in the SI system, and the Newton (force) is a derived unit. The formal definition of a Newton is ,"that force which gives to a mass of 1 kilogram an acceleration of 1 meter per second per second." From Newton's second law as expressed in Eq.(1.3), 9.81 newtons would be required to accelerate 1 kilogram at the constant acceleration rate of $g = 9.81\, m/\sec^2$, i.e.,

$$9.81\, N = 1\, kg \times 9.81\, m/\sec^2 = 9.81\, kg\ m/\sec^2$$

Hence, the newton has derived dimensions of $kg\ m/\sec^2$.

From Eq.(1.3), changing the length unit to the millimeter (mm) while retaining the kg as the mass unit gives $mg = m(kg) \times 9810\, mm/\sec^2$, which would imply a thousand fold increase in the weight force; however, 1 newton is still

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From the universal law of gravitation provided by Eq.(1.1), the acceleration of gravity varies with altitude; however, the standard value for *g* in Eq.(1.4) is used for most engineering analysis.

required to accelerate 1 kg at $g = 9.81\, m/\sec^2 = 9810\, mm/\sec^2$, and

$$m(kg) \times 9810\, mm/\sec^2 = w(10^{-3} N) = w(mN) \ .$$

Hence, for a kg-mm-second system of units, the derived force unit is
10^{-3} newton = 1 mN (1 milli newton).

Another view of units and dimensions is provided by the undamped-natural-frequency definition $\omega_n = \sqrt{K/M}$ of a mass M supported by a linear spring with spring coefficient K. Perturbing the mass from its equilibrium position causes harmonic motion at the frequency ω_n, and ω_n's dimension is $radians/second$, or $\sec^{-1}$ (since the radian is dimensionless). Using kg-meter-second system for length, mass, and time, the dimensions for ω_n^2 follow from

$$\omega_n^2 = \frac{K(N/m)}{M(kg)} = K\left(\frac{kg\, m}{\sec^2} \times \frac{1}{m}\right) \times \frac{1}{M(kg)} = \frac{K}{M}(\sec^{-2}) \ , \qquad \textbf{(1.5)}$$

confirming the expected dimensions.

Shifting to mm for the length unit while continuing to use the newton as the force unit would change the dimensions of the stiffness coefficient K to N/mm and reduce K by a factor of 1000. Specifically, the force required to deflect the spring 1 mm

should be smaller by a factor of 1000 than the force required to displace the same spring
1 m = 1000 mm. However, substituting *K* with dimensions of N/mm into Eq.(1.5), while retaining *M* in kg would cause an decrease in the undamped natural frequency by a factor of $\sqrt{1000}$. Obviously, changing the units should not change the undamped natural frequency; hence, this proposed dimensional set is wrong. The correct answer follows from using *mN* as the derived unit for force. This choice gives the dimensions of *mN /mm* for *K*, and leaves both *K* and ω_n unchanged. To confirm that *K* is unchanged (numerically) by this choice of units, suppose $K = 1000\, N/m = 1\, N/mm = 1000\, mN/mm$. The reaction force produced by the deflection $\delta = 1\, mm = 10^{-3} m$ is

$$f_s = K\delta = 1000\,\frac{N}{m} \times 10^{-3} m = 1\, N$$

$$= 1\,\frac{N}{mm} \times 1 mm = 1\, N$$

$$= 1000\,\frac{mN}{mm} \times 1\, mm = 10^3\, mN = 1\, N\ ,$$

confirming that *mN* is the appropriate derived force unit for a *kg-mm-sec* unit system.

Table 1.2 SI base and derived units

Base mass unit	Base time unit	Base length unit	Derived force unit	Derived force unit dimensions
kilogram (kg)	second (sec)	meter (m)	Newton (N)	kg m / sec^2
kilogram (kg)	second (sec)	millimeter (mm)	milli Newton (mN)	kg mm / sec^2

Shifting to the US Customary unit system, the unit of pounds for force is the base unit, and the mass dimensions are to be derived. Applying Eq.(1.3) to this situation gives

$$w = mg \;\Rightarrow\; m = \frac{w(lb)}{32.2\, ft/\sec^2} = \frac{w}{32.2}\,\frac{lb\sec^2}{ft}\;.$$

The derived mass unit has dimensions of $lb\sec^2/ft$ and is called a "slug." When acted on by a resultant force of 1 lb, a mass of one slug will accelerate at $1\,ft/\sec^2$. Alternatively, under standard conditions, a mass of one slug weighs 32.2 lbs.

If the inch-pound-second unit system is used for displacement, force, and time, respectively, Eq.(1.3) gives

$$w = mg \;\Rightarrow\; m = \frac{w(lb)}{386.\, in/\sec^2} = \frac{w}{386.}\,\frac{lb\sec^2}{in}\;,$$

and the mass has derived dimensions of $lb\sec^2/in$. Within the author's 1960's aerospace employer, a mass weighing one pound with the derived units of $lb\sec^2/in$ was called a "snail". To the author's knowledge, there is no commonly accepted name for this mass, so *snail* will be used in this discussion. When acted upon by a resultant 1 lb force, a mass of 1 snail will accelerate at 386. in/sec^2, and under standard conditions, a snail weighs 386. lbs.

Returning to the undamped natural frequency discussion, from Eq.(1.5) for a pound-ft-sec system,

$$\omega_n^2 = \frac{K(lb/ft)}{M(\dfrac{lb\sec^2}{ft})} = \frac{K}{M}(\sec^{-2}) \ .$$

Switching to the inch-pound-second unit system gives

$$\omega_n^2 = \frac{K(lb/in)}{M(\dfrac{lb\sec^2}{in})} = \frac{K}{M}(\sec^{-2}) \ .$$

Table 1.3 US Customary base and derived units

Base force unit	Base time unit	Base length unit	Derived mass unit	Derived mass unit dimensions
pound (lb)	second (sec)	foot (ft)	slug	lb sec^2/ ft
pound (lb)	second (sec)	inch (in)	snail	lb sec^2/ in

Many American students and engineers use the pound mass (lbm) unit in thermodynamics, fluid mechanics, and heat transfer. A one pound mass weighs one pound under standard conditions. However, the lbm unit makes no sense in dynamics. Inserting $m = 1\, lbm$ into $w = mg$ with $g = 32.2 ft/\sec^2$ would require a new force unit equal to 32.2 lbs, called the *Poundal*. *Stated briefly, unless you are prepared to use the poundal force unit, the pound-mass unit should never be used in dynamics.*

From one viewpoint, the US system makes more sense than the SI system in that a US-system scale states your weight (the force of gravity) in pounds, a force unit. A scale in SI units reports your weight in kilos (kilograms), the SI mass unit, rather than in Newtons, the SI force unit. Useful (and exact) conversion factors between the SI and USA systems are: 1 lbm = .4535 923 7* kg, 1 in = .0254* m, 1 ft = .3048* m, 1 lb = 4.448 221 615 260 5* N. The * in these definitions denotes internationally-agreed-upon *exact* conversion factors.

Conversions between SI and US Customary unit systems should be checked carefully. An article in the 4 October 1999 issue of *Aviation Week and Space Technology* states," Engineers have discovered that use of English instead of metric units in a navigation software table contributed to, if not caused, the loss of Mars Climate Orbiter during orbit injection on Sept. 23." This press report covers a highly visible and public failure; however, less spectacular mistakes are regularly made in unit conversions.