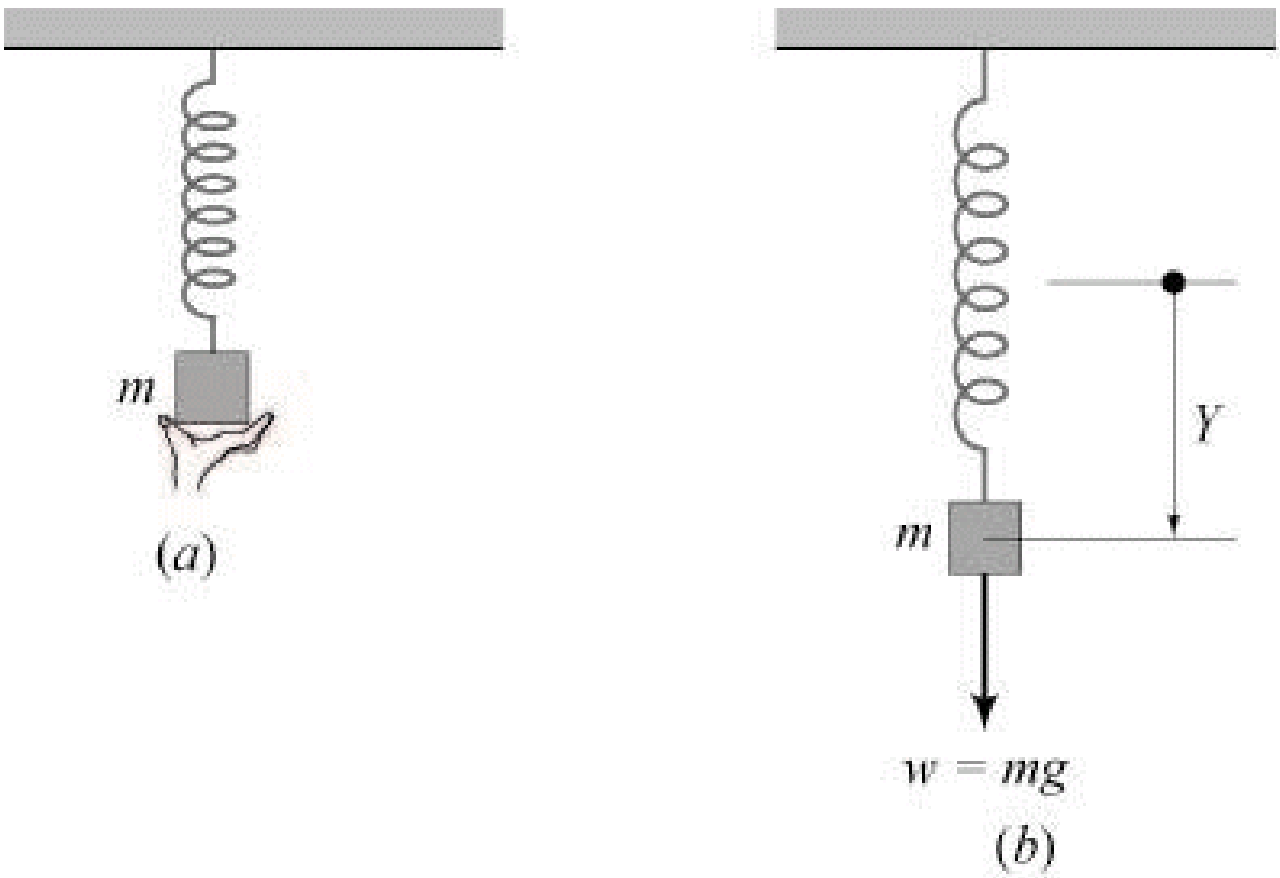
Lecture 6. MORE VIBRATIONS

Deriving the Equation of Motion, Starting From an Energy Equation



Assume that the spring is undeflected for $Y=0$ and the gravity potential-energy datum is also at $Y=0$. Starting with,

$$T+V=T_0+V_0,$$

we can state

$$\frac{m\dot{Y}^2}{2} - wY + \frac{kY^2}{2} = \frac{m\dot{Y}_0^2}{2} - w\ Y_0 + \frac{kY_0^2}{2}\ .$$

The negative sign applies for the gravity potential energy because *Y* is below the datum. Differentiating with respect to *Y* gives

$$\frac{d}{dY}\left(\frac{m\dot{Y}^2}{2}\right) - w + \frac{d}{dY}\left(\frac{kY^2}{2}\right) = 0 \Rightarrow \quad m\ddot{Y} = w - kY\ .$$

In many cases, the differential equation of motion is obtained more easily from an energy equation than from Newton's 2nd law.

Equilibrium Conditions. Equilibrium for the particle governed by the mass-spring differential equation,

$$\Sigma f_Y = m\ddot{Y} = w - kY\ ,$$

occurs for $\ddot{Y} = 0$, and defines the equilibrium position

$$0 = w - k\bar{Y} \Rightarrow \quad \bar{Y} = \frac{w}{k}\ .$$

We looked at motion about the equilibrium position by defining $Y = \bar{Y} + y \Rightarrow \ddot{Y} = \ddot{y}$. Substituting these results into the differential equation of motion gives the following *perturbed* differential equation of motion

$$m\ddot{y} + k(\bar{Y} + y) = w \Rightarrow m\ddot{y} + ky = 0 ,$$

since $k\bar{Y} = w$. This equation has the particular solution $Y_p = 0$ and the complete solution

$$Y = Y_h + Y_p = A \cos\omega_n t + B \sin\omega_n t + 0 .$$

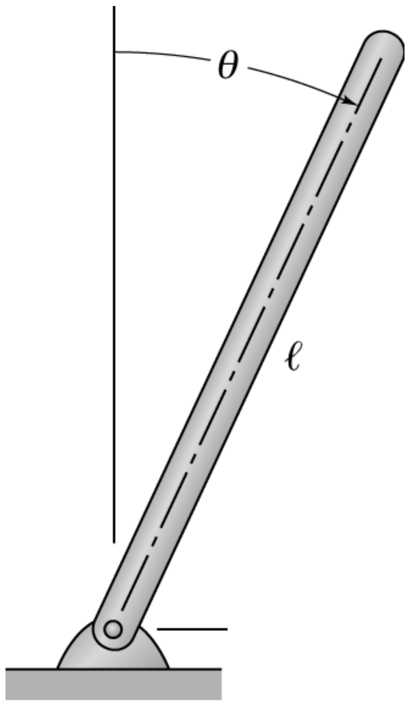
The motion is stable, oscillating about the equilibrium position at the undamped natural frequency $\omega_n = \sqrt{k/m}$; hence, $Y = \bar{Y} = w/k$ defines a *stable* equilibrium position.

For small motion about an unstable static equilibrium position, the perturbed differential equation of motion will have a "negative stiffness coefficient," such as

$$m\ddot{y} - ky = 0 ,$$

and an unstable time solution

$$Y = A \cosh\omega_n t + B \sinh\omega_n t$$



Inverted compound pendulum with an unstable static equilibrium position about $\theta = 0$.

For small motion about the equilibrium position $\theta = 0$, the inverted compound pendulum has the differential equation of motion

$$\ddot{\theta} - \frac{3g}{2l}\theta = 0 .$$

External time-varying force. Adding the external time-varying force $f(t) = f_0 \sin \omega t$ to the harmonic oscillator (mass-spring) system yields the differential equation

$$m\ddot{Y} = w - kY + f_0 \sin \omega t .$$

The energy-integral substitution,

$$\ddot{Y} = \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) ,$$

gives

$$m \frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = w - kY + f_0 \sin \omega t ,$$

and integration gives

$$\left(\frac{m\dot{Y}^2}{2} - wY + \frac{kY^2}{2}\right) - \left(\frac{m\dot{Y}_0^2}{2} - w Y_0 + \frac{kY_0^2}{2}\right) = \int_{Y_0}^{Y} f_0 \sin \omega t \, dy = \int_0^t f_0 \sin \omega \tau \times \dot{y}(\tau) \, d\tau .$$

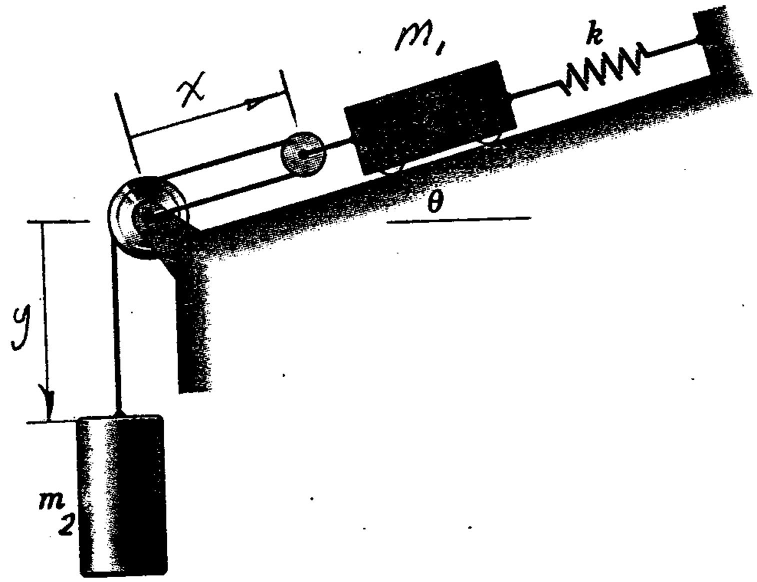
This equation is a specific example of the general equation

$$\Delta(T+V) = Work_{nonconservative} \ .$$

An external time-varying force is a nonconservative force and produces nonconservative work. The right-hand side integral cannot be evaluated unless the solution $Y(t)$ for the differential equation is known.

Hence, for nonconservative forces that are functions of time, neither the energy-integral substitution nor the work-energy equation is useful in solving the differential equation of motion. The substitution is normally helpful in solving the differential equation when the acceleration can be expressed as a function of displacement only.

Example 6.1

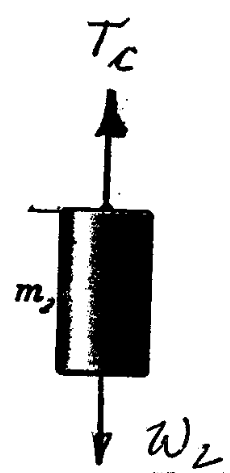


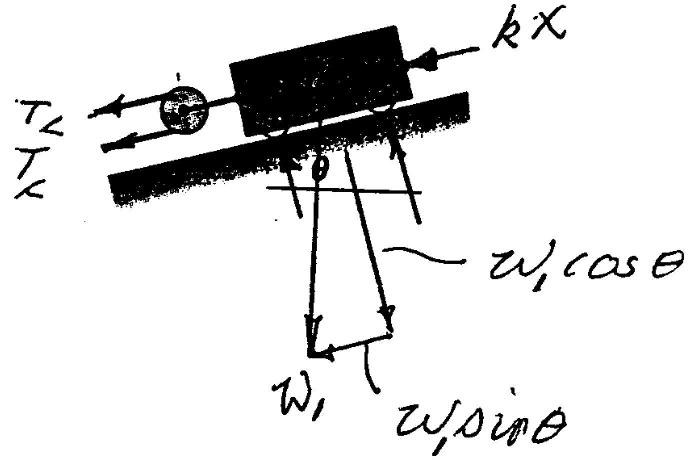
2 bodies, one degree of freedom
⇒ Kinematic constraint equations

$$l_c = cord\ length = 2x + y + a \Rightarrow \dot{y} = -2\dot{x} \Rightarrow \ddot{y} = -2\ddot{x},$$

where a is the amount of cord wrapped around the pulley.

Free-body diagrams





E

quations of Motion

$$\Sigma f_x = -2T_c - w_1 \sin\theta - kx = m_1 \ddot{x}$$
$$\Sigma f_y = w_2 - T_c = m\ddot{y} \quad \Rightarrow \quad T_c = w_2 - m_2 \ddot{y}.$$

Substituting for T_c, the cord tension, and using the kinematic constraint to eliminate $\ddot{y}$ gives

$$-2(w_2 - m_2\ddot{y}) - w_1 \sin\theta - kx = m_1 \ddot{x}$$

$$-2[w_2 - m_2(-2\ddot{x})] - w_1 \sin\theta - kx = m_1 \ddot{x}$$

$$\therefore \; (m_1 + 4m_2)\ddot{x} + kx = -2w_2 - w_1 \sin\theta \quad \Rightarrow \quad \omega_n^2 = k/(m_1 + 4m_2).$$

This equation of motion can be stated

$$m_{eq}\ddot{x} + k_{eq}x = f_{eq} \quad .$$

Note that both entries in $m_{eq} = m_1 + 4m_2$ are positive. If either were negative, the answer would be wrong. We don't have negative masses in dynamics, and ***a negative contribution in this type of coefficient always indicates a mistake in developing the equation of motion.***

Strategy for Deriving and Verifying the Equation of Motion from $\sum f_X = m\ddot{X}$:

1. Select Coordinates and draw them on your figure. Your choice for coordinates will establish the + and - signs for displacements, velocities, accelerations and forces. Check to see if there is a relation between your coordinates. If there is a relationship, write out the corresponding kinematic constraint equation(s).

2. Draw free-body diagrams corresponding to positive displacements and velocities for your bodies. In the present example a positive *x* displacement produced a compression force in the spring acting in the -*x* direction.

3. Use $\sum \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to state the equations of motion with + and - forces defined by the + and - signs of your coordinates.

4. Use the kinematic constraint equation(s) to eliminate excess variables to produce an equation of motion.

5. If your equation has the form $m_{eq}\ddot{x} + k_{eq}x = f\,(t)$, check to see that the individual contributions for m_{eq} and k_{eq} are positive.

Think about the degrees of freedom for your system. A one degree-of-freedom (1DOF) system needs one coordinate to define all of the bodies' positions. A 2DOF system needs two coordinate to define all of the bodies' positions, etc. Example 6.1 has two coordinates but only one degree of freedom.

Equation of Motion for motion about equilibrium

Equilibrium is defined by $\sum f_x = 0 \Rightarrow \ddot{x} = 0$ and implies

$$\bar{x} = -(2w_2 + w_1 \sin\theta)/k \; .$$

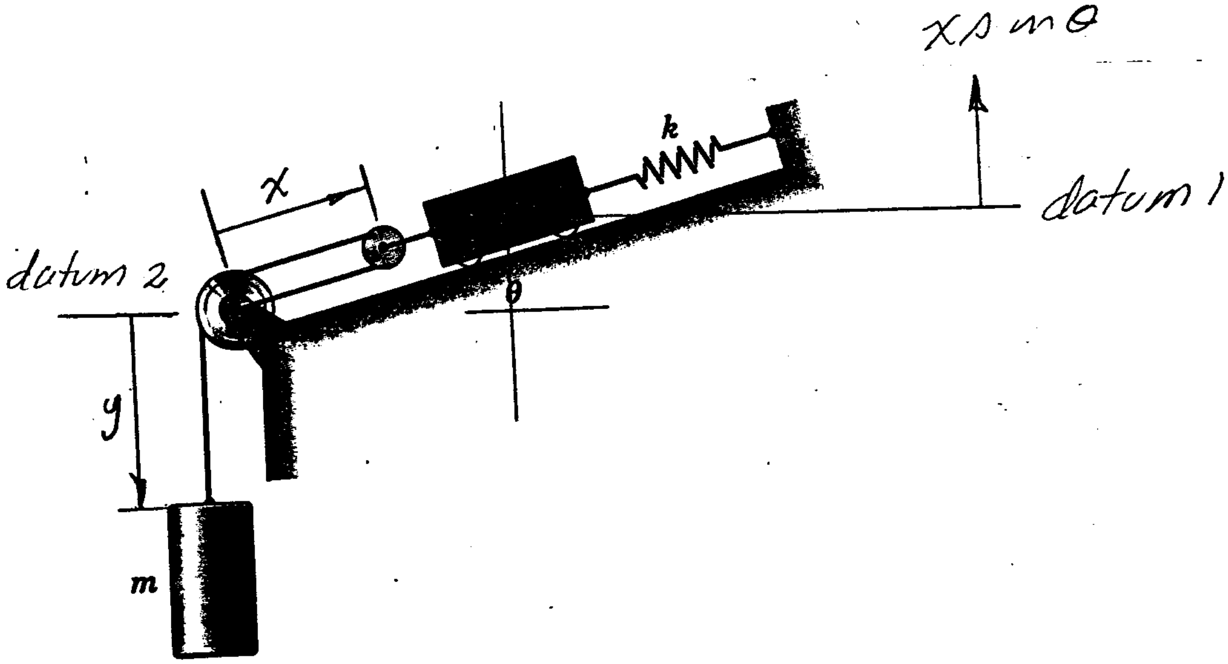
For motion about the equilibrium position defined by $x = \bar{x} + \delta x \Rightarrow \ddot{x} = \delta\ddot{x}$,

$$(m_1 + 4m_2)\delta\ddot{x} + k(\delta x + \bar{x}) = -2w_2 - w_1 \sin\theta$$

$$\therefore \; (m_1 + 4m_2)\delta\ddot{x} + k\delta x = 0 \; .$$

ENERGY APPROACH

Data for Potential Energy Functions



Conservation-of-Energy Equation: $T + V = T_0 + V_0$

$$m_2 \frac{\dot{y}^2}{2} + m_1 \frac{\dot{x}^2}{2} - w_2 y + w_1 x \sin\theta + k \frac{x^2}{2} = T_0 + V_0 \ , or$$

$$m_2 \frac{(-2\dot{x})^2}{2} + m_1 \frac{\dot{x}^2}{2} - w_2 (l_c - 2x) + w_1 x \sin\theta + k \frac{x^2}{2} = T_0 + V_0 , or$$

$$(m_1 + 4m_2) \frac{\dot{x}^2}{2} - w_2 (l_c - 2x) + w_1 x \sin\theta + k \frac{x^2}{2} = T_0 + V_0 \ .$$

Differentiating w.r.t. x

$$(m_1 + 4m_2)\frac{d}{dx}(\frac{\dot{x}^2}{2}) + 2w_2 + w_1\sin\theta + kx = 0$$

$$\therefore\ (m_1 + 4m_2)\ddot{x} + kx = -2w_2 - w_1\sin\theta$$