LECTURE 7. MORE VIBRATIONS

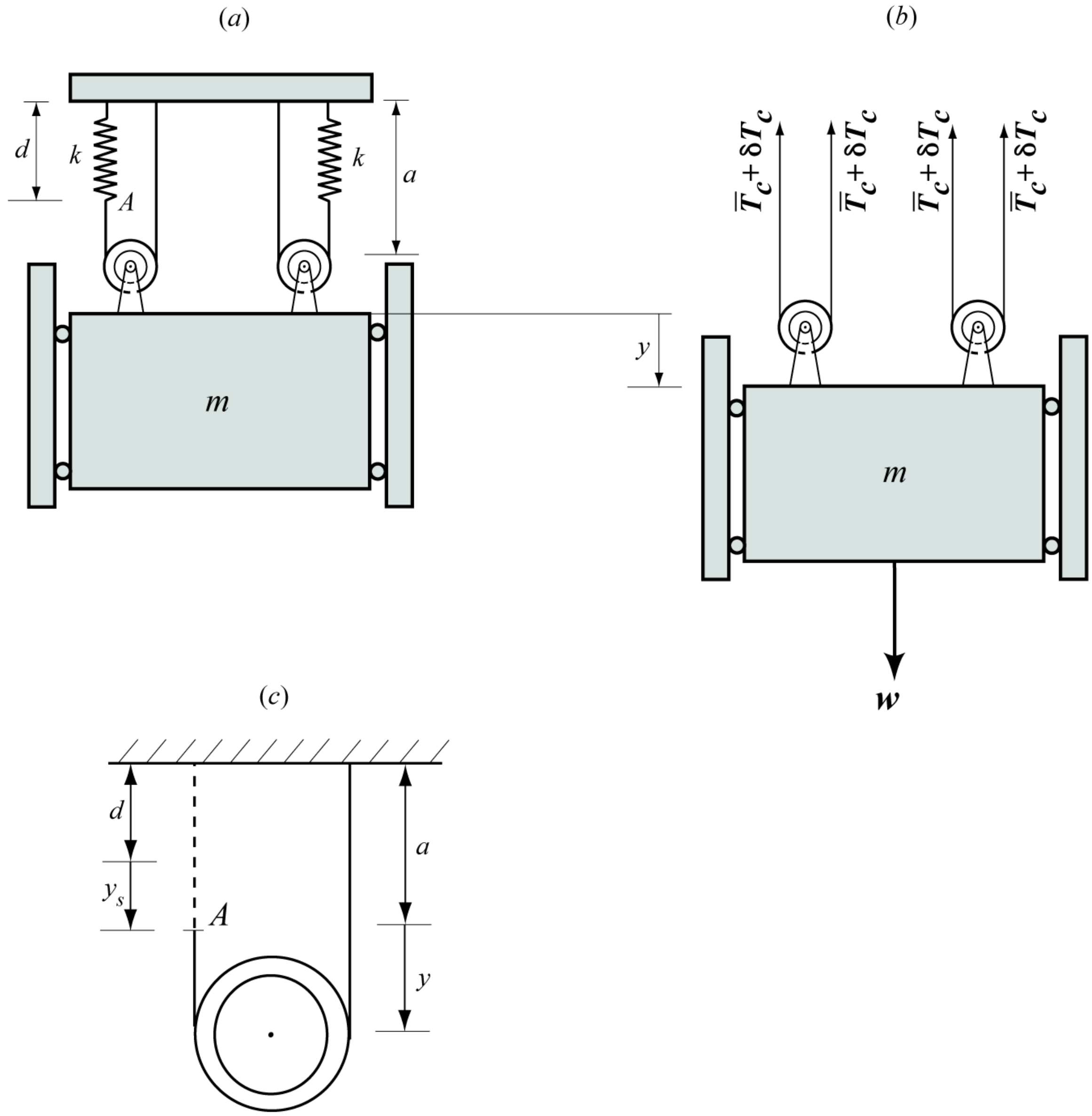


Figure XP3.1 (a) Mass in equilibrium, (b) Free-body diagram, (c) Kinematic constraint relation

Example Problem 3.1 Figure XP3.1 illustrates a mass *m* that is *in equilibrium* and supported by two spring-pulley combinations. **Tasks:** *a.* Draw a free-body diagram and derive the equations of motion, and *b.* For *w*=100 lbs, and $k = 200\, lb/in$. Determine the natural frequency.

For equilibrium,

$$\Sigma f_Y = w - 4\bar{T}_c$$

Applying $\Sigma f_y = m\ddot{y}$ to the free-body diagram gives

$$m\ddot{y} = \Sigma f_y = w - 4T_c = w - 4\bar{T}_c - 4\delta T_c = -4\delta T_c \ . \qquad \textbf{(i)}$$

The reaction forces are defined as $\delta T_c = k y_s$ where y_s is the change in length of the spring due to the displacement y. We need a relationship between y and y_s.

The cord wrapped around a pulley is inextensible with length l_c. Point A in figure XP3.1a locates the end of the cord. From this figure, $l_c = a + (a - d) = 2a - d$. Figure XP3.1c shows the displaced position and provides the following relationship,

$$\begin{aligned} l_c &= (a + y) + [(a + y) - (d + y_s)] = (2a - d) + 2y - y_s \\ &= l_c + 2y - y_s \end{aligned} \qquad \textbf{(ii)}$$

$$\therefore \ y_s = 2y \ .$$

Applying these results to Eq.(i) gives

$$n\ddot{y} = -4\delta T_c = -4ky_s = = -4k_s(2y) = -8ky \ \Rightarrow \ m\ddot{y} + 8ky = 0 \qquad \textbf{(iii)}$$

This step concludes *Task a.* Eq.(i) has the form

$m\ddot{y} + k_{eq} y = 0$ where $k_{eq} = 8k$ is the "equivalent stiffness."
Dividing through by m puts the equation into the form
$\ddot{y} + \omega_n^2 y = 0$; hence,

$$\omega_n^2 = \frac{k_{eq}}{m} = \frac{8 \times 100\, lb/in}{[100\, lb/(386.\, in/\sec^2)]} = 3088\left(\frac{rad}{\sec}\right)^2$$

$$\therefore\ \omega_n = 55.6\,\frac{rad}{\sec}$$

and

$$f_n = 55.6\,\frac{rad}{\sec} \times \left(\frac{1\, cycle}{2\pi\, rad}\right) = 8.84\,\frac{cycle}{\sec} = 8.84\, Hz$$

Note the conversion from weight to mass via $m = w/g$ where for the inch-pound-second system, $g = 386.\, in/\sec^2$.

$$Period = \tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = .113\sec\ .$$

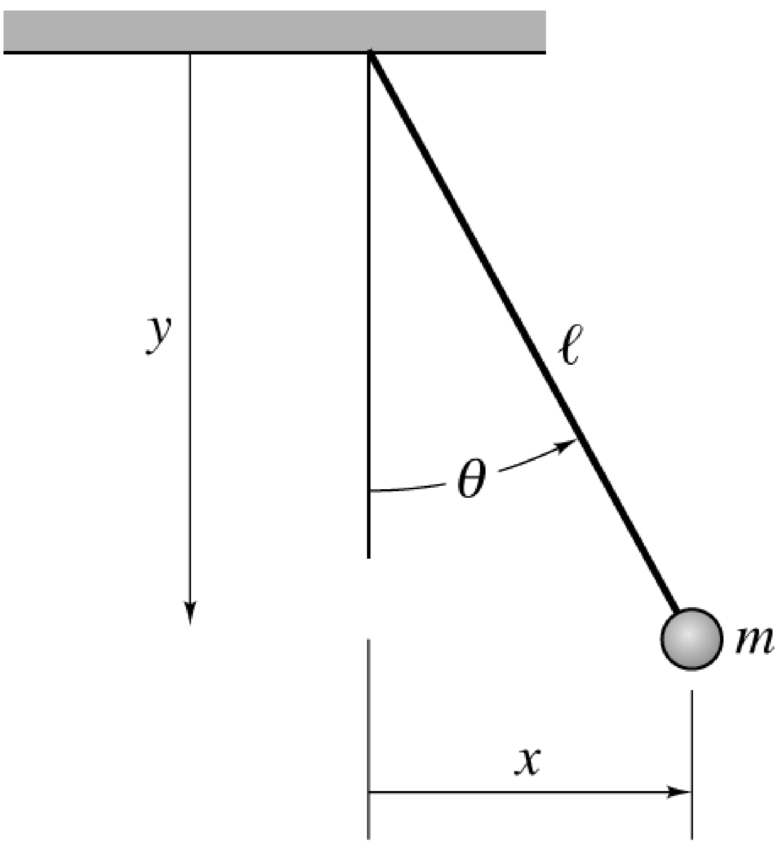
Deriving Equation of Motion From Conservation of Energy

$$T + V = T_0 + V_0 \quad \Rightarrow \quad m\frac{\dot{y}^2}{2} + 2\left[\frac{k}{2}(2y)^2\right] = Constant$$

$$m\frac{\dot{y}^2}{2} + 4y^2 = Constant$$

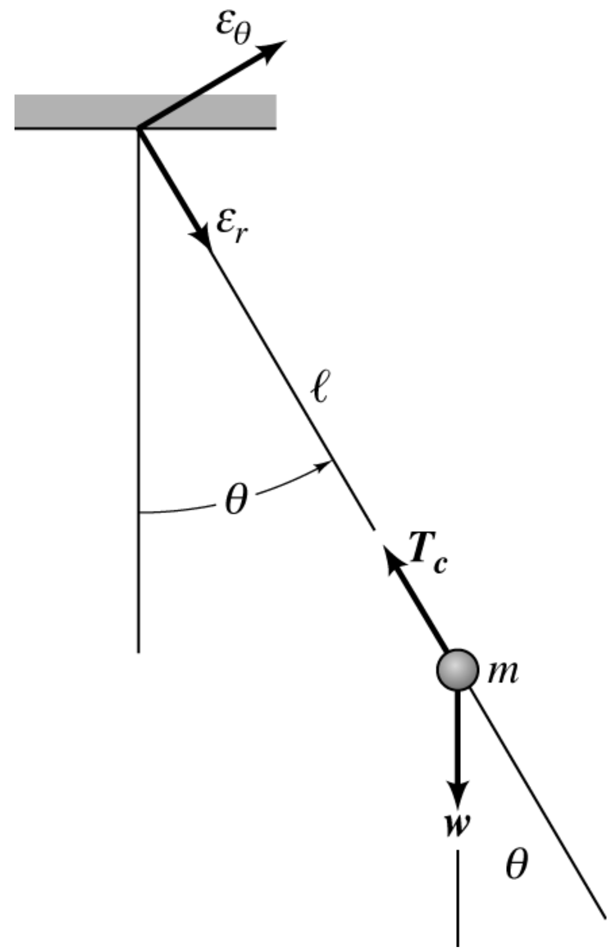
Differentiating w.r.t. y gives

$$m\frac{d}{dy}\left(\frac{\dot{y}^2}{2}\right) + 8ky = 0 \;\Rightarrow\; m\ddot{y} + 8ky = 0 \;\; .$$



The Simple Pendulum, Linearization of nonlinear Differential Equations for small motion about an equilibrium

Pendulum
Free-Body Diagram



The pendulum cord is inextensible.

Application of $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ in polar coordinates:

$$\begin{aligned} \Sigma f_r &= -T_c + w\cos\theta = m(\ddot{l} - ml\dot{\theta}^2) = -ml\dot{\theta}^2 \\ \Sigma f_\theta &= -w\sin\theta = m(l\ddot{\theta} + 2\dot{l}\dot{\theta}) = ml\ddot{\theta} \end{aligned} \qquad (3.82)$$

Nonlinear Equation of Motion

$$ml\ddot{\theta} = -w\sin\theta \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0\,. \qquad (3.83)$$

Equilibrium is defined by $\ddot{\theta} = 0 \Rightarrow \sin\overline{\theta} = 0 \Rightarrow \overline{\theta} = 0 \ or\ \pi$

Expanding the nonlinear $\sin\theta$ term in a Taylor series about $\theta = 0$ gives

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \ldots \tag{3.84}$$

Discarding second-order and higher terms in θ gives the linearized D.Eq.

$$\ddot{\theta} + \frac{g}{l}\theta \cong 0 \quad \Rightarrow \quad \ddot{\theta} + \omega_n^2\theta = 0 \ . \tag{3.85}$$

where

$$\omega_n = \sqrt{g/l} \ , \tag{3.86}$$

Linearization validity:

$$\begin{gathered} \sin 15° = \sin(.2618\, rad) = .2588 \\ \theta = .2618 \\ \theta^3/6 = .002990 \\ \theta^5/120 = .000010 \ . \end{gathered} \tag{3.87}$$

Linearized model is reasonable for $\theta \le 15°$.

Pendulum Differential Equation of Motion From Conservation of Energy

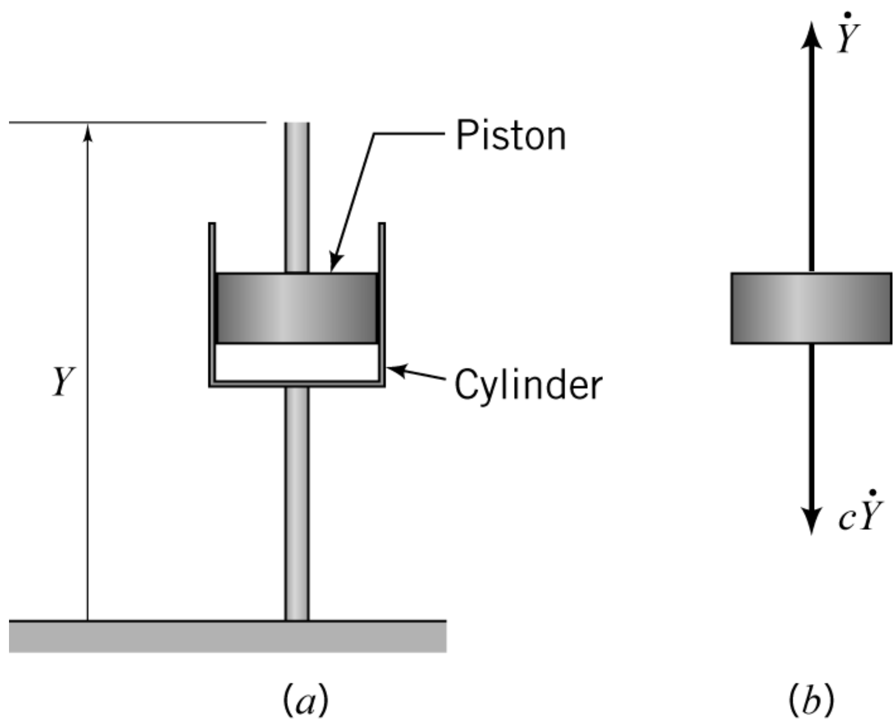
$$T + V = T_0 + V_0 \quad \Rightarrow \; m\frac{(l\dot{\theta})^2}{2} - wl\cos\theta = constant \; .$$

Differentiating w.r.t. θ gives

$$ml^2\frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) + wl\sin\theta = 0 \quad \Rightarrow \; \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \; .$$

ENERGY DISSIPATION—Viscous Damping

Automatic door-closers and shock absorbers for automobiles provide common examples of energy dissipation that is deliberately introduced into mechanical systems to limit peak response of motion.



As illustrated above, a thin layer of fluid lies between the piston and cylinder. As the piston is moved, shear flow is produced in the fluid which develops a resistance force that is proportional to the velocity of the piston relative to the cylinder. The flow between the piston and the cylinder is laminar, versus turbulent flow which occurs commonly in pipe flow and will be covered in your fluid mechanics course. The piston free-body diagram shows a <u>reaction force</u> that is proportional to the piston's velocity and acting in a direction that is <u>opposite</u> to the piston's velocity

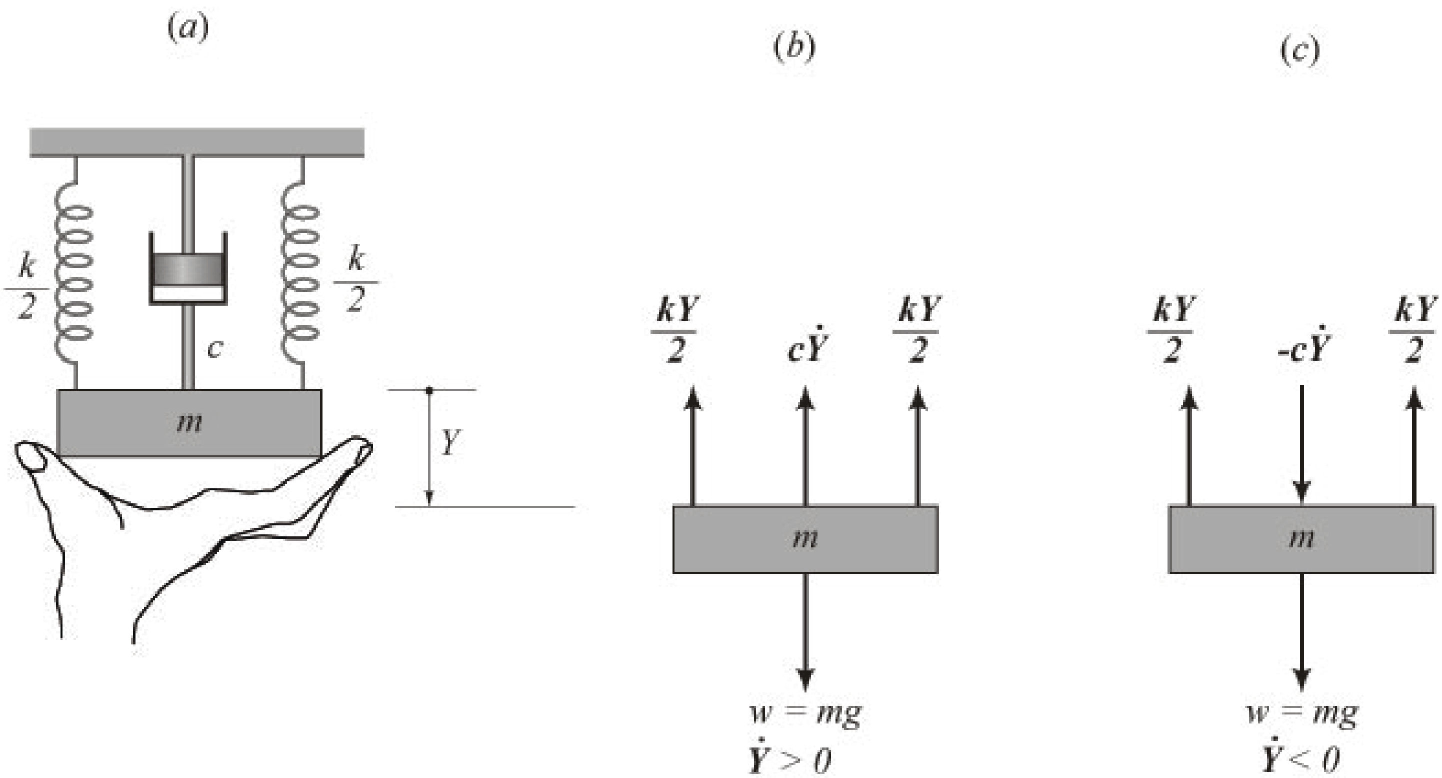


Figure 3.10 (a) Body acted on by its weight and supported by two springs and a viscous damper, (b) Free-body diagram for $Y>0, \dot{Y}>0$, (c) Free-body for $Y>0, \dot{Y}<0$

Figure 3.10a shows a body of mass *m* that is supported by two equal-stiffness linear springs with stiffness coefficients $k/2$ and a viscous damper with damping coefficient c. Figure 3.10b provides the free-body diagram for $Y>0, \dot{Y}>0$. The springs are undeflected at $Y=0$. Applying Newton's equation of motion to the free-body diagram gives:

$$m\ddot{Y} = \Sigma F_Y = w - \frac{kY}{2} - c\dot{Y} - \frac{kY}{2} ,$$

or

$$m\ddot{Y} + c\dot{Y} + kY = w \;. \qquad \textbf{(3.21)}$$

Dividing through by m gives

$$\ddot{Y} + 2\,\zeta\,\omega_n\,\dot{Y} + \omega_n^2\,Y = g \;,$$

where

$$\omega_n = \sqrt{k/m} \quad , \quad 2\,\zeta\,\omega_n = c/m \Rightarrow \zeta = \frac{c}{2\sqrt{km}}.$$

ζ is called the linear damping factor.

Transient Solution due to Initial Conditions and Weight.

Homogeneous differential Equation

$$\ddot{Y}_h + 2\,\zeta\,\omega_n\,\dot{Y}_h + \omega_n^2\,Y_h = 0 \qquad (3.22)$$

Assumed solution: $Y_h = Ae^{st}$ yields:

$$(s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2)\,Ae^{st} = 0 \;.$$

Since $A \neq 0$, and $e^{st} \neq 0$

$$s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2 = 0\,. \tag{3.23}$$

This is the characteristic equation. For $\zeta = 1$, the mass is critically damped and does not oscillate. For $\zeta<1$, the roots are

$$\begin{aligned} s &= -\,\zeta\,\omega_n \pm j\,\omega_n\,\sqrt{1-\zeta^2} \\ &= -\,\zeta\,\omega_n \pm j\,\omega_d\,. \end{aligned} \tag{3.24}$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ is called the *damped natural frequency*. The two roots defined by Eq.(3.24) are

$$s_1 = -\zeta\omega_n + \boldsymbol{j}\omega_d\ ,\ s_2 = -\zeta\omega_n - \boldsymbol{j}\omega_d\ .$$

The homogeneous solution looks like

$$\begin{aligned} Y_h &= A_1\,e^{s_1 t} + A_2\,e^{s_2 t} \\ &= A_1\,e^{(-\zeta\,\omega_n + j\,\omega_d)\,t} + A_2\,e^{(-\zeta\,\omega_n - j\,\omega_d)\,t}\ , \end{aligned} \tag{3.25}$$

where $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are *complex* coefficients. Substituting the identities

$$e^{j\omega_d t} = \cos\omega_d t + j\,\sin\omega_d t\ ,\quad e^{-j\omega_d t} = \cos\omega_d t - j\,\sin\omega_d t$$

into Eq.(3.25) yields a final homogeneous solution of the form

$$Y_h = e^{-\zeta \omega_n t} (A \cos\omega_d t + B \sin\omega_d t) . \qquad \textbf{(3.26)}$$

where A and B are *real* constants. The complete solution is

$$Y = Y_h + Y_P = e^{-\zeta \omega_n t} (A \cos\omega_d t + B \sin\omega_d t) + \frac{w}{k} . \qquad \textbf{(3.27)}$$

For I.C.'s $Y(0) = \dot{Y}(0) = 0$, starting with Eq.(3.27) yields

$$Y(0) = 0 = A + \frac{w}{k} \Rightarrow A = -\frac{w}{k} .$$

To evaluate B, we differentiate Eq.(3.27), obtaining

$$\dot{Y} = e^{-\zeta\omega_n t} \omega_d (-A \sin\omega_d t + B \cos\omega_d t)$$

$$-\zeta \omega_n e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t) .$$

Evaluating this expression at $t = 0$ gives

$$\dot{Y}(0) = 0 = B \omega_d - \zeta \omega_n A .$$

Hence,

$$B = \frac{\zeta \omega_n A}{\omega_d} = \frac{\zeta}{\sqrt{1-\zeta^2}} A = -\frac{w \zeta/k}{\sqrt{1-\zeta^2}} .$$

The complete solution is now given as

$$Y = \frac{w}{k}\left[1 - e^{-\zeta\,\omega_n\,t}\left(\cos\,\omega_d\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\sin\,\omega_d\,t\right)\right] . \quad \textbf{(3.28)}$$

Figure 3.10 illustrates this solution for the data

$$m = 1\ kg\ ,\quad k = 3{,}948.\ N/m\ ,\quad c = 12.57\ N\,\text{sec}/m\ ,$$

$$\omega_n = 62.8\ rd/\text{sec} \Rightarrow f_n = 10\ cycle/\text{sec} = 10\ Hz$$
$$\zeta = .10\ ,\quad \omega_d = 62.2\ rd/\text{sec}\ . \quad \textbf{(3.29)}$$

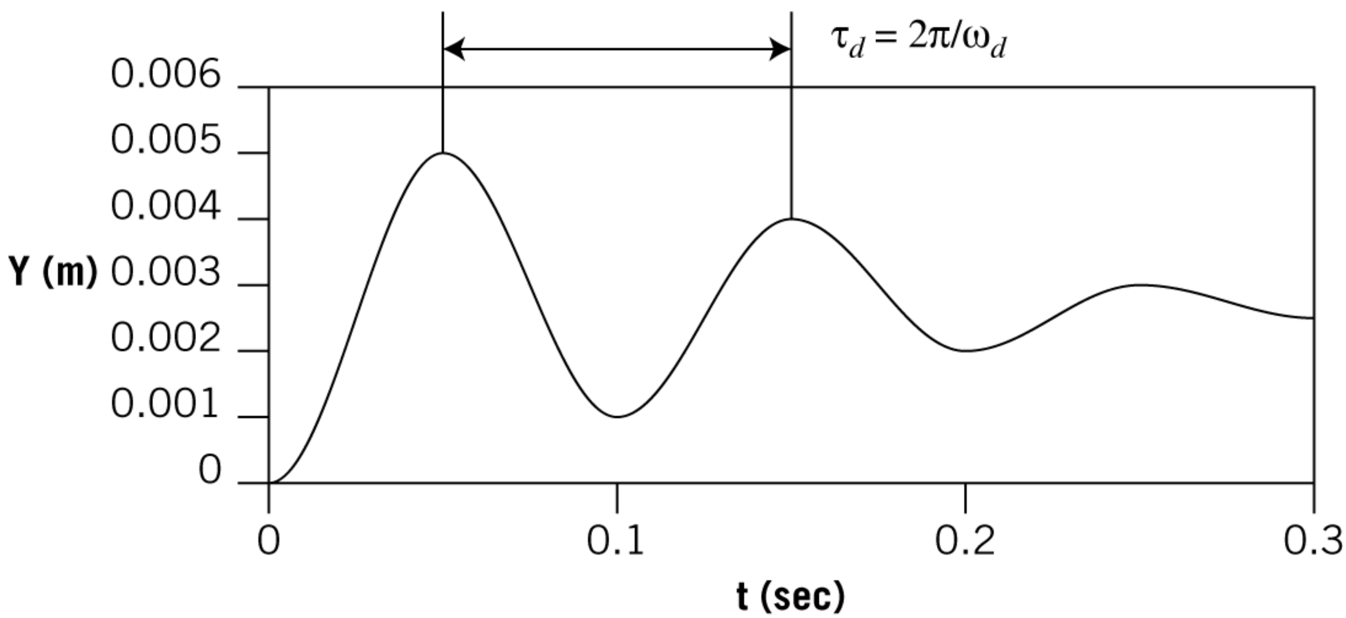


Figure 3.10 Time solution of Eq.(3.28) with the data of Eq.(3.29).

Substituting the "energy-integral" substitution $\ddot{Y}=d(\dot{Y}^2/2)/dY$ into Eq.(3.21)

$$m\,\frac{d}{dY}\left(\frac{\dot{Y}^2}{2}\right) = -kY + w - c\dot{Y}\,.$$

We can still multiply through by dY, but we can not execute the integral,

$$\int_{Y_0}^{Y} c\,\dot{Y}\,dY = \int_{t_0}^{t} c\,\dot{Y}\,\frac{dY}{dt}\,dt = \int_{t_0}^{t} c\,\dot{Y}^2 dt$$

since $\dot{Y}$ is generally a function of t, not Y. You need the *solution* $Y(t)$ to evaluate this integral and determine how much energy is dissipated.

Incorrect Sign for Damping.

Suppose you get the sign wrong on the damping force, netting,

$$m\ddot{Y} - c\dot{Y} + kY = w\,.$$

Dividing through by *m gives*

$$\ddot{Y} - 2\,\zeta\,\omega_n\,\dot{Y} + \omega_n^2\,Y = g\,.$$

the general solution is now

$$Y = e^{\zeta\omega_n t}\,(A\ \cos\omega_d t\ +\ B\ \sin\omega_d t)\ +\ \frac{w}{k}\ .$$

Instead of decaying exponentially with time, the solution grows exponentially with time.

UNITS — Damping Coefficients

The damping force is defined by $f_{damping} = -c\dot{x}$; hence, the appropriate dimensions for the damping coefficient c in SI units is

$$c = \frac{f(N)}{\dot{x}(m/\sec)} = c(\frac{N\sec}{m})\ .$$

In the US system with *lb-ft-sec* units,

$$c = \frac{f(lb)}{\dot{x}(ft/\sec)} = c(\frac{lb\sec}{ft})\ .$$

Similarly, in the US system with *lb-in-sec* units,

$$c = \frac{f(lb)}{\dot{x}(in/\sec)} = c(\frac{lb\sec}{in})\ .$$

The damping factor ζ is always dimensionless, defined by,

$$2\zeta\omega_n = \frac{c}{M} \Rightarrow \zeta = \frac{c}{2M\omega_n} \ ,$$

where M is the mass. In the SI system, we have

$$\zeta = c\left(\frac{N\sec}{m}\right) \times \frac{1}{2M(kg)} \times \frac{1}{\omega_n(\sec^{-1})} = \frac{c}{2M\omega_n}\left(\frac{N\sec^2}{m\,kg}\right) \ .$$

However, recall that the derived units for the newton are $kg\ \ m/\sec^2$; hence, ζ is dimensionless.

Using the *ft-lb-sec* US standard system of units,

$$\zeta = c\left(\frac{lb\sec}{ft}\right) \times \frac{1}{2M(slug)} \times \frac{1}{\omega_n(\sec^{-1})} = \frac{c}{2M\omega_n}\left(\frac{lb\sec^2}{ft\,slug}\right) \ .$$

However, the derived units for the slug is $lb\sec^2/ft$, so this result is also dimensionless.

Finally, Using the *in-lb-sec* US standard system of units,

$$\zeta = c\left(\frac{lb\sec}{in}\right) \times \frac{1}{2M(snail)} \times \frac{1}{\omega_n(\sec^{-1})} = \frac{c}{2M\omega_n}\left(\frac{lb\sec^2}{in\,snail}\right) \ .$$

However, the derived units for the snail is $lb\sec^2/in$, so this result is also dimensionless.

“Logarithmic Decrement” or simply “log dec” δ to Characterize Damping

The log dec can be determined directly from an experimentally-measured transient response. From Eq.(3.26), the motion about the equilibrium position can be stated

$$Y_h = e^{-\zeta\omega_n t} D\cos(\omega_d t - \varphi)\ ,$$

where $D = (A^2 + B^2)^{-1/2}$, and $\varphi = \tan^{-1}(-B/A)$. Peaks in the response curves occur when $\cos(\omega_d t - \varphi) = 1$, at time intervals equal to the damped period $\tau_d = 2\pi/\omega_d$. Hence, the ratio of two successive peaks would be

$$\frac{Y_1}{Y_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d}\ ,$$

and the log dec δ is defined as

$$\delta = \ln\left(\frac{Y_1}{Y_2}\right) = \zeta\omega_n\tau_d = \zeta\omega_n \frac{2\pi}{\omega_n(1-\zeta^2)^{1/2}} = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}\ . \qquad \textbf{(3.30)}$$

The log dec can also be defined in terms of the ratio of a peak to the *n*th successive peaks as:

$$\delta = \frac{1}{(n-1)} \ln(\frac{Y_1}{Y_n}) = \frac{1}{(n-1)} \ln[\frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + (n-1)\tau_d)}}]$$
$$= \zeta\omega_n \tau_d = \frac{2\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad . \tag{3.31}$$

Note in Eq.(3.30) that δ becomes unbounded as $\zeta \Rightarrow 1$. In many dynamic systems, the damping factor is small, $\zeta << 1$, and $\delta \cong 2\pi\zeta$.

In stability calculations for systems with unstable eigenvalues, negative log dec's are regularly stated.

Solving for the damping factor ζ in terms of δ from Eq.(3.30) gives

$$\zeta = \frac{\delta}{[(2\pi)^2 + \delta^2]^{1/2}} \quad . \tag{3.31}$$

The effect of damping in the model $m\ddot{x} + c\dot{x} + kx = 0$ can be characterized in terms of the damping factor ζ and the log-dec δ.

Example Problem 3.4

Figure XP3.4 illustrates a transient response result for a mass that has been disturbed from its equilibrium position. The first peak occurs at $t = 1.6\,\text{sec}$ with an amplitude of 0.018 *m*. The fourth peak occurs at $t = 10.6\,\text{sec}$ with an amplitude of .003 *m*.

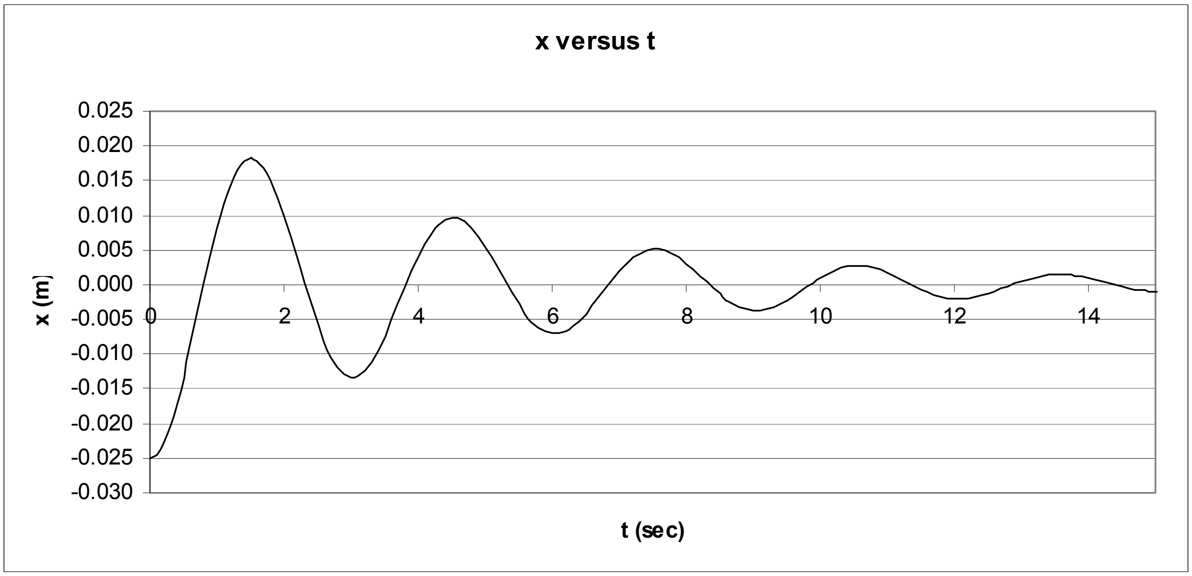


Figure XP3.4 Transient response

Tasks. Determine the log dec and the damping factor. Also, what is the damped natural frequency?

Solution Applying Eq.(3.31) gives

$$\delta = \frac{1}{n-1}\ln\left(\frac{Y_1}{Y_n}\right) = \frac{1}{3}\ln\left(\frac{Y_1}{Y_4}\right) = \frac{1}{3}\ln\left(\frac{.018}{.003}\right) = 0.597 \qquad \textbf{(3.31)}$$

for the log dec. From Table 3.1, the damping factor is

$$\zeta = \frac{\delta}{[(2\pi)^2 + \delta^2]^{1/2}} = \frac{0.597}{\sqrt{4\pi^2 + .597^2}} = 0.095$$

The damped period for the system is obtained as
$3\tau_d = 10.6 - 1.6 = 9\,\sec$. Hence,
$\tau_d = 3\,\sec \;\Rightarrow\; \omega_d = 2\pi/\tau_d = 2.09\,rad/\sec$.

Percent of Critical Damping

Recall that $\zeta = 1 \Rightarrow$ critical damping A "percent of critical damping" is also used to specify the amount of available damping. For example, $\zeta = 0.1$ implies 10% of critical damping.

Table 3.1 demonstrates how to proceed from one damping Characterization to another

Table 3.1 Relationship between damping characterizations

Output Column	ζ	δ	% of critical damping	q factor
ζ	1	$\delta/[(2\pi)^2+\delta^2]^{1/2}$	% of critical damping/ 100	$1/(2\,q)$
δ	$2\pi\zeta/(1-\zeta^2)^{1/2}$	1	find ζ first	find ζ first
% of critical damping	$100\times\zeta$	find ζ first	1	$50\,/\,q$
q factor	$1\,/\,(2\,\zeta)$	find ζ first	50/ % of critical damping	1

Important Concept and Knowledge Questions

How is the reaction force due to viscous damping defined?

In the differential equation of motion for a damped, spring-mass system, what is the correct sign for *c , the linear damping coefficient?*

What is implied by a negative damping coefficient *c?*

What are *c*'s dimensions in the inch-pound-second, foot-pound-second, newton-kilogram-second, newton-millimeter-second systems?

What is the damped natural frequency?

What is the damping factor ζ ?

How is the critical damping factor defined, and what does critical damping imply about free motion?

What is the log dec?