Lecture 8. TRANSIENT SOLUTIONS 1, FORCED RESPONSE AND INITIAL CONDITIONS

Introduction

Differential equations in dynamics normally arise from Newton's second law of motion, $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$. Accordingly, in dynamics, systems of coupled second order differential equations are the norm. Occasionally, the equation of motion for a particle or rigid body has the form, $\ddot{y} = d^2y/dt^2 = g(y)$, and the energy-integral substitution,

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{y}}{dy} = \frac{d}{dy}\left(\frac{\dot{y}^2}{2}\right) , \qquad \textbf{(B.1)}$$

reduces the second-order equation, with time *t* as the independent variable, to the first-order differential equation,

$$\frac{d}{dy}(\frac{\dot{y}^2}{2}) = g(y) , \qquad \textbf{(B.2)}$$

with displacement *y* as the independent variable and $\dot{y}^2/2$ as the dependent variable. Systems of first order equations arise in heat transfer, RC circuits, population studies, etc.

Undamped Spring-Mass Model

The undamped spring-mass system with weight acting as a forcing function can be stated

$$\ddot{Y} + \omega_n^2 Y = w/m = g \,. \qquad \textbf{(B.8)}$$

The homogeneous differential equation is obtained by setting the right-hand side to zero, netting in this case $\ddot{Y}_h + \omega_n^2 Y_h = 0$. In solving any linear second-order differential equation, we use the following steps:

(i). Solve the homogeneous equation for $Y_h(t)$, which will include two arbitrary constants A and B,

(ii). Solve for the particular solution $Y_p(t)$ that satisfies the right-hand side of the equation,

(iii). Form the complete solution $Y(t) = Y_h(t) + Y_p(t)$, and

(iv). Use the complete solution to solve for the two unknown constants A and B that satisfy the problem's initial conditions.

The formal solution to the homogeneous differential equation, $\ddot{Y}_h + \omega_n^2 Y_h = 0$, is obtained by assuming a solution of the form $Y_h = A e^{st} \Rightarrow \ddot{Y}_h = s^2 A e^{st}$. We have previously developed the homogeneous solution as

$$Y_h = A \cos \omega_n t + B \sin \omega_n t \ , \qquad \textbf{(B.9)}$$

where $\omega_n = \sqrt{k/m}$.

Any constant-coefficient linear ordinary differential equation can be solved by Laplace transforms including the particular solution. However, most particular solutions can be obtained by an inspection of the right-hand side terms and the differential equation itself. For Eq.(B.8), the right hand side is constant, and a guessed constant solution of the form $Y_p = c \Rightarrow \dot{Y}_p = \ddot{Y}_p = 0$ yields

$$(0 + c\omega_n^2) = g \Rightarrow c = \frac{g}{\omega_n^2} = \frac{w}{m\omega_n^2} \Rightarrow Y_p = \frac{w}{k} = Y_s \ ,$$

where $Y_s = w/k$ is the static deflection due to the weight *w*. Note that this particular solution is linearly proportional to the excitation on the right-hand side of Eq.(B.8).

The complete solution to $\ddot{Y} + \omega_n^2 Y = w/m = g$ is

$$Y = Y_h + Y_p = A\cos\omega_n t + B\sin\omega_n t + \frac{w}{k} \quad . \qquad \textbf{(B.10)}$$

Assuming that the initial conditions are $Y(0) = Y_0$, $\dot{Y}(0) = \dot{Y}_0$, we can first solve for the constant A via

$$Y_0 = A + \frac{g}{\omega_n^2} \Rightarrow A = Y_0 - \frac{w}{k} \quad .$$

Similarly, $\dot{Y} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t$, nets

$$\dot{Y}_0 = B\omega_n \Rightarrow B = \dot{Y}_0 / \omega_n \ ,$$

and the complete solution (satisfying the specified initial conditions) is

$$Y = Y_0 \cos\omega_n t + \frac{w}{k}(1 - \cos\omega_n t) + \frac{\dot{Y}_0}{\omega_n}\sin\omega_n t \quad . \qquad \textbf{(B.11)}$$

Suppose the spring-mass system is acted on by an external force that increases linearly with time, netting the differential equation of motion,

$$\ddot{Y} + \omega_n^2 Y = at \; . \tag{B.12}$$

This equation has the same homogeneous differential equation and solution; however, a new particular solution is required. By inspection, a solution of the form $Y_p = Dt \Rightarrow \dot{Y}_p = D \Rightarrow \ddot{Y}_p = 0$ will work. Substituting this guessed solution into Eq.(B.12) produces

$$(0 + \omega_n^2 Dt) = at \Rightarrow D = \frac{a}{\omega_n^2} \; , \; Y_p = \frac{at}{\omega_n^2} \; .$$

Eq.(B.12)'s complete solution is now

$$Y = A \cos\omega_n t + B \sin\omega_n t + \frac{at}{\omega_n^2} \; . \tag{B.13}$$

Table B.1 provides three particular solutions.

Table B.1. Particular solutions for $\ddot{Y}_p + \omega_n^2 Y_p = u(t)$.

Excitation, $u(t)$	$Y_p(t)$
$h = constant$	h/ω_n^2
$a\,t$	at/ω_n^2
bt^2	$-\dfrac{2b}{\omega_n^4} + \dfrac{bt^2}{\omega_n^2}$.

Consider the following version of Eq.(B.8)

$$\ddot{Y} + \omega_n^2 \, Y = c + at + bt^2 \ .$$

Since this equation is linear, from Table B.1 and the homogeneous solution defined by Eq.(B.9), from superposition, the complete solution is

$$Y = A \cos\omega_n t + B \sin\omega_n t + \frac{c}{\omega_n^2} + \frac{at}{\omega_n^2} + \left(-\frac{2b}{\omega_n^4} + \frac{bt^2}{\omega_n^2} \right) \ .$$

Spring-Mass-Damper Model

The equation of motion for a spring-mass-damper system acted upon by weight can be stated

$$\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = \frac{w}{m} = g . \qquad \textbf{(B.18)}$$

The homogeneous solution is obtained (again) by assuming a solution of the form $Y_h = Ae^{st} \Rightarrow \dot{Y} = sAe^{st} \Rightarrow \ddot{Y}_h = s^2 Ae^{st}$.
Substituting into the homogeneous differential equation nets

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) Ae^{st} = 0 \Rightarrow (s^2 + 2\zeta\omega_n s + \omega_n^2) = 0 \ . \qquad \textbf{(B.19)}$$

This result holds, since neither A nor e^{st} is zero.

The following three solution possibilities exist for Eq.(B.19):

(i). $\zeta = 1$, critically damped motion,

(ii). $\zeta > 1$, over-damped motion, and

(iii). $\zeta < 1$, under-damped motion.

For $\zeta > 1$, the roots to the characteristic Eq.(B.19) are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \;\Rightarrow\; s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} < 0$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} < 0 \;. \qquad \textbf{(B.20)}$$

Note that two real, negative roots are obtained, netting the homogeneous solution,

$$Y_h = A_1 e^{-|s_1|t} + A_2 e^{-|s_2|t} \;, \qquad \textbf{(B.21)}$$

where A_1, A_2 are real constants. The overdamped solution is the sum of two exponentially decaying terms.

For $\zeta = 1$ (critical damping), the characteristic Eq.(B.19) produces the single root $s = -\omega_n$, and single solution, $Y_h = A e^{-\omega_n t}$. This homogeneous solution (containing only one constant) is not adequate to satisfy two initial conditions (position and velocity). For less than obvious reasons, the complete homogeneous solution is

$$Y_h = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} \;, \qquad \textbf{(B.22)}$$

where A_1, A_2 are real constants. The second term in this solution also satisfies the differential equation as can be confirmed by substituting for Y_h . The critically-damped solution is interesting from a mathematical viewpoint as a limiting condition, but has minimal direct engineering value.

For $\zeta < 1$, the homogeneous equation, $\ddot{Y}_h + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y_h = 0$ has the solution form

$$Y_h \;=\; e^{-\zeta\omega_n t}(A \;\cos\omega_d t \;+\; B \;\sin\omega_d t) \;. \qquad \textbf{(B.25)}$$

where A, B are *real* constants, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

The particular solution for $\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = w/m = g$ is $Y_p = g/\omega_n^2 = w/k$, and the complete solution is

$$Y \;=\; e^{-\zeta\,\omega_n\,t}\;(A\;\cos\omega_d t \;+\; B\;\sin\omega_d t)\;+\;\frac{w}{k} \qquad \textbf{(B.26)}$$

For the initial conditions, $Y(0) = Y_0$, Eq.(B.26) yields

$$Y(0) = Y_0 = A + \frac{w}{k} \Rightarrow A = Y_0 - \frac{w}{k} . \tag{B.27}$$

Differentiating Eq.(B.26) gives

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t} (A \cos\omega_d t + B \sin\omega_d t)$$
$$+ e^{-\zeta\omega_n t} (-\omega_d A \sin\omega_d t + \omega_d B \cos\omega_d t) .$$

Hence, the initial condition, $\dot{Y}(0) = \dot{Y}_0$, defines B via,

$$\dot{Y}_0 = -\zeta\omega_n A + \omega_d B$$

$$\therefore B = \frac{\dot{Y}_0}{\omega_d} + \frac{\zeta\omega_n A}{\omega_d} = \frac{\dot{Y}_0}{\omega_d} + \frac{\zeta}{\sqrt{1 - \omega_n^2}} (Y_0 - \frac{w}{k}) , \tag{B.26}$$

and the complete solution is

$$Y = e^{-\zeta \omega_n t} \{ (Y_0 - \frac{w}{k}) \cos\omega_d t$$
$$+ [\frac{\dot{Y}_0}{\omega_d} + \frac{\zeta}{\sqrt{1 - \omega_n^2}} (Y_0 - \frac{w}{k})] \sin\omega_d t \} + \frac{w}{k} .$$

Table B.2 provides three particular solutions for the spring-mass-damper system.

Table B.2. Particular solutions for $\ddot{Y}_p + 2\zeta\omega_n \dot{Y}_p + \omega_n^2 Y_p = u(t)$.

Excitation, $u(t)$	$Y_p(t)$
c = constant	c/ω_n^2
$a\,t$	$\frac{a}{\omega_n^2}(t - \frac{2\zeta}{\omega_n})$
bt^2	$\frac{b}{\omega_n^2}[t^2 - \frac{4\zeta t}{\omega_n} - \frac{2}{\omega_n^2}(1 - 4\zeta^2)]$

As with the undamped equation, the following basic steps are taken to produce a complete solution for $\ddot{Y}+2\zeta\omega_n\dot{Y}+\omega_n^2 Y=u(t)$:

a. The homogeneous solution, $Y_h(t) = e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t)$, is developed for $\ddot{Y}_h+2\zeta\omega_n\dot{Y}_h+\omega_n^2 Y_h=0$ and involves the two constants A and B.

b. The particular solution $Y_p(t)$ is developed to satisfy the right hand side of the equation $\ddot{Y}+2\zeta\omega_n\dot{Y}+\omega_n^2 Y=u(t)$.

c. The complete solution, $Y(t)=Y_h(t)+Y_p(t)$, is formed, and the arbitrary constants A and B are determined <u>from the complete solution</u> such that the complete solution satisfies the initial conditions.

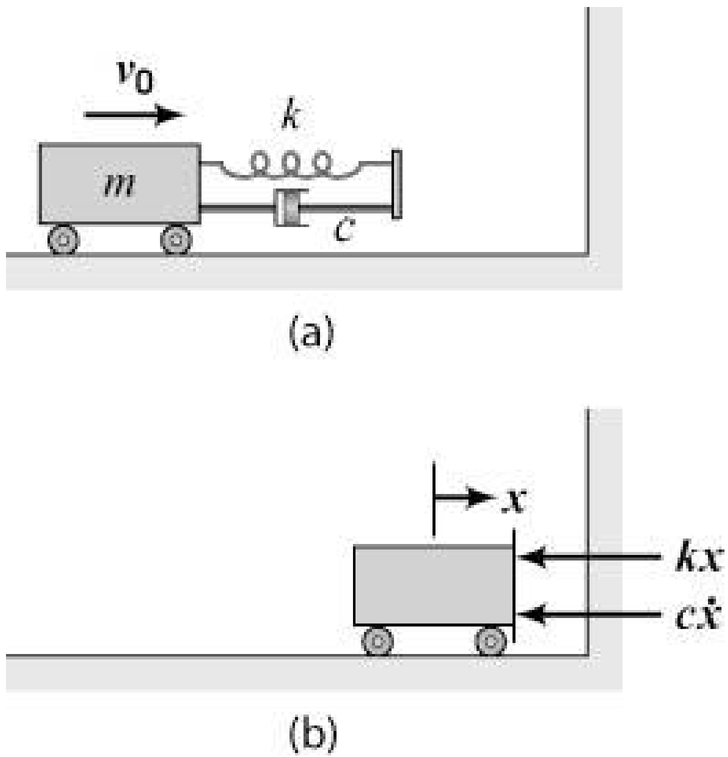

Figure XP3.2 (a). Cart approaching collision, (b). Free-body diagram during contact

Figure XP3.2a illustrates a cart that is rolling along at the speed v_0. It has a collision-absorption unit consisting of a spring and damper connected to a rigid plate of negligible mass. The engineering tasks for this example are listed below:

a. Select a coordinate and derive the equation of motion.

b. For $m = 100\,kg$, $k = 9.87E+04\ N/m$, $c = 3.1416E+03\,N\sec/m$, $v_0 = 10\,m/\sec$ determine the solution for motion while the cart remains in contact with the wall.

c. For arbitrary *k* and *m,* illustrate how a range of damping constants c producing $0 \le \zeta \le 1$ will reduce the stopping time and peak amplitude.

Solution: Figure XP3.2b shows the coordinate choice for x. Contact occurs for $x \ge 0$. The free-body diagram in figure XP3.2c corresponds to $x > 0$ and $\dot{x} > 0$, requiring compression in the spring and damper. Applying Newton's laws gives

$$m\ddot{x} = \sum \boldsymbol{f} = -kx - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0 \ ,$$

with the initial conditions, $x(0) = 0, \dot{x}(0) = v_0$. The spring and damper forces are negative in this equation because they are acting in the -*x* direction. There is no forcing function on the right-hand side of the equation; hence, the homogeneous solution of Eq.(B.25),

$$x = x_h + x_p = e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t) + 0, \qquad \textbf{(i)}$$

is the complete solution, and the velocity solution is

$$\begin{aligned} \dot{x} &= -\zeta\omega_n e^{-\zeta\omega_n t}(A \cos\omega_d t + B \sin\omega_d t) \\ &\quad + e^{-\zeta\omega_n t}\omega_d(-A \sin\omega_d t + B \cos\omega_d t) \ . \end{aligned} \qquad \textbf{(ii)}$$

Applying the initial conditions gives

$$x(0)=0=A\ ,\ \dot{x}(0)=v_0=B\omega_d\ \Rightarrow\ B=\frac{v_0}{\omega_d}\ .$$

Substituting back into (i) and (ii) gives

$$x=\frac{v_0}{\omega_d}\ e^{-\zeta\omega_n t}\ \sin\omega_d t$$

$$\dot{x}=-\frac{\zeta v_0}{\sqrt{1-\zeta^2}}\ e^{-\zeta\omega_n t}\ \sin\omega_d t+v_0\ e^{-\zeta\omega_n t}\ \cos\omega_d t\ , \qquad \textbf{(iii)}$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$.

The cart loses contact with the wall when $x(T_f)=0$. T_f is defined from the first of Eqs.(iii) by $\sin(\omega_d T_f)=0\ \Rightarrow\ \omega_d T_f=\pi$[1]. The problem parameters net

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The equation $\sin(\omega_d T_f)=0$ has an infinite number of solutions defined by: $\omega_d T_f=n\pi\ ;n=0,1,2,..$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.87E+4\,N/m}{100\,kg}} = 31.416\,\frac{rad}{\sec} \;\Rightarrow\; f_n = \frac{\omega_n}{2\pi} = 5\,\frac{cy}{\sec} = 5\,Hz$$

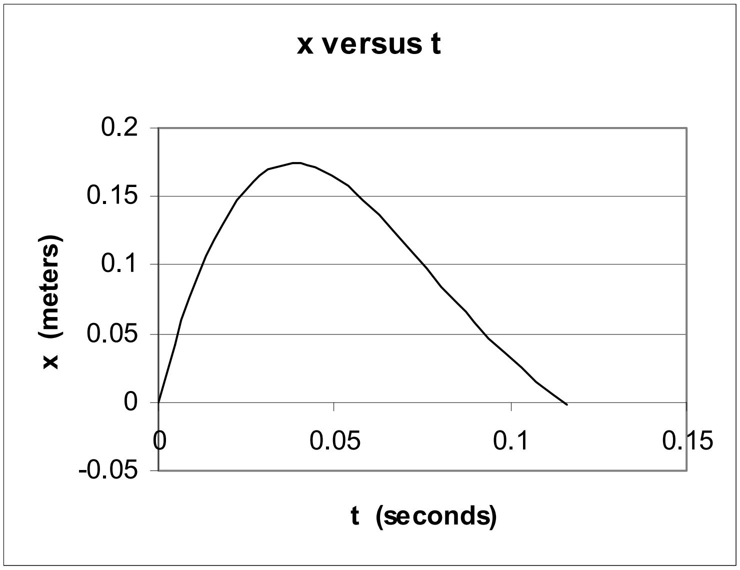
$$2\zeta\omega_n = \frac{c}{m} \;\Rightarrow\; \zeta = \frac{3.1416E+3\,Nsec/m}{2\times 31.146\sec^{-1}\times 100\,kg} = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.416\sqrt{1-0.25} = 26.97\,\frac{rad}{\sec}$$

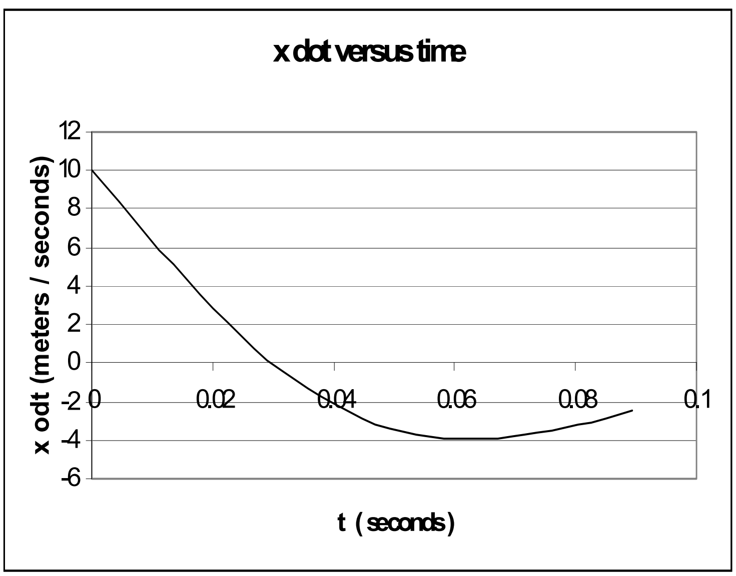
$$f_d = \frac{\omega_d}{2\pi} = 4.29Hz \;\Rightarrow\; \tau_d = \frac{1}{f_d} = 0.233\sec \;\;;\;\; T_f = \pi/\omega_d = .116\sec.$$

Plots for x and x(t) are provided below for $0 \le t \le T_f$.

Figure XP3.2 a. Position and b. Velocity solution following contact



a. Position



b. Velocity

Moving to *Task c*, the cart's forward motion stops, and the peak deflection occurs at the time T^* defined by $\dot{x}(T^*)=0$. From the second of Eq.(iii), this "stopping time" T^* is defined by

$$\dot{x}(T^*) = 0 = -\frac{\zeta v_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T^*} \sin\omega_d T^* + v_0 e^{-\zeta\omega_n T^*} \cos\omega_d T^*, \qquad \textbf{(iv)}$$

$$\therefore \tan(\omega_d T^*) = \frac{\sqrt{1-\zeta^2}}{\zeta} \Rightarrow \omega_d T^* = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The maximum value for T^* occurs at $\zeta = 0$ and is defined by

$$\omega_d T^*_{\max} = \omega_n T^*_{\max} = \pi/2 \;\; ;$$

i.e., one fourth of the natural period. From Eq.(iv), $\sin(\omega_d T^*) = \sqrt{1-\zeta^2}$. The peak amplitude is defined by substituting this result into the first of Eq.(iii), netting

$$x(T^*) = x_{\max} = \frac{v_0}{\omega_d} e^{-\zeta\omega_n T^*} \sqrt{1-\zeta^2} = \frac{v_0}{\omega_n} e^{-\zeta\omega_n T^*} \; . \qquad \textbf{(v)}$$

For $\zeta = 0$, the peak deflection is

$$x(T^*)\big|_{\zeta=0} = \frac{v_0}{\omega_n}$$

Figure XP3.2c illustrates the stopping-time ratio

$$\frac{T^*}{T^*_{\max}} = \frac{2}{\pi} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

versus ζ. Increasing the damping ratio ζ from 0 to 1 reduces the stopping time by 32% .

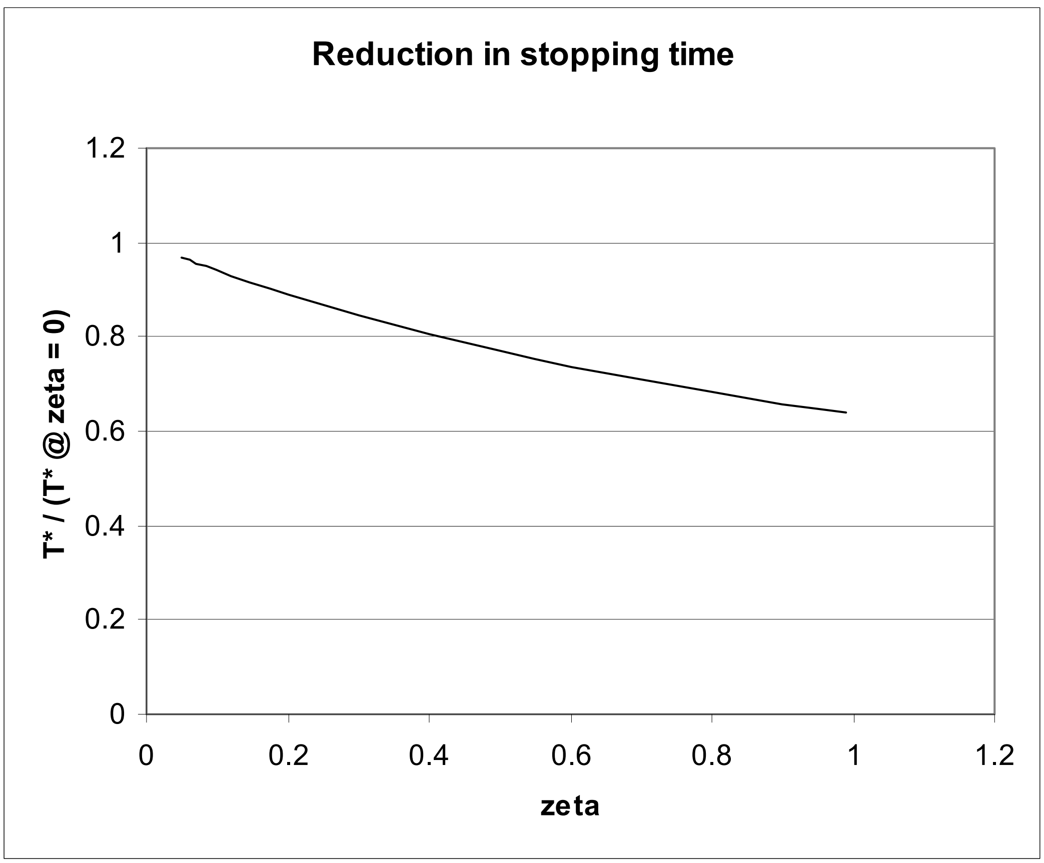


Figure XP3.2c Ratio of stopping time for $0<\zeta<1$.

Figure XPL10.d illustrates the peak deflection ratio

$$\frac{x_{\max}}{x_{\max}(\zeta=0)} = e^{-\zeta\omega_n T*}$$

versus ζ. Increasing the damping ratio to $\zeta \cong 1$ decreases the peak amplitude by 56%.

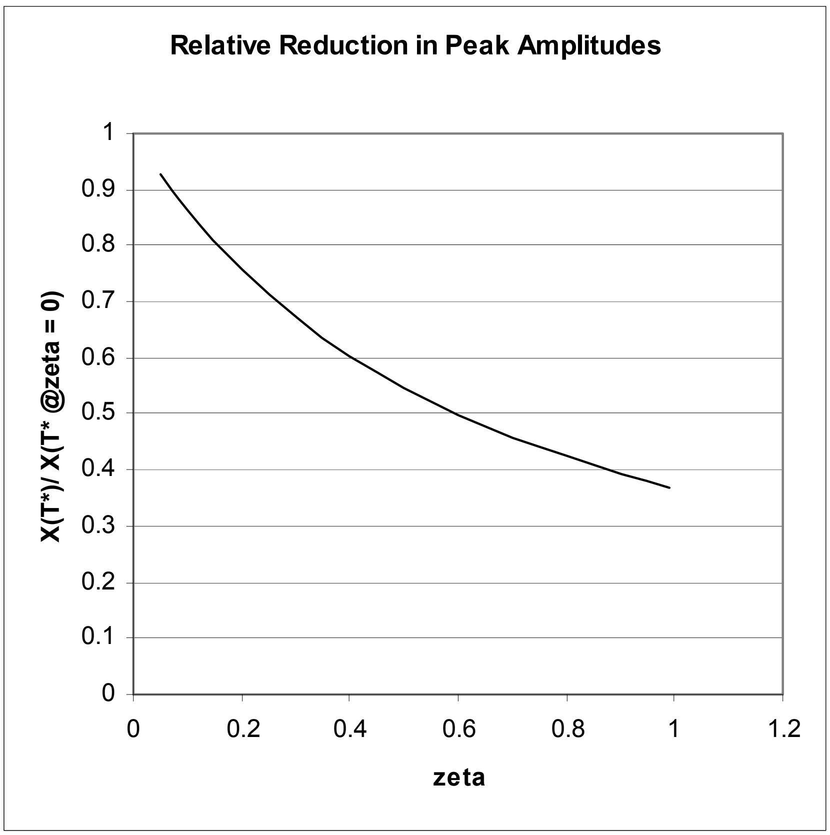


Figure XP3.2d. Ratio of maximum deflection for $0<\zeta<1$ to the $\zeta=0$ value.

We could increase the damping constant *c* such that $\zeta \geq 1$; however, the solution provided by Eq.(i) is no longer valid. For $\zeta > 1$, the over-damped solution of Eq.(B.21) applies. For $\zeta = 1$, the critically damped solution of Eq.(B.22) applies.