Lecture 9. More Transient Solutions, Base Excitation

Example Problem L9.1

Consider the model $m\ddot{Y} + c\dot{Y} + kY = f(t)$ with initial conditions $Y(0) = \dot{Y}(0) = 0$. The parameters *k*, *m,* and *c* are defined by $k = 9.87E+04\ N/m$, $m = 100\,Kg$, and $c = 314.16\ N\sec/m$. These data produce:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.87E+4\,N/m}{100\,kg}} = 31.416\,\frac{rad}{\sec}$$

$$\therefore\ f_n = \frac{\omega_n}{2\pi} = 5\,\frac{cy}{\sec} = 5\,Hz$$

$$2\zeta\omega_n = \frac{c}{m}\ \Rightarrow\ \zeta = \frac{314.16\,Nsec/m}{2\times 31.146\sec^{-1}\times 100\,kg} = 0.05$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.416\sqrt{1-0.0025} = 31.38\,\frac{rad}{\sec}$$

$$f_d = \frac{\omega_n}{2\pi} = 5\,\frac{cyc}{\sec} = 4.993\,Hz\ ,\ \tau_d = \frac{1}{f_d} = .200\sec\ .$$

The force $f(t)$ is defined by the half-sine wave pulse illustrated below, and defined by

$$f(t) = f_0 \sin \omega_n t \ , \ 0 \le t \le t_1 = \pi / \omega_n$$

$$f(t) = 0 \ , \ t \ge t_1 \ .$$

where $f_0 = w = m \times g = 100\,kg \times 9.81\ m/\sec^2 = 981\,N$.

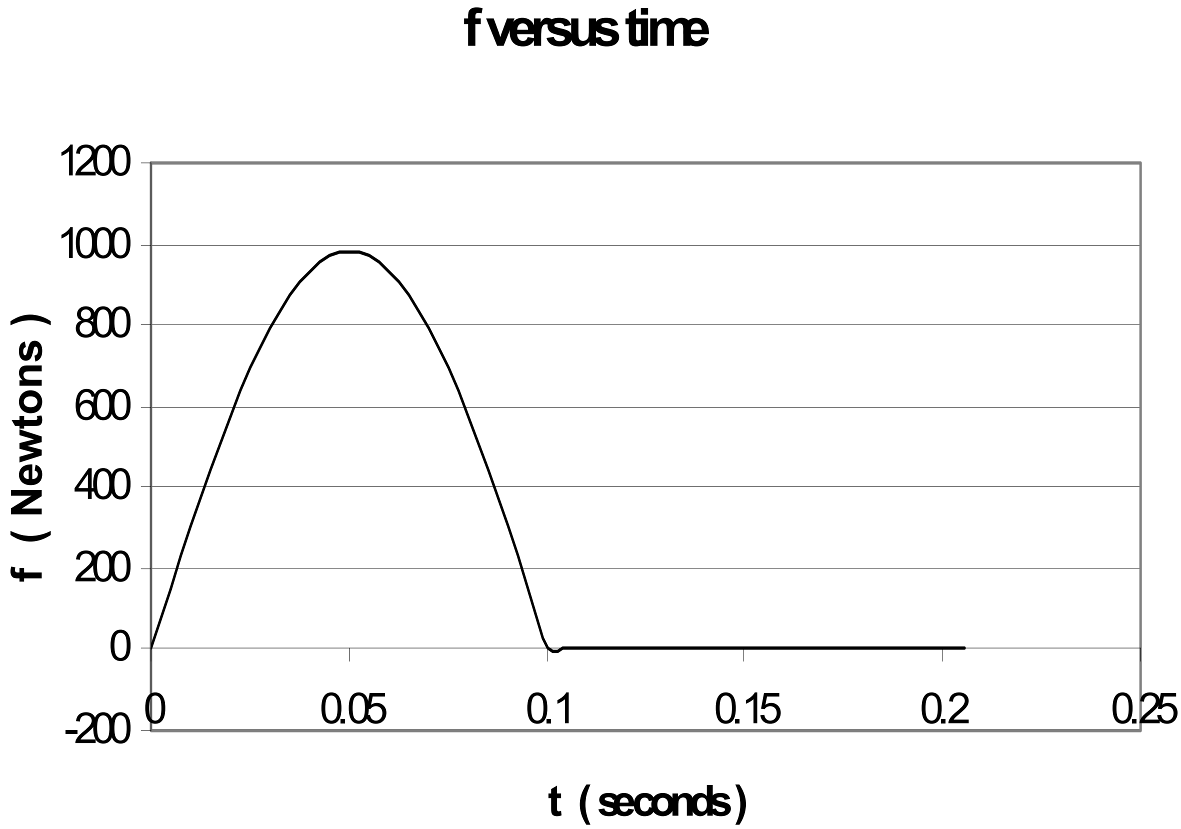


Figure XP9.1a Force excitation $f(t) = 981 \sin \omega_n t$ (*Newtons*), $0 \le t \le t_1 = \pi / \omega_n$; $f(t) = 0$, $t \ge t_1$.

Complete the following engineering analysis tasks:

a. Determine the solution $Y(t)$ for $0 \le t \le t_1 = \pi/\omega_n$.

b. Determine the solution $Y(t)$ for $t \ge t_1$.

c. Plot the solution for $0 \le t \le 3\tau_d$

Solution. In lecture 11, we will develop the particular solution for $m\ddot{Y} + c\dot{Y} + kY = f_0 \sin\omega t$ as $Y_p = Y_{op}\sin(\omega t + \psi)$, where

$$Y_{op} = \frac{f_o}{k}\frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}} \;,\; r = \omega/\omega_n \;,$$

$$\psi(r) = -\tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \;.$$

For $\omega = \omega_n \Rightarrow r = 1$, $Y_{op} = (f_0/k)(1/2\zeta) = f_0/(2k\zeta)$, $\psi = -\pi/2$, and the particular solution is

$$Y_p = \frac{f_0}{2k\zeta}\sin(\omega_n t - \frac{\pi}{2}) = -\frac{f_0}{2k\zeta}\cos(\omega_n t)$$

The complete solution is

$$Y = Y_h + Y_p = e^{-\zeta\omega_n t} \; (A\cos\omega_d t + B\sin\omega_d t)$$

$$- \frac{f_0}{2k\zeta}\cos\omega_n t \;\; .$$

A is solved via

$$Y(0) = 0 = A - \frac{f_0}{2k\zeta} \;\Rightarrow\; A = \frac{f_0}{2k\zeta}$$

To determine B, we start with

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t} \; (A\cos\omega_d t + B\sin\omega_d t)$$
$$+ \omega_d e^{-\zeta\omega_n t} \; (-A\sin\omega_d t + B\cos\omega_d t)$$
$$+ \omega_n \frac{f_0}{2k\zeta}\sin\omega_n t \;\; .$$

Evaluating,

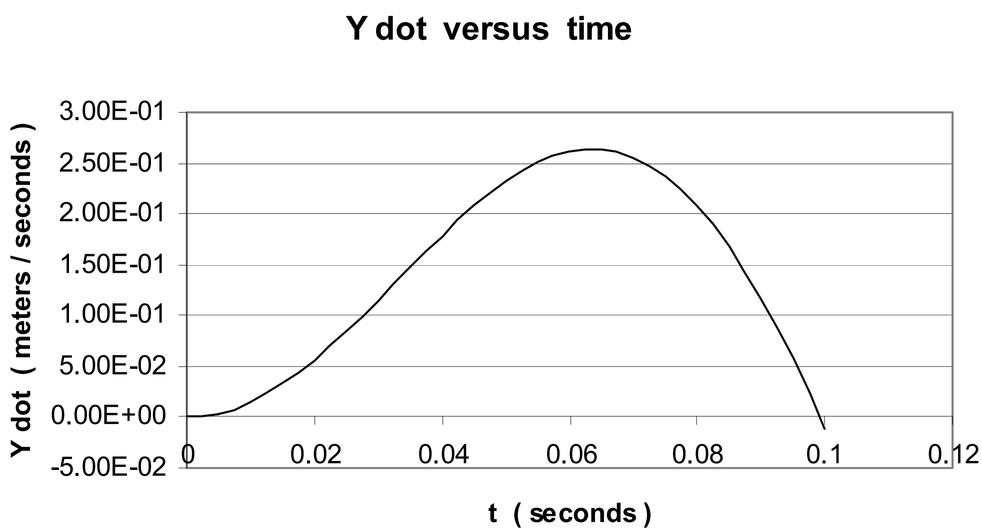
$$\dot{Y}(0) = 0 = -\zeta\omega_n A + B\omega_d \;\Rightarrow\; B = \frac{\zeta A}{\sqrt{1-\zeta^2}} = \frac{f_0}{2k}\frac{1}{\sqrt{1-\zeta^2}}$$

The complete solution satisfying the initial conditions is

$$Y = \frac{f_0}{2k\zeta}[e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t) - \cos\omega_n t] \ ,$$

concluding Task a. Starting on Task b, from the data provided: The solution for $0 \le t \le t_1 = \pi/\omega_n = 0.100\,\sec$ is presented below. At $t = t_1$, the values for the position and velocity are

$$Y(t_1) = .0145\,m, and \ \dot{Y}(t_1) = -.0105\,m/\sec$$



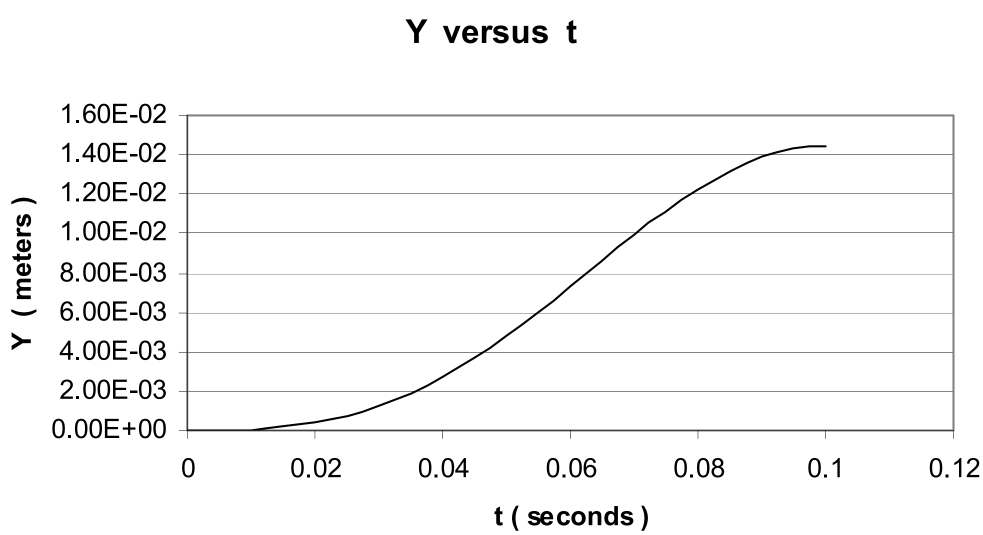


Figure XPL9.1b Solution for $Y(t)$ and $\dot{Y}(t)$ for $0 \le t \le t_1 = \pi/\omega_d$.

For $t \ge t_1$, the model reverts to $m\ddot{Y} + c\dot{Y} + kY = 0$, and the solution is now

$$Y = Y_h = e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)$$

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t}\ (A\cos\omega_d t + B\sin\omega_d t)$$
$$+\omega_d e^{-\zeta\omega_n t}\ (-A\sin\omega_d t + B\cos\omega_d t).$$

We can solve for *A* and *B* via the $t = t_1$ results $Y(t_1) = .0145\,m$, $\dot{Y}(t_1) = -.0105\ m/\sec$ from:

$$Y(t_1) = .0145 = e^{-\zeta\omega_n t_1}\ (A\cos\omega_d t_1 + B\sin\omega_d t_1)$$

$$\dot{Y}(t_1) = -.0105 = -\zeta\omega_n e^{-\zeta\omega_n t_1}\ (A\cos\omega_d t_1 + B\sin\omega_d t_1)$$
$$+\omega_d e^{-\zeta\omega_n t_1}\ (-A\sin\omega_d t_1 + B\cos\omega_d t_1).$$

Two equations for the two unknowns *A* and *B*.

We are going to use $t' = t - t_1$, and “restart “ time for $t \ge t_1 \;\Rightarrow\; t' \ge 0$. We previously developed the solution in terms of arbitrary initial conditions as

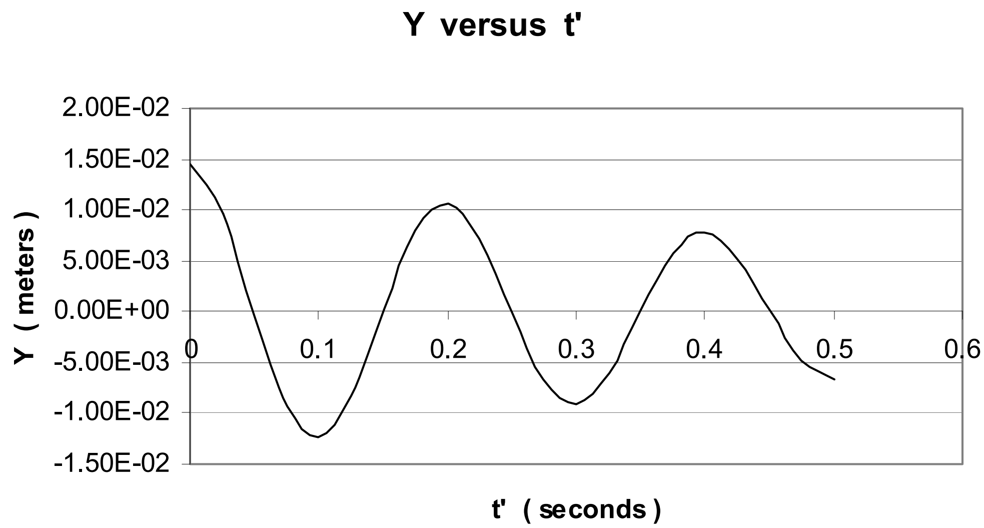
$$Y = e^{-\zeta \omega_n t'} \left[Y_0 \cos \omega_d t' + \frac{(\dot{Y}_0 + \zeta \omega_n Y_0)}{\omega_d} \sin \omega_d t' \right] .$$

The initial conditions for this solution are:

$$Y(t' = 0) = Y(t = t_1) = Y_0 = = .0145\, m$$

$$\dot{Y}(t' = 0) = \dot{Y}(t = t_1) = \dot{Y}_0 = -.0105\, m/\sec \;\; .$$

The solution for $0 \le t' \le (3\tau_d - t_1),\; t_1 \le t \le 3\tau_d$ is presented below.



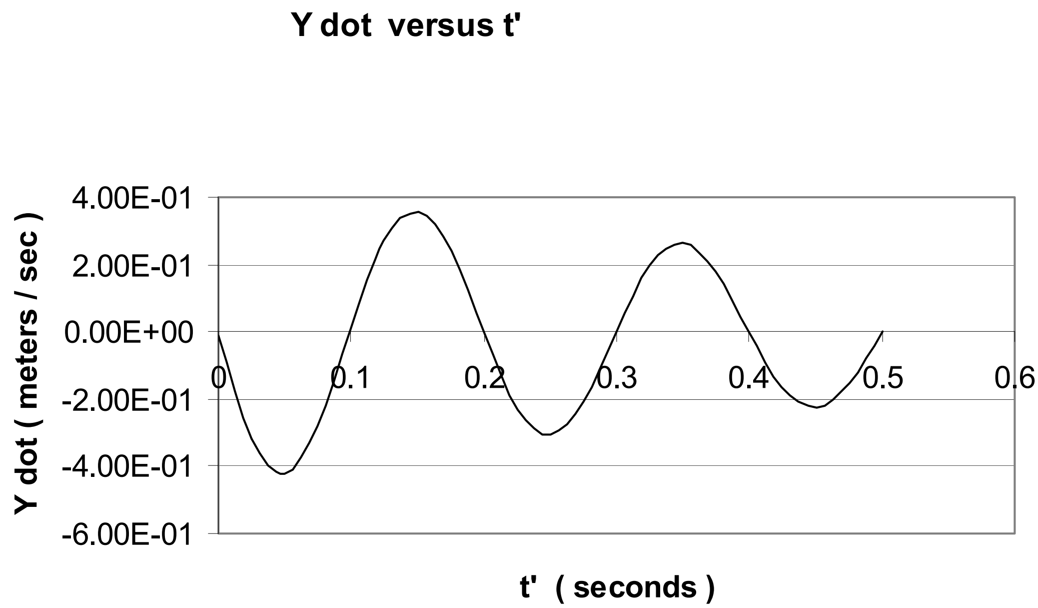


Figure XPL9.1c $Y(t')$ and $\dot{Y}(t')$ for $0 \le t' \le (3\tau_d - t_1)$.

Base Excitation

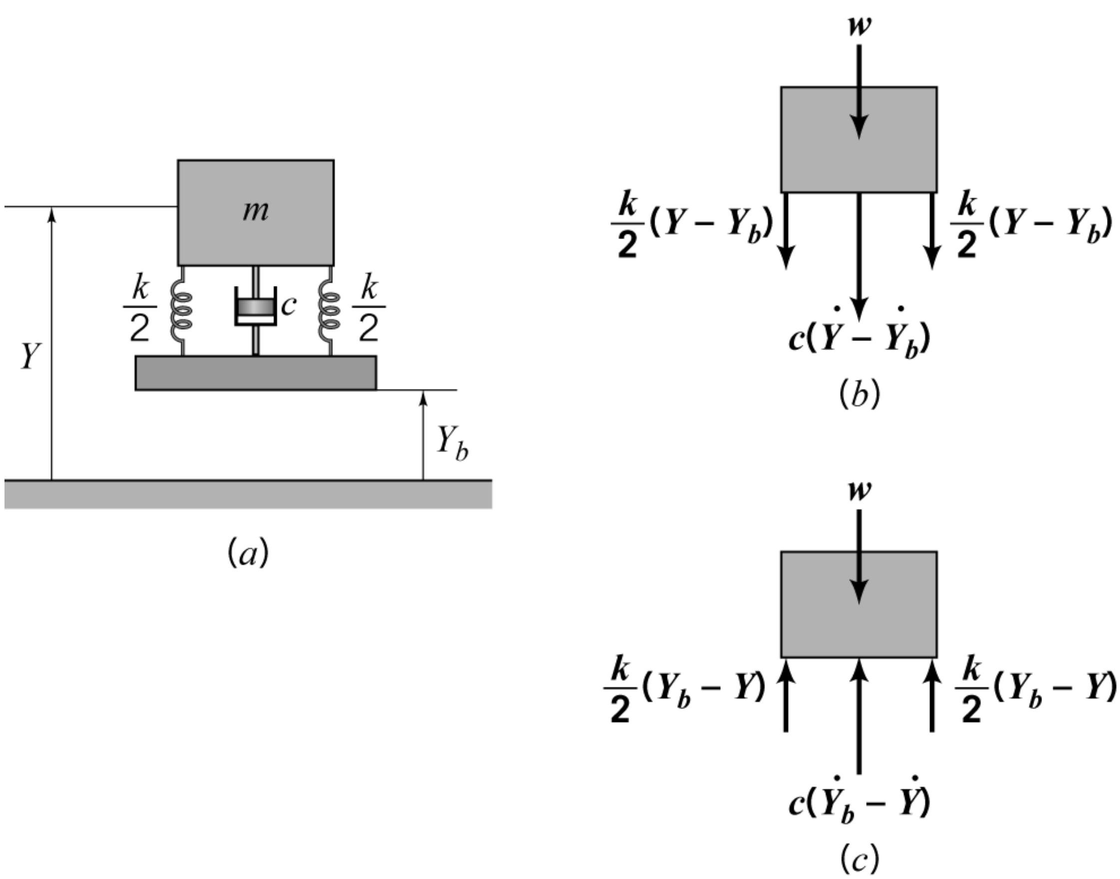


Figure 3.15. (a). Suspended mass and movable base. (b). Free-body diagram for tension in the springs and damper, (c). Free-body diagram for compression in the springs and damper.

For free-body diagram b, assume that the base and the mass m have positive displacements $(Y > 0, Y_b > 0)$ and velocities $(\dot{Y} > 0, \dot{Y}_b > 0)$.

For zero base motion $(Y_b = 0)$, the springs are undeflected when $Y = 0$.

Assuming that the m's displacement is greater than the base displacement $(Y > Y_b)$ means that the springs are in tension, and

the spring forces are defined by $f_s = (k/2)(Y - Y_b)$.

Assuming that the m's velocity is greater than the base velocity $(\dot{Y} > \dot{Y}_b)$, the damper is also in tension, and the damping force is defined by $f_d = c(\dot{Y} - \dot{Y}_b)$.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the free-body diagram of figure 3.15b gives the differential equation of motion

$$m\ddot{Y} = \Sigma \boldsymbol{f_Y} = -w - 2 \times \frac{k}{2}(Y - Y_b) - c(\dot{Y} - \dot{Y}_b) \quad ,or$$

$$m\ddot{Y} + c\dot{Y} + kY = -w + c\dot{Y}_b + kY_b \ .$$

For free-body diagram c , the base displacement is greater than the mass displacement $(Y_b > Y)$. With this assumed relative motion, the springs are in compression, and the spring forces are defined by $f_s = (k/2)(Y_b - Y)$. Similarly, assuming that the base velocity is greater than m's $(\dot{Y}_b > \dot{Y})$ causes the damper to also be in compression, and the damper force is defined by $f_d = c(\dot{Y}_b - \dot{Y})$. The free-body diagram of figure 3.15C is consistent with this assumed motion and leads to the differential equation of motion

$$m\ddot{Y} = \Sigma f_Y = -w + c(\dot{Y}_b - \dot{Y}) + k(Y_b - Y) \ ,or$$

$$m\ddot{Y} + c\dot{Y} + kY = -w + kY_b + c\dot{Y}_b \ ,$$

The short lesson from these developments is that the same governing equation should result for any assumed motion, since the governing equation applies for any position and velocity. Note that the following procedural steps were taken in arriving at the equation of motion:

a. The nature of the motion was assumed e.g.,$(Y_b > Y)$,

b. The spring or damper force was stated in a manner that was consistent with the assumed motion, e.g., $f_s = k/2 \times (Y_b - Y)$,

c. The free-body diagram was drawn in a manner that was consistent with the assumed motion and its resultant spring and damper forces., i.e., in tension or compression, and

d. Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ was applied to the free-body diagram to obtain the equation of motion.

Note: We have looked at only one arrangement for base excitation. Your homework has several examples for which the same procedure works, but the final equations are different.