LECTURE 14: DEVELOPING THE EQUATIONS OF MOTION FOR TWO-MASS VIBRATION EXAMPLES

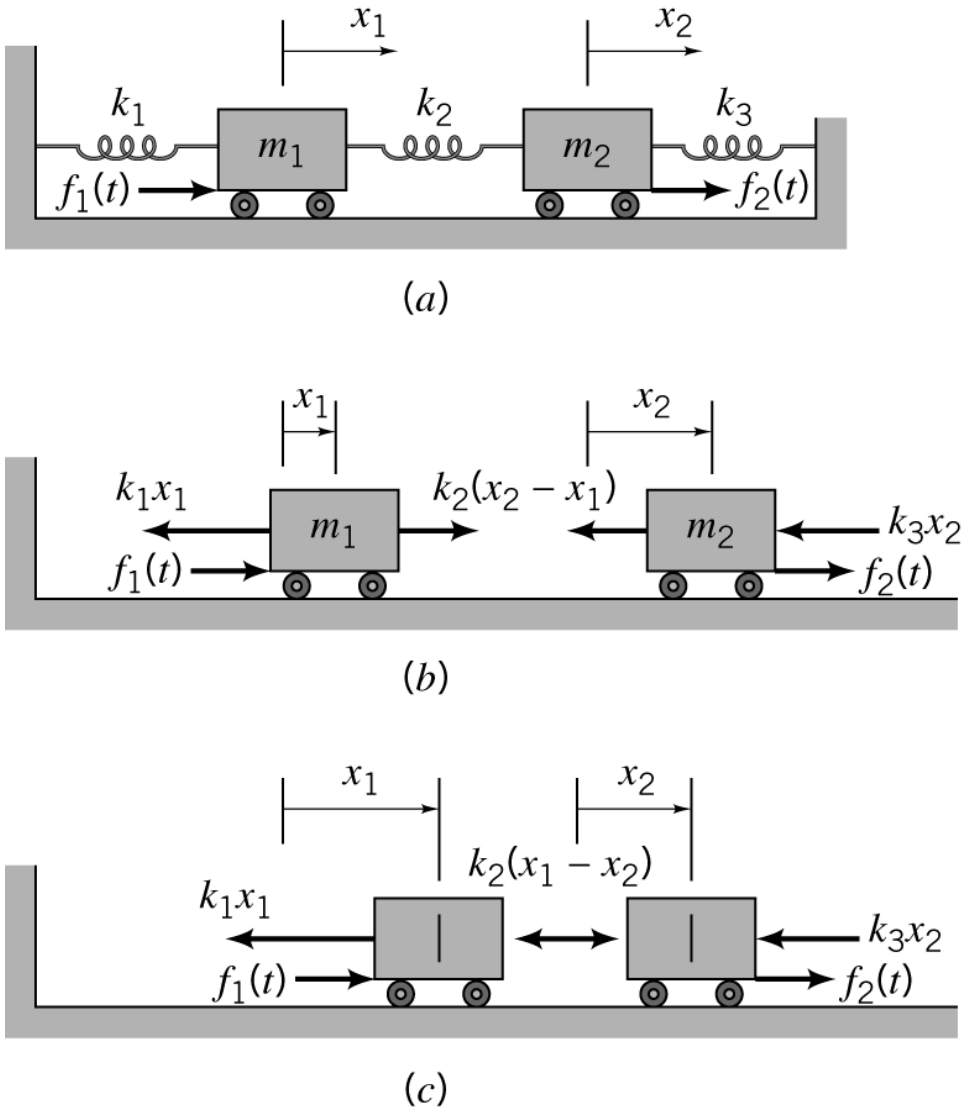


Figure 3.47 a. Two-mass, linear vibration system with spring connections. b. Free-body diagrams. c. Alternative free-body diagram.

***Equations of Motion Assuming*:** $x_1, x_2 > 0\,; x_2 > x_1$

The connecting spring is in tension, and the connecting spring-force magnitude is $f_{12} = k_2(x_2 - x_1)$. From figure 3.47B:

$$\begin{aligned} mass\ m_1 &: \Sigma f_{x1} = f_1(t) - k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1 \\ mass\ m_2 &: \Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2\,, \end{aligned} \qquad (3.122)$$

with the resultant differential equations:

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= f_1(t) \\ m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 &= f_2(t)\ . \end{aligned} \qquad (3.123)$$

***Equations of Motion Assuming*:** $x_1, x_2 > 0\,; x_2 < x_1$

The spring is in compression, and the connecting-spring force magnitude is $f_{12} = k_2(x_1 - x_2)$. From figure 3.47C:

$$mass\ m_1 : \Sigma f_{x1} = f_1(t) - k_1 x_1 - k_2(x_1 - x_2) = m_1 \ddot{x}_1$$

$$mass\ m_2 : \Sigma f_{x2} = f_2(t) + k_2(x_1 - x_2) - k_3 x_2 = m_2 \ddot{x}_2\ .$$

Rearranging these differential equations gives Eqs.(3.122).

Steps for obtaining the correct differential equations of motion:

a. Assume displaced positions for the bodies and decide whether the connecting spring forces are in tension or compression.

b. Draw free-body diagrams that conform to the assumed displacement positions and their resultant reaction forces (i.e., tension or compression).

c. Apply $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the free body diagrams to obtain the governing equations of motion.

The matrix statement of Eqs.(3.123) is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} . \tag{3.124}$$

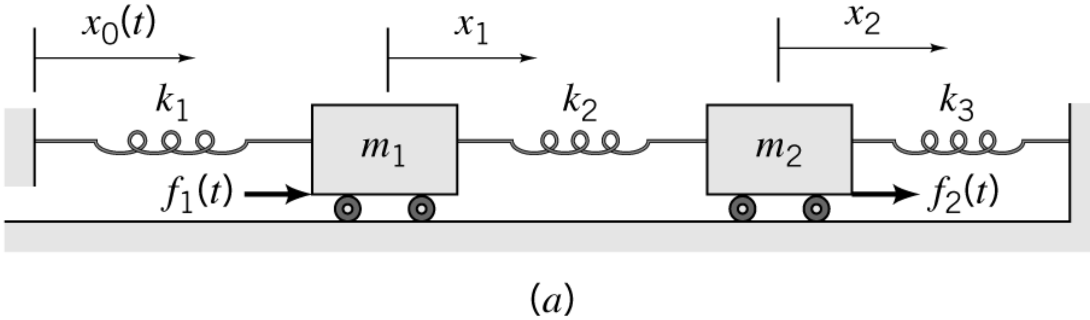
The mass matrix is diagonal, and the stiffness matrix is symmetric. A stiffness matrix that is not symmetric and cannot be made symmetric by multiplying one or more of its rows by constants indicates a system that is or can be dynamically unstable. You have made a mistake, if in working through the example problems, you arrive at a nonsymmetric stiffness matrix. Also, for a neutrally-stable system, the diagonal entries for the mass and stiffness matrices must be greater than zero.

The center spring "**couples**" the two coordinates. If $k_2 = 0$, the following "**uncoupled**" equations result

$$m_1 \ddot{x}_1 + k_1 x_1 = f_1(t)$$

$$m_2 \ddot{x}_2 + k_3 x_2 = f_2(t) \ .$$

These uncoupled equations of motion can be solved separately using the same procedures of the preceding section.



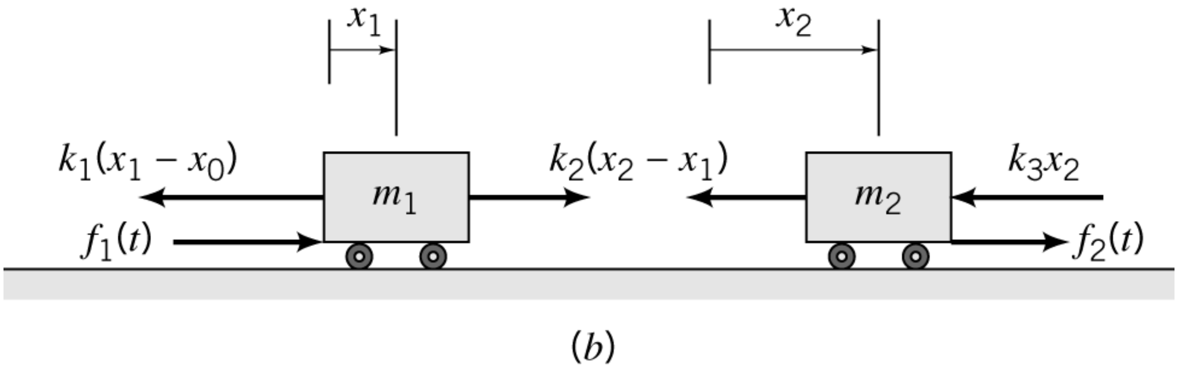


Figure 3.48 a. Two-mass, linear vibration system with motion of the left-hand support. b. Free-body diagram for assumed motion $x_2 > x_1 > x_0 > 0$.

Base Excitation from the Left-Hand Wall

Assume that the left-hand wall is moving creating base excitation via $x_0(t)$. From the free-body diagram for assumed motion $x_2 > x_1 > x_0 > 0$,

$$\begin{aligned} &mass\ m_1 : \Sigma f_{x1} = f_1(t) - k_1(x_1 - x_0) + k_2(x_2 - x_1) = m_1\ddot{x}_1 \\ &mass\ m_2 : \Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 = m_2\ddot{x}_2 , \end{aligned} \tag{3.125}$$

In matrix notation

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) + k_1 x_0(t) \\ f_2(t) \end{Bmatrix} . \tag{3.126}$$

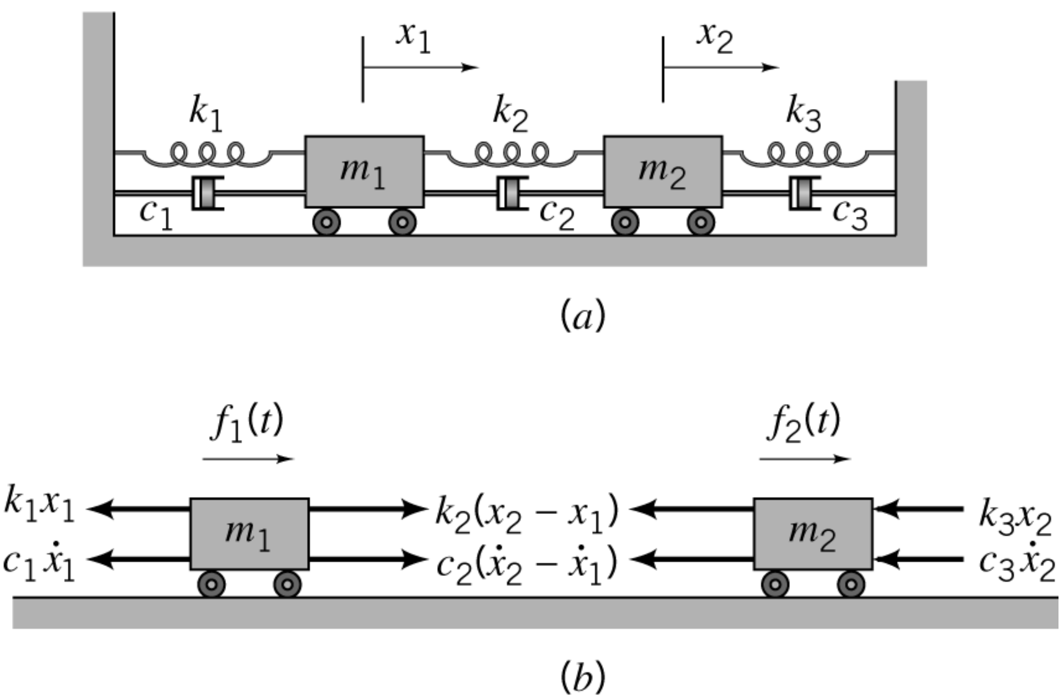


Figure 3.49 a. Two-mass, linear vibration system with spring and damper connections. b. Free-body diagram for $x_2 > x_1 > 0$, $\dot{x}_2 > \dot{x}_1 > 0$

Connection with Dampers

Assumed motion conditions:

a. Both m_1 and m_2 are moving to the right $(\dot{x}_1>0;\dot{x}_2>0)$, and

b. The velocity of m_2 is greater than the velocity of m_1 $(\dot{x}_2>\dot{x}_1)$.

Based on this assumed motion, tension is developed in left and center dampers, but compression is developed in the right damper. The tension in damper 1 is $c_1\dot{x}_1$, the tension in damper 2 is $c_2(\dot{x}_2-\dot{x}_1)$, and the compression in damper 3 is $c_3\dot{x}_2$.

Applying $\Sigma \boldsymbol{f}=m\ddot{\boldsymbol{r}}$ to the free body diagrams of figure 3.52B gives:

$$mass\ m_1: \Sigma f_{x1}=f_1(t)-k_1x_1+k_2(x_2-x_1)-c_1\dot{x}_1+c_2(\dot{x}_2-\dot{x}_1)=m_1\ddot{x}_1$$

$$mass\ m_2: \Sigma f_{x2}=f_2(t)-k_2(x_2-x_1)-k_3x_2-c_2(\dot{x}_2-\dot{x}_1)-c_3\dot{x}_2=m_2\ddot{x}_2$$

These equations can be rearranged as

$$\begin{aligned} &m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=f_1(t)\\ &m_2\ddot{x}_2-c_2\dot{x}_1+(c_2+c_3)\dot{x}_2-k_2x_1+(k_2+k_3)x_2=f_2(t)\,. \end{aligned} \tag{3.128}$$

In Matrix format

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & (c_2 + c_3) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} . \tag{3.129}$$

Similar steps are involved in the development of equations of motion for systems connected by dampers that held for spring connections; namely,

a. Assume relative magnitudes for the bodies' velocities and decide whether the connecting damper forces are in tension or compression.

b. Draw free-body diagrams that conform to the assumed velocity conditions and their resultant damper forces (i.e., tension or compression).

c. Apply $\Sigma \boldsymbol{f} = m \ddot{\boldsymbol{r}}$ to the free-body diagrams to obtain the governing equations of motion.

The spring and damper forces can be developed sequentially.

Base Excitation via Right-hand wall motion

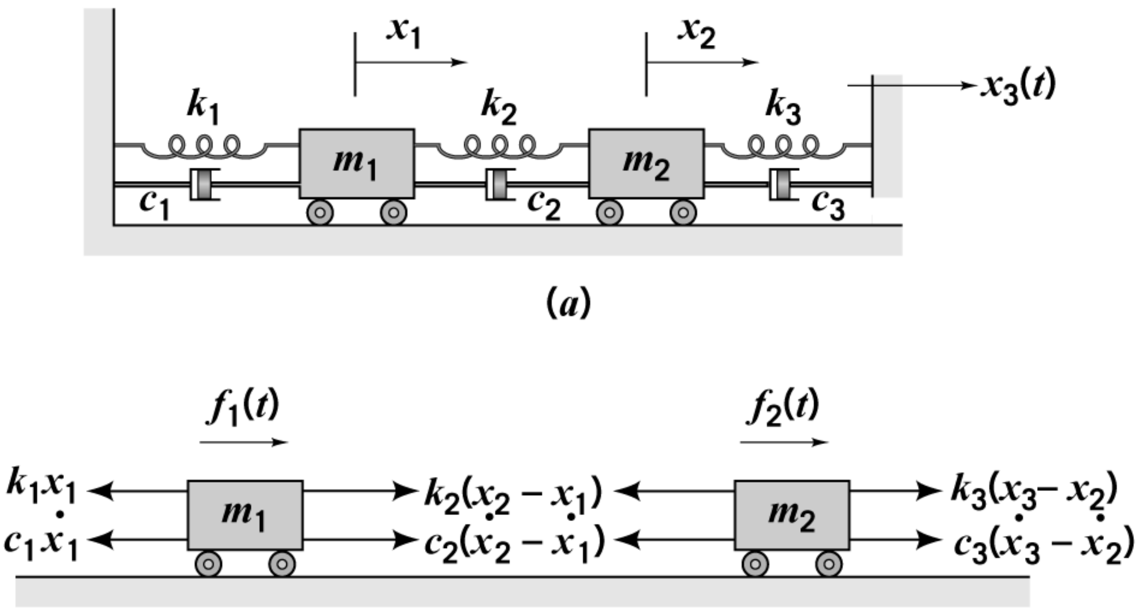


Figure 3.50 a. Coupled two-mass system with motion of the right-hand support defined by $x_3(t)$. b. Free-body diagram corresponding to assumed motion defined by $x_3 > x_2 > x_1 > 0$ and $\dot{x}_3 > \dot{x}_2 > \dot{x}_1 > 0$.

mass m_1,

$$\Sigma f_{x1} = f_1(t) - k_1 x_1 + k_2(x_2 - x_1) - c_1 \dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1)$$
$$= m_1 \ddot{x}_1$$

mass m_2,

$$\Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) + k_3(x_3 - x_2) - c_2(\dot{x}_2 - \dot{x}_1) + c_3(\dot{x}_3 - \dot{x}_2)$$
$$= m_2 \ddot{x}_2 \,.$$

Matrix Statement

Base excitation causes the additional forcing functions on the right. The stiffness and damping matrices should always be symmetric. If they are not, you have made a mistake.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & (c_2 + c_3) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) + k_3 x_3(t) + c_3 \dot{x}_3(t) \end{Bmatrix} . \tag{3.131}$$

Developing the Equations of Motion for a Double Pendulum

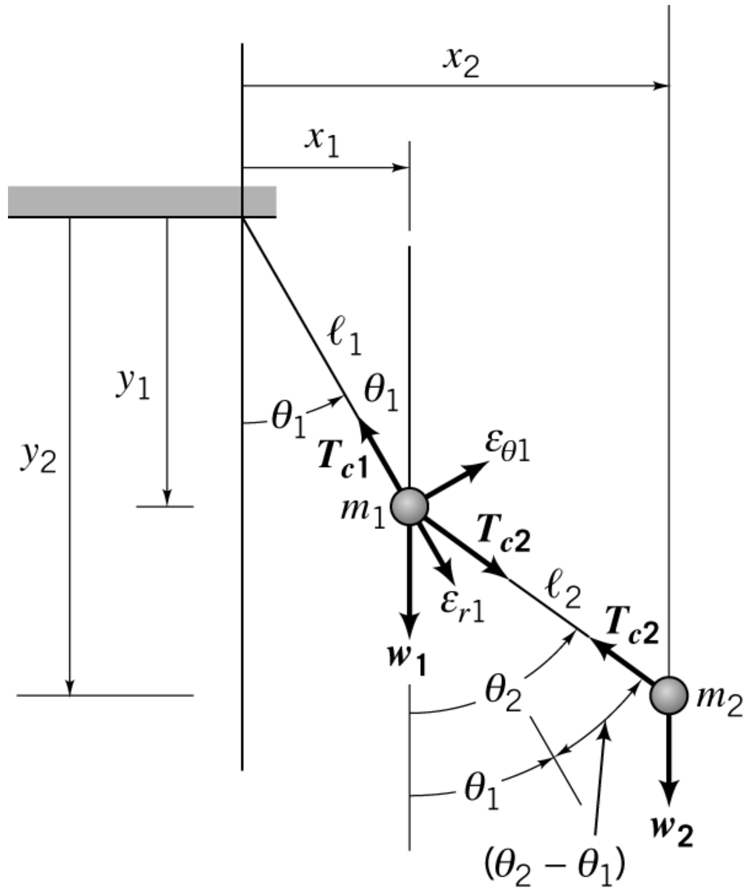


Figure 3.51 Free-body diagram for the double pendulum of figure 3.25.

Equations of motion for mass m_1 :

$$\begin{aligned}\Sigma f_{r1} &= w_1 \cos\theta_1 + T_{c2}\cos(\theta_2 - \theta_1) - T_{c1} \\ &= m_1(\ddot{r} - r\dot{\theta}_1^2) = -m_1 l_1 \dot{\theta}_1^2 \\ \Sigma f_{\theta 1} &= T_{c2}\sin(\theta_2 - \theta_1) - w_1 \sin\theta_1 \\ &= m_1(r\ddot{\theta} + 2\dot{r}\dot{\theta}_1) = m_1 l_1 \ddot{\theta}_1 \ .\end{aligned} \tag{3.131}$$

The second equation provides one equation in the two unknowns ($T_{c2}, \ddot{\theta}_1$).

Simple -pendulum equations of motion,

$$\begin{aligned} \Sigma f_r &= w\cos\theta - T_c = -ml\dot{\theta}^2 \\ \Sigma f_\theta &= -w\sin\theta = ml\ddot{\theta} \end{aligned} \tag{3.82}$$

Equations of motion for mass m $_{2:}$

$$\begin{aligned} \Sigma f_{x2} &= -T_{c2}\sin\theta_2 = m_2\ddot{x}_2 \\ \Sigma f_{y2} &= w_2 - T_{c2}\cos\theta_2 = m_2\ddot{y}_2 \ . \end{aligned} \tag{3.132}$$

We now have two additional unknowns $(\ddot{y}_2, \ddot{x}_2)$.

Kinematics from figure 3.54:

$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \ , \quad y_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \ . \tag{3.133a}$$

Differentiating with respect to time gives:

$$\begin{aligned} \dot{x}_2 &= l_1\cos\theta_1\dot{\theta}_1 + l_2\cos\theta_2\dot{\theta}_2 \\ \dot{y}_2 &= -l_1\sin\theta_1\dot{\theta}_1 - l_2\sin\theta_2\dot{\theta}_2 \ . \end{aligned} \tag{3.133b}$$

Differentiating again gives:

$$\ddot{x}_2 = l_1 \cos\theta_1 \ddot{\theta}_1 - l_1 \sin\theta_1 \dot{\theta}_1^2 + l_2 \cos\theta_2 \ddot{\theta}_2 - l_2 \sin\theta_2 \dot{\theta}_2^2$$

$$\ddot{y}_2 = -l_1 \sin\theta_1 \ddot{\theta}_1 - l_1 \cos\theta_1 \dot{\theta}_1^2 - l_2 \sin\theta_2 \ddot{\theta}_2 - l_2 \cos\theta_2 \dot{\theta}_2^2 \quad . \tag{3.133c}$$

Substituting these results into Eq.(3.132) gives:

$$\begin{aligned} & -T_{c2} \sin\theta_2 \\ & = m_2 (l_1 \cos\theta_1 \ddot{\theta}_1 - l_1 \sin\theta_1 \dot{\theta}_1^2 + l_2 \cos\theta_2 \ddot{\theta}_2 - l_2 \sin\theta_2 \dot{\theta}_2^2) \\ & w_2 - T_{c2} \cos\theta_2 \\ & = m_2 (-l_1 \sin\theta_1 \ddot{\theta}_1 - l_1 \cos\theta_1 \dot{\theta}_1^2 - l_2 \sin\theta_2 \ddot{\theta}_2 - l_2 \cos\theta_2 \dot{\theta}_2^2) \end{aligned} \tag{3.134}$$

The second of Eq.(3.131) and Eqs.(134) provide three equations for the three unknowns $(T_{c2}, \ddot{\theta}_1, \ddot{\theta}_2)$.

Eqs.(3.134-a) $\times$ $\cos\theta_2$ - Eqs.(3.134-b) $\times$ $\sin\theta_2$ gives

$$\begin{aligned} -w_2 \sin\theta_2 = m_2 [& l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) \\ & + l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)] \ , \end{aligned} \tag{3.135}$$

Eqs.(3.134-a) $\times$ $\cos\theta_1$ - Eqs.(3.134-b) $\times$ $\sin\theta_1$ gives $T_{c2} \sin(\theta_2 - \theta_1) + ..$ Substituting for $T_{c2} \sin(\theta_2 - \theta_1)$ into the second of Eq.(3.131) gives (with a lots of algebra) :

$$-w_2 \sin\theta_1 + m_2[-l_1\ddot{\theta}_1 - l_2\ddot{\theta}_2\cos(\theta_2-\theta_1)$$
$$+ l_2\dot{\theta}_2^2\sin(\theta_2-\theta_1)] - w_1\sin\theta_1 = m_1 l_1\ddot{\theta}\ .$$

This the second of the two required differential equations. In matrix format the model is

$$\begin{bmatrix} l_1(m_1+m_2) & m_2 l_2\cos(\theta_2-\theta_1) \\ m_2 l_1\cos(\theta_2-\theta_1) & m_2 l_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$
$$= \begin{Bmatrix} -(w_1+w_2)\sin\theta_1 + m_2 l_2\dot{\theta}_2^2\sin(\theta_2-\theta_1) \\ -w_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin(\theta_2-\theta_1) \end{Bmatrix}. \tag{3.137}$$

Note that this inertia matrix is neither diagonal nor symmetric, but it can be made symmetric; e.g., multiply the first equation by l_1 and the second equation by l_2. As with the stiffness matrix, the inertia matrix should be either symmetric, or capable of being made symmetric. Also, correct diagonal entries are positive.

The linearized version of this equation is obtained by assuming that both θ_1 and θ_2 are small (i.e., $\sin\theta_1 \cong \theta_1$; $\cos\theta_1 \cong 1$, etc.) and can be stated

$$\begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_2 l_1 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} (w_1 + w_2)l_1 & 0 \\ 0 & w_2 l_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \ .$$

We have "symmetricized" the inertia matrix and now have a diagonal stiffness matrix. The inertia matrix couples these two degrees of freedom.