Lecture 15. EIGENANALYSIS FOR 2DOF VIBRATION EXAMPLES

Thinking about solving coupled linear differential equations by considering the problem of developing a solution to the following homogeneous version of Eq.(3.124)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 . \quad (3.139)$$

To find a solution to the one-degree-of-freedom problem $m\ddot{x} + kx = 0$, we "guessed" a solution of the form $x = A\cos\omega t \Rightarrow \ddot{x} = -\omega^2 A \cos\omega t$. Substituting this guess netted

$$(-\omega^2 + k/m) A \cos\omega t = 0 \ .$$

A nontrivial solution ($A \neq 0$) requires that $\omega = \omega_n = \sqrt{k/m}$.

For Eq.(3.139) we will guess

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos\omega t \Rightarrow \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos\omega t \ . \quad (3.140)$$

Substituting this guessed solution gives

$$\left[-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \right] \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos\omega t = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} ,$$

or

$$\begin{bmatrix} [-m_1\omega^2 + (k_1 + k_2)] & -k_2 \\ -k_2 & [-m_2\omega^2 + (k_2+k_3)] \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.141)$$

Solving for a_1, a_2 via Cramer's rule gives:

$$a_1 = \frac{1}{\Delta} \begin{vmatrix} 0 & -k_2 \\ 0 & -m_2\omega^2 + (k_2+k_3) \end{vmatrix} = \frac{0}{\Delta} ,$$

$$a_2 = \frac{1}{\Delta} \begin{vmatrix} -m_1\omega^2 + (k_1+k_2) & 0 \\ -k_2 & 0 \end{vmatrix} = \frac{0}{\Delta} ,$$

where Δ is the determinant of the coefficient matrix. For a nontrivial solution $(a_1, a_2 \neq 0)$, $\Delta = 0$; i.e., the coefficient matrix must be singular.

$$\Delta = [(k_1 + k_2) - m_1\omega^2]\,[(k_2+k_3) - m_2\omega^2] - k_2^2$$

$$= m_1 m_2 \omega^4 - [m_1(k_2+k_3) + m_2(k_1+k_2)]\,\omega^2 + (k_1+k_2)(k_2+k_3) - k_2^2 = 0. \qquad (3.142)$$

This is the characteristic equation. It is quadratic and defines two natural frequencies $\omega_{n1}^2, \omega_{n2}^2$ versus the single natural frequency for the one-degree-of-freedom vibration examples.

Numerical Example:

$$m_1 = 1\,kg,\ m_2 = 2\,kg\ ,\ k_1 = k_2 = k_3 = 1 N/m. \qquad (3.143)$$

For these data, the differential Eq.(3.139) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0, \qquad (3.144)$$

and the frequency Eq.(3.142) becomes

$$\omega^4 - 3\,\omega^2 + \frac{3}{2} = 0 \Rightarrow \omega^2 = \frac{3}{2} \pm \frac{\sqrt{3}}{2}\ ;$$

with the solutions:

$$\omega_{n1}^2 = .633975\sec^{-2} \Rightarrow \omega_{n1} = .7962\sec^{-1}$$
$$\omega_{n2}^2 = 2.36603\sec^{-2} \Rightarrow \omega_{n2} = 1.538\sec^{-1}\ . \qquad (3.145)$$

The first (lowest) root $\omega_{n1}^2 = .634\sec^{-2}$ is the first eigenvalue and defines the first natural frequency $\omega_{n1} = .796\,rad/\sec$. The next root $\omega_{n2}^2 = 2.366\sec^{-2}$ is the second eigenvalue and defines the second natural frequency $\omega_{n2} = 1.510\,rad/\sec$.

Solving for the a_1, a_2 coefficients. Substituting the data of Eq.(3.145) into Eq.(3.141) gives

$$\begin{bmatrix} -\omega^2+2 & -1 \\ -1 & -2\omega^2+2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

Now substituting $\omega^2 = \omega_{n1}^2 = .634\sec^{-2}$ gives

$$\begin{bmatrix} -.633975+2 & -1 \\ -1 & -1.26796+2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = 0$$

The coefficient-matrix determinant is zero, which implies that there is only one independent equation for the two unknowns. Hence, we can use either equation to solve for the ratios of the two unknowns. Setting $a_{11} = 1$ gives:

$$1.36603(1) - a_{21} = 0 \;\Rightarrow\; a_{21} = 1.36603$$

$$-1(1) + .732050\, a_{21} = 0 \;\Rightarrow\; a_{21} = 1.36603 \;.$$

Hence, the first "eigenvector" is

$$(a_{i1}) = \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.36603 \end{Bmatrix} . \qquad (3.146)$$

Multiplying this vector by any finite constant (positive or negative) will yield an equally valid first eigenvector, since the vector is defined only in terms of the ratio of its components. In vibration problems, an eigenvector is also called a "mode shape."

Substituting $\omega^2 = \omega_{n2}^2 = 2.3360\,\mathrm{sec}^{-2}$ into Eq.(3.141), nets

$$(a_{i2}) = \begin{Bmatrix} a_{21} \\ a_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -.36603 \end{Bmatrix} .$$

The matrix of eigenvectors is

$$[A] = \begin{bmatrix} 1.0 & 1.0 \\ 1.36603 & -.36603 \end{bmatrix} . \qquad (3.147)$$

Figure 3.52 illustrates the two eigenvectors.

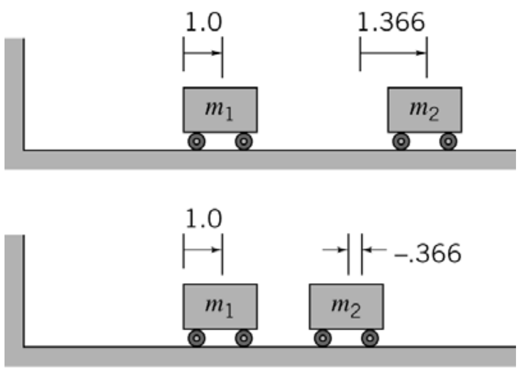

Figure 3.52 Eigenvectors for the two-mass system of figure 3.47, with the numerical values of Eq.(3.143).

Consider the following coordinate transformation for Eq.(3.139)

$$(x_i) = [A](q_i) \;\Rightarrow\; \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0 \\ 1.36603 & -.36603 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} ,(3.148a)$$

$$(\ddot{x}_i) = [A](\ddot{q}_i) \;. \qquad (3.148b)$$

where the right vector (q_i) is the vector of *modal* coordinates.

Substituting from Eqs.(3.148) into Eq.(3.124) gives

$$[M][A](\ddot{q}_i) + [K][A](q_i) = (f_i) \ . \qquad (3.149)$$

Premultiplying Eq.(3.149) by the transpose of $[A]$ gives

$$[A]^T[M][A](\ddot{q}_i) + [A]^T[K][A](q_i) = [A]^T(f_i) \ . \qquad (3.150)$$

We can now show by substitution (for this example problem) that

$$\begin{aligned} [A]^T[M][A] &= [M_q] \\ [A]^T[K][A] &= [K_q] \ , \end{aligned} \qquad (3.148)$$

where the "modal mass matrix" $[M_q]$ and "modal stiffness matrix $[K_q]$ are diagonal. Note

$$[A]^T[M][A] = \begin{bmatrix} 1.0 & 1.36603 \\ 1.0 & -.36603 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1.0 & 1.0 \\ 1.36603 & -.36603 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 1.36603 \\ 1.0 & -.36603 \end{bmatrix} \begin{bmatrix} 1.0 & 1.0 \\ 2.73206 & -.73206 \end{bmatrix} \qquad (3.149)$$

$$= \begin{bmatrix} 4.732 & .0000 \\ .0000 & 1.268 \end{bmatrix} = [M_q] \ .$$

The modal mass matrix $[M_q]$ is diagonal, with the first and second "modal masses" defined by $m_{q1} = 4.732 \ ; \ m_{q2} = 1.268$.

We want to "normalize" the eigenvectors *with respect to the mass matrix* such that the modal mass matrix $[M_q]$ reduces to the identity matrix [I]. The modal-mass coefficient for the jth mode is defined by $(a_{ji})^T[M](a_{ji}) = m_{qj}$. Dividing the jth eigenvector (a_{ji}) by $m_{qj}^{1/2}$ will yield an eigenvector with a modal mass equal to 1, yielding

$$[A^*]^T[M][A^*] = [I] \ . \qquad (3.150)$$

Normalizing the current eigenvector set means dividing the first and second eigenvectors by $(M_{q1})^{1/2} = \sqrt{4.732} = 2.175$ and $(M_{q2})^{1/2} = \sqrt{1.268} = 1.126$, respectively, obtaining

$$[A^*] = \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix} . \qquad (3.151)$$

You may want to repeat calculations for this set of eigenvectors to confirm that the modal mass matrix is now the identity matrix. Proceeding with this normalized version of the eigenvector matrix to verify that the modal stiffness matrix is diagonal yields

$$[A^*]^T[K][A^*] = [A^*]^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix}$$

$$= \begin{bmatrix} .45970 & .62796 \\ .88807 & -.32506 \end{bmatrix} \begin{bmatrix} .29144 & 2.10120 \\ .79622 & -1.53819 \end{bmatrix}$$

$$= \begin{bmatrix} .63345 & -.00000 \\ -.00000 & 2.3660 \end{bmatrix} = [K_q].$$

The normalized matrix of eigenvectors yields a diagonalized modal stiffness matrix [K_q]; moreover, the diagonal entries are the eigenvalues defined in Eq.(3.142); i.e.,

$$[A^*]^T[K][A^*] = [\Lambda] = \begin{bmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix}, \qquad (3.152)$$

where $[\Lambda]$ is the diagonal matrix of eigenvalues.
The resultant modal equations are:

$$\ddot{q}_1 + .63345\, q_1 = (A_1)^T(f_1) = Q_1$$
$$\ddot{q}_2 + 2.3660\, q_1 = (A_2)^T(f_2) = Q_2 \ .$$

The transformation from modal to physical coordinates is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}.$$

Modal Units

Given that $[A^*]^T[M][A^*] = [I]$, and $[A^*]^T[K][A^*] = [\Lambda]$, the units for an entry in normalized eigenvector matrix $[A^*]$ is $mass^{-1/2}$. Hence, for the SI system, the eigenvector units are $kg^{-1/2}$; for the USA standard unit system, the units are $slug^{-1/2}$. From the coordinate transformation $(x)_i = [A^*](q)_i$, the units for a modal coordinate is $mass^{1/2} \times length$. For the SI and USA standard systems, the appropriate units are, respectively, $meter\, kg^{1/2}$ and $ft\, slug^{1/2}$. Looking at the first of Eq.(3,154), a dimensional analysis yields

$$\ddot{q}_1(Lm^{1/2}T^{-2}) + \omega_{n1}^2(T^{-2})\, q_1(Lm^{1/2}) = a_{11}(m^{-1/2}) f_1(F)$$
$$+ a_{12}(m^{-1/2}) f_2(F)$$

$$\therefore\ mLT^{-2} = F\ .$$

confirming the correctness of these dimensions.

Lessons:

a. Vibration problems can have multiple degrees of freedom.

b. Multiple-degree-of-freedom (MDOF) vibration problems can be coupled by either the stiffness (linear spring-mass system) or inertia (double pendulum) matrices.

c. For a neutrally stable system, the inertia and stiffness matrices should be symmetric and the diagonal elements should be positive.

d. Free vibrations of a MDOF vibration problem leads to an eigenvalue problem. The solution to the eigenvalue problem yields eigenvalues, ω_{ni}^2, which define the natural frequencies ω_{ni}, and eigenvectors that define the system mode shapes.

e. The matrix of eigenvectors $[A]$ can be normalized such that it diagonalizes the original inertia and stiffness matrices as

$$[A^*]^T[M][A^*] = [I] \ , \quad [A^*]^T[K][A^*] = [\Lambda] \ ,$$

where $[I]$ is the identity matrix, and $[\Lambda]$ is the diagonal matrix of eigenvalues.