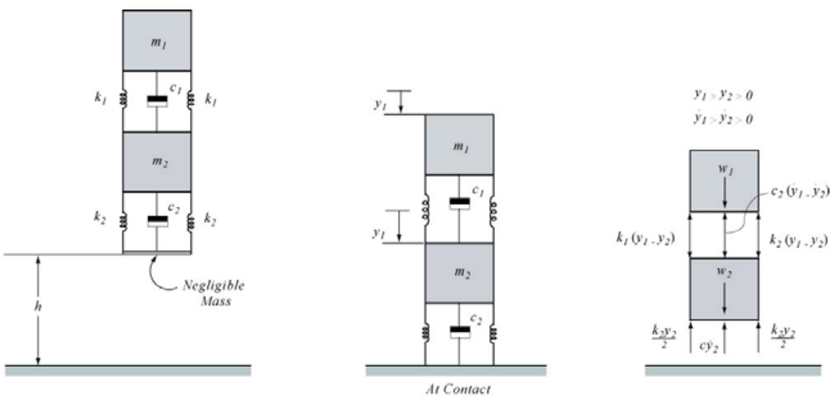
Lecture 17. MORE TRANSIENT MOTION USING MODAL COORDINATES

Example



At the left is an assembly that is released from a height $h=2ft$ above the ground. In the middle, the assembly has just contacted the ground. The subsequent positions of m_1 and m_2 are defined, respectively, by y_1 and y_2. At the time of contact, $y_1(0)=y_2(0)=0$, and $\dot{y}_1(0)=\dot{y}_2(0)=v_0=\sqrt{2gh}=\sqrt{2\times 32.2\times 2}=11.35 ft/\sec$ ..

Engineering-analysis tasks:

a. Draw free-body diagrams and derive the equations of motion.

b. State the matrix equations of motion.

c. Solve for two cycles of motion for the lowest natural frequency.

Equation of Motion from Free-body diagrams:

$$m_1\ddot{y}_1=\sum f_{y_1}=w_1-k_1(y_1-y_2)-c_1(\dot{y}_1-\dot{y}_2)$$
$$m_2\ddot{y}_2=\sum f_{y_2}=w_2-k_2y_2-c_2\dot{y}_2+k_1(y_1-y_2)+c_1(\dot{y}_1-\dot{y}_2)\ . \tag{1}$$

Matrix Format:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{Bmatrix}+\begin{bmatrix} c_1 & -c_1 \\ -c_1 & (c_1+c_2)\end{bmatrix}\begin{Bmatrix}\dot{y}_1\\ \dot{y}_2\end{Bmatrix}+ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1+k_2)\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix}=\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}. \tag{2}$$

Equations for Modal Coordinates using Modal damping to account for internal damping

$$(\ddot{q})_i+[2\zeta\omega_n](\dot{q}_i)+[\Lambda](q_i)=(Q_i)=\ [A^*]^T(f_i)$$
$$=\begin{bmatrix}A^*_{11} & A^*_{21}\\ A^*_{12} & A^*_{22}\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}=\begin{Bmatrix}A^*_{11}w_1 + A^*_{21}w_2\\ A^*_{12}w_1 + A^*_{22}w_2\end{Bmatrix}.$$

Component modal differential equations:

$$\ddot{q}_1 + 2\zeta_1\omega_{n1}\dot{q}_1 + \omega_{n1}^2 q_1 = Q_1 = A_{11}^* w_1 + A_{21}^* w_2$$
$$\ddot{q}_2 + 2\zeta_2\omega_{n2}\dot{q}_2 + \omega_{n2}^2 q_2 = Q_2 = A_{12}^* w_1 + A_{22}^* w_2 \quad . \qquad (3)$$

The homogeneous version of Eq.(3) is

$$\ddot{q}_{1h} + 2\zeta_1\omega_{n1}\dot{q}_{1h} + \omega_{n1}^2 q_{1h} = 0 \ , \quad \ddot{q}_{2h} + 2\zeta_2\omega_{n2}\dot{q}_{2h} + \omega_{n2}^2 q_{2h} = 0 \ ,$$

with solutions:

$$q_{1h}(t) = e^{-\zeta_1\omega_{n1}t}(A_1\cos\omega_{d1}t + B_1\sin\omega_{d1}t)$$
$$q_{2h}(t) = e^{-\zeta_2\omega_{n2}t}(A_2\cos\omega_{d2}t + B_2\sin\omega_{d2}t) \quad .$$

The particular solutions $q_{1p}(t), q_{2p}(t)$ corresponding to Eq.(3) are

$$q_{1p} = (A_{11}^* w_1 + A_{21}^* w_2)/\omega_{n1}^2 \ , \ q_{2p} = (A_{12}^* w_1 + A_{22}^* w_2)/\omega_{n2}^2 \ ,$$

yielding the complete modal-coordinate solutions

$$q_1(t) = \frac{(A_{11}^* w_1 + A_{21}^* w_2)}{\omega_{n1}^2} + e^{-\zeta_1\omega_{n1}t}(A_1\cos\omega_{d1}t + B_1\sin\omega_{d1}t)$$
$$q_2(t) = \frac{(A_{12}^* w_1 + A_{22}^* w_2)}{\omega_{n2}^2} + e^{-\zeta_2\omega_{n2}t}(A_2\cos\omega_{d2}t + B_2\sin\omega_{d2}t) \quad .$$

The constants A_i, B_i must be determined from the modal-coordinate initial conditions.

The modal-coordinate initial conditions are defined by

$$(q_0) = [A^*]^T[M](y_0) \quad .$$

Similarly, the modal-velocity initial conditions are defined by

$$(\dot{q}_0) = [A^*]^T[M](\dot{y}_0) \quad .$$

A previous undamped model had the physical parameters:

$$m_1 = 150\,kg, m_2 = 100\,kg \;,\; k_1 = k_2 = 1.5\times 10^4 N/m,$$

yielding

$$[M] = \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix}, \quad [K] = \begin{bmatrix} 1.5\times 10^4 & -1.5\times 10^4 \\ -1.5\times 10^4 & 3.0\times 10^4 \end{bmatrix}$$

$$\{w_i\} = \begin{Bmatrix} 1457 \\ 981 \end{Bmatrix} .$$

The eigenvalues and natural frequencies are:

$$\omega_{n1}^2 = 41.886\,\sec^{-2} \Rightarrow \omega_{n1} = 6.4720\,\sec^{-1}$$

$$\omega_{n2}^2 = 358.11\,\sec^{-2} \Rightarrow \omega_{n2} = 18.924\,\sec^{-1} \;.$$

The matrix of normalized eigenvectors is

$$[A^*] = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix} .$$

From,

$$\ddot{q}_1 + 2\zeta_1\omega_{n1}\dot{q}_1 + \omega_{n1}^2 q_1 = Q_1 = A_{11}^* w_1 + A_{21}^* w_2$$
$$\ddot{q}_2 + 2\zeta_2\omega_{n2}\dot{q}_2 + \omega_{n2}^2 q_2 = Q_2 = A_{12}^* w_1 + A_{22}^* w_2 \;, \qquad (3)$$

the model (with 5% modal damping) is

$$\ddot{q}_1 + 6.47\,\dot{q}_1 + 41.886\,q_1 = 149.5$$

$$\ddot{q}_2 + 18.92\,\dot{q}_2 + 358.11\,q_2 = -37.63$$

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} .$$

From $(q_0) = [A^*]^T[M](y_0)$, the modal-coordinate initial conditions are zero. From $(\dot{q}_0) = [A^*]^T[M](\dot{y}_0)$, the modal-velocity initial conditions are

$$\begin{Bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{Bmatrix} = \begin{bmatrix} .073767 & .042866 \\ .035002 & -.090344 \end{bmatrix} \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{Bmatrix} 11.35 \\ 11.35 \end{Bmatrix}$$

$$= \begin{bmatrix} 11.065 & 4.2866 \\ 5.2503 & -9.0344 \end{bmatrix} \begin{Bmatrix} 11.35 \\ 11.35 \end{Bmatrix} = \begin{Bmatrix} 174.22 \\ -42.95 \end{Bmatrix} .$$

Substituting into,

$$q_1(t)=\frac{(A_{11}^* w_1 + A_{21}^* w_2)}{\omega_{n1}^2}+e^{-\zeta_1\omega_{n1}t}(A_1\cos\omega_{d1}t+B_1\sin\omega_{d1}t)$$

$$q_2(t)=\frac{(A_{12}^* w_1 + A_{22}^* w_2)}{\omega_{n2}^2}+e^{-\zeta_2\omega_{n2}t}(A_2\cos\omega_{d2}t+B_2\sin\omega_{d2}t)\ ,$$

nets

$$q_1(t)=\frac{149.5}{41.866}+e^{-0.3235t}(A_1\cos 6.462t+B_1\sin 6.462t)$$

$$q_2(t)=\frac{-37.63}{358.11}+e^{-0.946t}(A_2\cos 18.90t+B_2\sin 18.90t)\ ,$$

where

$$\omega_{d1}=\omega_{n1}\sqrt{1-\zeta_1^2}=6.470\sqrt{1-0.0025}=6.462$$

$$\omega_{d2}=\omega_{n2}\sqrt{1-\zeta_2^2}=18.92\sqrt{1-0.0025}=18.90\ .$$

Imposing initial conditions for $q_1(t)$

$$q_1(0)=0=\frac{149.5}{41.866}+A_1\ \Rightarrow\ A_1=-3.571$$

Further

$$\dot q_1(t)=-0.3235\,e^{-0.3235t}(A_1\cos 6.462t+B_1\sin 6.462t)$$

$$+6.462\,e^{-0.3235t}(-A_1\sin 6.462t+B_1\cos 6.462t)$$

$$\therefore\ \dot q_1(0)=174.22=-0.3253\,A_1+6.462B_1$$

$$B_1=\frac{174.22+0.3253\times-3.571}{6.462}=26.78$$

The complete solution satisfying the initial conditions is

$$q_1(t)=3.571+e^{-0.3235t}(-3.571\cos 6.462t+26.78\sin 6.462t)$$

Similarly, the complete solution for $q_2(t)$ is

$$q_2(t)=-0.1051+e^{-0.946t}(0.1051\cos 18.90t+B_2\sin 18.90t)$$

$$\dot q_2(t)=-0.946\,e^{-0.946t}(0.1051\cos 18.90t+B_2\sin 18.90t)$$
$$+e^{-0.946t}\,18.90(0.1051\sin 18.90t-B_2\cos 18.90t)$$

$$\therefore\ \dot q_2(0)=-42.95=-0.946\times 0.1051-18.90B_2\ \Rightarrow\ B_2=2.267$$

The complete solution for $q_2(t)$ satisfying the initial conditions is

$$q_2(t) = -0.1051 + e^{-0.946t}(0.1051 \cos 18.90t + 2.267 \sin 18.90t)$$

The physical solution is

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = q_1(t)\begin{Bmatrix} .073767 \\ .042866 \end{Bmatrix} + q_2(t)\begin{Bmatrix} 0.035002 \\ -0.090344 \end{Bmatrix} .$$