MEEN 363 Notes
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Lecture 1. PARTICLE KINEMATICS IN A PLANE

Kinematics: Geometric in nature, defines motion without regard to forces that cause motion or result from motion.

Kinetics: Defines the motion of particles or rigid bodies that are caused by forces. Generally based on Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$.

Motion in a Straight Line

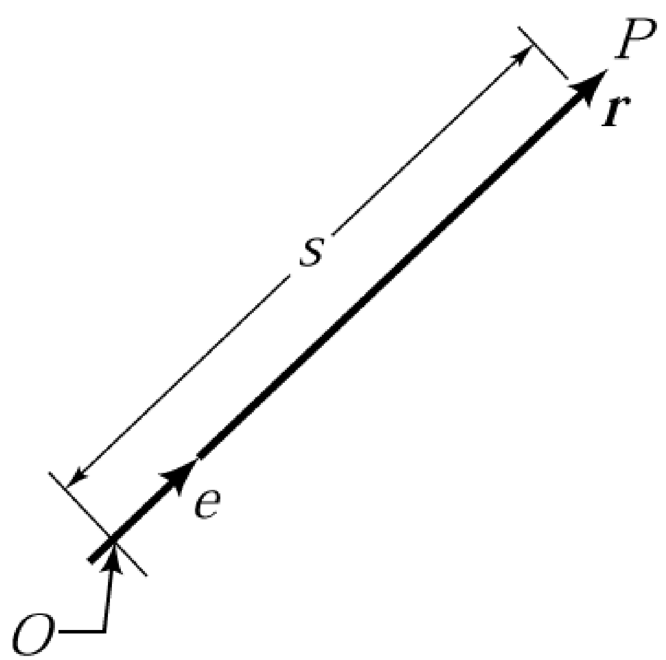


Figure 2.1 Displacement vector along the unit vector $\boldsymbol{e}$.

Position: $\boldsymbol{r} = \boldsymbol{e}s(t)$

Velocity: $\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{e}\dot{s} = \boldsymbol{e}v$

Acceleration: $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \boldsymbol{e}\ddot{s} = \boldsymbol{e}\dot{v} = \boldsymbol{e}a$

Particle Motion in a Plane: Cartesian Coordinates

Figure 2.3 illustrates the position of a particle *P* in a Cartesian *X, Y* coordinate system. The position vector locating *P* is

$$\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y , \qquad \textbf{(2.8)}$$

with ***I*** and ***J*** being vectors of unit magnitudes pointed along the orthogonal *X* and *Y* axes. r_X and r_Y are *components* of the position *vector* ***r*** in the *X, Y* coordinate system.

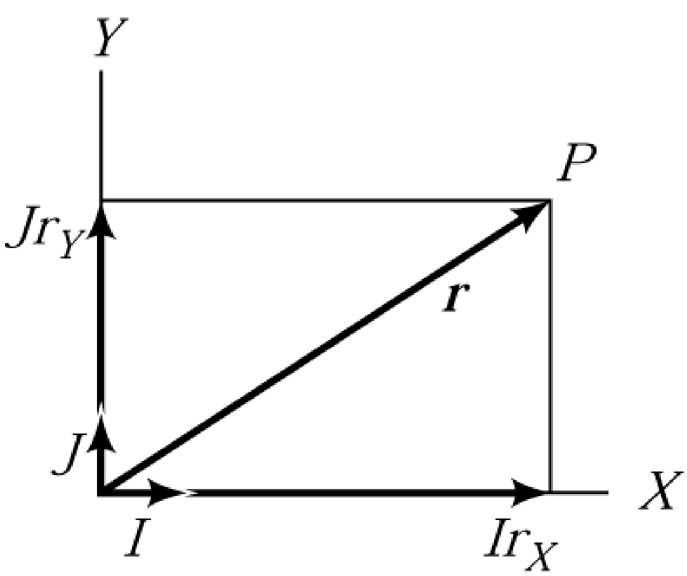


Figure 2.3 Particle *P* located in the *X, Y* system by the vector ***r***.

The velocity of point *P* <u>with respect</u> to the *X, Y* coordinate system is

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}\Big|_{X,Y} = \dot{\boldsymbol{r}} = \boldsymbol{I}\dot{r}_X + \boldsymbol{J}\dot{r}_Y = \boldsymbol{I}v_X + \boldsymbol{J}v_Y . \qquad \textbf{(2.9)}$$

The velocity vector's magnitude is

$$|v| = (v_X^2 + v_Y^2)^{1/2} . \qquad \textbf{(2.10)}$$

Q: How do you find the derivative of a vector ***B*** with respect to a coordinate system?"
A: The time derivative of any vector ***B*** with respect to a specified coordinate system is found by writing ***B*** out in terms of its components in the specified coordinate system,

$$\boldsymbol{B} = \boldsymbol{I}\, B_X + \boldsymbol{J}\, B_Y ,$$

and then differentiating with respect to time *while holding the unit vectors constant.*

$$\dot{\boldsymbol{B}} = \frac{d\boldsymbol{B}}{dt}\Bigg|_{X,Y} = \boldsymbol{I}\,\dot{B}_X + \boldsymbol{J}\,\dot{B}_Y .$$

These are exactly the steps that are performed for ***r*** in Eqs. (2.8) and (2.9) to obtain $\boldsymbol{v} = d\boldsymbol{r}/dt|_{X,Y,Z}$.

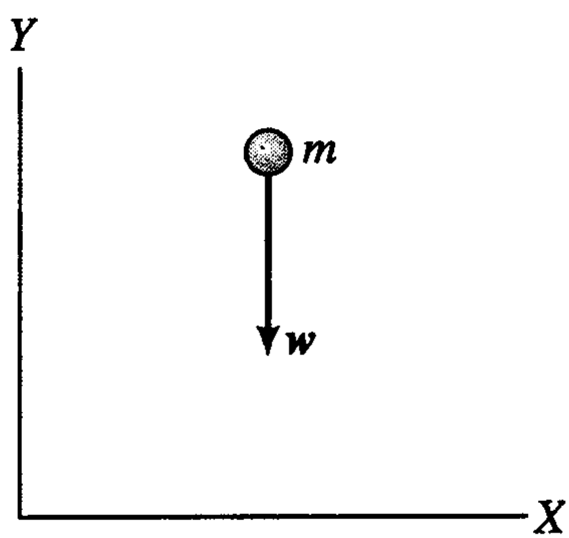
The acceleration of point *P* with respect to the *X, Y* system is

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}\Big|_{X,Y} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \boldsymbol{I}\dot{v}_X + \boldsymbol{J}\dot{v}_Y = \boldsymbol{I}a_X + \boldsymbol{J}a_Y \ . \qquad (2.11)$$

The magnitude of the acceleration vector is

$$|a| = (a_X^2 + a_Y^2)^{1/2} \ .$$

Drag-Free Motion of a Particle in a Plane

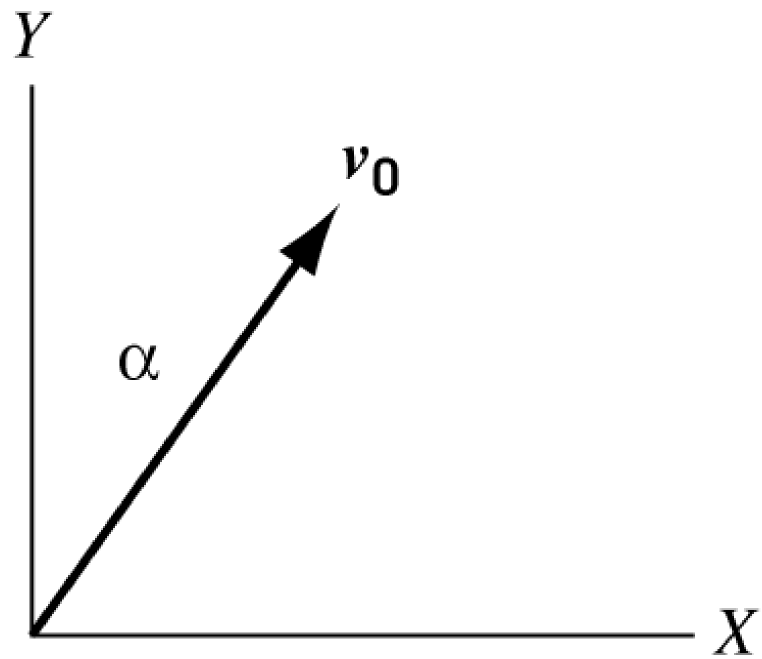


Free-body diagram for a particle that is falling, neglecting drag forces.

The figure below provides a free-body diagram for a particle acted on by gravity, neglecting aerodynamic drag. Applying Newton's second law of motion yields the following differential equations of motion:

$$\Sigma f_X = 0 = m\ddot{r}_X \Rightarrow \ddot{r}_X = 0$$

$$\Sigma f_Y = -w = m\ddot{r}_Y \Rightarrow \ddot{r}_Y = -\frac{w}{m} = -g$$



Example Problem 2.4 The acceleration components of a particle that is falling under the influence of gravity (neglecting aerodynamic drag forces) are:

$$a_X = \ddot{r}_X = 0 \ , \quad a_Y = \ddot{r}_Y = -g \ . \tag{i}$$

This definition of the acceleration components has the particle accelerating straight down in the -*Y* direction. *For the initial conditions illustrated above, namely:*
$v_X(0) = v_0 \sin\alpha$, $v_Y(0) = v_0 \cos\alpha$ *and* $r_X(0) = r_Y(0) = 0$, *find the components of the position and velocity vectors.*

Solution: Integrating Eqs.(i) once with respect to time yields:

$$v_X = \dot{r}_X = v_X(0) = v_0 \sin\alpha$$

$$v_Y = \dot{r}_Y = v_Y(0) - gt = v_0 \cos\alpha - gt \ .$$

A second integration gives:

$$r_X = r_Y(0) + t\, v_0 \sin\alpha = t\, v_0 \sin\alpha$$

$$r_Y = r_Y(0) + t\, v_0 \cos\alpha - \frac{gt^2}{2} = t\, v_0 \cos\alpha - \frac{gt^2}{2} \ .$$

c

b

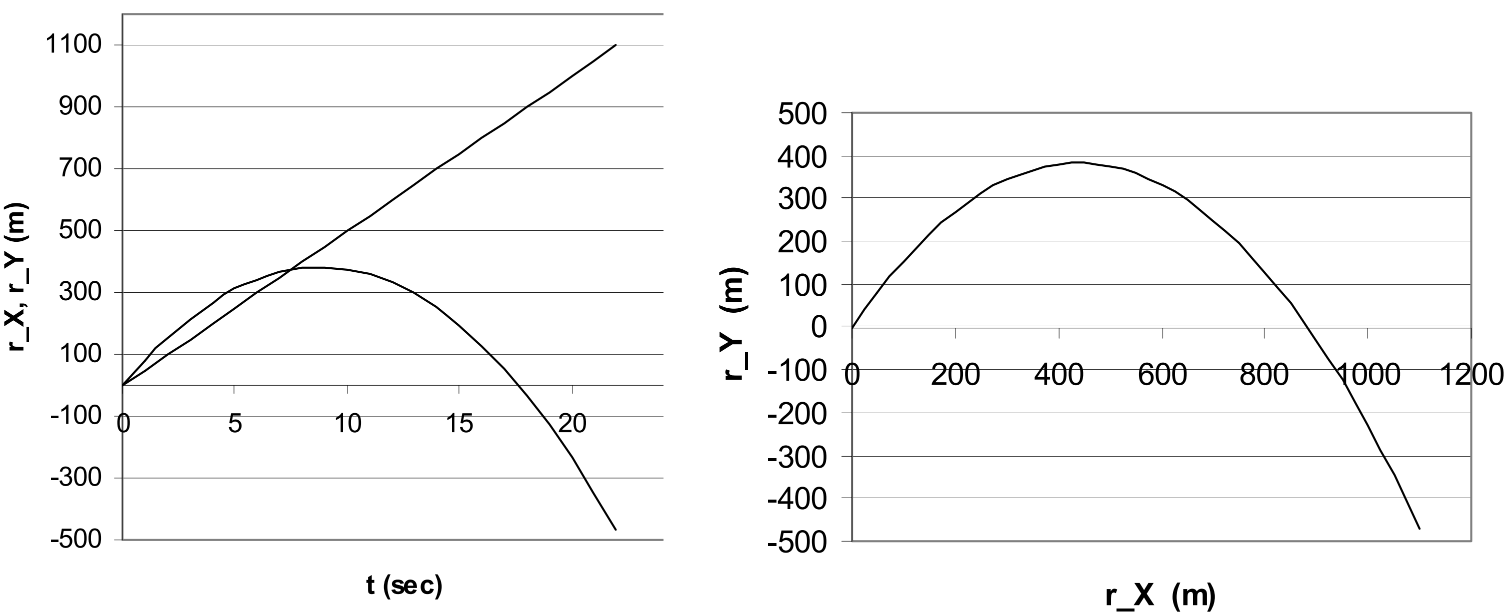


Figure XP 2.4 (b). r_X and r_Y versus time. (c). r_Y versus r_X for $v_0 = 100\, m/\sec$, $\alpha = 30°$, and $g = 9.81\ m/\sec^2$.

Matrix Algebra

$$[A] = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & & \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

m rows n columns

a_{ij} = entry in *ith* row and *jth* column

Matrix Multiplication, $[\boldsymbol{a}][\boldsymbol{b}]=[\boldsymbol{c}]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = [c] =$$

$$3 \times 3 \qquad\qquad 3 \times 2$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

$$3 \times 2$$

In general: $[\boldsymbol{A}][\boldsymbol{B}] \neq [\boldsymbol{B}][\boldsymbol{A}]$

Matrix inverse

$$[A]^{-1}\,[A] = [I] = \begin{bmatrix} 1 & 0 & . & . & . \\ 0 & 1 & & & \\ 0 & & & & \\ . & & & & \\ . & & & & 1 \end{bmatrix}$$

$\mathbf{[A]^{-1}}$ exists if the determinant of $[A]$ is nonzero; i.e., $|\boldsymbol{A}| \neq \mathbf{0}$

Matrix transpose

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \; \boldsymbol{a_{ij} = a_{ji}}$$

Orthogonal Matrix: $[A]^T = [A]^{-1}$.

Cramer’s Rule for solving 2 simultaneous equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

Solve for the determinant D

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Solve for x_1

$$Dx_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - b_2 a_{12}$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

Solve for x_2

$$Dx_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = b_2 a_{11} - b_1 a_{21}$$

$$x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

Coordinate Transformations: Relationships Between Components of a Vector in Two Coordinate Systems

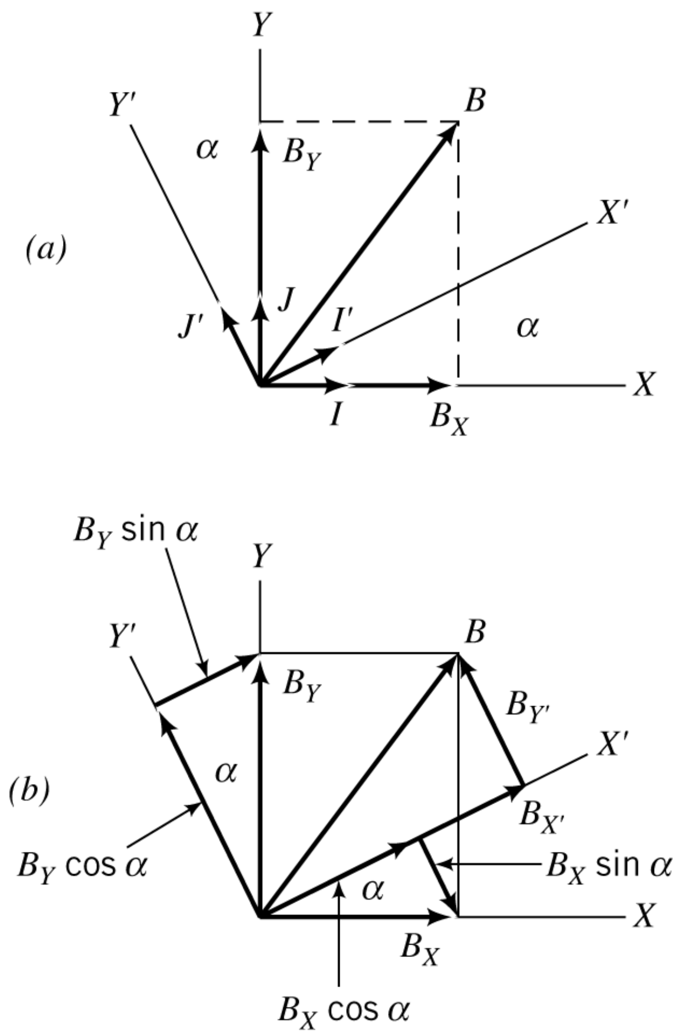


Figure 2.4 Vector *B* in two coordinate systems. (a). Vector ***B*** in terms of its X, Y components. (b). Two coordinate definitions.

Figure 2.4a illustrates a vector $\boldsymbol{B}$ in the "original" *X, Y* coordinate system and a "new" X', Y' coordinate system. The orientation of the *X, Y* and X', Y' coordinate systems is defined by the *constant* angle α.

Given: B_X, B_Y and α , **Find:** $B_{X'}, B_{Y'}$

From Figure 2.4b

$$\begin{aligned} \boldsymbol{I}B_X &= \boldsymbol{I}'B_X\cos\alpha - \boldsymbol{J}'B_X\sin\alpha \\ \boldsymbol{J}B_Y &= \boldsymbol{I}'B_Y\sin\alpha + \boldsymbol{J}'B_Y\cos\alpha\ . \end{aligned} \tag{2.12}$$

Since, $\boldsymbol{B} = \boldsymbol{I}B_X + \boldsymbol{J}B_Y$,

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{I}'(B_X\cos\alpha + B_Y\sin\alpha) + \boldsymbol{J}'(-B_X\sin\alpha + B_Y\cos\alpha) \\ &= \boldsymbol{I}'B_{X'} + \boldsymbol{J}'B_{Y'}\ . \end{aligned} \tag{2.13}$$

Equating coefficients of $\boldsymbol{I}'$, and $\boldsymbol{J}'$ in Eqs. (2.13) yields:

$$\begin{aligned} B_{X'} &= B_X\cos\alpha + B_Y\sin\alpha \\ B_{Y'} &= -B_X\sin\alpha + B_Y\cos\alpha\ . \end{aligned} \tag{2.14}$$

In matrix format,

$$\begin{Bmatrix} B_{X'} \\ B_{Y'} \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} . \qquad \textbf{(2.15)}$$

or, symbolically,

$$(B)_{I'} = [A](B)_{\boldsymbol{I}} . \qquad \textbf{(2.16)}$$

$[A]$ is called the "direction-cosine matrix," and can be formally defined as

$$[A] = \begin{bmatrix} \cos(X', X) & \cos(X', Y) \\ \cos(Y', X) & \cos(Y', Y) \end{bmatrix} , \qquad \textbf{(2.17)}$$

where $\cos(X', X)$ is the cosine of the angle between X and X', $\cos(X', Y)$ is the cosine of the angle between X' and Y, etc. Returning to figure 2.4B,

$$\begin{aligned} \cos(X', X) &= \cos(Y', Y) = \cos\alpha \\ \cos(X', Y) &= \cos(\frac{\pi}{2} - \alpha) = \cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha \\ &= \sin\alpha \qquad \textbf{(2.18)} \\ \cos(Y', X) &= \cos(\frac{\pi}{2} + \alpha) = \cos\frac{\pi}{2}\cos\alpha - \sin\frac{\pi}{2}\sin\alpha \\ &= -\sin\alpha . \end{aligned}$$

Substituting from Eqs. (2.18) into (2.17) gives

$$[A] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix},$$

which is the same result provided earlier by Eq.(2.15). $[A]$ is orthogonal; i.e., its inverse $[A]^{-1} = [A]^T$.

$$[A]^T[A] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given $B_{X'}$ *and* $B_{Y'}$ *what are* B_X *and* B_Y ? Premultiplying Eq.(2.15) by $[A]^T$ gives

$$[A]^T(B)_{I'} = [A]^T[A](B)_I = (B)_I .$$

Hence,

$$(B)_I = [A]^T(B)_{I'} ,$$

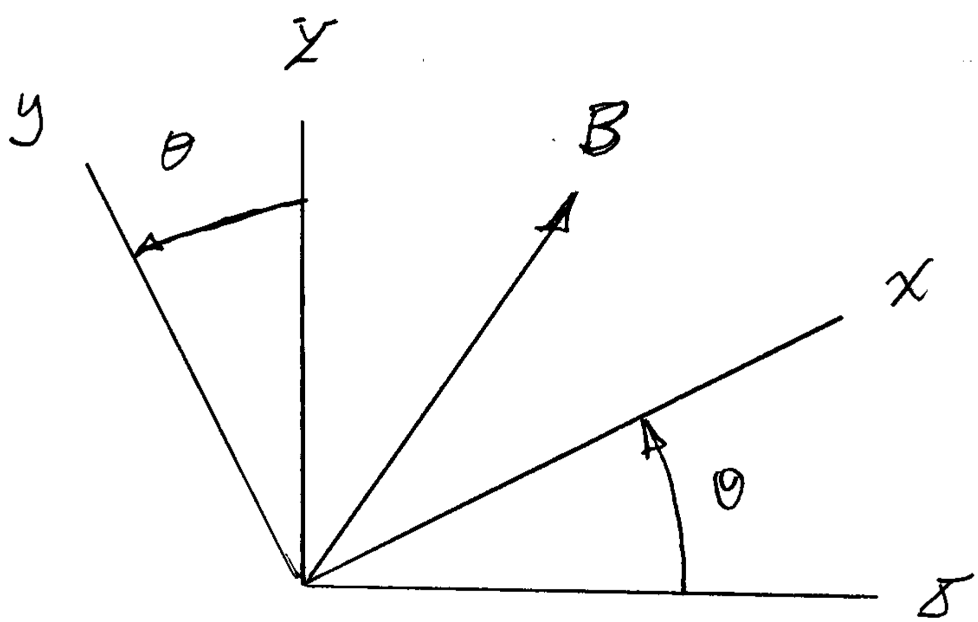
or

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} B_{X'} \\ B_{Y'} \end{Bmatrix}. \qquad \textbf{(2.19)}$$

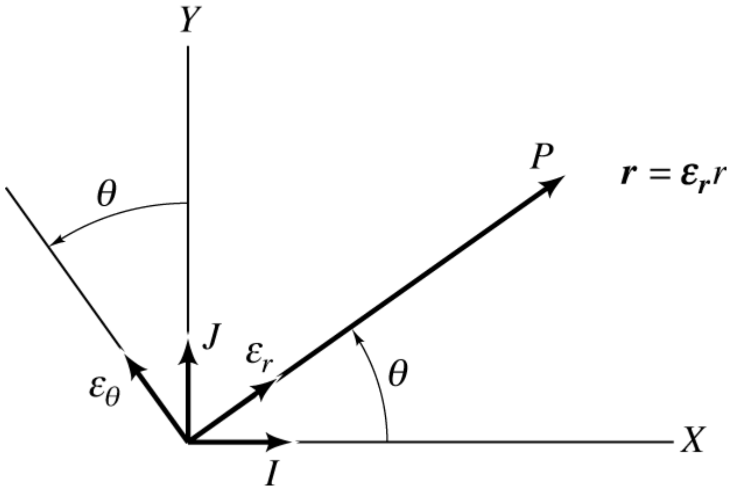
Homework Problem

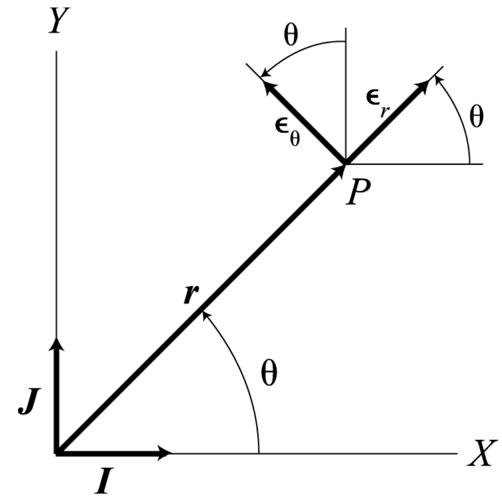
Given: $B_x = 10$, $B_y = 5$

Find: B_X, B_Y for $\theta = 0$, 30, 45, 90,120, 135, 153.4, 180 (degrees).



PARTICLE MOTION IN A PLANE: POLAR COORDINATES





Developing useful expressions for velocity and acceleration using polar coordinates is our present task.

Unit-Vector Definition

$$\boldsymbol{\varepsilon}_{\boldsymbol{r}} = \boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta$$

$$\boldsymbol{\varepsilon}_{\boldsymbol{\theta}} = -\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta\ .$$

Start with

$$\boldsymbol{r} = r\boldsymbol{\varepsilon}_{\boldsymbol{r}}\ . \qquad \textbf{(2.20)}$$

Differentiating w.r.t. time relative to the *X, Y* coordinate system

$$\dot{\boldsymbol{r}} = \left.\frac{d\boldsymbol{r}}{dt}\right|_{X,Y} = \dot{r}\boldsymbol{\varepsilon}_r + r\left.\frac{d\boldsymbol{\varepsilon}_r}{dt}\right|_{X,Y}$$

$$= \dot{r}\boldsymbol{\varepsilon}_r + r\dot{\boldsymbol{\varepsilon}}_r \ . \tag{2.22}$$

Differentiating $\boldsymbol{\varepsilon}_r = \boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta$ while holding $\boldsymbol{I}$ and $\boldsymbol{J}$ constant

$$\dot{\boldsymbol{\varepsilon}}_r = \left.\frac{d\boldsymbol{\varepsilon}_r}{dt}\right|_{X,Y} = (-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)\dot{\theta} = \dot{\theta}\boldsymbol{\varepsilon}_\theta \ , \tag{2.25}$$

and

$$\dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{\varepsilon}_r + r\dot{\theta}\,\boldsymbol{\varepsilon}_\theta$$

$$= v_r\,\boldsymbol{\varepsilon}_r + v_\theta\,\boldsymbol{\varepsilon}_\theta \ . \tag{2.27}$$

This expression provides $\dot{\boldsymbol{r}}$, the time derivative of $\boldsymbol{r}$ with respect to the *X, Y* coordinate system, but the answer is given in terms of *components* in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

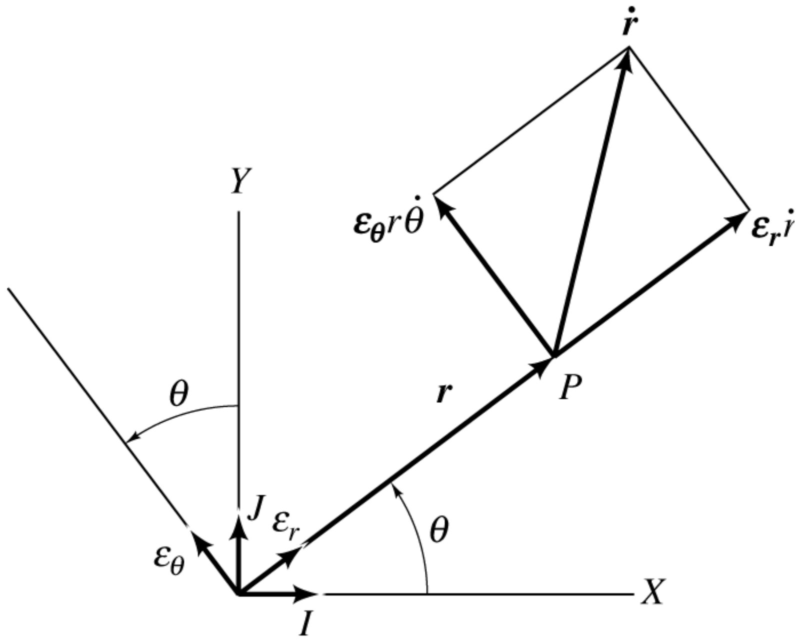


Figure 2.6 Components of $\dot{\boldsymbol{r}}$ in $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ coordinate system.

Given the velocity $\dot{\boldsymbol{r}}$, find acceleration $\ddot{\boldsymbol{r}}$.

$$\ddot{\boldsymbol{r}} = \left.\frac{d\dot{\boldsymbol{r}}}{dt}\right|_{X,Y} = \ddot{r}\boldsymbol{\varepsilon_r} + \dot{r}\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{r}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{\varepsilon_\theta} + r\dot{\theta}\,\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{\theta}} \;. \quad \textbf{(2.28)}$$

Differentiating $\boldsymbol{\varepsilon_\theta} = -\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta$ with respect to time, holding $\boldsymbol{I}$ and $\boldsymbol{J}$ constant,

$$\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{\theta}} = \left.\frac{d\boldsymbol{\varepsilon_\theta}}{dt}\right|_{X,Y} = -(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)\dot{\theta} = -\boldsymbol{\varepsilon_r}\dot{\theta} \;. \quad \textbf{(2.29)}$$

Substituting for $\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{r}}$ into Eq.(2.28) gives

$$\ddot{\boldsymbol{r}} = \ddot{r}\boldsymbol{\varepsilon}_r + \dot{r}\dot{\theta}\boldsymbol{\varepsilon}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{\varepsilon}_\theta - r\dot{\theta}^2\boldsymbol{\varepsilon}_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{\varepsilon}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\varepsilon}_\theta \qquad \textbf{(2.30)}$$

$$= a_r\boldsymbol{\varepsilon}_r + a_\theta\boldsymbol{\varepsilon}_\theta \ .$$

Eq.(2.30) provides a definition for the acceleration of point *P*, with respect to the *X, Y* coordinate system, in terms of components in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

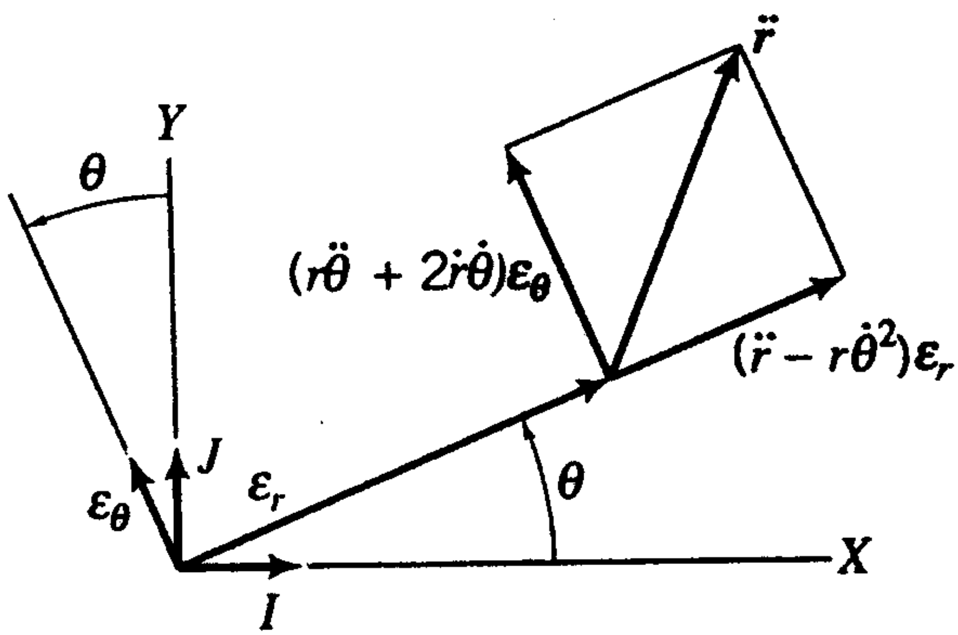


Figure 2.7 Components of $\ddot{\boldsymbol{r}}$ in $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

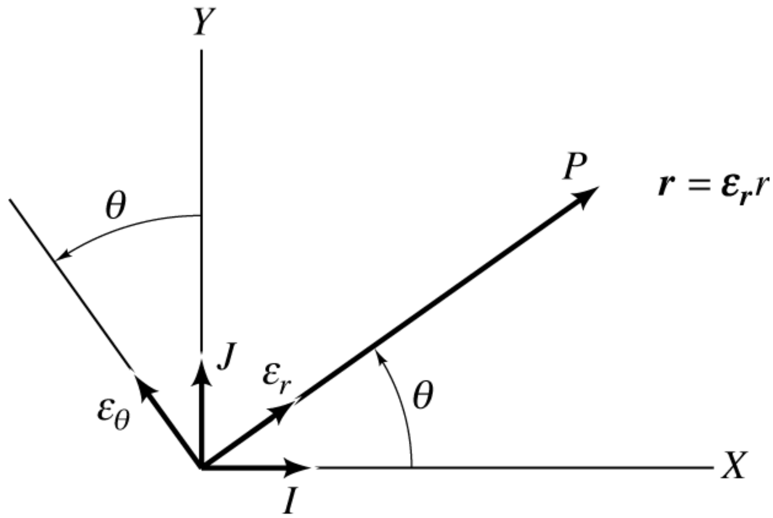
The polar velocity and acceleration components are:

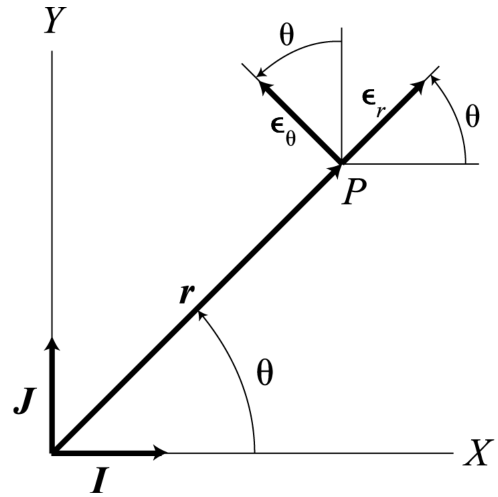
$$v_r = \dot{r}\ ,\ v_\theta = r\dot{\theta}$$
$$a_r = \ddot{r} - r\dot{\theta}^2\ ,\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}\ . \tag{2.31}$$

Gustave Coriolis (1792-1843) was assistant professor of mathematics at the Ecole Polytechnique, Paris from 1816 to 1838 and studied mechanics and engineering mathematics. He is best remembered for the Coriolis force which appears in the paper *Sur les équations du mouvement relatif des systèmes de corps* (1835). He showed that the laws of motion could be used in a rotating frame of reference if an extra "force term" incorporating the Coriolis acceleration is added to the equations of motion.

Coriolis also introduced the terms "work" and "kinetic energy" with their present scientific meaning. (Website, school of Mathematics and Statistics, Saint Andrews University, Scotland)

Lecture 2. MORE PARTICLE MOTION IN A PLANE — POLAR COORDINATES





The polar velocity and acceleration components are:

$$v_r = \dot{r}\ ,\ v_\theta = r\dot{\theta}$$
$$a_r = \ddot{r} - r\dot{\theta}^2\ ,\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}\ . \qquad \textbf{(2.31)}$$

Example Problem 2.6 As illustrated in figure XP 2.6a, a mass is sliding freely along a bar that is rotating at a constant 50 cycles per minute (cpm). At $r = .38\,m$ and $\theta = 135^o$, $\ddot{r} = 6.92\,m/\sec^2$, and $\dot{r} = .785\,m/\sec$. The engineering-analysis tasks are:

a. Determine the velocity and acceleration components of the mass in the rotating $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system, and

b. Determine the components of ***v*** and ***a*** in the stationary *X, Y* system.

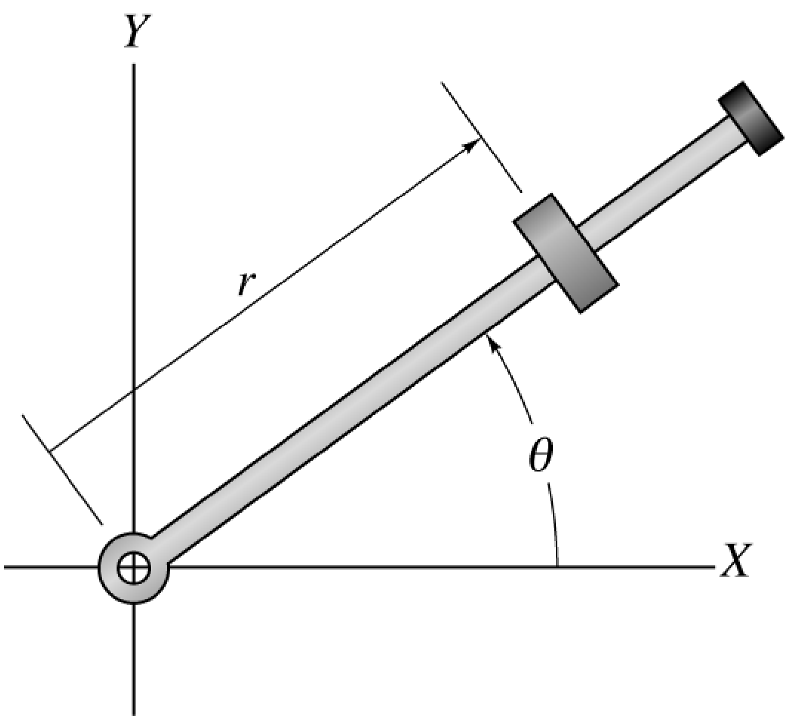


Figure XP2.6a. Mass sliding along a smooth rotating bar.

Solution. In applying Eqs.(2.31) to find the polar components of velocity and acceleration, $\dot{\theta}$ is the angular velocity of the bar; however, we need to convert its given dimensions from cpm to radians per second via

$$\dot{\theta} = 50\,\frac{cycle}{\min} \times \frac{1\min}{60\sec} \times \frac{2\pi\, rads}{1\, cycle} = 5.24\,\frac{rad}{\sec} \quad .$$

Direct application of Eqs.(2.31) gives

$$v_r = \dot{r} = .785\, m/\sec$$

$$v_\theta = r\,\dot{\theta} = .38\, m \times 5.24\, rad/\sec = 1.99\, m/\sec$$

$$\begin{aligned} a_r &= (\ddot{r} - r\,\dot{\theta}^2) = 6.92\, m/\sec^2 - .38\, m \times 5.24^2\, rad^2/\sec^2 \\ &= -3.51\, m/\sec^2 \end{aligned} \qquad \text{(i)}$$

$$\begin{aligned} a_\theta &= (r\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}) = .38\, m \times 0 + 2 \times .785\, m/\sec \times 5.24\, rad/\sec \\ &= 8.23\, m/\sec^2 \quad , \end{aligned}$$

and concludes *Task a.*

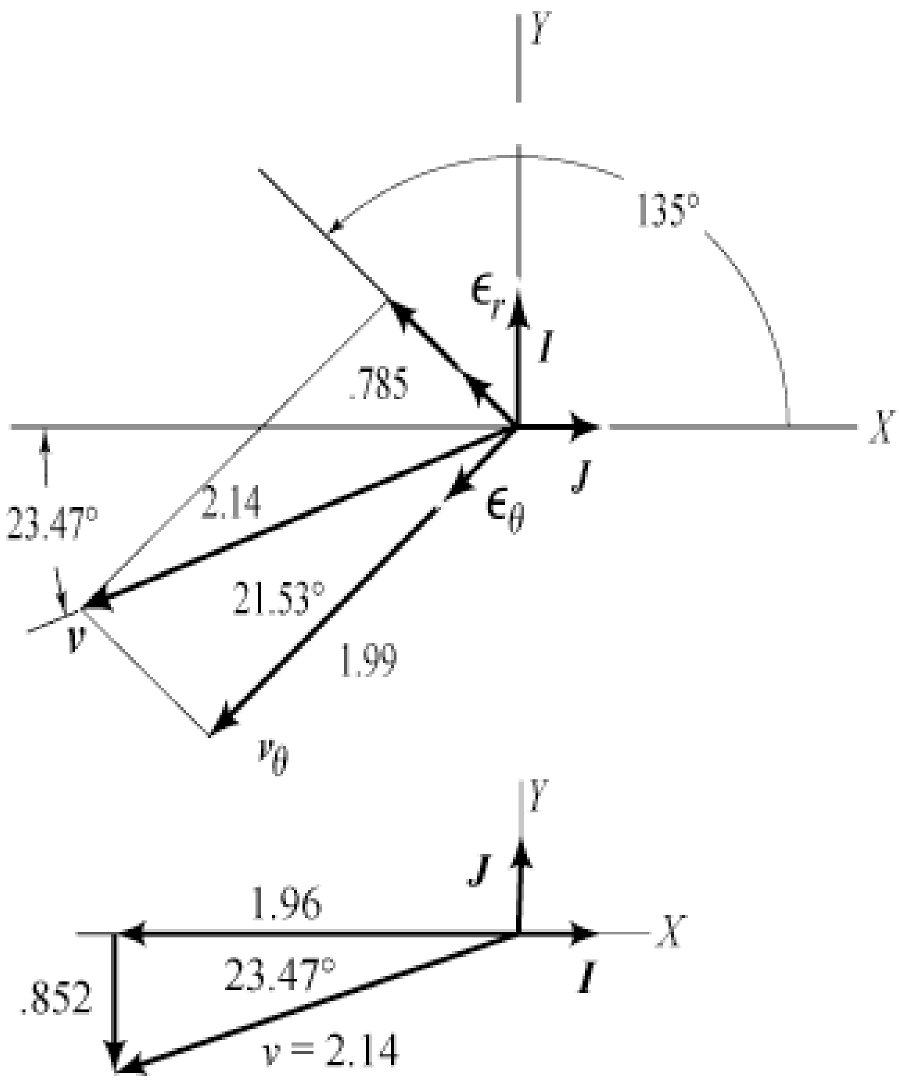


Figure XP 2.6b Velocity components in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ and $X-Y$ systems

Figure XP2.6b illustrates the velocity components in the rotated $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system. There are several ways to develop the X and Y components of $\boldsymbol{v}$ and $\boldsymbol{a}$. We will first consider a development based directly on geometry and then use coordinate transformations.

The velocity vector magnitude is $v = (.785^2 + 1.99^2)^{1/2} = 2.1\ m/\text{sec}$, and $\boldsymbol{v}$ is oriented at the angle $21.53° = \tan^{-1}(.785/1.99)$ with respect to $\boldsymbol{\varepsilon}_\theta$. Since $\boldsymbol{\varepsilon}_\theta$ is directed 45° below the - X axis, $\boldsymbol{v}$ is at

$23.47^\circ = 45^o - 21.53^\circ$ below the - X axis, and

$$v_X = -2.1\cos 23.47^\circ = -1.96\, m/\sec$$
$$v_Y = -2.1\sin 23.47^\circ = -.852\, m/\sec \tag{ii}$$

Moving along to the acceleration components, figure XP 2.6c illustrates the rotated $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system with the a_r and a_θ components. Note that $a_r = -3.51\, m/\sec^2$ is directed in the $-\boldsymbol{\varepsilon}_r$ direction. The acceleration magnitude is $a = (8.23^2 + 3.51^2)^{1/2} = 8.95\, m/\sec^2$. The acceleration vector $\boldsymbol{a}$ is rotated $23.1^o = \tan^{-1}(3.51/8.23)$ counterclockwise from $\boldsymbol{\varepsilon}_\theta$; hence, $\boldsymbol{a}$ is directed at $23.1^\circ + 45^\circ = 68.1^\circ$ below the -X axis. Accordingly,

$$a_X = -8.95\cos 68.1^\circ = -3.34\, m/\sec^2$$
$$a_Y = -8.95\sin 68.1^\circ = -8.30\, m/\sec^2 \quad . \tag{iii}$$

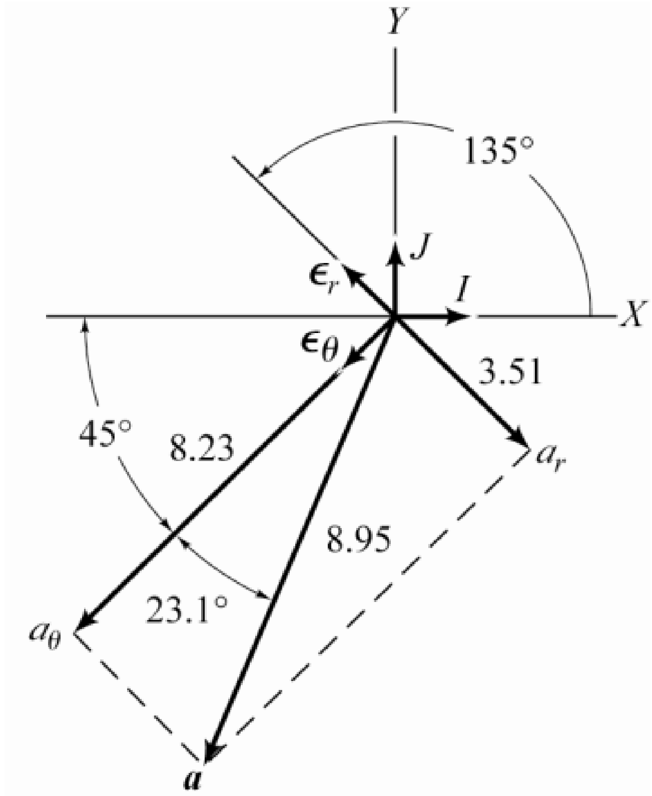


Figure XP2.6 (c)
Acceleration components
in the $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ system

These velocity and acceleration component results can also be obtained via a coordinate transformation. Figure XP 2.6d illustrates a vector ***B*** defined in terms of its components B_r, B_θ in the $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ coordinate system. Projecting these components into the *X*, *Y* system and adding the results along the *X* and *Y* axes gives

$$B_X = B_r \cos\theta - B_\theta \sin\theta \ , \quad B_Y = B_r \sin\theta + B_\theta \cos\theta \ .$$

In matrix notation, these results become

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} . \tag{iv}$$

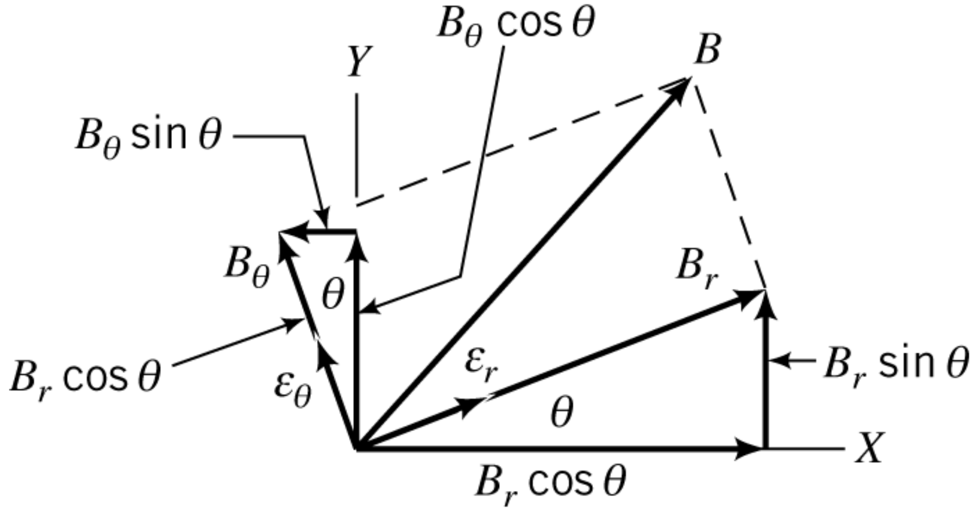


Figure XP 2.6d. Coordinate transformation development to move from the $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ coordinate frame to the X, Y frame.

Note that this result basically coincides with Eq.(2.19) obtained earlier in discussing coordinate transformations. The $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ coordinate system of figure XP 2.6d replaces the $X^/, Y^/$system of figure 2.4. Applying the transformation to the components of $\boldsymbol{v}$ and $\boldsymbol{a}$ gives

$$\begin{Bmatrix} v_X \\ v_Y \end{Bmatrix} = \begin{bmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{bmatrix} \begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} -.707 & -.707 \\ .707 & -.707 \end{bmatrix} \begin{Bmatrix} .785 \\ 1.99 \end{Bmatrix} \tag{v}$$

$$= \begin{Bmatrix} -1.96 \\ -.852 \end{Bmatrix} \frac{m}{\sec} ,$$

and

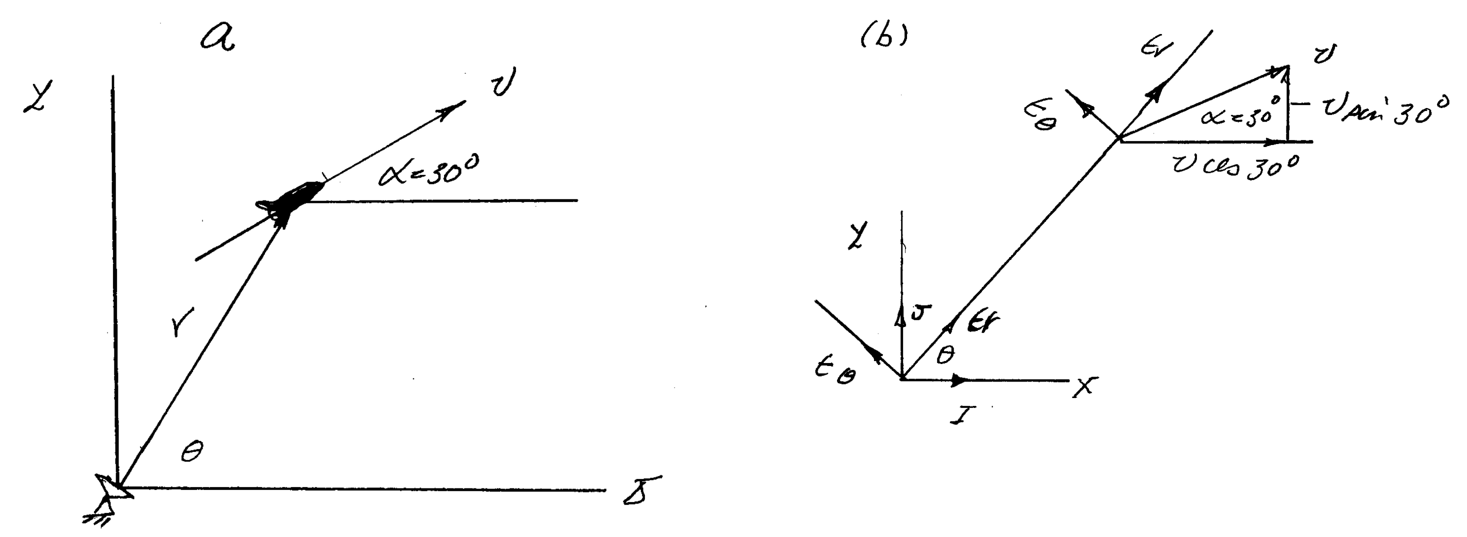
$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{bmatrix} \begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} -.707 & -.707 \\ .707 & -.707 \end{bmatrix} \begin{Bmatrix} -3.31 \\ 8.23 \end{Bmatrix} \qquad \textbf{(vi)}$$

$$= \begin{Bmatrix} -3.34 \\ -8.30 \end{Bmatrix} \frac{m}{\sec^2} \ .$$

Note that the same (numerical) coordinate transformation applies for converting $\boldsymbol{v}$ and $\boldsymbol{a}$.

Task a ⇒ direct substitution into Eqs.(2.31).

The coordinate transformation result of Eq.(iv) applies for any vector and any angle θ.



Example Problem

An airplane is flying at a constant speed v with a constant pitch angle $\alpha = 30^{\circ}$ with respect to the horizontal. It is being tracked by radar that shows a range of 9.24 km and the tracking angle and angular rate: $\theta = 60^{\circ}$, $\dot{\theta} = -0.0387\, rad/\sec$. Tasks:

a. Determine $\dot{r}$ and the plane's speed v.

b. Determine $\ddot{\theta}$, $\ddot{r}$.

Solution. From figure b, the components of $\boldsymbol{v}$ can be stated:

$$v_X = v \cos 30^{\circ} \ , \ v_Y = v \sin 30^{\circ} \ .$$

The coordinate transformation from the ***I***, ***J*** coordinate system to the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system,

$$\begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix},$$

can be used for the velocity components as

$$\begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{Bmatrix} \dot{r} \\ r\dot{\theta} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v\cos 30^\circ \\ v\sin 30^\circ \end{Bmatrix},$$

netting,

$$\dot{r} = v\,(\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta) = v\cos(\theta - 30^\circ)$$

$$r\dot{\theta} = v\,(-\cos 30^\circ \sin\theta + \sin 30^\circ \cos\theta) = -v\sin(\theta - 30^\circ)$$

Substituting for $r = 9.24\,km$, $\dot{\theta} = -0.0387\,rad/\sec$, and $\theta = 60^\circ$ gives:

$$\dot{r} = v\cos 30^\circ = .866\,v$$

$$r\dot{\theta} = -v\sin(30^\circ) = -0.5\,v \;\Rightarrow\; v = -2\times(-0.0387\,\frac{rad}{\sec})\times 9.24\,km$$

$$\therefore\; v = .715\,km/\sec \;\; and \;\; \dot{r} = .866\times .715 = .619\,km/\sec$$

Check, $v = \sqrt{v_r^2 + v_\theta^2}$?

$$v = [\dot{r}^2 + (r\dot{\theta})^2] = [0.619^2 + (9.24 \times -0.0387)^2]^{1/2}$$
$$= (.3844 + .1278)^{1/2} = .715\, km/\sec$$

The plane is flying at constant speed; hence, $\boldsymbol{a} = 0$, and

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \;\Rightarrow\; \ddot{r} = 9.24\, km(\,-0.0387\, rad/\sec)^2$$
$$= .0138\, km/\sec^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \;\Rightarrow\; \ddot{\theta} = -\frac{2 \times 0.62\, km/\sec \times -0.0387\, rad/\sec}{9.24\, km}$$
$$= 5.19E{-}03\, rad/\sec^2$$

Lecture 3. PARTICLE MOTION IN A PLANE, NORMAL-TANGENTIAL (PATH) COORDINATES

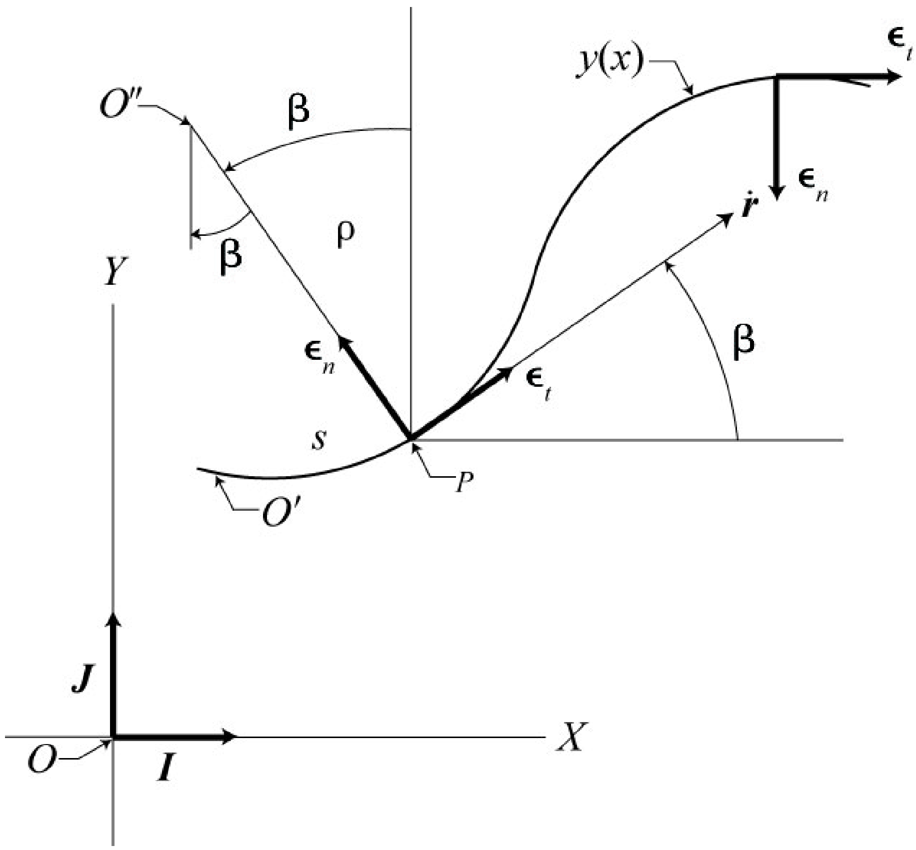


Figure 2.8 Path-coordinate unit vectors; $\tan\beta = dy/dx$

Unit Vectors:

$$\boldsymbol{\varepsilon}_t = \boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta$$

$$\boldsymbol{\varepsilon}_n = -\boldsymbol{I}\sin\beta + \boldsymbol{J}\cos\beta \ .$$

Radius of curvature of a path $y = f(x)$ is

$$\frac{1}{\rho} = \frac{|y''|}{[\,1 + (y')^2\,]^{3/2}} \quad ,$$

where $y' = df/dx = \tan\beta$, and $y'' = d^2f/dx^2$.

Velocity of point *P* with respect to the *X, Y* system

$$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt}\Big|_{X,Y} = \dot{s}\ \boldsymbol{\varepsilon_t} = \rho\dot{\beta}\ \boldsymbol{\varepsilon_t} = v\ \boldsymbol{\varepsilon_t}\ , \tag{2.32}$$

where *s* defines the distance traveled along the path from some arbitrary reference point *O*.

Note that

$$\dot{s} = v = \rho\ \dot{\beta}\ . \tag{2.33}$$

Acceleration of point *P* with respect to the *X, Y* system.
Taking the time derivative of $\dot{\boldsymbol{r}} = v\boldsymbol{\varepsilon_t}$ with respect to the *X, Y* coordinate system gives

$$\begin{aligned}\ddot{\boldsymbol{r}} = \frac{d\dot{\boldsymbol{r}}}{dt}\Big|_{X,Y} &= \dot{v}\ \boldsymbol{\varepsilon_t} + v\ \frac{d\boldsymbol{\varepsilon_t}}{dt}\Big|_{X,Y} \\ &= \dot{v}\ \boldsymbol{\varepsilon_t} + v\ \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{t}}\ .\end{aligned} \tag{2.34}$$

This result requires $\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{t}}$, the time derivative of $\boldsymbol{\varepsilon_t}$ with respect to the *X, Y* system. Differentiating $\boldsymbol{\varepsilon_t} = \boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta$ while holding ***I*** and ***J*** constant yields

$$\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{t}} = \frac{d\boldsymbol{\varepsilon}_{\boldsymbol{t}}}{dt}\Big|_{X,Y} = \dot{\beta}(-\,\boldsymbol{I}\sin\beta + \boldsymbol{J}\,\cos\beta) = \dot{\beta}\;\boldsymbol{\varepsilon}_{\boldsymbol{n}}\;.$$

Substituting this result into Eq.(2.34) then gives

$$\ddot{\boldsymbol{r}} = \dot{v}\;\boldsymbol{\varepsilon}_{\boldsymbol{t}} + v\;\dot{\beta}\;\boldsymbol{\varepsilon}_{\boldsymbol{n}}\;, \tag{2.35}$$

Substituting $v = \rho\,\dot{\beta}$ from Eq.(2.33) gives the following alternative expressions for $\ddot{\boldsymbol{r}}$:

$$\ddot{\boldsymbol{r}} = \dot{v}\;\boldsymbol{\varepsilon}_{\boldsymbol{t}} + \frac{v^2}{\rho}\;\boldsymbol{\varepsilon}_{\boldsymbol{n}}\;, \tag{2.36}$$

or

$$\ddot{\boldsymbol{r}} = \dot{v}\;\boldsymbol{\varepsilon}_{\boldsymbol{t}} + \rho\;\dot{\beta}^2\;\boldsymbol{\varepsilon}_{\boldsymbol{n}}\;. \tag{2.37}$$

Since

$$\ddot{\boldsymbol{r}} = a_t\;\boldsymbol{\varepsilon}_{\boldsymbol{t}} + a_n\;\boldsymbol{\varepsilon}_{\boldsymbol{n}}\;,$$

Eqs.(2.35) through (2.37) provide the following component definitions for a_t and a_n:

$$\boxed{a_t = \dot{v}\;,\; a_n = v\;\dot{\beta} = \frac{v^2}{\rho} = \rho\;\dot{\beta}^2\;.} \tag{2.38}$$

$a_n = v^2/\rho$ is the more generally useful expression.

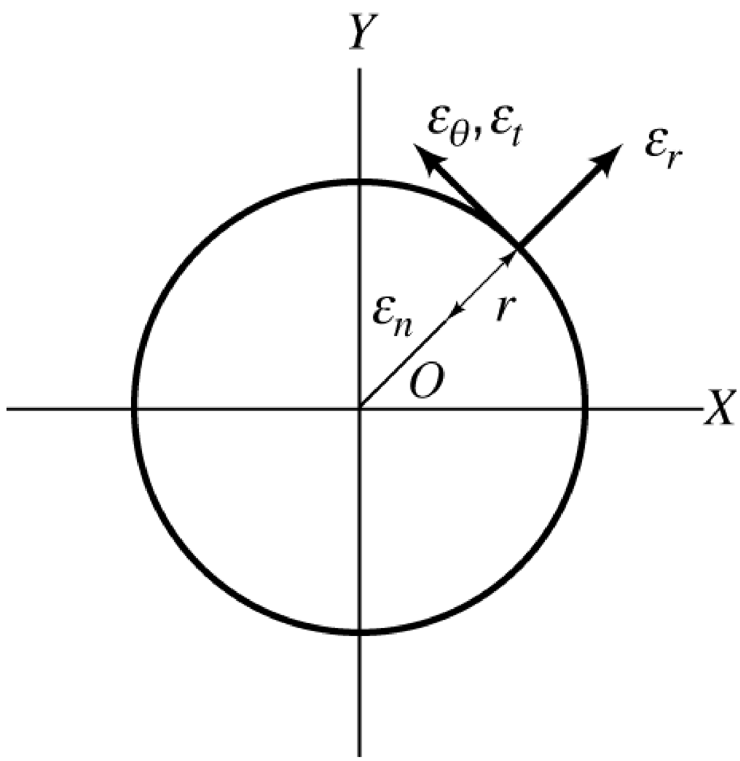


Figure 2.9 Constant-radius circular motion with $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ and $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ unit vectors.

Polar and path coordinate relationships. For $\dot{r}=0$, $\rho = r$, $\dot{\theta} = \dot{\beta}$, $\dot{r}=0$, and $\boldsymbol{\varepsilon_n}$ and $\boldsymbol{\varepsilon_r}$ are oppositely directed. Hence the polar coordinate model gives

$$v_r = 0, \; v_\theta = r\,\dot{\theta}$$
$$a_r = -\,r\,\dot{\theta}^2, \; a_\theta = r\ddot{\theta}\ .$$

For this reduced case, $r = \rho$, and

$$v = r\dot{\theta}$$

$$a_n = -a_r \Rightarrow r\dot{\theta}^2 = \frac{v^2}{r}$$

$$a_t = a_\theta \Rightarrow \dot{v} = \ddot{s} = r\ddot{\theta}$$

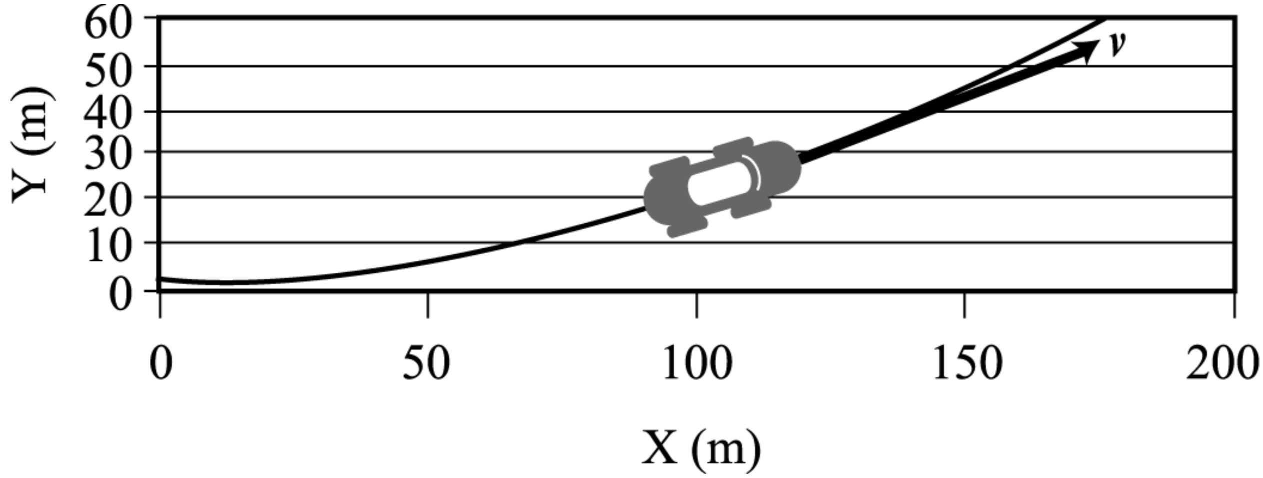


Figure XP2.7a Track segment in the horizontal *X*-*Y* system

Example Problem 2.7 As illustrated in figure XP 2.7a, a track lies in the horizontal plane and is defined by $Y = kX^2$ with X and Y in meters and $k = 1/400\ m^{-1}$. At $X = 100\,m$, the velocity and acceleration components *along the path* are $20\ m/\text{sec}$ and $-2\ m/\text{sec}^2$, respectively. The relevant engineering-analysis tasks are:

a. Determine the normal and tangential components of $\boldsymbol{v}$

and $\boldsymbol{a}$, and

b. Determine $\boldsymbol{v}$ and $\boldsymbol{a}$'s components in the *X, Y* system.

Solution. From the definitions of the $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ coordinate system, $\boldsymbol{v}$ and $\boldsymbol{\varepsilon_t}$ are colinear, and both are directed along the tangent of the path. Hence, $\boldsymbol{v} = \boldsymbol{\varepsilon_t}\, 20\ m/\sec$. The velocity vector $\boldsymbol{v}$ has no component along $\boldsymbol{\varepsilon_n}$. The problem statement gives $a_t = \dot{v} = -2\ m/\sec^2$. From Eq.(2.38), the normal component is $a_n = v^2/\rho$. We are given $v = 20\ m/\sec$; however, we need to define the radius of curvature.

For $Y = kX^2, Y^{/} = 2kX,\ Y^{//} = 2k$ and

$$\frac{1}{\rho} = \frac{|\, Y^{//} \,|}{[\ 1 + (\, Y^{/})^2\,]^{3/2}} \quad ,$$

gives

$$\frac{1}{\rho}\Big|_{X=100m} = \frac{2/400}{[1 + (2 \times 100/400)^2]^{3/2}} = 3.57 \times 10^{-3} m^{-1}$$

$$\therefore\ \rho = 280\ \ m\ \ ,$$

and

$$a_n = 20^2/280. = 1.43\ m/\sec^2\ .$$

The answer for *Task a* is stated

$$\boldsymbol{v} = 20\,\boldsymbol{\varepsilon_t}\,m/\sec\ ,\quad \boldsymbol{a} = -2\,\boldsymbol{\varepsilon_t} + 1.43\,\boldsymbol{\varepsilon_n}\,m/\sec^2\ .$$

Moving to *Task b* , the first question to answer in finding the components of $\boldsymbol{v}$ and $\boldsymbol{a}$ in the X, Y system is, "How are $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ oriented in the X, Y system?" Since $\boldsymbol{\varepsilon_t}$ is directed along the tangent of the path, we can find the orientation of $\boldsymbol{\varepsilon_t}$ with respect to the X axis via,

$$Y'\big|_{X=100m} = 2kX\big|_{X=100m} = 2\times\frac{1}{400}\times 100 = .5$$

$$\therefore\ \beta = \tan^{-1}(.5) = 26.6^\circ\ .$$

Figure XP 2.7b shows this orientation of the $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ coordinate system at $X = 100\,m$.

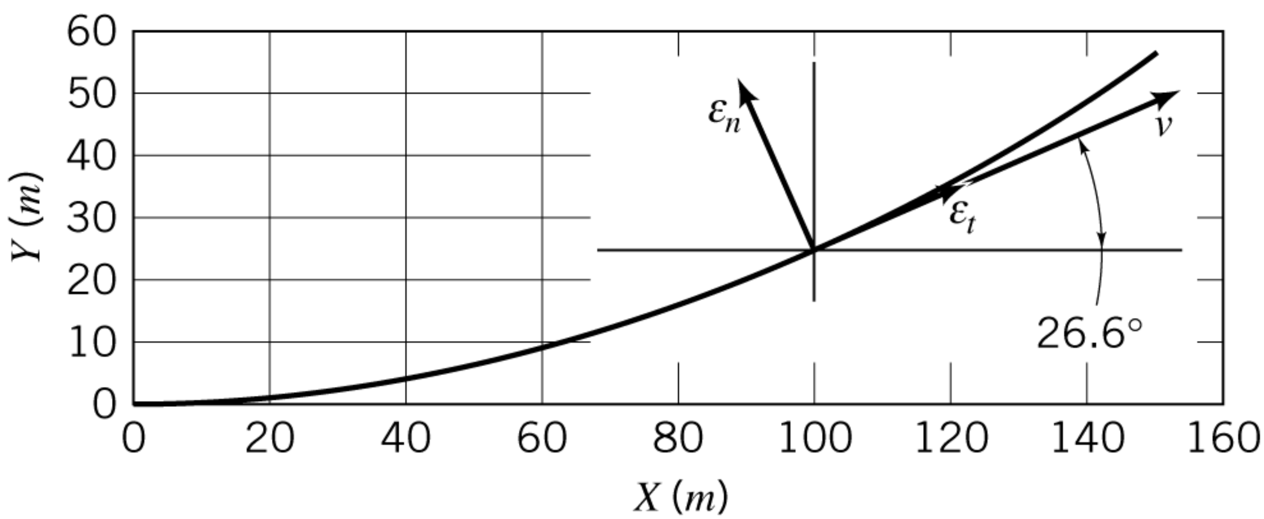


Figure XP2.7b $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ orientation at $X = 100\,m$

From this figure, $\boldsymbol{v}$'s X and Y components are:

$$v_X = v\cos 26.6^o = 20 \times .894 = 17.9\ m/\sec$$

$$v_Y = v\sin 26.6^o = 20 \times .447 = 8.95\ m/\sec\ .$$

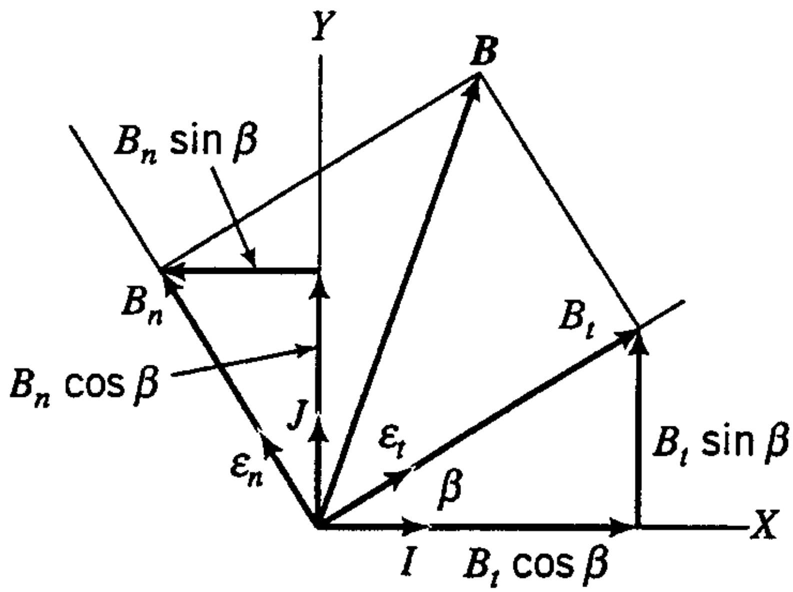


Figure 2.10 Coordinate transformation development to move from components in the $\boldsymbol{\varepsilon}_t$, $\boldsymbol{\varepsilon}_n$ coordinate system to the X, Y coordinate system.

To find $\boldsymbol{a}$'s components in the X, Y system, consider the components of the arbitrary vector $\boldsymbol{B}$ in Figure 2.10. This figure is very similar to figure XP2.7d (page 27, Lecture 2) that we developed to move from components in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ system to components in the X, Y system. Summing components in the X and Y directions gives

$$B_X = B_t\cos\beta - B_n\sin\beta\ ,\quad B_Y = B_t\sin\beta + B_n\cos\beta\ .$$

In matrix notation, these results become

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} B_t \\ B_n \end{Bmatrix} .$$

Substituting the acceleration components and $\beta = 26.6°$. gives

$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos 26.6^o & -\sin 26.6^o \\ \sin 26.6^o & \cos 26.6^o \end{bmatrix} \begin{Bmatrix} a_t \\ a_n \end{Bmatrix} = \begin{bmatrix} .894 & -.447 \\ .447 & .894 \end{bmatrix} \begin{Bmatrix} -2. \\ 1.43 \end{Bmatrix}$$

$$= \begin{Bmatrix} -2.43 \\ .384 \end{Bmatrix} \frac{m}{\sec^2} .$$

This step concludes Task b. Note that ***a***'s magnitude is unchanged by the transformation.

In reviewing the steps involved in working out this example, applying the definitions to find the components of ***v*** and ***a*** in the path coordinate system is relatively straightforward, except for a modest effort to determine the radius of curvature ρ. The essential first step in finding ***v*** and ***a***'s components in the *X, Y* system is in recognizing that $\boldsymbol{\varepsilon}_t$ lies along the path's tangent. Following this insight, projecting ***v***'s components into the *X, Y* system is simple, as is finding ***a***'s components via the coordinate transformation.

Lecture 4. MOVING BETWEEN CARTESIAN, POLAR, AND PATH COORDINATE DEFINITIONS FOR VELOCITY AND ACCELERATION COMPONENTS

An Example That is Naturally Analyzed with Cartesian Components

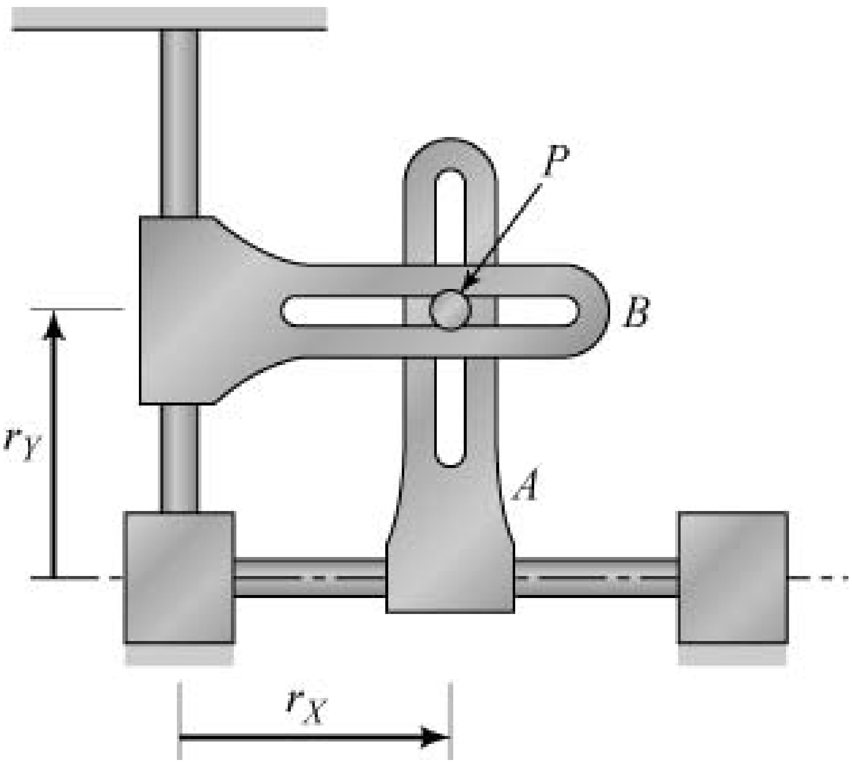


Figure 2.11 A lag-screw-driven mechanism.

Assume that control is applied to the screws such that,

$$r_X(t) = A + a\cos(\omega t)\ ,\quad r_Y(t) = B + b\sin(2\omega t)$$

$$A = B = 1\,mm,\ a = 1.25\ mm\ ,\ b = 2.5\ mm \tag{2.39}$$

$$\omega = 3.1416\, rad/\sec\ .$$

The engineering-analysis task associated with this system is: *At*

$\omega t = 30^o$, *determine the components of P's velocity and acceleration vectors in the* [(X, Y), (r, θ), *and* $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n)$] *coordinate systems.*

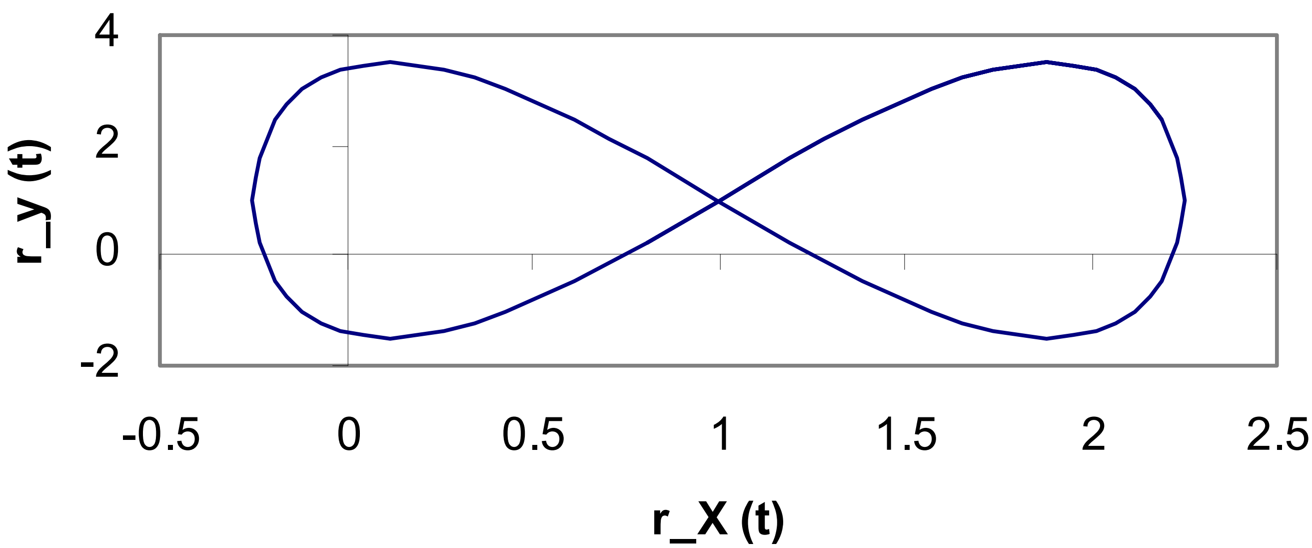


Figure 2.12 Lissajous[1] figure produced by the motion defined in Eq.(2.39); r_Y versus r_X.

Differentiating the components of the position vector nets:

$$v_X = \dot{r}_X = -a\omega\sin(\omega t) \ , \quad v_Y = \dot{r}_Y = 2b\omega\cos(2\omega t) \ . \qquad \text{(i)}$$

Differentiating again yields:

1

Named for Jules Antoine Lissajous, March 1822-June 1880. Primarily noted for accomplishments in acoustics.

$$a_X = \ddot{r}_X = -a\omega^2 \cos(\omega t)\ , \quad a_Y = \ddot{r}_Y = -4b\omega^2 \sin(2\omega t)\ . \qquad \text{(ii)}$$

Plugging in the numbers from Eq.(2.39) at $\omega t = 30^o$ nets:

$$\begin{aligned}
r_X &= 1 + 1.25 \times .866 = 2.08\ mm \\
r_Y &= 1 + 2.5 \times .866 = 3.165\ mm \\
v_X &= -1.25 \times 3.1416 \times .5 = -1.96\ mm/\sec \\
v_Y &= 2 \times 2.5 \times 3.1416 \times .5 = 7.85\ mm/\sec \\
a_X &= -1.25 \times (3.1416)^2 \times .866 = -10.68\ mm/\sec^2 \\
a_Y &= -4 \times 2.5(3.1416)^2 \times .866 = -85.47\ mm/\sec^2\ .
\end{aligned} \qquad \text{(iii)}$$

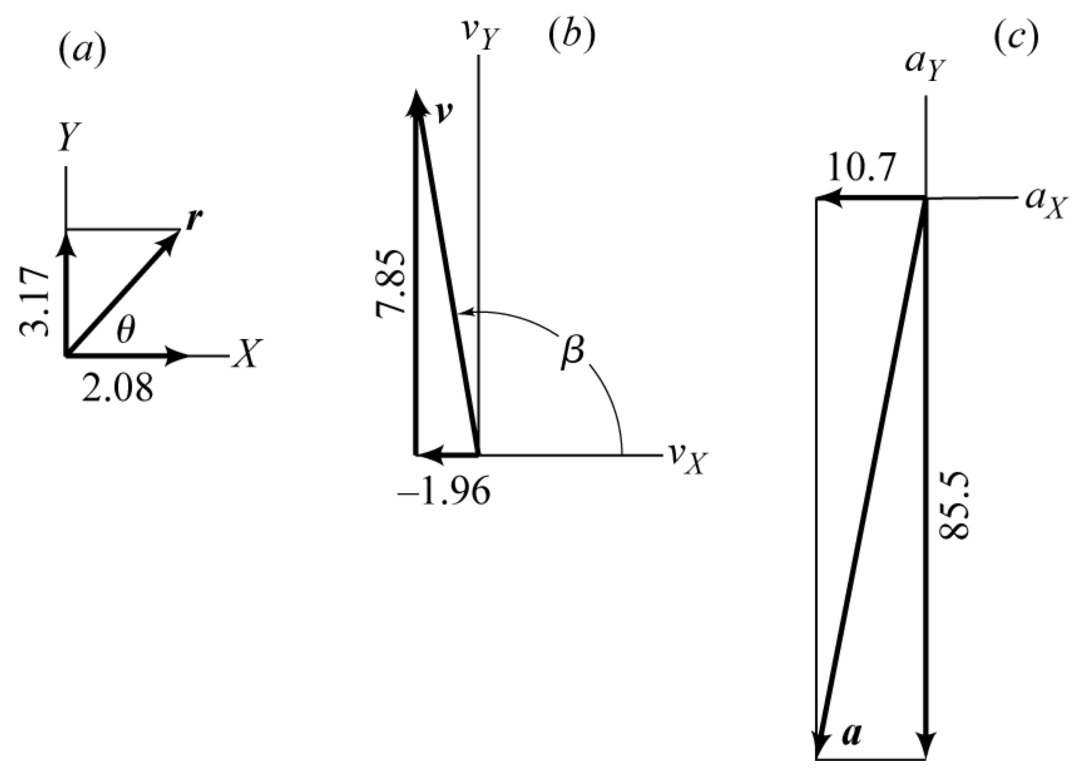


Figure 2.13 (a) Position, (b) Velocity, and (c) Acceleration vectors in the *X,Y* system at $\omega t = 30°$; mm-sec units

Given *X, Y* components, find *r*-θ components

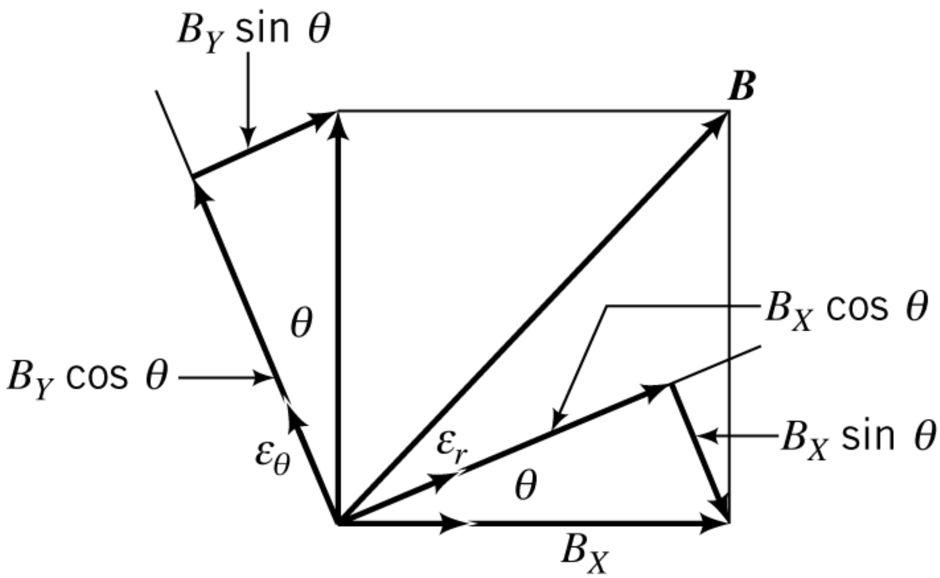


Figure 2.14 Components of the vector ***B*** in the *X, Y* system, projected along the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ unit vectors.

From figure 2.14

$$B_r = B_X \cos\theta + B_Y \sin\theta \ , \quad B_\theta = -B_X \sin\theta + B_Y \cos\theta \ ,$$

or

$$\begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} . \qquad \textbf{(2.41)}$$

From figure 2.13a and Eq.(iii),
$\theta = \tan^{-1}(r_Y/r_X) = \tan^{-1}(3.17/2.08) = 56.65^o$. Hence,

$$\begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} \cos 56.65^o & \sin 56.65^o \\ -\sin 56.65^o & \cos 56.65^o \end{bmatrix} \begin{Bmatrix} v_X \\ v_Y \end{Bmatrix}$$

$$= \begin{bmatrix} 0.550 & 0.835 \\ -0.835 & 0.550 \end{bmatrix} \begin{Bmatrix} -1.96 \\ 7.85 \end{Bmatrix} = \begin{Bmatrix} 5.48 \\ 5.95 \end{Bmatrix} mm/\text{sec}. \quad \textbf{(iv)}$$

Continuing,

$$\begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} 0.550 & 0.835 \\ -0.835 & 0.550 \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} 0.550 & 0.835 \\ -0.835 & 0.550 \end{bmatrix} \begin{Bmatrix} -10.68 \\ -85.47 \end{Bmatrix}$$

$$= \begin{Bmatrix} -77.2 \\ -38.2 \end{Bmatrix} mm/\text{sec}^2. \quad \textbf{(vi)}$$

The transformation has not changed either ***v*** or ***a***'s magnitude.

Given *X, Y* components, find *n-t* (path)components

Where are $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\varepsilon}_n$? Since $\boldsymbol{\varepsilon}_t$ is directed along the path of the trajectory and ***v*** must also be pointed along the particle's trajectory; hence, $\boldsymbol{\varepsilon}_t$ must be collinear with ***v***. From figure 2.12b, $\boldsymbol{\varepsilon}_t$ is pointed at the angle $\beta = \tan^{-1}(v_Y/v_X) = \tan^{-1}(7.85/-1.96) = 104.^o$ relative to the *X* axis. Also,

$$v_t = v = (v_X^2 + v_Y^2)^{1/2} = 8.090 \ mm/\sec \ ; \ v_n = 0 \ .$$

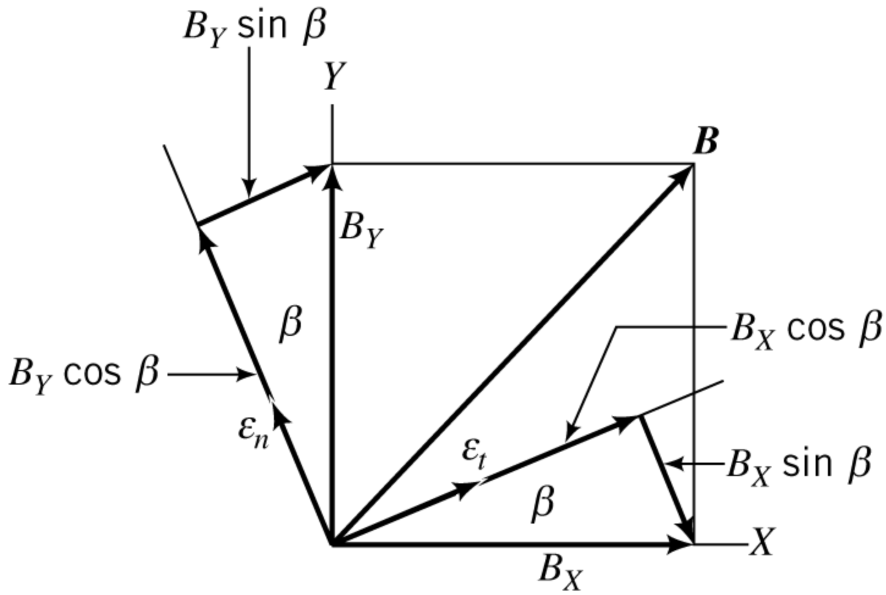


Figure 2.15 Components of the vector ***B*** in the *X, Y* system, projected along the $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ unit vectors.

Find: a_t, a_n. From figure 2.15:

$$B_t = B_X \cos\beta + B_Y \sin\beta \ , \quad B_n = -B_X \sin\beta + B_Y \cos\beta \ ,$$

or,

$$\begin{Bmatrix} B_t \\ B_n \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} . \qquad \textbf{(vii)}$$

Applying Eq.**(vii)** to find a_t, a_n gives

$$\begin{Bmatrix} a_t \\ a_n \end{Bmatrix} = \begin{bmatrix} \cos 104.^o & \sin 104.^o \\ -\sin 104.^o & \cos 104.^o \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix}$$

$$= \begin{bmatrix} -0.242 & 0.970 \\ -0.970 & -0.242 \end{bmatrix} \begin{Bmatrix} -10.68 \\ -85.47 \end{Bmatrix} \qquad \textbf{(viii)}$$

$$= \begin{Bmatrix} -80.3 \\ 31.0 \end{Bmatrix} \frac{mm}{\sec^2}.$$

The positive sign for a_n implies that (at this instant) the direction drawn for $\boldsymbol{\varepsilon_n}$ in figure 2.14 is correct. A negative sign would imply that $\boldsymbol{\varepsilon_n}$ (at this instant) is actually directed negatively from the direction shown in figure 2.14.

LESSON: We are looking at the *same* ***a*** and ***v*** vectors in three different coordinate systems. The component definitions change, but the vectors do not.

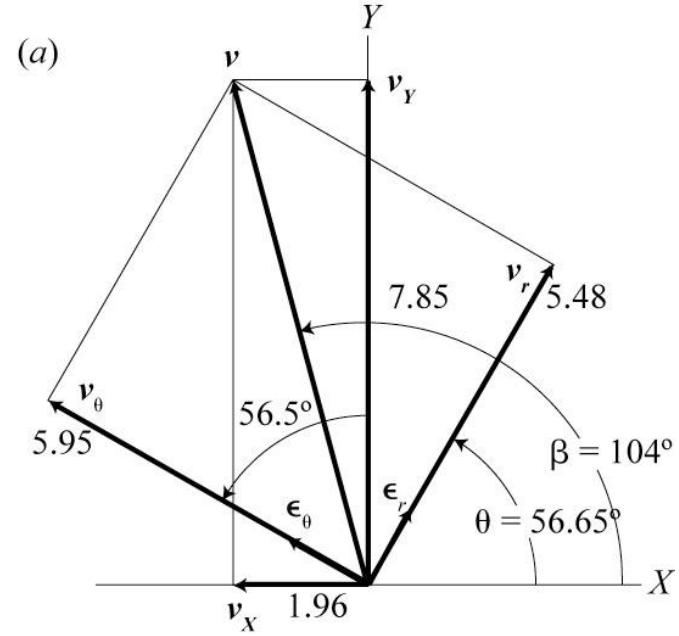


Figure 2.16a Components of ***v*** in the three coordinate systems; mm/sec

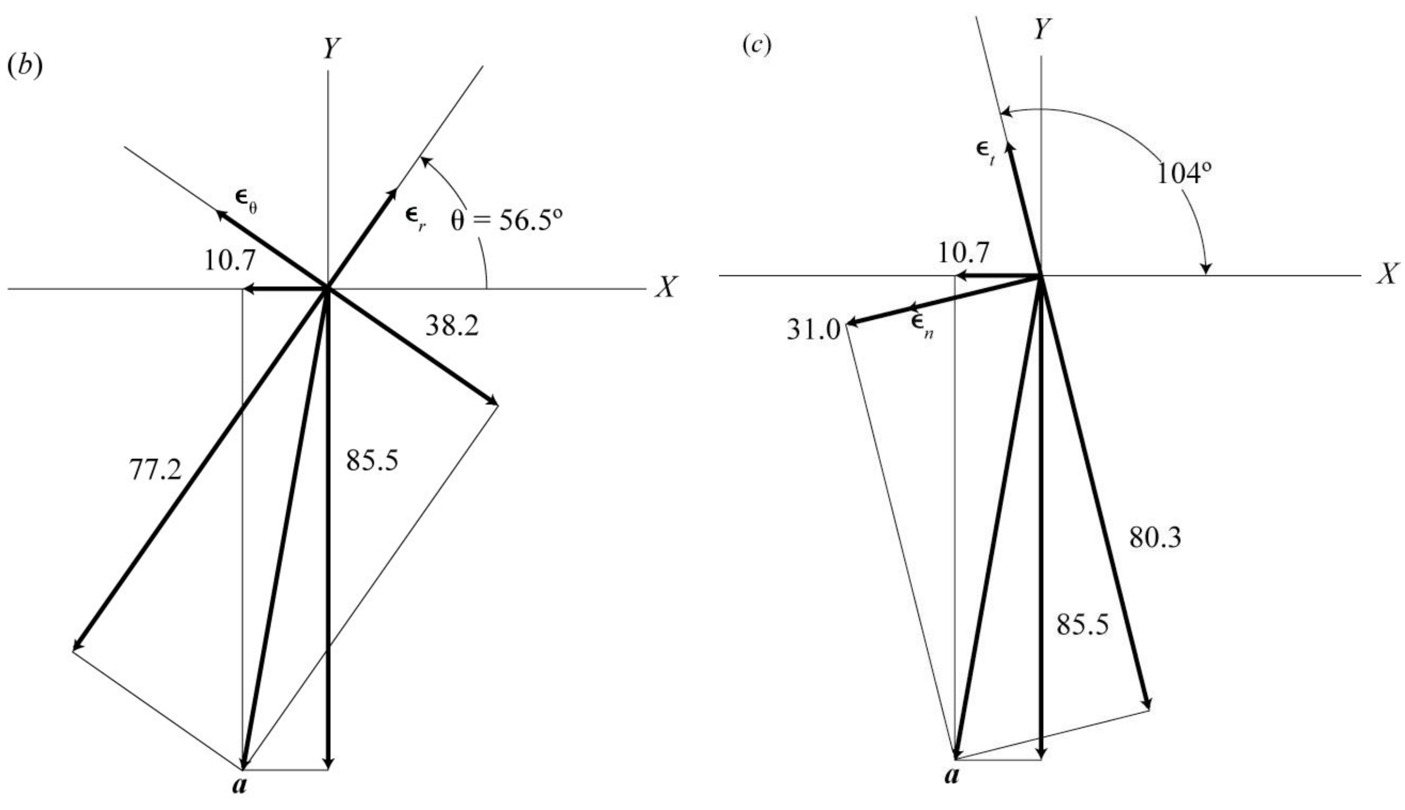


Figure 2.16 Acceleration components: (b) *X*-*Y* and *r*-θ systems, and (c) *X*-*Y* and path systems; mm/sec^2

2.7b An Example that is Naturally Analyzed Using Polar Coordinates

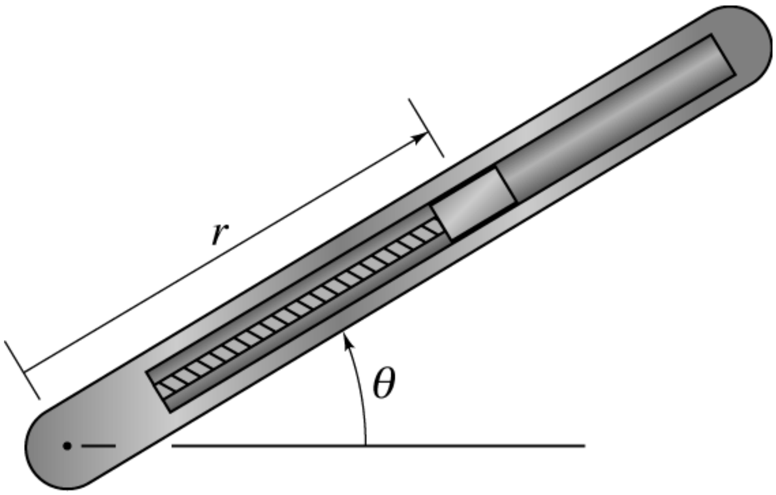


Figure 2.17 A rotating-bar/lag-screw mechanism.

Assuming that the motion is defined by

$$\theta = \frac{\pi}{4}\cos(\omega t) \ , \quad r = 10 + 5\cos(2\omega t)\ mm \ ,$$

where $\omega = 2\pi\ rad/\sec$, the engineering-analysis task is: *At* $\omega t = \pi/6 = 30^o$*, determine the components of the velocity and acceleration vector for point P, and state your results in the* [(*X, Y*), (*r*, θ), *and* $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n)$] *coordinate systems.*

First,

$$\dot{\theta} = -\omega \frac{\pi}{4} \sin(\omega t) \ \ rad/\sec, \quad \dot{r} = -10\,\omega \sin(2\,\omega t)\, mm/\sec$$

$$\ddot{\theta} = -\omega^2 \frac{\pi}{4} \cos(\omega t) \ \ rad/\sec^2, \quad \ddot{r} = -20\,\omega^2 \cos(2\,\omega t)\, mm/\sec^2 \ .$$

Hence, for $\omega t = 30^o$ and,

$$\theta = .680\, radians = 39.0^o \ , \quad r = 12.5\, mm$$
$$\dot{\theta} = -2.467\, rad/\sec \ , \quad \dot{r} = -54.41\, mm/\sec$$
$$\ddot{\theta} = -26.85\, rad/\sec^2 \ , \quad \ddot{r} = -394.8\, mm/\sec^2 \ .$$

From Eqs.(2.31),

$$v_\theta = r\dot{\theta} = 12.5 \times -2.467 = -30.83\, mm/\sec$$
$$v_r = \dot{r} = -54.41\, mm/\sec$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 12.5 \times -26.85 + 2 \times -54.41 \times -2.467$$
$$= -67.17\, mm/\sec^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = -394.8 - 12.5 \times (-2.467)^2 = -470.9\, mm/\sec^2 \ .$$

Cartesian Components. Since $[A]$ in Eq.(2.40) is orthogonal, $([A]^T = [A]^{-1})$,

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} . \qquad \textbf{(i)}$$

Hence,

$$\begin{Bmatrix} v_X \\ v_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} \cos 39.0^o & -\sin 39.0^o \\ \sin 39.0^o & \cos 39.0^o \end{bmatrix} \begin{Bmatrix} -54.41 \\ -30.83 \end{Bmatrix}$$

$$= \begin{Bmatrix} -22.88 \\ -58.20 \end{Bmatrix} \frac{mm}{\sec} . \qquad \textbf{(ii)}$$

Similarly,

$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} \cos 39.0^o & -\sin 39.0^o \\ \sin 39.0^o & \cos 39.0^o \end{bmatrix} \begin{Bmatrix} -470.9 \\ -67.17 \end{Bmatrix}$$

$$= \begin{Bmatrix} -323.7 \\ -348.5 \end{Bmatrix} \frac{mm}{\sec^2} . \qquad \textbf{(iii)}$$

Find Path Components: We can use the results of Eq.(ii) to define ***v***'s orientation in the *X, Y* system via

$\beta = \tan^{-1}(v_Y/v_X) = \tan^{-1}(-58.20/-22.8) = 248.6^o$. The components of $\boldsymbol{v}$ in the path system are:

$$v_t = (58.2^2 + 22.88^2)^{1/2} = 62.54\, mm/\sec \;,\; v_n = 0$$

Using the direct transformation result from Eq.(vii),

$$\begin{Bmatrix} B_t \\ B_n \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix}, \qquad \textbf{(vii)}$$

$$\begin{Bmatrix} a_t \\ a_n \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix}$$

$$= \begin{bmatrix} \cos 248.6^o & \sin 248.6^o \\ -\sin 248.6^o & \cos 248.6^o \end{bmatrix} \begin{Bmatrix} -323.7 \\ -348.5 \end{Bmatrix} \qquad \textbf{(iv)}$$

$$= \begin{Bmatrix} 442.6 \\ -174.2 \end{Bmatrix} \frac{mm}{\sec^2} \;.$$

Note in comparing figures 2.15 and 2.18a with 2.18b that $\boldsymbol{\varepsilon}_{\boldsymbol{n}}$ has reversed directions in accordance with the negative sign for a_n in Eq.(iv).

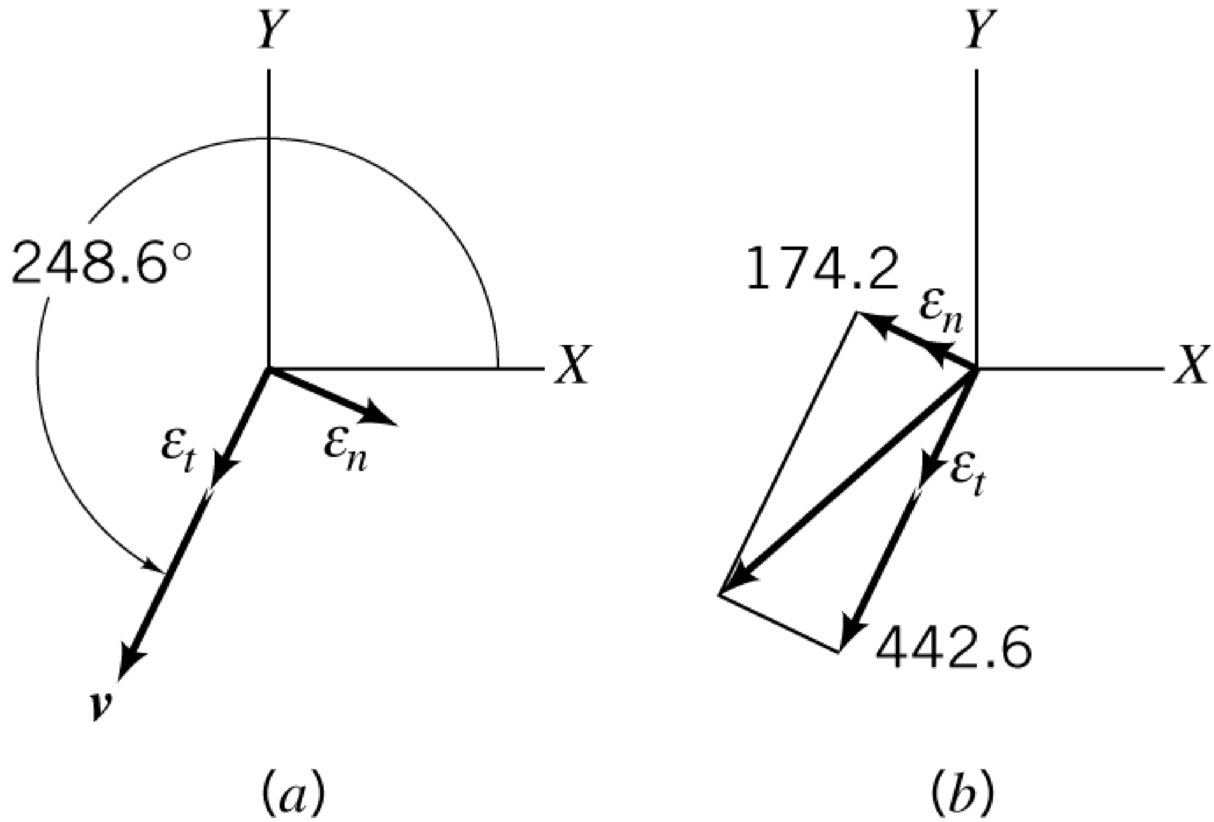


Figure 2.18 (a). Velocity ***v*** from Eq. (ii). (b). Acceleration ***a*** from Eq.(iv).

2.7c An Example That is Naturally Analyzed with Path-Coordinate Components

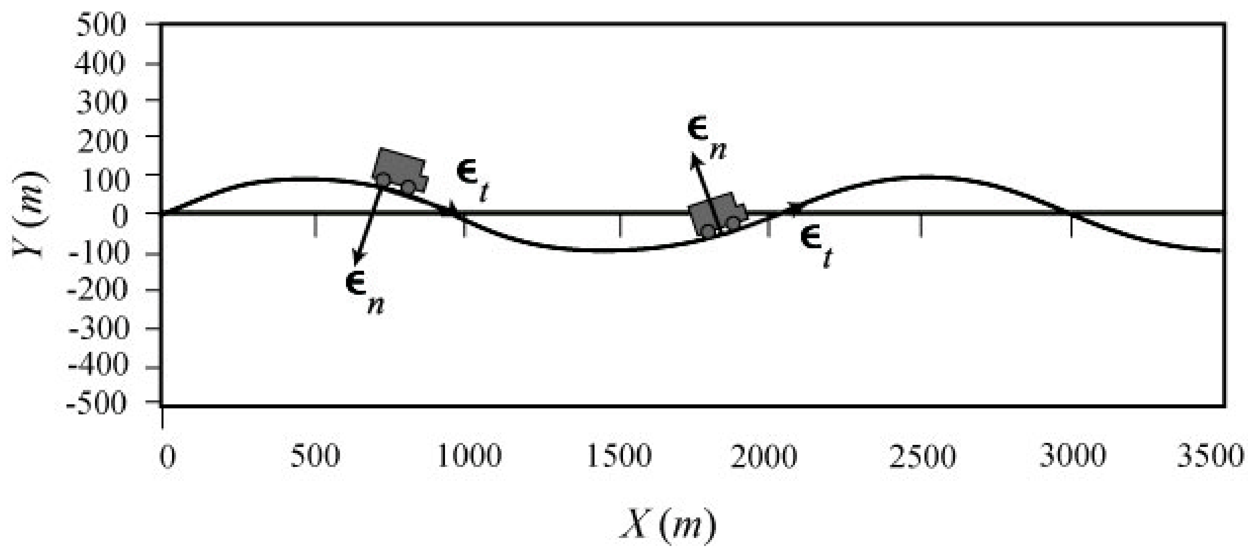


Figure 2.19 Vehicle following the curved path $Y = A\sin(2\pi X/L)$, $A = 100\,m$, $L = 2000\,m$

Figure 2.19 illustrates a particle traveling along a path in a vertical plane defined by,

$$y = A\sin\left(\frac{2\pi x}{L}\right)\ ,\ A = 100\,m\ ,\ L = 2000\,m\ . \qquad \text{(i)}$$

At $x = 750\,m$, the velocity and acceleration of the vehicle along the path are $v = 100\,km/hr$, and $a_t = 2\,m/\text{sec}^2$. The engineering-analysis task associated with this figure is: *Determine the components of the velocity and acceleration vectors at this position and state the components in the* [(X, Y), (r, θ), *and* $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n)$] *coordinate systems.*

To find the velocity direction in the X, Y system at $x = 750\,m$, we differentiate Eq.(i) with respect to x, obtaining

$$y' = \frac{dy}{dx} = \frac{2A\pi}{L}\cos(\frac{2\pi x}{L}) = \frac{2\times 100\pi}{2000}\times\cos(\frac{2\pi 750}{2000}) = -.222$$

$$\beta = \tan^{-1}(-.222) = -12.52^{o}\ .$$

Hence, for the position of interest, $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{v}$ are pointed at 12.52° below the horizontal, and the velocity components in the X, Y system are:

$$v_X = v\cos\beta = 27.78\,m/\sec\times\cos(-12.52^{o}) = 27.12\,m/\sec$$

$$v_Y = v\sin\beta = 27.78\,m/\sec\times\sin(-12.52^{o}) = -6.022\,m/\sec\ ,$$

where
$$v = (100\,km/hr)\times(1000\,m/km)\times(1\,hr/3600\sec) = 27.78\,m/\sec.$$

To find $a_n = v^2/\rho$, we need ρ. We have obtained $y' = dy/dx$ above but still need $y'' = d^2y/dx^2$, defined by

$$y'' = -A(\frac{2\pi}{L})^2\sin(\frac{2\pi x}{L}) = -100(\frac{2\pi}{2000})^2\times\sin(\frac{2\pi\,750}{2000})$$

$$= -6.98\times 10^{-4}\ m^{-1.}$$

Hence,

$$\frac{1}{\rho} = \frac{|y^{//}|}{[1+(y^{/})^2]^{3/2}} = \frac{6.98\times 10^{-4} m^{-1}}{[1+(-.222)^2]^{3/2}} = 6.49\times 10^{-4} m-1$$

$$\rho = 1540\, m \ ,$$

and

$$a_n = 27.78^2\,(\frac{m}{\sec})^2 / 1540\, m = .501\, m/\sec^2 \ .$$

Figure 2.20a and 2.20b illustrate, respectively, the components of ***v*** and ***a*** in terms of path coordinates. Summing components in the *X* and *Y* directions gives:

$$a_X = a_t \cos 12.5^o - a_n \sin 12.5^o = 2\times .976 - .501\times .216$$
$$= 1.84\, m/\sec^2$$

$$a_Y = -a_t \sin 12.5^\circ - a_n \cos 12.5^\circ = -2\times .216 - .501\times .976$$
$$= -.921\, m/\sec^2 \ ,$$

and concludes the Cartesian coordinate definition requirements.

Polar-coordinate definitions. We need to first find $r_Y = y(x=750\,m) = 100\,m \times \sin(2\times 750\pi/2000) = 70.7\,m$ to define θ as

$$\theta = \tan^{-1}(r_Y/r_X) = \tan^{-1}(70.7/750) = 5.386^o \ .$$

Applying the coordinate transformation of Eq.(2.40) to the Cartesian velocity and acceleration components gives:

$$\begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} \cos 5.39^o & \sin 5.39^o \\ -\sin 5.39^o & \cos 5.39^o \end{bmatrix} \begin{Bmatrix} v_X \\ v_Y \end{Bmatrix}$$

$$= \begin{bmatrix} 0.996 & 0.0939 \\ -0.0939 & 0.996 \end{bmatrix} \begin{Bmatrix} 27.12 \\ -6.02 \end{Bmatrix} = \begin{Bmatrix} 26.44 \\ -8.55 \end{Bmatrix} m/\sec \ , \qquad \textbf{(v)}$$

and

$$\begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} \cos 5.39^o & \sin 5.39^o \\ -\sin 5.39^o & \cos 5.39^o \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix}$$

$$= \begin{bmatrix} 0.996 & 0.0939 \\ -0.0939 & 0.996 \end{bmatrix} \begin{Bmatrix} 1.84 \\ -.921 \end{Bmatrix} = \begin{Bmatrix} 1.746 \\ -1.09 \end{Bmatrix} m/\sec^2 \ . \qquad \textbf{(vi)}$$

These results conclude the engineering-analysis tasks.

The $X = 1750\,m$ location results start by finding β from

$$Y' = \frac{dY}{dX} = \frac{2A\pi}{L}\cos(\frac{2\pi X}{l}) = \frac{2\times 100\pi}{2000}\times\cos(\frac{2\pi 1750}{2000}) = .222$$

$$\therefore\ \beta = \tan^{-1}(.222) = 12.52°\ .$$

$\boldsymbol{v}$ and $\boldsymbol{\varepsilon_t}$ are pointed $12.52°$ above the horizontal; $\boldsymbol{\varepsilon_n}$ has flipped directions by $180°$; v, a_t, a_n, ρ are unchanged. Hence:

$$v_X = v\cos\beta = 27.78\,m/\sec\times\cos(12.52°) = 27.12\,m/\sec$$

$$v_Y = v\sin\beta = 27.78\,m/\sec\times\sin(12.52°) = 6.02\,m/\sec\ ,$$

The orientation of $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ is the same in figure 2.19 at $X = 1750\,m$ as in figure 2.10; hence, Eq.(2.40) applies and can be used to obtain $\boldsymbol{a}$'s components as

$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} a_t \\ a_n \end{Bmatrix}$$
$$= \begin{bmatrix} \cos 12.52° & -\sin 12.52° \\ \sin 12.52° & \cos 12.52° \end{bmatrix} \begin{Bmatrix} 2 \\ .501 \end{Bmatrix} = \begin{Bmatrix} 1.844 \\ .9226 \end{Bmatrix} \frac{m}{\sec^2}$$

This concludes the Cartesian-coordinate results. Obtaining the polar-coordinate components from the Cartesian components starts with,

$$\theta = \tan^{-1}(r_Y/r_X) = \tan^{-1}(70.7/1750) = 2.313° \ ,$$

and then follows the same steps used earlier.

Lesson (again): The **same** vectors ***v*** and ***a*** have different components in the three different coordinate systems.

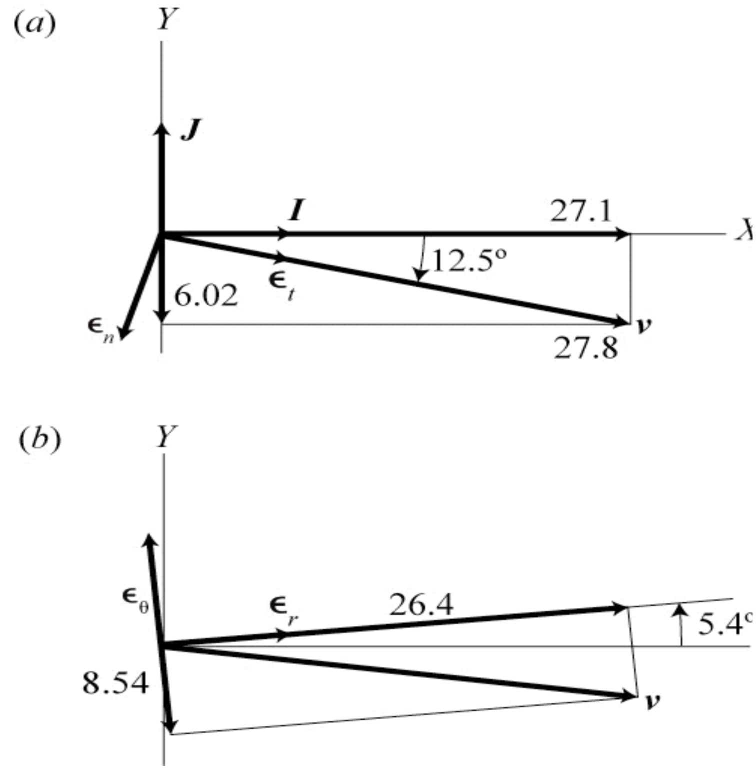


Figure 2.20 Velocity components (m/sec) at $X = 750\,m$, (a) Path and Cartesian components, (b) Pola components

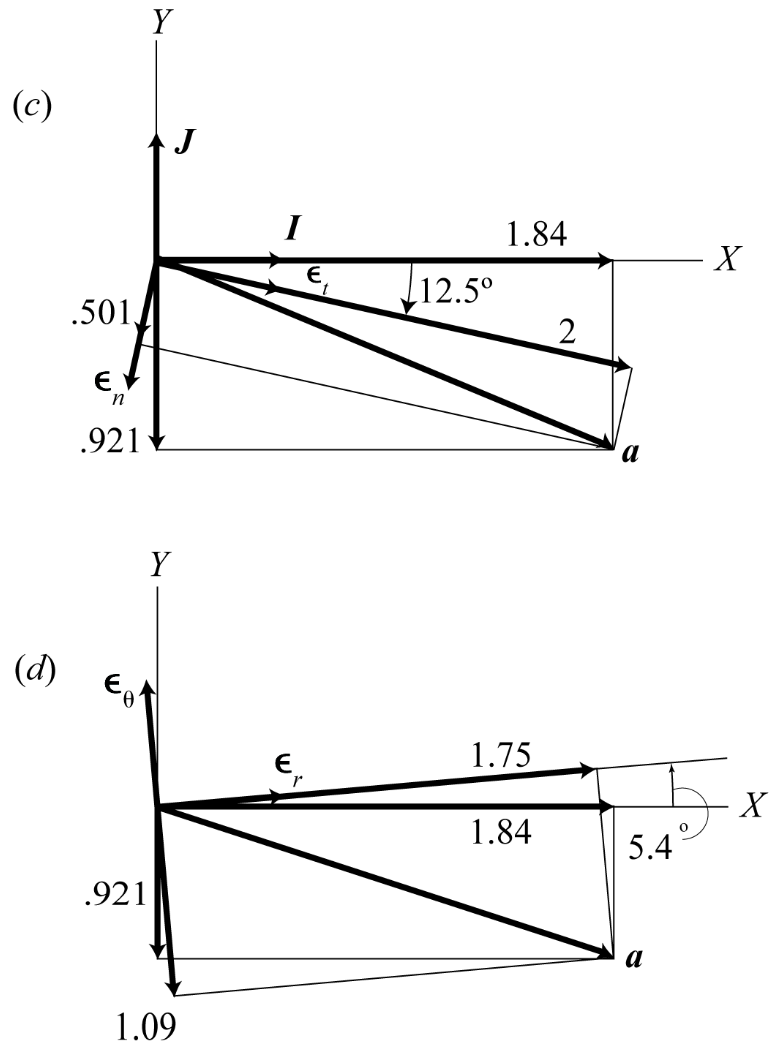


Figure 2.20 Acceleration components (m/sec^2) for $X = 750\,m$, (c) Path and Cartesian components, and (d) Polar and Cartesian components