MEEN 363 Notes
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Lecture 1. PARTICLE KINEMATICS IN A PLANE

Kinematics: Geometric in nature, defines motion without regard to forces that cause motion or result from motion.

Kinetics: Defines the motion of particles or rigid bodies that are caused by forces. Generally based on Newton's second law of motion $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$.

Motion in a Straight Line

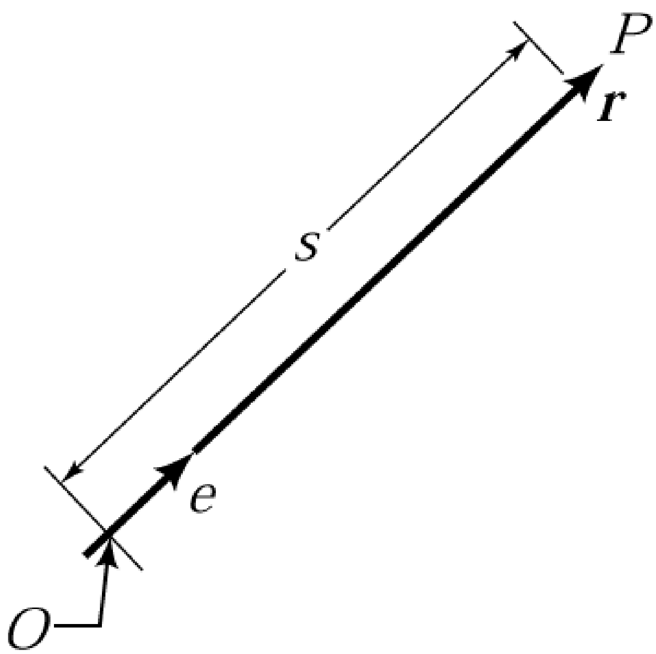


Figure 2.1 Displacement vector along the unit vector $\boldsymbol{e}$.

Position: $\boldsymbol{r} = \boldsymbol{e}s(t)$

Velocity: $\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{e}\dot{s} = \boldsymbol{e}v$

Acceleration: $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \boldsymbol{e}\ddot{s} = \boldsymbol{e}\dot{v} = \boldsymbol{e}a$

Particle Motion in a Plane: Cartesian Coordinates

Figure 2.3 illustrates the position of a particle P in a Cartesian X, Y coordinate system. The position vector locating P is

$$\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y , \qquad \textbf{(2.8)}$$

with $\boldsymbol{I}$ and $\boldsymbol{J}$ being vectors of unit magnitudes pointed along the orthogonal X and Y axes. r_X and r_Y are *components* of the position *vector* $\boldsymbol{r}$ in the X, Y coordinate system.

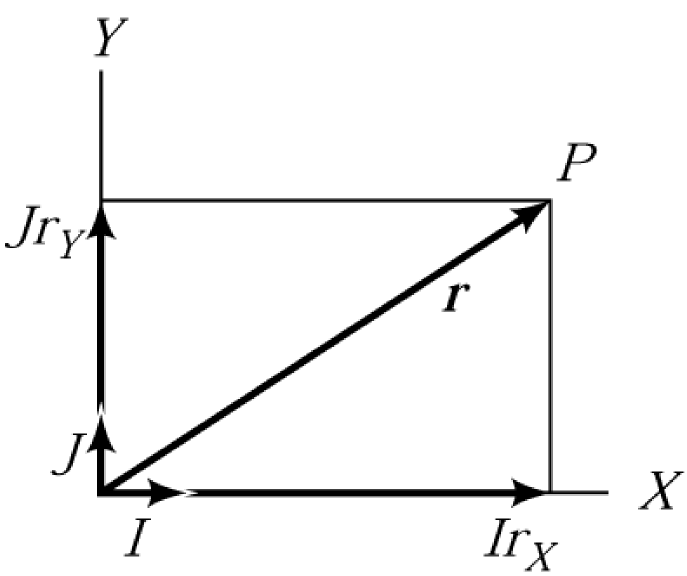


Figure 2.3 Particle P located in the X, Y system by the vector $\boldsymbol{r}$.

The velocity of point P <u>with respect</u> to the X, Y coordinate system is

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}\Big|_{X,Y} = \dot{\boldsymbol{r}} = \boldsymbol{I}\dot{r}_X + \boldsymbol{J}\dot{r}_Y = \boldsymbol{I}v_X + \boldsymbol{J}v_Y \,. \qquad \textbf{(2.9)}$$

The velocity vector's magnitude is

$$|v| = (v_X^2 + v_Y^2)^{1/2} \,. \qquad \textbf{(2.10)}$$

Q: How do you find the derivative of a vector ***B*** with respect to a coordinate system?"

A: The time derivative of any vector ***B*** with respect to a specified coordinate system is found by writing ***B*** out in terms of its components in the specified coordinate system,

$$\boldsymbol{B} = \boldsymbol{I}\, B_X + \boldsymbol{J}\, B_Y \,,$$

and then differentiating with respect to time *while holding the unit vectors constant.*

$$\dot{\boldsymbol{B}} = \frac{d\boldsymbol{B}}{dt}\Bigg|_{X,Y} = \boldsymbol{I}\, \dot{B}_X + \boldsymbol{J}\, \dot{B}_Y \,.$$

These are exactly the steps that are performed for ***r*** in Eqs. (2.8) and (2.9) to obtain $\boldsymbol{v} = d\boldsymbol{r}/dt|_{X,Y,Z}$.

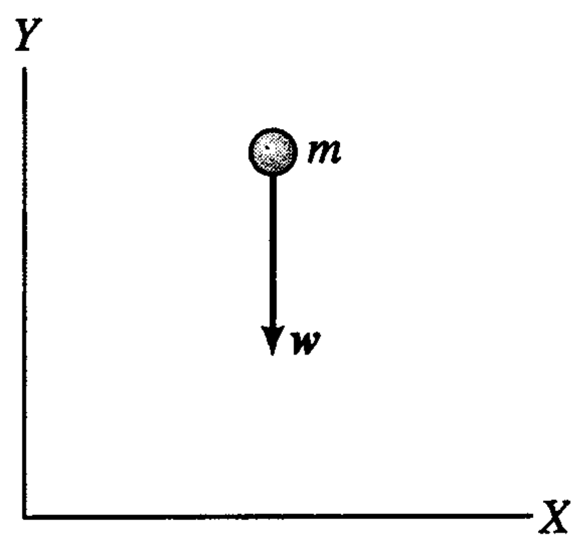
The acceleration of point *P* with respect to the *X, Y* system is

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}\Big|_{X,Y} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \boldsymbol{I}\dot{v}_X + \boldsymbol{J}\dot{v}_Y = \boldsymbol{I}a_X + \boldsymbol{J}a_Y \,. \quad \textbf{(2.11)}$$

The magnitude of the acceleration vector is

$$|a| = (a_X^2 + a_Y^2)^{1/2} \,.$$

Drag-Free Motion of a Particle in a Plane

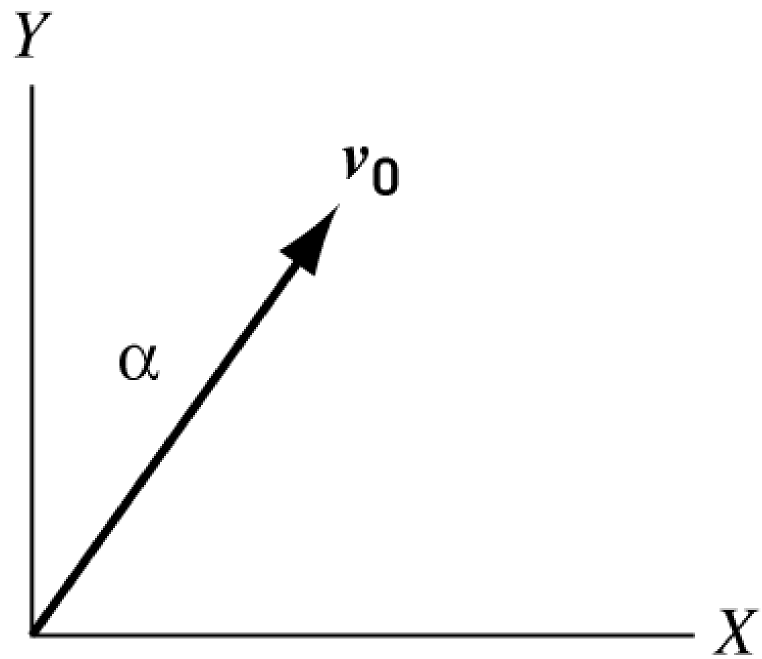


Free-body diagram for a particle that is falling, neglecting drag forces.

The figure below provides a free-body diagram for a particle acted on by gravity, neglecting aerodynamic drag. Applying Newton's second law of motion yields the following differential equations of motion:

$$\Sigma f_X = 0 = m\ddot{r}_X \;\Rightarrow\; \ddot{r}_X = 0$$

$$\Sigma f_Y = -w = m\ddot{r}_Y \;\Rightarrow\; \ddot{r}_Y = -\frac{w}{m} = -g$$



Example Problem 2.4 The acceleration components of a particle that is falling under the influence of gravity (neglecting aerodynamic drag forces) are:

$$a_X = \ddot{r}_X = 0 \;,\quad a_Y = \ddot{r}_Y = -g \;. \tag{i}$$

This definition of the acceleration components has the particle accelerating straight down in the -*Y* direction. *For the initial conditions illustrated above, namely:*
$v_X(0) = v_0 \sin\alpha$, $v_Y(0) = v_0 \cos\alpha$ *and* $r_X(0) = r_Y(0) = 0$, *find the components of the position and velocity vectors.*

Solution: Integrating Eqs.(i) once with respect to time yields:

$$v_X = \dot{r}_X = v_X(0) = v_0 \sin\alpha$$

$$v_Y = \dot{r}_Y = v_Y(0) - gt = v_0 \cos\alpha - gt \ .$$

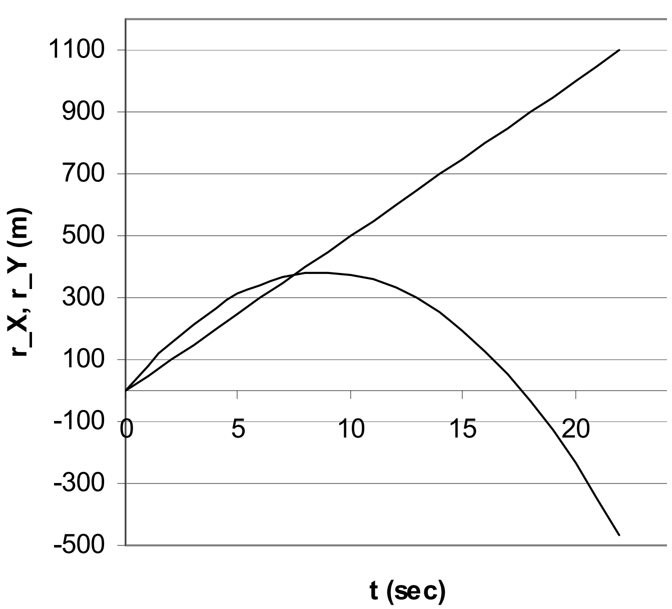
A second integration gives:

$$r_X = r_Y(0) + t\,v_0 \sin\alpha = t\,v_0 \sin\alpha$$

$$r_Y = r_Y(0) + t\,v_0 \cos\alpha - \frac{gt^2}{2} = t\,v_0 \cos\alpha - \frac{gt^2}{2} \ .$$

c

b



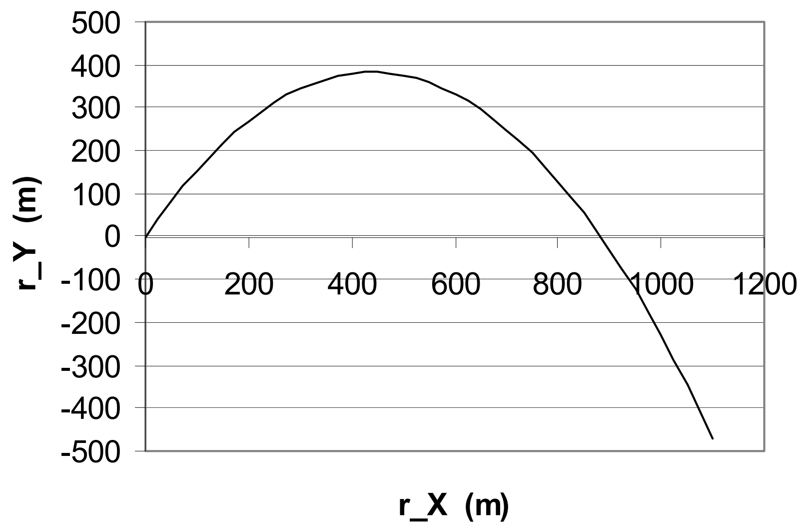


Figure XP 2.4 (b). r_X and r_Y versus time. (c). r_Y versus r_X for $v_0 = 100\,m/\sec$, $\alpha = 30°$, and $g = 9.81\ m/\sec^2$.

Matrix Algebra

$$[A] = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & & \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

m rows n columns

a_{ij} = entry in *ith* row and *jth* column

Matrix Multiplication, $[\boldsymbol{a}][\boldsymbol{b}]=[\boldsymbol{c}]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = [c] =$$

$$3 \times 3 \qquad\qquad 3 \times 2$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

$$3 \times 2$$

In general: $[\boldsymbol{A}][\boldsymbol{B}] \neq [\boldsymbol{B}][\boldsymbol{A}]$

Matrix inverse

$$[A]^{-1}\,[A] = [I] = \begin{bmatrix} 1 & 0 & . & . & . \\ 0 & 1 & & & \\ 0 & & & & \\ . & & & & \\ . & & & & 1 \end{bmatrix}$$

$\mathbf{[A]^{-1}}$ exists if the determinant of $[A]$ is nonzero; i.e., $\mathbf{|A| \neq 0}$

Matrix transpose

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \; \boldsymbol{a_{ij} = a_{ji}}$$

Orthogonal Matrix: $[A]^T = [A]^{-1}$.

Cramer's Rule for solving 2 simultaneous equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

Solve for the determinant D

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Solve for x_1

$$Dx_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - b_2a_{12}$$

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

Solve for x_2

$$Dx_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = b_2 a_{11} - b_1 a_{21}$$

$$x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

Coordinate Transformations: Relationships Between Components of a Vector in Two Coordinate Systems

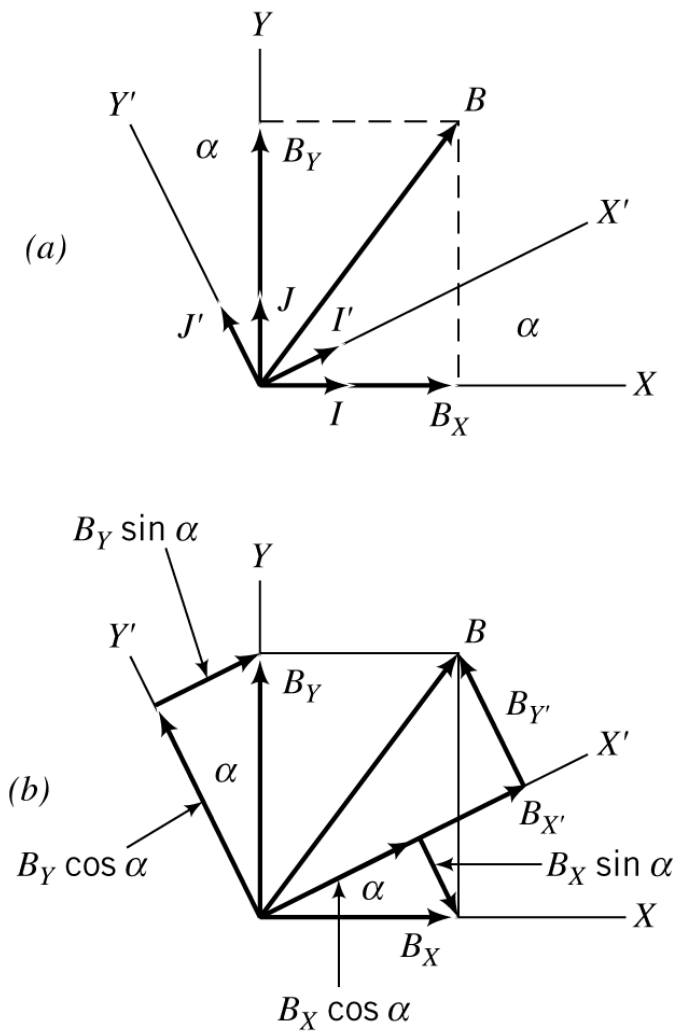


Figure 2.4 Vector *B* in two coordinate systems. (a). Vector ***B*** in terms of its X, Y components. (b). Two coordinate definitions.

Figure 2.4a illustrates a vector $\boldsymbol{B}$ in the "original" *X, Y* coordinate system and a "new" X', Y' coordinate system. The orientation of the *X, Y* and X', Y' coordinate systems is defined by the *constant* angle α.

Given: B_X, B_Y and α , **Find:** $B_{X'}, B_{Y'}$

From Figure 2.4b

$$\boldsymbol{I}B_X = \boldsymbol{I}'B_X\cos\alpha - \boldsymbol{J}'B_X\sin\alpha$$
$$\boldsymbol{J}B_Y = \boldsymbol{I}'B_Y\sin\alpha + \boldsymbol{J}'B_Y\cos\alpha \ . \qquad \textbf{(2.12)}$$

Since, $\boldsymbol{B} = \boldsymbol{I}B_X + \boldsymbol{J}B_Y$,

$$\boldsymbol{B} = \boldsymbol{I}'(B_X\cos\alpha + B_Y\sin\alpha) + \boldsymbol{J}'(-B_X\sin\alpha + B_Y\cos\alpha)$$
$$= \boldsymbol{I}'B_{X'} + \boldsymbol{J}'B_{Y'} \ . \qquad \textbf{(2.13)}$$

Equating coefficients of $\boldsymbol{I}'$, and $\boldsymbol{J}'$ in Eqs. (2.13) yields:

$$B_{X'} = B_X\cos\alpha + B_Y\sin\alpha$$
$$B_{Y'} = -B_X\sin\alpha + B_Y\cos\alpha \ . \qquad \textbf{(2.14)}$$

In matrix format,

$$\begin{Bmatrix} B_{X'} \\ B_{Y'} \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} . \qquad \textbf{(2.15)}$$

or, symbolically,

$$(B)_{I'} = [A](B)_{\boldsymbol{I}} \; . \qquad \textbf{(2.16)}$$

$[A]$ is called the “direction-cosine matrix,” and can be formally defined as

$$[A] = \begin{bmatrix} \cos(X', X) & \cos(X', Y) \\ \cos(Y', X) & \cos(Y', Y) \end{bmatrix} , \qquad \textbf{(2.17)}$$

where cos (X',X) is the cosine of the angle between X and X', cos (X',Y) is the cosine of the angle between X' and Y, etc. Returning to figure 2.4B,

$$\begin{aligned} \cos(X', X) &= \cos(Y', Y) = \cos\alpha \\ \cos(X', Y) &= \cos(\frac{\pi}{2} - \alpha) = \cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha \\ &= \sin\alpha \qquad \textbf{(2.18)} \\ \cos(Y', X) &= \cos(\frac{\pi}{2} + \alpha) = \cos\frac{\pi}{2}\cos\alpha - \sin\frac{\pi}{2}\sin\alpha \\ &= -\sin\alpha . \end{aligned}$$

Substituting from Eqs. (2.18) into (2.17) gives

$$[A] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix},$$

which is the same result provided earlier by Eq.(2.15). $[A]$ is orthogonal; i.e., its inverse $[A]^{-1} = [A]^T$.

$$[A]^T[A] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given $B_{X'}$ and $B_{Y'}$ what are B_X and B_Y ? Premultiplying Eq.(2.15) by $[A]^T$ gives

$$[A]^T(B)_{I'} = [A]^T[A](B)_I = (B)_I .$$

Hence,

$$(B)_I = [A]^T(B)_{I'} ,$$

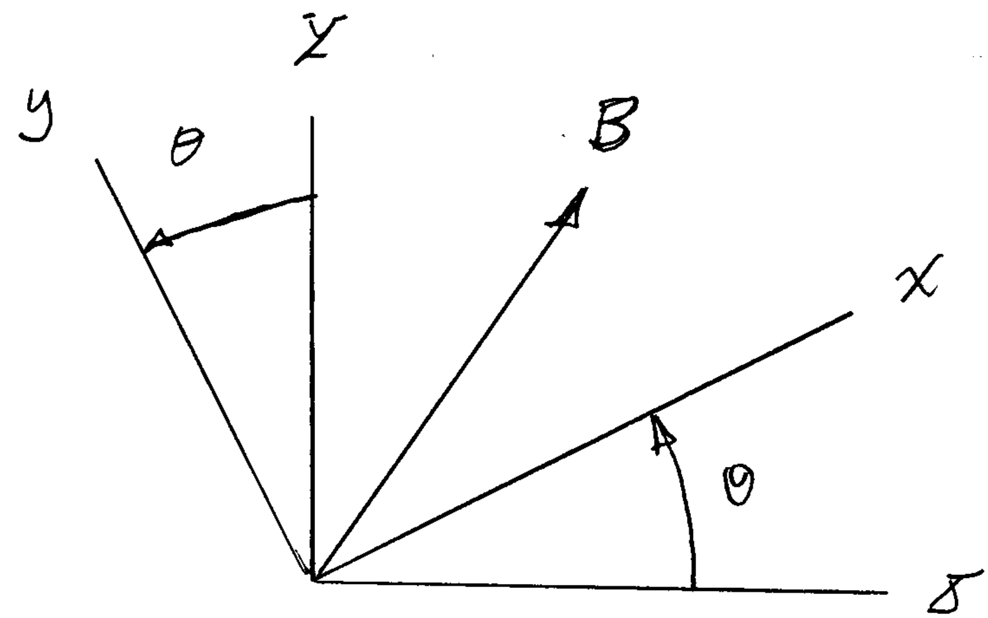
or

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} B_{X'} \\ B_{Y'} \end{Bmatrix}. \qquad \textbf{(2.19)}$$

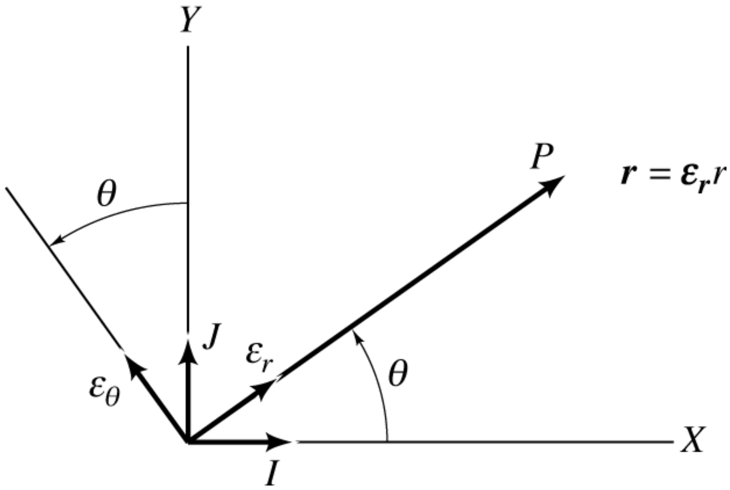
Homework Problem

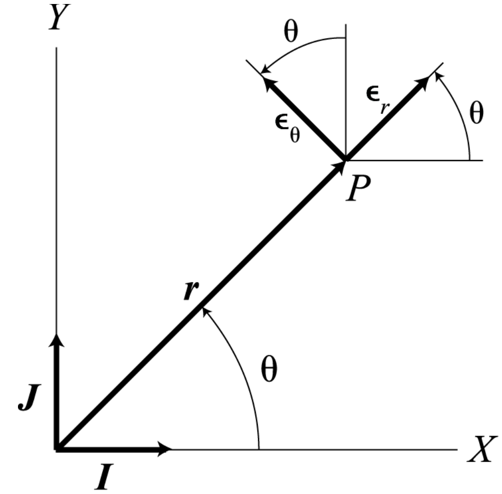
Given: $B_x = 10$, $B_y = 5$

Find: B_X, B_Y for $\theta = 0, 30, 45, 90, 120, 135, 153.4, 180$ (degrees).



PARTICLE MOTION IN A PLANE: POLAR COORDINATES





Developing useful expressions for velocity and acceleration using polar coordinates is our present task.

Unit-Vector Definition

$$\boldsymbol{\varepsilon_r} = \boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta$$

$$\boldsymbol{\varepsilon_\theta} = -\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta\ .$$

Start with

$$\boldsymbol{r} = r\boldsymbol{\varepsilon_r}\ . \tag{2.20}$$

Differentiating w.r.t. time relative to the *X, Y* coordinate system

$$\dot{\boldsymbol{r}} = \left.\frac{d\boldsymbol{r}}{dt}\right|_{X,Y} = \dot{r}\boldsymbol{\varepsilon}_r + r\left.\frac{d\boldsymbol{\varepsilon}_r}{dt}\right|_{X,Y}$$

$$= \dot{r}\boldsymbol{\varepsilon}_r + r\dot{\boldsymbol{\varepsilon}}_r \ . \qquad \textbf{(2.22)}$$

Differentiating $\boldsymbol{\varepsilon}_r = \boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta$ while holding $\boldsymbol{I}$ and $\boldsymbol{J}$ constant

$$\dot{\boldsymbol{\varepsilon}}_r = \left.\frac{d\boldsymbol{\varepsilon}_r}{dt}\right|_{X,Y} = (-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)\dot{\theta} = \dot{\theta}\,\boldsymbol{\varepsilon}_\theta \ , \qquad \textbf{(2.25)}$$

and

$$\dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{\varepsilon}_r + r\dot{\theta}\,\boldsymbol{\varepsilon}_\theta$$

$$= v_r\,\boldsymbol{\varepsilon}_r + v_\theta\,\boldsymbol{\varepsilon}_\theta \ . \qquad \textbf{(2.27)}$$

This expression provides $\dot{\boldsymbol{r}}$, the time derivative of $\boldsymbol{r}$ with respect to the *X, Y* coordinate system, but the answer is given in terms of *components* in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

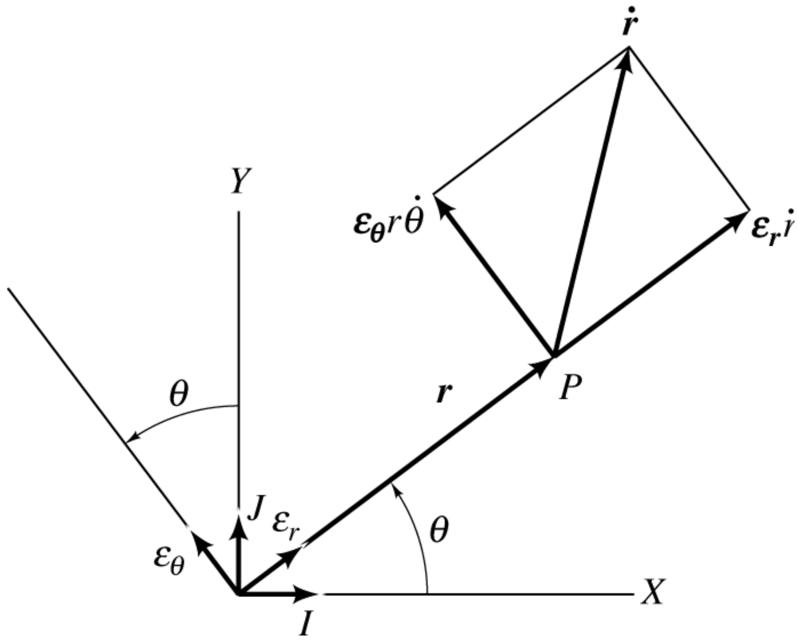


Figure 2.6 Components of $\dot{\boldsymbol{r}}$ in $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

Given the velocity $\dot{\boldsymbol{r}}$, find acceleration $\ddot{\boldsymbol{r}}$.

$$\ddot{\boldsymbol{r}} = \left.\frac{d\dot{\boldsymbol{r}}}{dt}\right|_{X,Y} = \ddot{r}\boldsymbol{\varepsilon}_r + \dot{r}\dot{\boldsymbol{\varepsilon}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{\varepsilon}_\theta + r\dot{\theta}\,\dot{\boldsymbol{\varepsilon}}_\theta \ . \quad \textbf{(2.28)}$$

Differentiating $\boldsymbol{\varepsilon}_\theta = -\boldsymbol{I}\ \sin\theta + \boldsymbol{J}\ \cos\theta$ with respect to time, holding $\boldsymbol{I}$ and $\boldsymbol{J}$ constant,

$$\dot{\boldsymbol{\varepsilon}}_\theta = \left.\frac{d\boldsymbol{\varepsilon}_\theta}{dt}\right|_{X,Y} = -(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)\dot{\theta} = -\boldsymbol{\varepsilon}_r\dot{\theta} \ . \quad \textbf{(2.29)}$$

Substituting for $\dot{\boldsymbol{\varepsilon}}_\theta$ and $\dot{\boldsymbol{\varepsilon}}_r$ into Eq.(2.28) gives

$$\ddot{\boldsymbol{r}} = \ddot{r}\boldsymbol{\varepsilon}_r + \dot{r}\dot{\theta}\boldsymbol{\varepsilon}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{\varepsilon}_\theta - r\dot{\theta}^2\boldsymbol{\varepsilon}_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{\varepsilon}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\varepsilon}_\theta \qquad \textbf{(2.30)}$$

$$= a_r\boldsymbol{\varepsilon}_r + a_\theta\boldsymbol{\varepsilon}_\theta .$$

Eq.(2.30) provides a definition for the acceleration of point *P*, with respect to the *X, Y* coordinate system, in terms of components in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

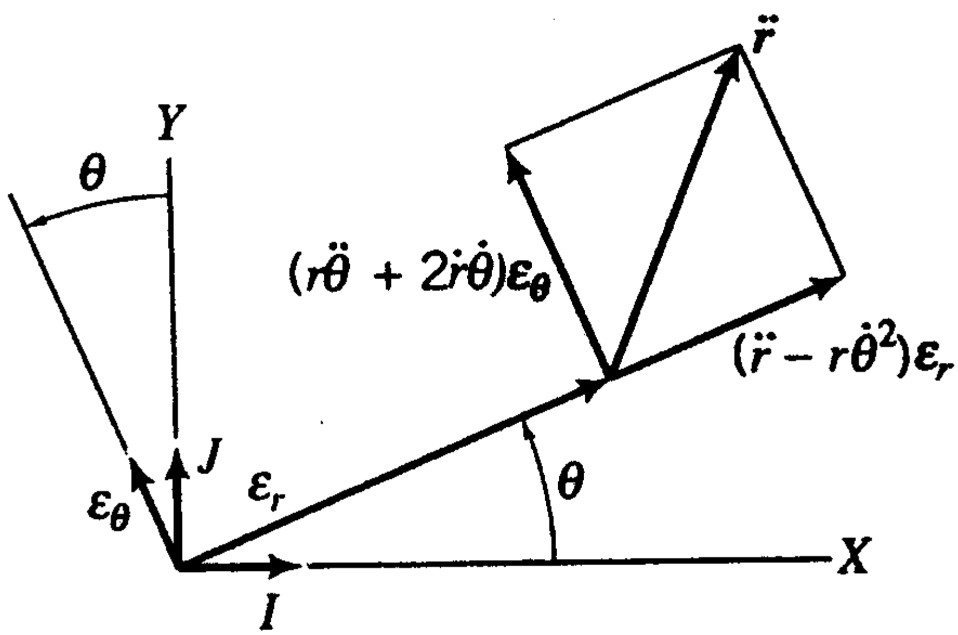


Figure 2.7 Components of $\ddot{\boldsymbol{r}}$ in $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system.

The polar velocity and acceleration components are:

$$v_r = \dot{r} \, , \; v_\theta = r\dot{\theta}$$
$$a_r = \ddot{r} - r\dot{\theta}^2 \, , \; a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \, . \tag{2.31}$$

Gustave Coriolis (1792-1843) was assistant professor of mathematics at the Ecole Polytechnique, Paris from 1816 to 1838 and studied mechanics and engineering mathematics. He is best remembered for the Coriolis force which appears in the paper *Sur les équations du mouvement relatif des systèmes de corps* (1835). He showed that the laws of motion could be used in a rotating frame of reference if an extra "force term" incorporating the Coriolis acceleration is added to the equations of motion.

Coriolis also introduced the terms "work" and "kinetic energy" with their present scientific meaning. (Website, school of Mathematics and Statistics, Saint Andrews University, Scotland)