Lecture 2. MORE PARTICLE MOTION IN A PLANE — POLAR COORDINATES

The polar velocity and acceleration components are:

\begin{align}
  v_r &= \dot{r} , \quad v_\theta = r \dot{\theta} \\
  a_r &= \ddot{r} - r \dot{\theta}^2 , \quad a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} .
\end{align}
Example Problem 2.6  As illustrated in figure XP 2.6a, a mass is sliding freely along a bar that is rotating at a constant 50 cycles per minute (cpm). At \( r = 0.38 \, m \) and \( \theta = 135^\circ \), \( \dot{\theta} = 6.92 \, m/\text{sec}^2 \), and \( \ddot{r} = 0.785 \, m/\text{sec} \). The engineering-analysis tasks are:

a. Determine the velocity and acceleration components of the mass in the rotating \( \varepsilon_r, \varepsilon_\theta \) coordinate system, and

b. Determine the components of \( v \) and \( a \) in the stationary \( X, Y \) system.

[Figure XP2.6a. Mass sliding along a smooth rotating bar.]

Solution. In applying Eqs.(2.31) to find the polar components of velocity and acceleration, \( \dot{\theta} \) is the angular velocity of the bar; however, we need to convert its given dimensions from cpm to radians per second via
\[ \dot{\theta} = 50 \frac{\text{cycle}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{2\pi \text{ rads}}{1 \text{ cycle}} = 5.24 \frac{\text{rad}}{\text{sec}}. \]

Direct application of Eqs.(2.31) gives

\[ v_r = \dot{r} = 0.785 \text{ m/sec} \]

\[ v_\theta = r \dot{\theta} = 0.38 \text{ m} \times 5.24 \text{ rad/sec} = 1.99 \text{ m/sec} \]

\[ a_r = (\ddot{r} - r \ddot{\theta}^2) = 6.92 \text{ m/sec}^2 - 0.38 \text{ m} \times 5.24^2 \text{ rad}^2/\text{sec}^2 \]

\[ = -3.51 \text{ m/sec}^2 \quad \text{(i)} \]

\[ a_\theta = (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) = 0.38 \text{ m} \times 0 + 2 \times 0.785 \text{ m/sec} \times 5.24 \text{ rad/sec} \]

\[ = 8.23 \text{ m/sec}^2, \]

and concludes Task a.
Figure XP 2.6b Velocity components in the $\varepsilon_r, \varepsilon_\theta$ and $X-Y$ systems

Figure XP2.6b illustrates the velocity components in the rotated $\varepsilon_r, \varepsilon_\theta$ coordinate system. There are several ways to develop the $X$ and $Y$ components of $v$ and $a$. We will first consider a development based directly on geometry and then use coordinate transformations.

The velocity vector magnitude is $v = (0.785^2 + 1.99^2)^{1/2} = 2.1 \text{ m/sec}$, and $v$ is oriented at the angle $21.53^\circ = \tan^{-1}(0.785/1.99)$ with respect to $\varepsilon_\theta$. Since $\varepsilon_\theta$ is directed $45^\circ$ below the -$X$ axis, $v$ is at
Moving along to the acceleration components, figure XP 2.6c illustrates the rotated $\varepsilon_r, \varepsilon_\theta$ coordinate system with the $a_r$ and $a_\theta$ components. Note that $a_r = -3.51 \text{ m/sec}^2$ is directed in the $-\varepsilon_r$ direction. The acceleration magnitude is 

$$a = (8.23^2 + 3.51^2)^{1/2} = 8.95 \text{ m/sec}^2.$$ 

The acceleration vector $\mathbf{a}$ is rotated $23.1^\circ = \tan^{-1}(3.51/8.23)$ counterclockwise from $\varepsilon_\theta$; hence, $\mathbf{a}$ is directed at $23.1^\circ + 45^\circ = 68.1^\circ$ below the $-X$ axis. Accordingly,

$$a_x = -8.95 \cos 68.1^\circ = -3.34 \text{ m/sec}^2$$

and

$$a_y = -8.95 \sin 68.1^\circ = -8.30 \text{ m/sec}^2.$$
These velocity and acceleration component results can also be obtained via a coordinate transformation. Figure XP 2.6d illustrates a vector $\mathbf{B}$ defined in terms of its components $B_r, B_\theta$ in the $\mathbf{e}_r, \mathbf{e}_\theta$ coordinate system. Projecting these components into the $X, Y$ system and adding the results along the $X$ and $Y$ axes gives

$$B_X = B_r \cos \theta - B_\theta \sin \theta, \quad B_Y = B_r \sin \theta + B_\theta \cos \theta.$$ 

In matrix notation, these results become
\[
\begin{bmatrix}
B_X \\
B_Y
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
B_r \\
B_\theta
\end{bmatrix}.
\]

(Fig. XP 2.6d) Coordinate transformation development to move from the \(\varepsilon_r, \varepsilon_\theta\) coordinate frame to the \(X, Y\) frame.

Note that this result basically coincides with Eq.(2.19) obtained earlier in discussing coordinate transformations. The \(\varepsilon_r, \varepsilon_\theta\) coordinate system of figure XP 2.6d replaces the \(X', Y'\) system of figure 2.4. Applying the transformation to the components of \(v\) and \(a\) gives

\[
\begin{bmatrix}
v_X \\
v_Y
\end{bmatrix}
= \begin{bmatrix}
\cos 135^\circ & -\sin 135^\circ \\
\sin 135^\circ & \cos 135^\circ
\end{bmatrix}
\begin{bmatrix}
v_r \\
v_\theta
\end{bmatrix}
= \begin{bmatrix}
-.707 & -.707 \\
.707 & -.707
\end{bmatrix}
\begin{bmatrix}
.785 \\
1.99
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
-1.96 \\
-.852
\end{bmatrix}
\frac{m}{\text{sec}},
\]
\[
\begin{align*}
\begin{bmatrix} a_X \\ a_Y \end{bmatrix} = & \begin{bmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{bmatrix} \begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \\
= & \begin{bmatrix} -3.34 \\ -8.30 \end{bmatrix} \frac{m}{\text{sec}^2}.
\end{align*}
\]

Note that the same (numerical) coordinate transformation applies for converting \( v \) and \( a \).

*Task a* ⇒ direct substitution into Eqs.(2.31).

The coordinate transformation result of Eq.(iv) applies for any vector and any angle \( \theta \).
Example Problem

An airplane is flying at a constant speed $v$ with a constant pitch angle $\alpha = 30^\circ$ with respect to the horizontal. It is being tracked by radar that shows a range of 9.24 km and the tracking angle and angular rate: $\theta = 60^\circ$, $\dot{\theta} = -0.0387 \text{ rad/sec}$. Tasks:

1. Determine $\dot{r}$ and the plane’s speed $v$.
2. Determine $\ddot{\theta}$, $\ddot{r}$.

**Solution.** From figure b, the components of $v$ can be stated:

$$v_x = v \cos 30^\circ, \quad v_y = v \sin 30^\circ.$$  

The coordinate transformation from the $I, J$ coordinate system to the $\epsilon_r, \epsilon_\theta$ coordinate system,
\[
\begin{bmatrix}
B_r \\
B_\theta
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
B_X \\
B_Y
\end{bmatrix},
\]

can be used for the velocity components as
\[
\begin{bmatrix}
v_r \\
v_\theta
\end{bmatrix} = \begin{bmatrix}
\dot{r} \\
r\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
v \cos 30^\circ \\
v \sin 30^\circ
\end{bmatrix},
\]

netting,
\[
\dot{r} = v (\cos 30^\circ \cos \theta + \sin 30^\circ \sin \theta) = v \cos (\theta - 30^\circ)
\]
\[
r\dot{\theta} = v (-\cos 30^\circ \sin \theta + \sin 30^\circ \cos \theta) = -v \sin (\theta - 30^\circ)
\]

Substituting for \( r = 9.24 \text{ km} \), \( \dot{\theta} = -0.0387 \text{ rad/sec} \), and \( \theta = 60^\circ \) gives:
\[
\dot{r} = v \cos 30^\circ = 0.866v
\]
\[
r\dot{\theta} = -v \sin(30^\circ) = -0.5v \Rightarrow v = -2 \times \left(-0.0387 \frac{\text{rad}}{\text{sec}}\right) \times 9.24 \text{ km}
\]
\[
\therefore v = 0.715 \text{ km/sec and } \dot{r} = 0.866 \times 0.715 = 0.619 \text{ km/sec}
\]

Check, \( v = \sqrt{v_r^2 + v_\theta^2} \) ?
\[ v = \sqrt{\left( \dot{r} \right)^2 + (r \dot{\theta})^2} = \sqrt{0.619^2 + (9.24 \times -0.0387)^2} \]
\[ = \sqrt{(0.3844 + 0.1278)^2} = 0.715 \text{ km/sec} \]

The plane is flying at constant speed; hence, \( a = 0 \), and

\[ a_r = \ddot{r} - r \ddot{\theta} = 0 \Rightarrow \ddot{r} = 9.24 \text{ km} (-0.0387 \text{ rad/sec})^2 \]
\[ = 0.0138 \text{ km/sec}^2 \]

\[ a_{\theta} = r \dddot{\theta} + 2 \dot{r} \ddot{\theta} = 0 \Rightarrow \dddot{\theta} = -\frac{2 \times 0.62 \text{ km/sec} \times -0.0387 \text{ rad/sec}}{9.24 \text{ km}} \]
\[ = 5.19 \times 10^{-3} \text{ rad/sec}^2 \]