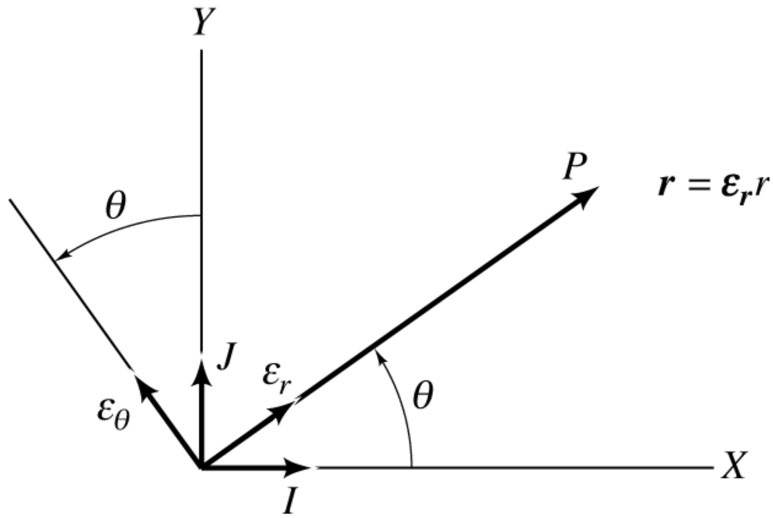
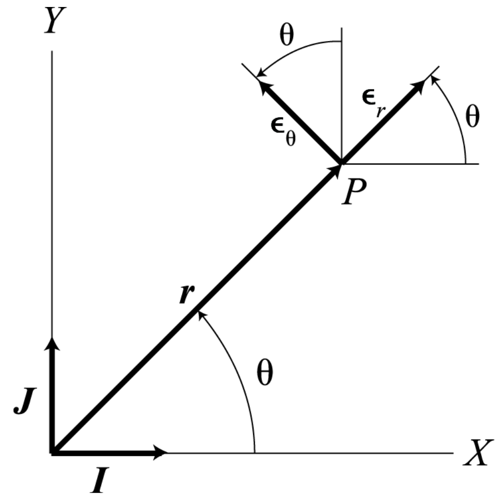
Lecture 2. MORE PARTICLE MOTION IN A PLANE — POLAR COORDINATES





The polar velocity and acceleration components are:

$$v_r = \dot{r}\ ,\ v_\theta = r\dot{\theta}$$
$$a_r = \ddot{r} - r\dot{\theta}^2\ ,\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}\ . \qquad \textbf{(2.31)}$$

Example Problem 2.6 As illustrated in figure XP 2.6a, a mass is sliding freely along a bar that is rotating at a constant 50 cycles per minute (cpm). At $r = .38\,m$ and $\theta = 135^o$, $\ddot{r} = 6.92\,m/\sec^2$, and $\dot{r} = .785\,m/\sec$. The engineering-analysis tasks are:

a. Determine the velocity and acceleration components of the mass in the rotating $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system, and

b. Determine the components of ***v*** and ***a*** in the stationary *X, Y* system.

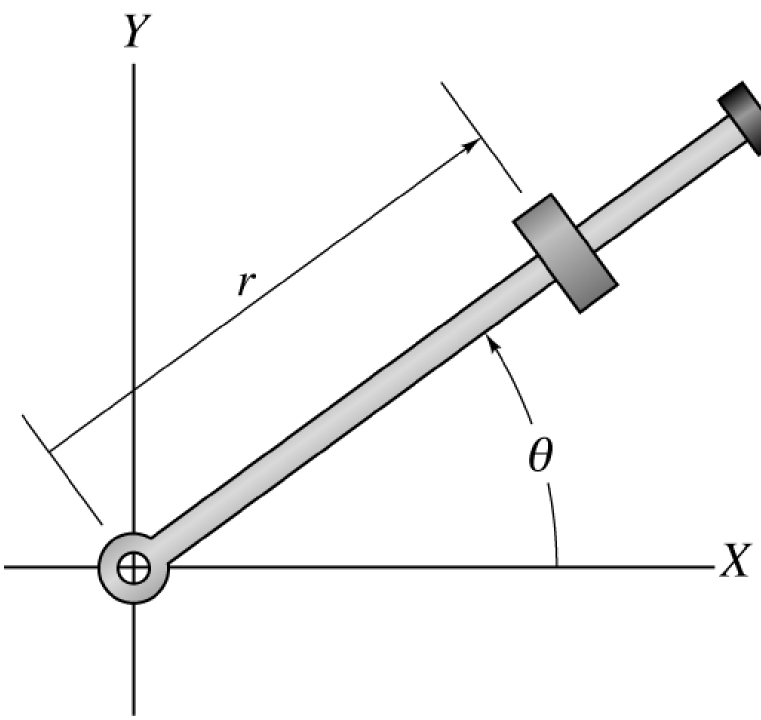


Figure XP2.6a. Mass sliding along a smooth rotating bar.

Solution. In applying Eqs.(2.31) to find the polar components of velocity and acceleration, $\dot{\theta}$ is the angular velocity of the bar; however, we need to convert its given dimensions from cpm to radians per second via

$$\dot{\theta} = 50\,\frac{cycle}{\min} \times \frac{1\min}{60\sec} \times \frac{2\pi\, rads}{1\, cycle} = 5.24\,\frac{rad}{\sec} \quad .$$

Direct application of Eqs.(2.31) gives

$$v_r = \dot{r} = .785\, m/\sec$$

$$v_\theta = r\,\dot{\theta} = .38\, m \times 5.24\, rad/\sec = 1.99\, m/\sec$$

$$\begin{aligned} a_r &= (\ddot{r} - r\,\dot{\theta}^2) = 6.92\, m/\sec^2 - .38\, m \times 5.24^2\, rad^2/\sec^2 \\ &= -3.51\, m/\sec^2 \end{aligned} \qquad \text{(i)}$$

$$\begin{aligned} a_\theta &= (r\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}) = .38\, m \times 0 + 2 \times .785\, m/\sec \times 5.24\, rad/\sec \\ &= 8.23\, m/\sec^2 \quad , \end{aligned}$$

and concludes *Task a.*

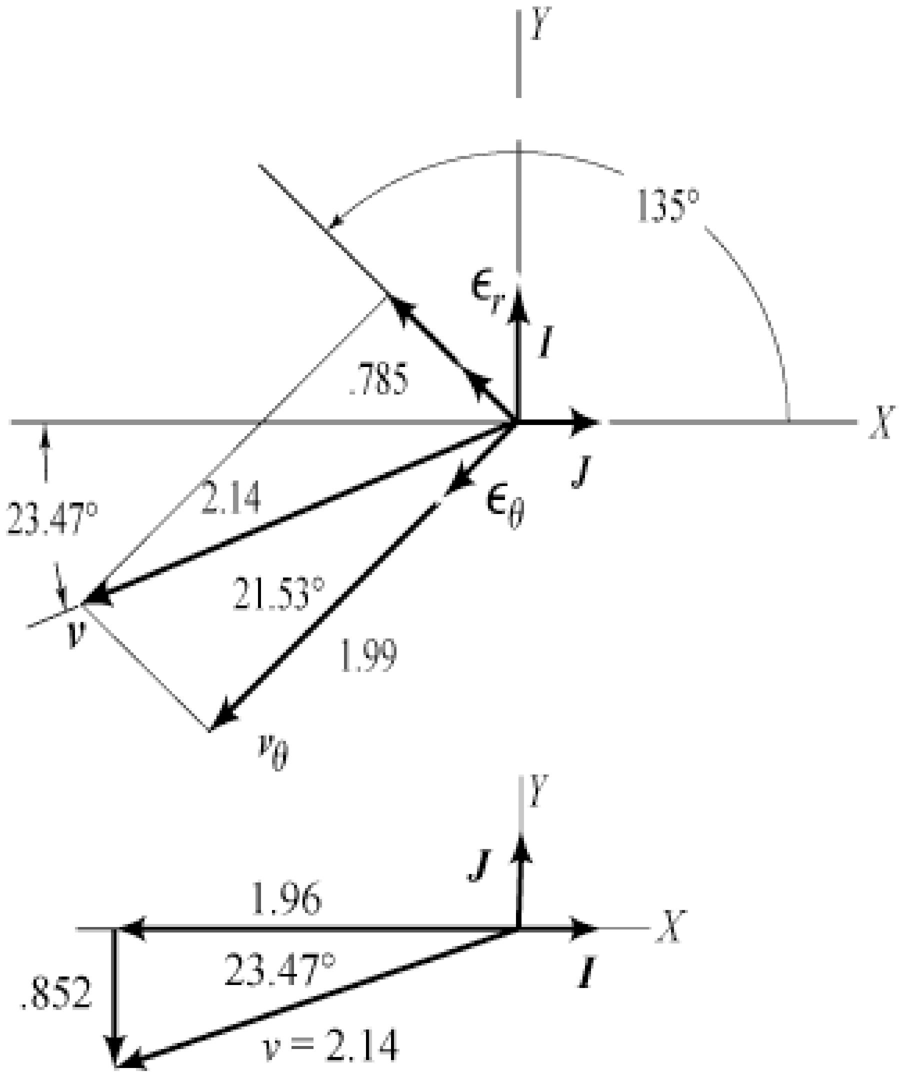


Figure XP 2.6b Velocity components in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ and $X-Y$ systems

Figure XP2.6b illustrates the velocity components in the rotated $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system. There are several ways to develop the *X* and *Y* components of $\boldsymbol{v}$ and $\boldsymbol{a}$. We will first consider a development based directly on geometry and then use coordinate transformations.

The velocity vector magnitude is $v = (.785^2 + 1.99^2)^{1/2} = 2.1\ m/\text{sec}$, and $\boldsymbol{v}$ is oriented at the angle $21.53° = \tan^{-1}(.785/1.99)$ with respect to $\boldsymbol{\varepsilon}_\theta$. Since $\boldsymbol{\varepsilon}_\theta$ is directed 45° below the - *X* axis, $\boldsymbol{v}$ is at

$23.47° = 45^o - 21.53°$ below the - X axis, and

$$v_X = -2.1\cos 23.47° = -1.96\, m/\sec$$
$$v_Y = -2.1\sin 23.47° = -.852\, m/\sec \qquad \text{(ii)}$$

Moving along to the acceleration components, figure XP 2.6c illustrates the rotated $\boldsymbol{\varepsilon_r}, \boldsymbol{\varepsilon_\theta}$ coordinate system with the a_r and a_θ components. Note that $a_r = -3.51\, m/\sec^2$ is directed in the $-\boldsymbol{\varepsilon_r}$ direction. The acceleration magnitude is $a = (8.23^2 + 3.51^2)^{1/2} = 8.95\, m/\sec^2$. The acceleration vector $\boldsymbol{a}$ is rotated $23.1^o = \tan^{-1}(3.51/8.23)$ counterclockwise from $\boldsymbol{\varepsilon_\theta}$; hence, $\boldsymbol{a}$ is directed at $23.1° + 45° = 68.1°$ below the -X axis. Accordingly,

$$a_X = -8.95\cos 68.1° = -3.34\, m/\sec^2$$
$$a_Y = -8.95\sin 68.1° = -8.30\, m/\sec^2 \ . \qquad \text{(iii)}$$

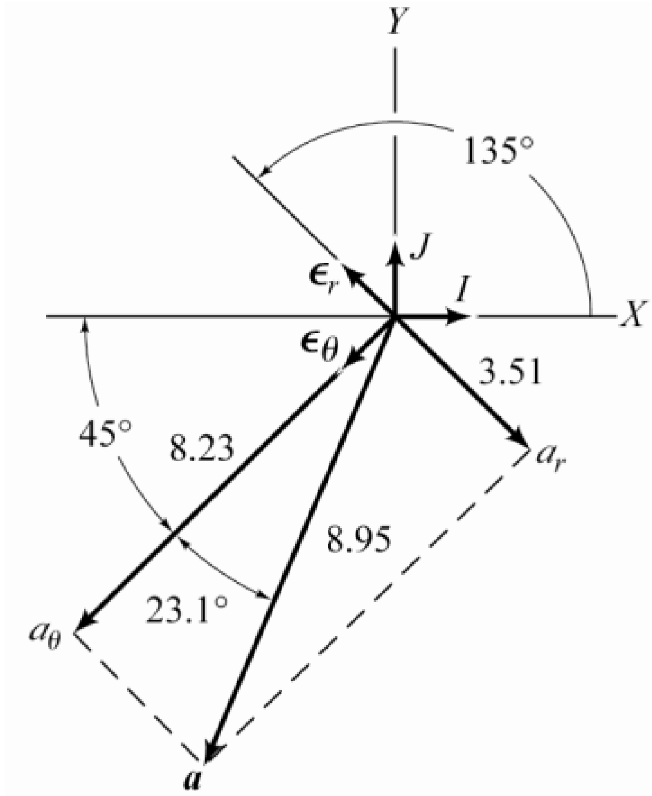


Figure XP2.6 (c)
Acceleration components
in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ system

These velocity and acceleration component results can also be obtained via a coordinate transformation. Figure XP 2.6d illustrates a vector $\boldsymbol{B}$ defined in terms of its components B_r, B_θ in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system. Projecting these components into the *X*, *Y* system and adding the results along the *X* and *Y* axes gives

$$B_X = B_r \cos\theta - B_\theta \sin\theta \ , \quad B_Y = B_r \sin\theta + B_\theta \cos\theta \ .$$

In matrix notation, these results become

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix}. \tag{iv}$$

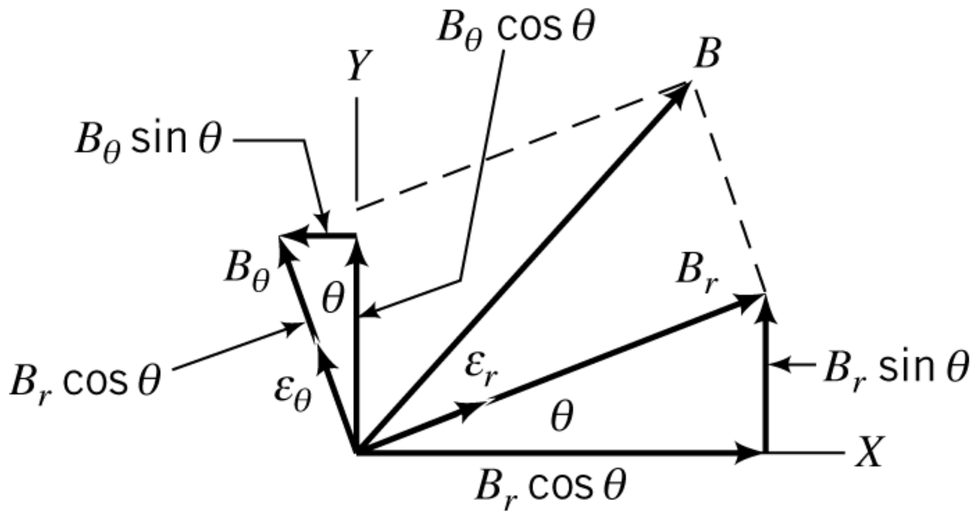


Figure XP 2.6d. Coordinate transformation development to move from the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate frame to the X, Y frame.

Note that this result basically coincides with Eq.(2.19) obtained earlier in discussing coordinate transformations. The $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system of figure XP 2.6d replaces the X', Y' system of figure 2.4. Applying the transformation to the components of $\boldsymbol{v}$ and $\boldsymbol{a}$ gives

$$\begin{Bmatrix} v_X \\ v_Y \end{Bmatrix} = \begin{bmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{bmatrix} \begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} -.707 & -.707 \\ .707 & -.707 \end{bmatrix} \begin{Bmatrix} .785 \\ 1.99 \end{Bmatrix} \tag{v}$$
$$= \begin{Bmatrix} -1.96 \\ -.852 \end{Bmatrix} \frac{m}{\sec},$$

and

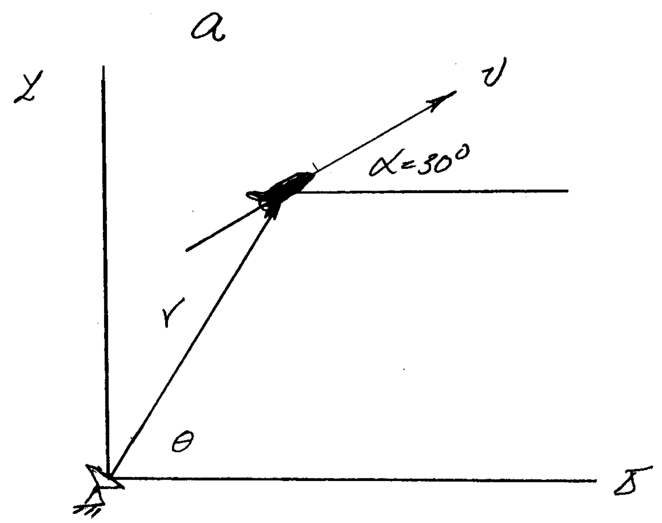
$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{bmatrix} \begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} -.707 & -.707 \\ .707 & -.707 \end{bmatrix} \begin{Bmatrix} -3.31 \\ 8.23 \end{Bmatrix} \quad \textbf{(vi)}$$

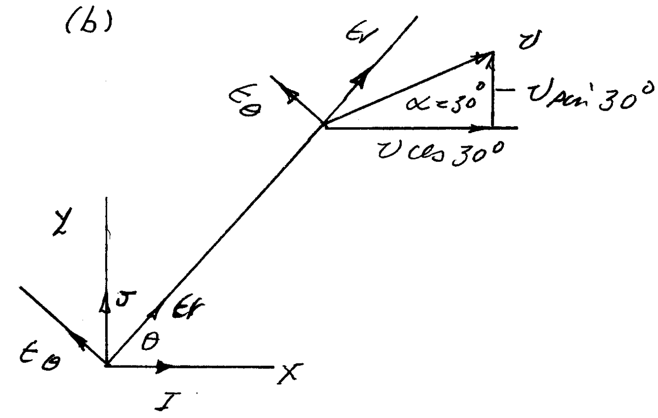
$$= \begin{Bmatrix} -3.34 \\ -8.30 \end{Bmatrix} \frac{m}{\sec^2} \ .$$

Note that the same (numerical) coordinate transformation applies for converting ***v*** and ***a*** .

Task a ⇒ direct substitution into Eqs.(2.31).

The coordinate transformation result of Eq.(iv) applies for any vector and any angle θ.





Example Problem

An airplane is flying at a constant speed v with a constant pitch angle $\alpha = 30^{\circ}$ with respect to the horizontal. It is being tracked by radar that shows a range of 9.24 km and the tracking angle and angular rate: $\theta = 60°$, $\dot{\theta} = -0.0387\, rad/\sec$. Tasks:

a. Determine $\dot{r}$ and the plane's speed v.

b. Determine $\ddot{\theta}$, $\ddot{r}$.

Solution. From figure b, the components of $\boldsymbol{v}$ can be stated:

$$v_X = v\cos 30° \ , \ v_Y = v\sin 30° \ .$$

The coordinate transformation from the ***I***, ***J*** coordinate system to the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ coordinate system,

$$\begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix},$$

can be used for the velocity components as

$$\begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{Bmatrix} \dot{r} \\ r\dot{\theta} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v\cos 30^\circ \\ v\sin 30^\circ \end{Bmatrix},$$

netting,

$$\dot{r} = v\,(\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta) = v\cos(\theta - 30^\circ)$$

$$r\dot{\theta} = v\,(-\cos 30^\circ \sin\theta + \sin 30^\circ \cos\theta) = -v\sin(\theta - 30^\circ)$$

Substituting for $r = 9.24\,km$, $\dot{\theta} = -0.0387\,rad/\sec$, and $\theta = 60^\circ$ gives:

$$\dot{r} = v\cos 30^\circ = .866\,v$$

$$r\dot{\theta} = -v\sin(30^\circ) = -0.5\,v \;\Rightarrow\; v = -2\times(-0.0387\,\frac{rad}{\sec})\times 9.24\,km$$

$$\therefore\; v = .715\,km/\sec \;\; and \;\; \dot{r} = .866\times .715 = .619\,km/\sec$$

Check, $v = \sqrt{v_r^2 + v_\theta^2}$?

$$v = [\dot{r}^2 + (r\dot{\theta})^2] = [0.619^2 + (9.24 \times -0.0387)^2]^{1/2}$$
$$= (.3844 + .1278)^{1/2} = .715\, km/\sec \quad .$$

The plane is flying at constant speed; hence, $\boldsymbol{a} = 0$, and

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \;\Rightarrow\; \ddot{r} = 9.24\, km(\,-0.0387\, rad/\sec)^2$$
$$= .0138\, km/\sec^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \;\Rightarrow\; \ddot{\theta} = -\frac{2 \times 0.62\, km/\sec \times -0.0387\, rad/\sec}{9.24\, km}$$
$$= 5.19\, E{-}03\, rad/\sec^2$$