Lecture 3. PARTICLE MOTION IN A PLANE, NORMAL-TANGENTIAL (PATH) COORDINATES

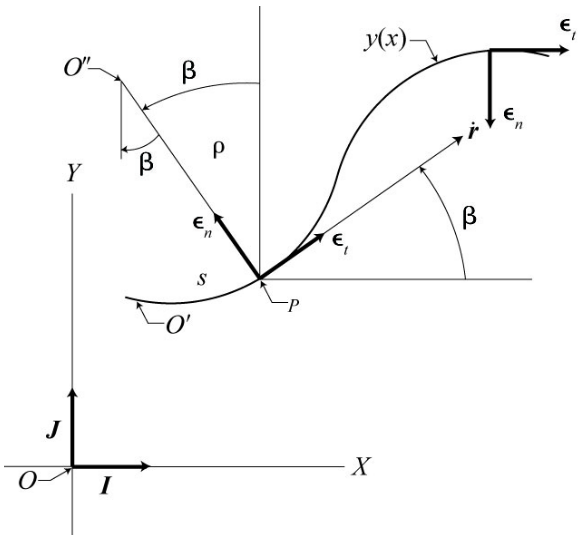

Figure 2.8 Path-coordinate unit vectors; $\tan\beta = dy/dx$

Unit Vectors:

$$\boldsymbol{\varepsilon}_t = \boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta$$

$$\boldsymbol{\varepsilon}_n = -\boldsymbol{I}\sin\beta + \boldsymbol{J}\cos\beta \ .$$

Radius of curvature of a path $y = f(x)$ is

$$\frac{1}{\rho} = \frac{|y''|}{[\,1 + (y')^2\,]^{3/2}} \quad ,$$

where $y' = df/dx = \tan\beta$, and $y'' = d^2f/dx^2$.

Velocity of point *P* with respect to the *X*, *Y* system

$$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt}\Big|_{X,Y} = \dot{s}\,\boldsymbol{\varepsilon}_t = \rho\dot{\beta}\,\boldsymbol{\varepsilon}_t = v\,\boldsymbol{\varepsilon}_t \ , \tag{2.32}$$

where s defines the distance traveled along the path from some arbitrary reference point O.

Note that

$$\dot{s} = v = \rho\,\dot{\beta} \ . \tag{2.33}$$

Acceleration of point *P* with respect to the *X*, *Y* system. Taking the time derivative of $\dot{\boldsymbol{r}} = v\boldsymbol{\varepsilon}_t$ with respect to the X, Y coordinate system gives

$$\begin{aligned}\ddot{\boldsymbol{r}} = \frac{d\dot{\boldsymbol{r}}}{dt}\Big|_{X,Y} &= \dot{v}\,\boldsymbol{\varepsilon}_t + v\,\frac{d\boldsymbol{\varepsilon}_t}{dt}\Big|_{X,Y} \\ &= \dot{v}\,\boldsymbol{\varepsilon}_t + v\,\dot{\boldsymbol{\varepsilon}}_t \ .\end{aligned} \tag{2.34}$$

This result requires $\dot{\boldsymbol{\varepsilon}}_t$, the time derivative of $\boldsymbol{\varepsilon}_t$ with respect to the X, Y system. Differentiating $\boldsymbol{\varepsilon}_t = \boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta$ while holding $\boldsymbol{I}$ and $\boldsymbol{J}$ constant yields

$$\dot{\boldsymbol{\varepsilon}}_t = \frac{d\boldsymbol{\varepsilon}_t}{dt}\Big|_{X,Y} = \dot{\beta}(-\boldsymbol{I}\sin\beta + \boldsymbol{J}\cos\beta) = \dot{\beta}\ \boldsymbol{\varepsilon}_n\ .$$

Substituting this result into Eq.(2.34) then gives

$$\ddot{\boldsymbol{r}} = \dot{v}\ \boldsymbol{\varepsilon}_t + v\ \dot{\beta}\ \boldsymbol{\varepsilon}_n\ , \qquad \textbf{(2.35)}$$

Substituting $v = \rho\,\dot{\beta}$ from Eq.(2.33) gives the following alternative expressions for $\ddot{\boldsymbol{r}}$:

$$\ddot{\boldsymbol{r}} = \dot{v}\ \boldsymbol{\varepsilon}_t + \frac{v^2}{\rho}\ \boldsymbol{\varepsilon}_n\ , \qquad \textbf{(2.36)}$$

or

$$\ddot{\boldsymbol{r}} = \dot{v}\ \boldsymbol{\varepsilon}_t + \rho\ \dot{\beta}^2\ \boldsymbol{\varepsilon}_n\ . \qquad \textbf{(2.37)}$$

Since

$$\ddot{\boldsymbol{r}} = a_t\ \boldsymbol{\varepsilon}_t + a_n\ \boldsymbol{\varepsilon}_n\ ,$$

Eqs.(2.35) through (2.37) provide the following component definitions for a_t and a_n:

$$\boxed{a_t = \dot{v}\ ,\ a_n = v\ \dot{\beta} = \frac{v^2}{\rho} = \rho\ \dot{\beta}^2\ .} \qquad \textbf{(2.38)}$$

$a_n = v^2/\rho$ is the more generally useful expression.

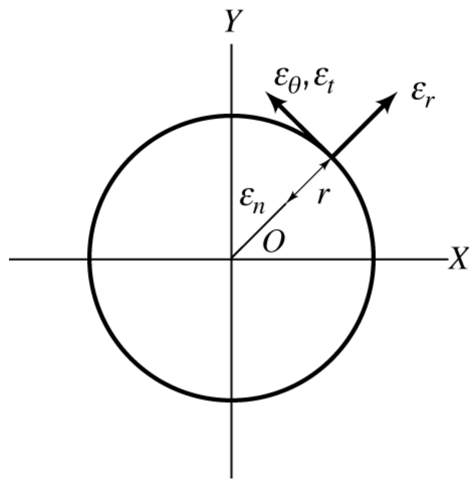

Figure 2.9 Constant-radius circular motion with $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ and $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ unit vectors.

Polar and path coordinate relationships. For $\dot{r}=0$, $\rho=r$, $\dot{\theta}=\dot{\beta}$, $\dot{r}=0$, and $\boldsymbol{\varepsilon}_n$ and $\boldsymbol{\varepsilon}_r$ are oppositely directed. Hence the polar coordinate model gives

$$v_r = 0,\ v_\theta = r\ \dot{\theta}$$
$$a_r = -\,r\ \dot{\theta}^2,\ a_\theta = r\ddot{\theta}\ .$$

For this reduced case, $r = \rho$, and

$$v = r\dot{\theta}$$

$$a_n = -a_r \Rightarrow r\dot{\theta}^2 = \frac{v^2}{r}$$

$$a_t = a_\theta \Rightarrow \dot{v} = \ddot{s} = r\ddot{\theta}$$

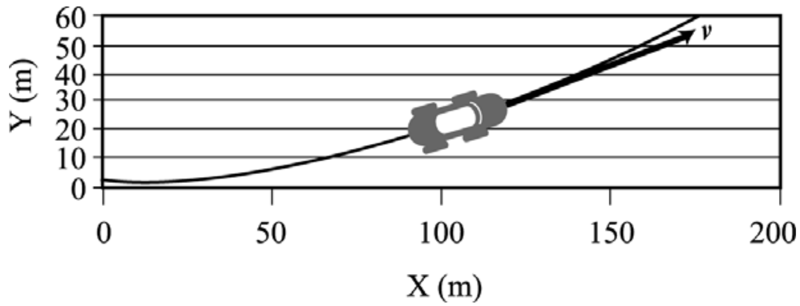

Figure XP2.7a Track segment in the horizontal *X*-*Y* system

Example Problem 2.7 As illustrated in figure XP 2.7a, a track lies in the horizontal plane and is defined by $Y = kX^2$ with X and Y in meters and $k = 1/400\ m^{-1}$. At $X = 100\,m$, the velocity and acceleration components *along the path* are $20\ m/\sec$ and $-2\ m/\sec^2$, respectively. The relevant engineering-analysis tasks are:

a. Determine the normal and tangential components of $\boldsymbol{v}$

and $\boldsymbol{a}$, and

b. Determine $\boldsymbol{v}$ and $\boldsymbol{a}$'s components in the *X, Y* system.

Solution. From the definitions of the $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ coordinate system, $\boldsymbol{v}$ and $\boldsymbol{\varepsilon}_t$ are colinear, and both are directed along the tangent of the path. Hence, $\boldsymbol{v} = \boldsymbol{\varepsilon}_t 20\ m/\sec$. The velocity vector $\boldsymbol{v}$ has no component along $\boldsymbol{\varepsilon}_n$. The problem statement gives $a_t = \dot{v} = -2\ m/\sec^2$. From Eq.(2.38), the normal component is $a_n = v^2/\rho$. We are given $v = 20\ m/\sec$; however, we need to define the radius of curvature.

For $Y = kX^2, Y' = 2kX,\ Y'' = 2k$ and

$$\frac{1}{\rho} = \frac{|Y''|}{[\ 1 + (Y')^2\ ]^{3/2}}\ ,$$

gives

$$\frac{1}{\rho}\Big|_{X=100m} = \frac{2/400}{[1 + (2\times 100/400)^2]^{3/2}} = 3.57\times 10^{-3}\,m^{-1}$$

$$\therefore\ \rho = 280\ m\ ,$$

and

$$a_n = 20^2/280. = 1.43\ m/\sec^2\ .$$

The answer for *Task a* is stated

$$\boldsymbol{v} = 20\ \boldsymbol{\varepsilon}_t\ m/\sec\ ,\quad \boldsymbol{a} = -2\,\boldsymbol{\varepsilon}_t + 1.43\ \boldsymbol{\varepsilon}_n\ m/\sec^2\ .$$

Moving to *Task b* , the first question to answer in finding the components of **v** and ***a*** in the *X, Y* system is, "How are $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ oriented in the *X, Y* system?" Since $\boldsymbol{\varepsilon}_t$ is directed along the tangent of the path, we can find the orientation of $\boldsymbol{\varepsilon}_t$ with respect to the *X* axis via,

$$Y'|_{X=100m} = 2kX|_{X=100m} = 2\times\frac{1}{400}\times 100 = .5$$

$$\therefore\ \beta = \tan^{-1}(.5) = 26.6°\ .$$

Figure XP 2.7b shows this orientation of the $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ coordinate system at $X = 100\,m$.

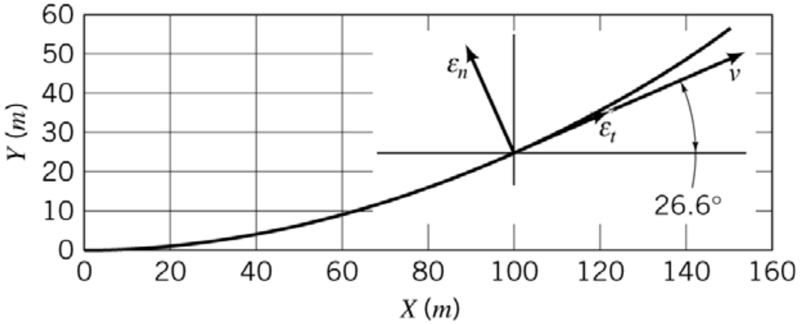

Figure XP2.7b $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ orientation at $X = 100\,m$

From this figure, **v**'s *X* and *Y* components are:

$$v_X = v\cos 26.6^o = 20\times .894 = 17.9\ m/\sec$$

$$v_Y = v\sin 26.6^o = 20\times .447 = 8.95\ m/\sec\ .$$

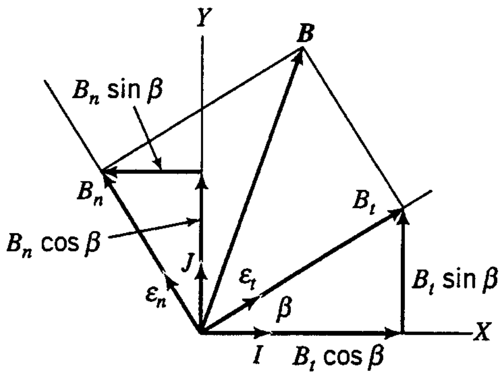

Figure 2.10 Coordinate transformation development to move from components in the $\boldsymbol{\varepsilon}_t$, $\boldsymbol{\varepsilon}_n$ coordinate system to the *X, Y* coordinate system.

To find ***a***'s components in the *X, Y* system, consider the components of the arbitrary vector ***B*** in Figure 2.10. This figure is very similar to figure XP2.7d (page 27, Lecture 2) that we developed to move from components in the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ system to components in the *X, Y* system. Summing components in the *X* and *Y* directions gives

$$B_X = B_t\cos\beta - B_n\sin\beta\ ,\quad B_Y = B_t\sin\beta + B_n\cos\beta\ .$$

In matrix notation, these results become

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} B_t \\ B_n \end{Bmatrix} .$$

Substituting the acceleration components and $\beta = 26.6°$. gives

$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos 26.6^o & -\sin 26.6^o \\ \sin 26.6^o & \cos 26.6^o \end{bmatrix} \begin{Bmatrix} a_t \\ a_n \end{Bmatrix} = \begin{bmatrix} .894 & -.447 \\ .447 & .894 \end{bmatrix} \begin{Bmatrix} -2. \\ 1.43 \end{Bmatrix}$$

$$= \begin{Bmatrix} -2.43 \\ .384 \end{Bmatrix} \frac{m}{\sec^2} .$$

This step concludes Task b. Note that ***a***'s magnitude is unchanged by the transformation.

In reviewing the steps involved in working out this example, applying the definitions to find the components of ***v*** and ***a*** in the path coordinate system is relatively straightforward, except for a modest effort to determine the radius of curvature ρ. The essential first step in finding ***v*** and ***a***'s components in the *X, Y* system is in recognizing that $\boldsymbol{\varepsilon}_t$ lies along the path's tangent. Following this insight, projecting ***v***'s components into the *X, Y* system is simple, as is finding ***a***'s components via the coordinate transformation.