Lecture 4. MOVING BETWEEN CARTESIAN, POLAR, AND PATH COORDINATE DEFINITIONS FOR VELOCITY AND ACCELERATION COMPONENTS

An Example That is Naturally Analyzed with Cartesian Components

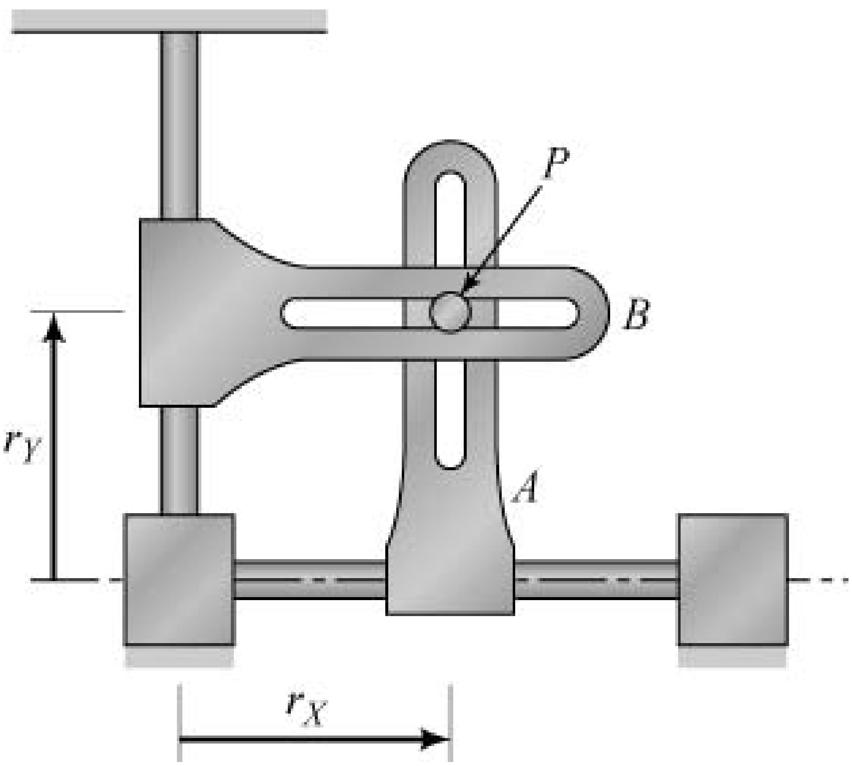

Figure 2.11 A lag-screw-driven mechanism.

Assume that control is applied to the screws such that,

$$r_X(t) = A + a\cos(\omega t)\ ,\quad r_Y(t) = B + b\sin(2\omega t)$$

$$A = B = 1\,mm,\ a = 1.25\ mm\ ,\ b = 2.5\ mm \qquad (2.39)$$

$$\omega = 3.1416\, rad/\sec\ .$$

The engineering-analysis task associated with this system is: *At*

$\omega t = 30^o$, determine the components of P's velocity and acceleration vectors in the [(X, Y), (r, θ), *and* $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n)$] *coordinate systems.*

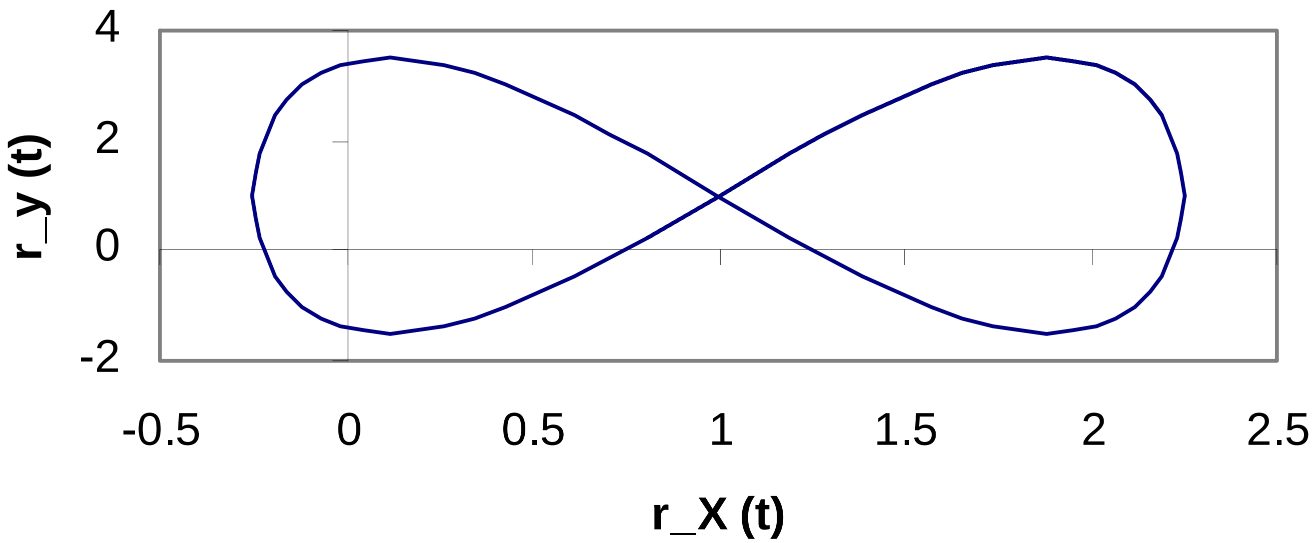

Figure 2.12 Lissajous[1] figure produced by the motion defined in Eq.(2.39); r_Y versus r_X.

Differentiating the components of the position vector nets:

$$v_X = \dot{r}_X = -a\omega\sin(\omega t) \ , \quad v_Y = \dot{r}_Y = 2b\omega\cos(2\omega t) \ . \qquad \text{(i)}$$

Differentiating again yields:

1

Named for Jules Antoine Lissajous, March 1822-June 1880. Primarily noted for accomplishments in acoustics.

$$a_X = \ddot{r}_X = -a\omega^2 \cos(\omega t) \ , \quad a_Y = \ddot{r}_Y = -4b\omega^2 \sin(2\omega t) \ . \qquad \text{(ii)}$$

Plugging in the numbers from Eq.(2.39) at $\omega t = 30^o$ nets:

$$\begin{aligned}
r_X &= 1 + 1.25 \times .866 = 2.08 \ \ mm \\
r_Y &= 1 + 2.5 \times .866 = 3.165 \ \ mm \\
v_X &= -1.25 \times 3.1416 \times .5 = -1.96 \ \ mm/\sec \\
v_Y &= 2 \times 2.5 \times 3.1416 \times .5 = 7.85 \ \ mm/\sec \\
a_X &= -1.25 \times (3.1416)^2 \times .866 = -10.68 \ \ mm/\sec^2 \\
a_Y &= -4 \times 2.5 (3.1416)^2 \times .866 = -85.47 \ \ mm/\sec^2 \ .
\end{aligned} \qquad \text{(iii)}$$

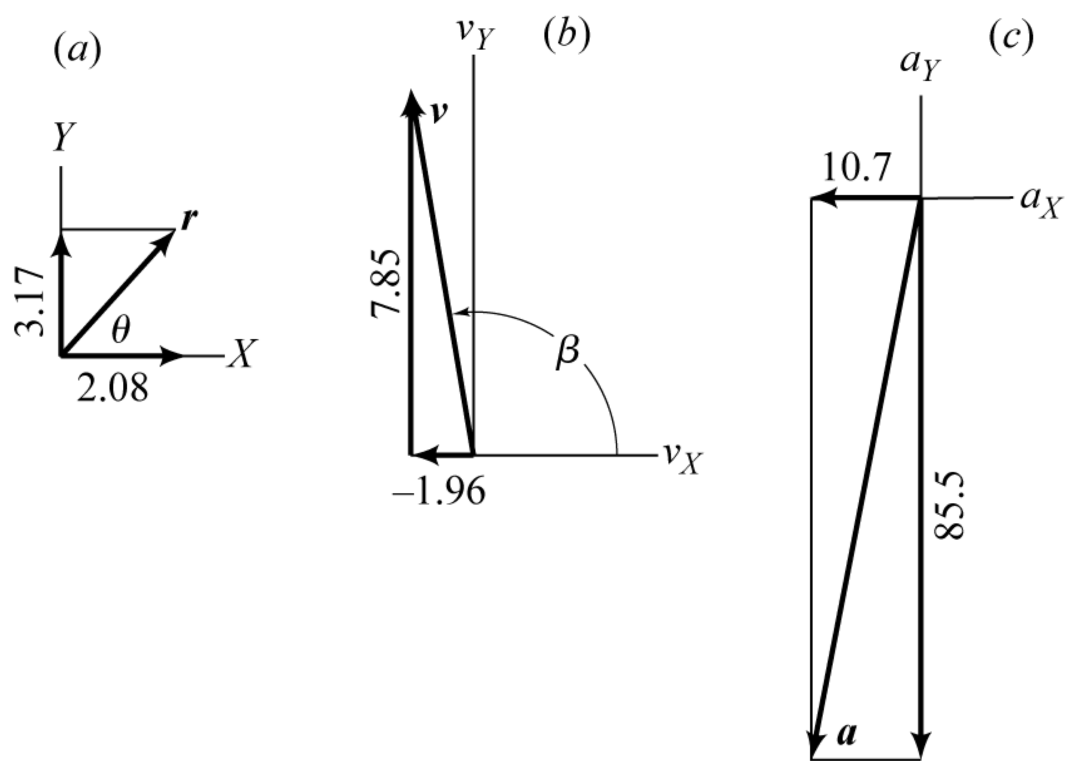

Figure 2.13 (a) Position, (b) Velocity, and (c) Acceleration vectors in the X,Y system at $\omega t = 30°$; mm-sec units

Given *X, Y* components, find *r*-θ components

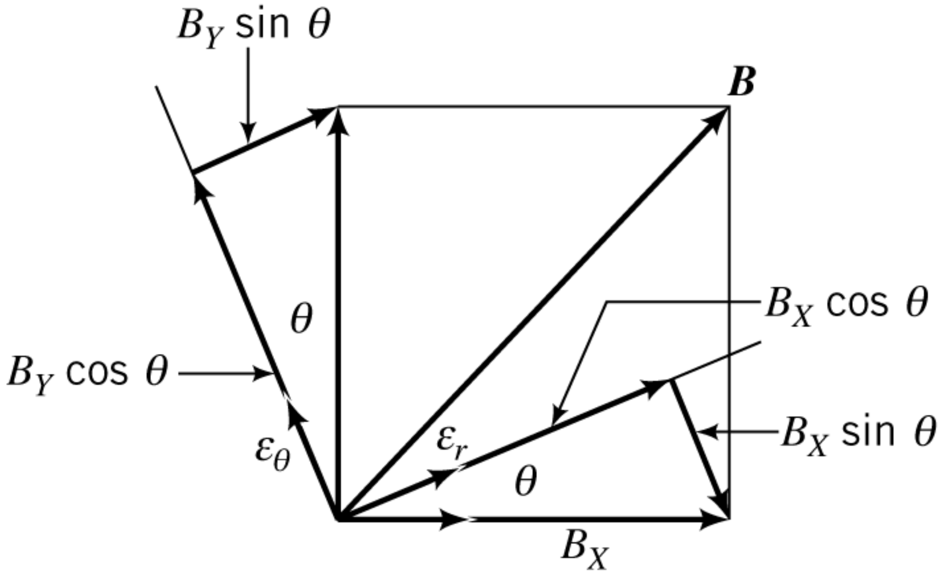

Figure 2.14 Components of the vector ***B*** in the *X*, *Y* system, projected along the $\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_\theta$ unit vectors.

From figure 2.14

$$B_r = B_X \cos\theta + B_Y \sin\theta \ , \quad B_\theta = -B_X \sin\theta + B_Y \cos\theta \ ,$$

or

$$\begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} . \qquad \textbf{(2.41)}$$

From figure 2.13a and Eq.(iii),
$\theta = \tan^{-1}(r_Y/r_X) = \tan^{-1}(3.17/2.08) = 56.65^o$. Hence,

$$\begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} \cos 56.65^o & \sin 56.65^o \\ -\sin 56.65^o & \cos 56.65^o \end{bmatrix} \begin{Bmatrix} v_X \\ v_Y \end{Bmatrix}$$

(iv)

$$= \begin{bmatrix} 0.550 & 0.835 \\ -0.835 & 0.550 \end{bmatrix} \begin{Bmatrix} -1.96 \\ 7.85 \end{Bmatrix} = \begin{Bmatrix} 5.48 \\ 5.95 \end{Bmatrix} mm/\text{sec.}$$

Continuing,

$$\begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} 0.550 & 0.835 \\ -0.835 & 0.550 \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} 0.550 & 0.835 \\ -0.835 & 0.550 \end{bmatrix} \begin{Bmatrix} -10.68 \\ -85.47 \end{Bmatrix}$$

(vi)

$$= \begin{Bmatrix} -77.2 \\ -38.2 \end{Bmatrix} mm/\text{sec}^2.$$

The transformation has not changed either ***v*** or ***a***'s magnitude.

Given *X, Y* components, find *n-t* (path)components

Where are $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\varepsilon}_n$? Since $\boldsymbol{\varepsilon}_t$ is directed along the path of the trajectory and ***v*** must also be pointed along the particle's trajectory; hence, $\boldsymbol{\varepsilon}_t$ must be collinear with ***v***. From figure 2.12b, $\boldsymbol{\varepsilon}_t$ is pointed at the angle $\beta = \tan^{-1}(v_Y/v_X) = \tan^{-1}(7.85/-1.96) = 104.^o$ relative to the X axis. Also,

$$v_t = v = (v_X^2 + v_Y^2)^{1/2} = 8.090\ mm/\sec\ ;\ v_n = 0\ .$$

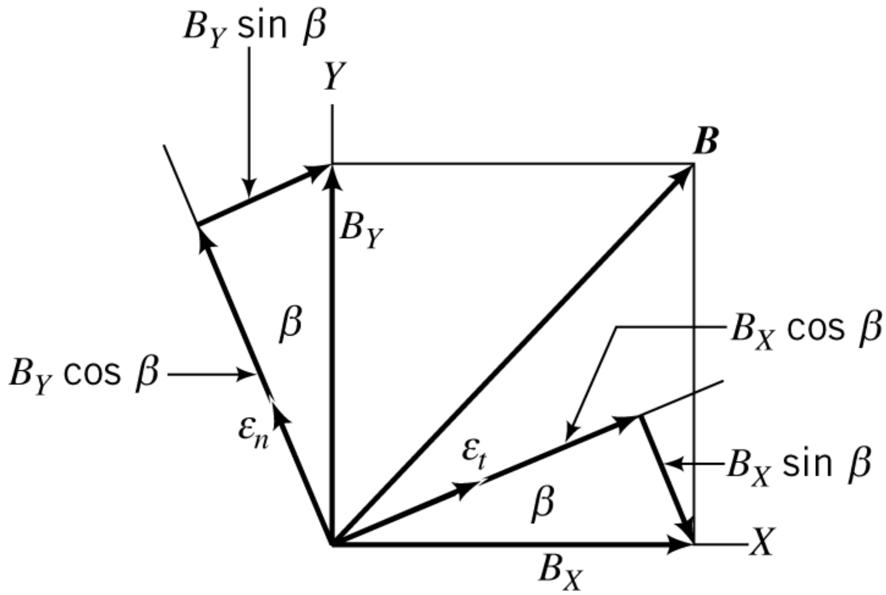

Figure 2.15 Components of the vector ***B*** in the *X*, *Y* system, projected along the $\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n$ unit vectors.

Find: a_t, a_n. From figure 2.15:

$$B_t = B_X \cos\beta + B_Y \sin\beta\ , \quad B_n = -B_X \sin\beta + B_Y \cos\beta\ ,$$

or,

$$\begin{Bmatrix} B_t \\ B_n \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} . \qquad \textbf{(vii)}$$

Applying Eq.**(vii)** to find a_t, a_n gives

$$\begin{Bmatrix} a_t \\ a_n \end{Bmatrix} = \begin{bmatrix} \cos 104.^o & \sin 104.^o \\ -\sin 104.^o & \cos 104.^o \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix}$$

$$= \begin{bmatrix} -0.242 & 0.970 \\ -0.970 & -0.242 \end{bmatrix} \begin{Bmatrix} -10.68 \\ -85.47 \end{Bmatrix} \qquad \textbf{(viii)}$$

$$= \begin{Bmatrix} -80.3 \\ 31.0 \end{Bmatrix} \frac{mm}{\sec^2}.$$

The positive sign for a_n implies that (at this instant) the direction drawn for $\boldsymbol{\varepsilon}_{\boldsymbol{n}}$ in figure 2.14 is correct. A negative sign would imply that $\boldsymbol{\varepsilon}_{\boldsymbol{n}}$ (at this instant) is actually directed negatively from the direction shown in figure 2.14.

LESSON: We are looking at the *same* ***a*** and ***v*** vectors in three different coordinate systems. The component definitions change, but the vectors do not.

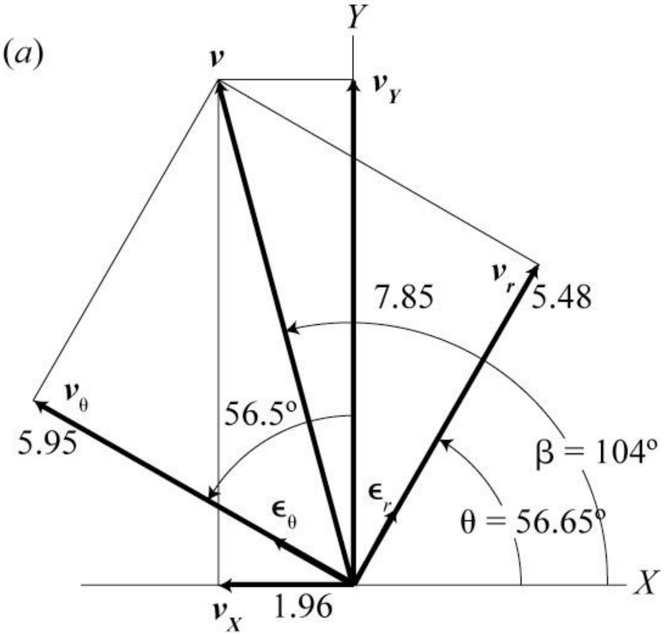

Figure 2.16a Components of ***v*** in the three coordinate systems; mm/sec

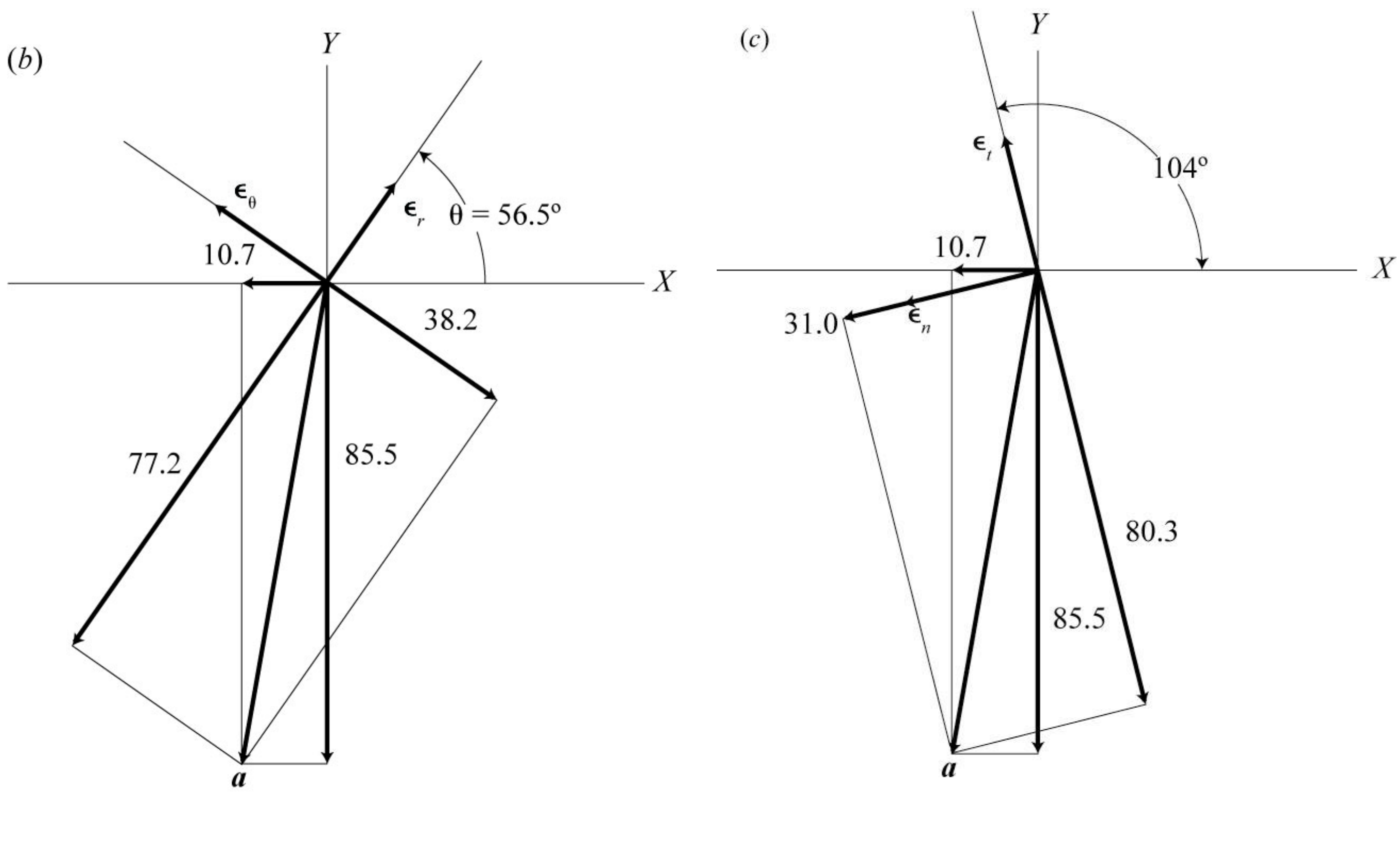

Figure 2.16 Acceleration components: (b) *X*-*Y* and *r*-θ systems, and (c) *X*-*Y* and path systems; mm/sec^2

2.7b An Example that is Naturally Analyzed Using Polar Coordinates

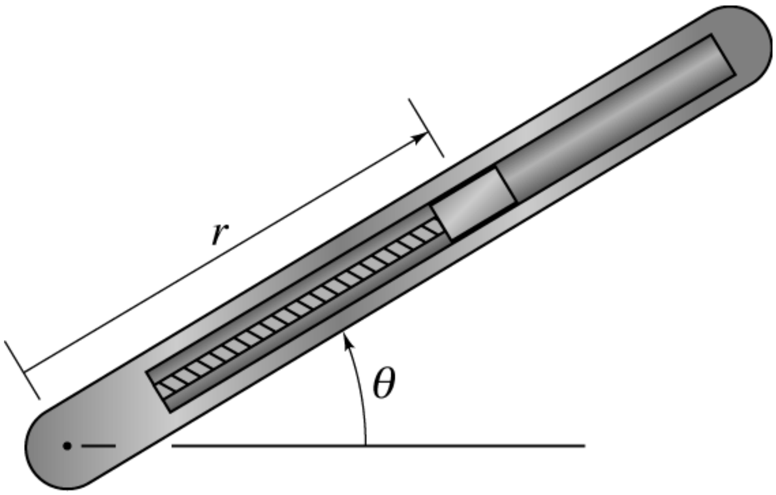

Figure 2.17 A rotating-bar/lag-screw mechanism.

Assuming that the motion is defined by

$$\theta = \frac{\pi}{4}\cos(\omega t) \ , \quad r = 10 + 5\cos(2\omega t)\ mm \ ,$$

where $\omega = 2\pi\ rad/\sec$, the engineering-analysis task is: *At* $\omega t = \pi/6 = 30^o$*, determine the components of the velocity and acceleration vector for point P, and state your results in the* [(X, Y), (r, θ), *and* $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n)$] *coordinate systems.*

First,

$$\dot{\theta} = -\omega\frac{\pi}{4}\sin(\omega t)\ \ rad/\sec, \quad \dot{r} = -10\,\omega\sin(2\,\omega t)\ mm/\sec$$

$$\ddot{\theta} = -\omega^2\frac{\pi}{4}\cos(\omega t)\ \ rad/\sec^2, \quad \ddot{r} = -20\,\omega^2\cos(2\,\omega t)\ mm/\sec^2\ .$$

Hence, for $\omega t = 30^o$ and,

$$\theta = .680\,radians = 39.0^o \ \ , \quad r = 12.5\,mm$$
$$\dot{\theta} = -2.467\,rad/\sec \ \ , \quad \dot{r} = -54.41\,mm/\sec$$
$$\ddot{\theta} = -26.85\,rad/\sec^2 \ \ , \quad \ddot{r} = -394.8\,mm/\sec^2\ .$$

From Eqs.(2.31),

$$v_\theta = r\dot{\theta} = 12.5\times-2.467 = -30.83\,mm/\sec$$
$$v_r = \dot{r} = -54.41\,mm/\sec$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 12.5\times-26.85 + 2\times-54.41\times-2.467$$
$$= -67.17\,mm/\sec^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = -394.8 - 12.5\times(-2.467)^2 = -470.9\,mm/\sec^2\ .$$

Cartesian Components. Since $[A]$ in Eq.(2.40) is orthogonal, $([A]^T = [A]^{-1})$,

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_r \\ B_\theta \end{Bmatrix} . \qquad \textbf{(i)}$$

Hence,

$$\begin{Bmatrix} v_X \\ v_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} \cos 39.0^o & -\sin 39.0^o \\ \sin 39.0^o & \cos 39.0^o \end{bmatrix} \begin{Bmatrix} -54.41 \\ -30.83 \end{Bmatrix}$$

$$= \begin{Bmatrix} -22.88 \\ -58.20 \end{Bmatrix} \frac{mm}{\sec} . \qquad \textbf{(ii)}$$

Similarly,

$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} \cos 39.0^o & -\sin 39.0^o \\ \sin 39.0^o & \cos 39.0^o \end{bmatrix} \begin{Bmatrix} -470.9 \\ -67.17 \end{Bmatrix}$$

$$= \begin{Bmatrix} -323.7 \\ -348.5 \end{Bmatrix} \frac{mm}{\sec^2} . \qquad \textbf{(iii)}$$

Find Path Components: We can use the results of Eq.(ii) to define $\boldsymbol{v}$'s orientation in the X, Y system via

$\beta = \tan^{-1}(v_Y/v_X) = \tan^{-1}(-58.20/-22.8) = 248.6^o$. The components of $\boldsymbol{v}$ in the path system are:

$$v_t = (58.2^2 + 22.88^2)^{1/2} = 62.54\, mm/\sec \ , \ v_n = 0$$

Using the direct transformation result from Eq.(vii),

$$\begin{Bmatrix} B_t \\ B_n \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix}, \qquad \textbf{(vii)}$$

$$\begin{Bmatrix} a_t \\ a_n \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix}$$

$$= \begin{bmatrix} \cos 248.6^o & \sin 248.6^o \\ -\sin 248.6^o & \cos 248.6^o \end{bmatrix} \begin{Bmatrix} -323.7 \\ -348.5 \end{Bmatrix} \qquad \textbf{(iv)}$$

$$= \begin{Bmatrix} 442.6 \\ -174.2 \end{Bmatrix} \frac{mm}{\sec^2} \ .$$

Note in comparing figures 2.15 and 2.18a with 2.18b that $\boldsymbol{\varepsilon}_n$ has reversed directions in accordance with the negative sign for a_n in Eq.(iv).

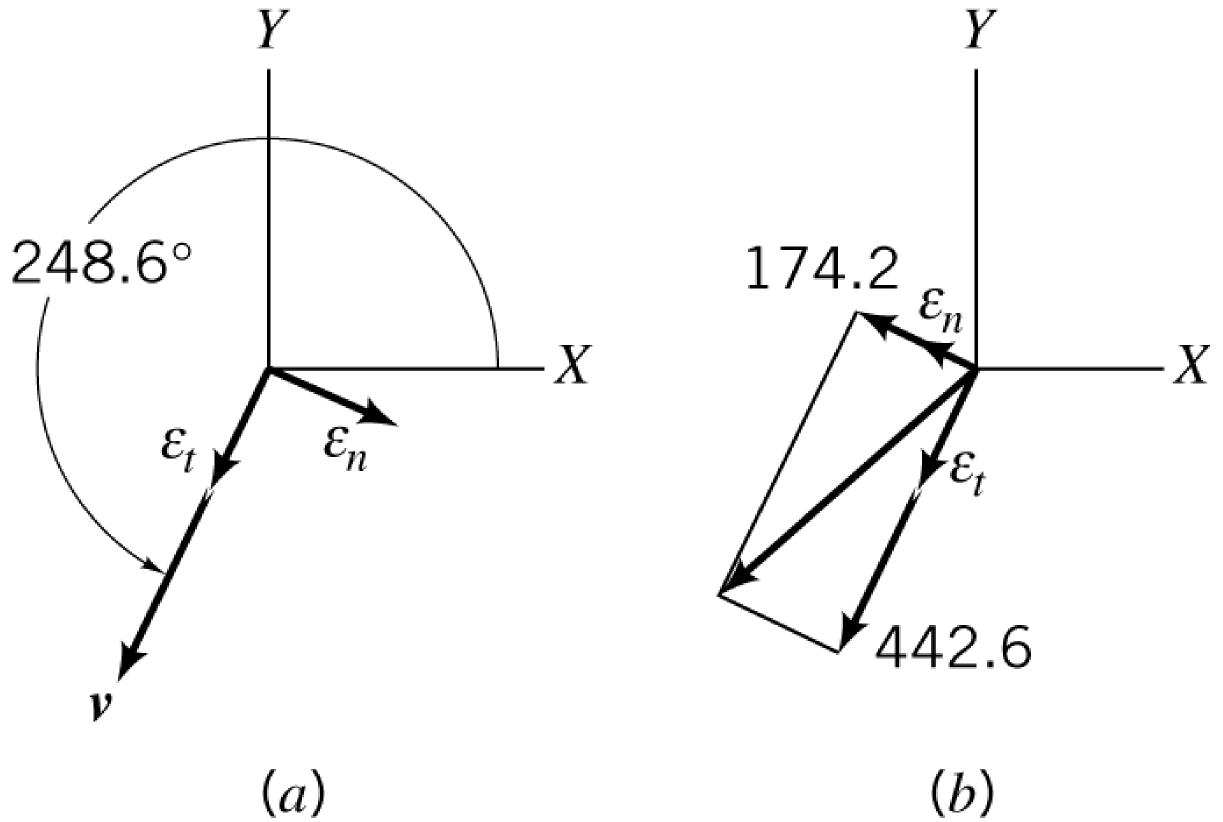

Figure 2.18 (a). Velocity $\boldsymbol{v}$ from Eq. (ii). (b). Acceleration $\boldsymbol{a}$ from Eq.(iv).

2.7c An Example That is Naturally Analyzed with Path-Coordinate Components

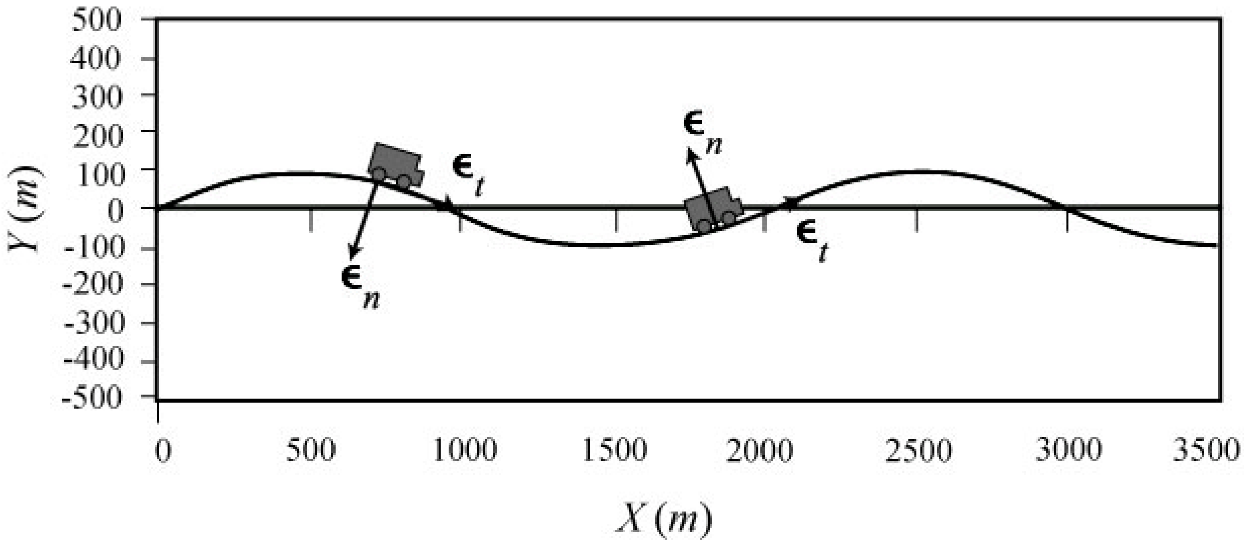

Figure 2.19 Vehicle following the curved path $Y = A\sin(2\pi X/L)$, $A = 100\,m$, $L = 2000\,m$

Figure 2.19 illustrates a particle traveling along a path in a vertical plane defined by,

$$y = A\sin\left(\frac{2\pi x}{L}\right)\ ,\ A = 100\,m\ ,\ L = 2000\,m\ . \tag{i}$$

At $x = 750\,m$, the velocity and acceleration of the vehicle along the path are $v = 100\,km/hr$, and $a_t = 2\,m/\sec^2$. The engineering-analysis task associated with this figure is: *Determine the components of the velocity and acceleration vectors at this position and state the components in the* [(*X*, *Y*), (*r*, θ), *and* $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_n)$] *coordinate systems.*

To find the velocity direction in the X, Y system at $x = 750\,m$, we differentiate Eq.(i) with respect to x, obtaining

$$y' = \frac{dy}{dx} = \frac{2A\pi}{L}\cos\left(\frac{2\pi x}{L}\right) = \frac{2\times 100\pi}{2000}\times\cos\left(\frac{2\pi 750}{2000}\right) = -.222$$

$$\beta = \tan^{-1}(-.222) = -12.52^o \;.$$

Hence, for the position of interest, $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{v}$ are pointed at 12.52^o below the horizontal, and the velocity components in the X, Y system are:

$$v_X = v\cos\beta = 27.78\,m/\sec\times\cos(-12.52^o) = 27.12\,m/\sec$$

$$v_Y = v\sin\beta = 27.78\,m/\sec\times\sin(-12.52^o) = -6.022\,m/\sec \;,$$

where
$v = (100\,km/hr)\times(1000\,m/km)\times(1\,hr/3600\sec) = 27.78\,m/\sec$.

To find $a_n = v^2/\rho$, we need ρ. We have obtained $y' = dy/dx$ above but still need $y'' = d^2y/dx^2$, defined by

$$y'' = -A\left(\frac{2\pi}{L}\right)^2\sin\left(\frac{2\pi x}{L}\right) = -100\left(\frac{2\pi}{2000}\right)^2\times\sin\left(\frac{2\pi\,750}{2000}\right)$$

$$= -6.98\times 10^{-4}\;m^{-1.}$$

Hence,

$$\frac{1}{\rho} = \frac{|y^{//}|}{[1+(y^{/})^2]^{3/2}} = \frac{6.98\times 10^{-4} m^{-1}}{[1+(-.222)^2]^{3/2}} = 6.49\times 10^{-4} m-1$$

$$\rho = 1540 m \ ,$$

and

$$a_n = 27.78^2 \left(\frac{m}{\sec}\right)^2 / 1540 m = .501 m/\sec^2 \ .$$

Figure 2.20a and 2.20b illustrate, respectively, the components of ***v*** and ***a*** in terms of path coordinates. Summing components in the *X* and *Y* directions gives:

$$a_X = a_t \cos 12.5^o - a_n \sin 12.5^o = 2\times .976 - .501\times .216$$
$$= 1.84 m/\sec^2$$

$$a_Y = -a_t \sin 12.5^\circ - a_n \cos 12.5^\circ = -2\times .216 - .501\times .976$$
$$= -.921 m/\sec^2 \ ,$$

and concludes the Cartesian coordinate definition requirements.

Polar-coordinate definitions. We need to first find $r_Y = y(x = 750\,m) = 100\,m \times \sin(2 \times 750\pi/2000) = 70.7\,m$ to define θ as

$$\theta = \tan^{-1}(r_Y/r_X) = \tan^{-1}(70.7/750) = 5.386^o \ .$$

Applying the coordinate transformation of Eq.(2.40) to the Cartesian velocity and acceleration components gives:

$$\begin{Bmatrix} v_r \\ v_\theta \end{Bmatrix} = \begin{bmatrix} \cos 5.39^o & \sin 5.39^o \\ -\sin 5.39^o & \cos 5.39^o \end{bmatrix} \begin{Bmatrix} v_X \\ v_Y \end{Bmatrix}$$
$$= \begin{bmatrix} 0.996 & 0.0939 \\ -0.0939 & 0.996 \end{bmatrix} \begin{Bmatrix} 27.12 \\ -6.02 \end{Bmatrix} = \begin{Bmatrix} 26.44 \\ -8.55 \end{Bmatrix} m/\sec \ , \tag{v}$$

and

$$\begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \begin{bmatrix} \cos 5.39^o & \sin 5.39^o \\ -\sin 5.39^o & \cos 5.39^o \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \end{Bmatrix}$$
$$= \begin{bmatrix} 0.996 & 0.0939 \\ -0.0939 & 0.996 \end{bmatrix} \begin{Bmatrix} 1.84 \\ -.921 \end{Bmatrix} = \begin{Bmatrix} 1.746 \\ -1.09 \end{Bmatrix} m/\sec^2 \ . \tag{vi}$$

These results conclude the engineering-analysis tasks.

The $X = 1750\,m$ location results start by finding β from

$$Y' = \frac{dY}{dX} = \frac{2A\pi}{L}\cos(\frac{2\pi X}{l}) = \frac{2\times 100\pi}{2000}\times\cos(\frac{2\pi 1750}{2000}) = .222$$

$$\therefore\ \beta = \tan^{-1}(.222) = 12.52^\circ\ .$$

v and $\boldsymbol{\varepsilon_t}$ are pointed 12.52° above the horizontal; $\boldsymbol{\varepsilon_n}$ has flipped directions by 180°; v, a_t, a_n, ρ are unchanged. Hence:

$$v_X = v\cos\beta = 27.78\,m/\sec\times\cos(12.52^\circ) = 27.12\,m/\sec$$

$$v_Y = v\sin\beta = 27.78\,m/\sec\times\sin(12.52^\circ) = 6.02\,m/\sec\ ,$$

The orientation of $\boldsymbol{\varepsilon_t}, \boldsymbol{\varepsilon_n}$ is the same in figure 2.19 at $X = 1750\,m$ as in figure 2.10; hence, Eq.(2.40) applies and can be used to obtain ***a***'s components as

$$\begin{Bmatrix} a_X \\ a_Y \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} a_t \\ a_n \end{Bmatrix}$$
$$= \begin{bmatrix} \cos 12.52^\circ & -\sin 12.52^\circ \\ \sin 12.52^\circ & \cos 12.52^\circ \end{bmatrix} \begin{Bmatrix} 2 \\ .501 \end{Bmatrix} = \begin{Bmatrix} 1.844 \\ .9226 \end{Bmatrix} \frac{m}{\sec^2}$$

This concludes the Cartesian-coordinate results. Obtaining the polar-coordinate components from the Cartesian components starts with,

$$\theta = \tan^{-1}(r_Y/r_X) = \tan^{-1}(-70.7/1750) = -2.313° \quad ,$$

and then follows the same steps used earlier.

Lesson (again): The **same** vectors ***v*** and ***a*** have different components in the three different coordinate systems.

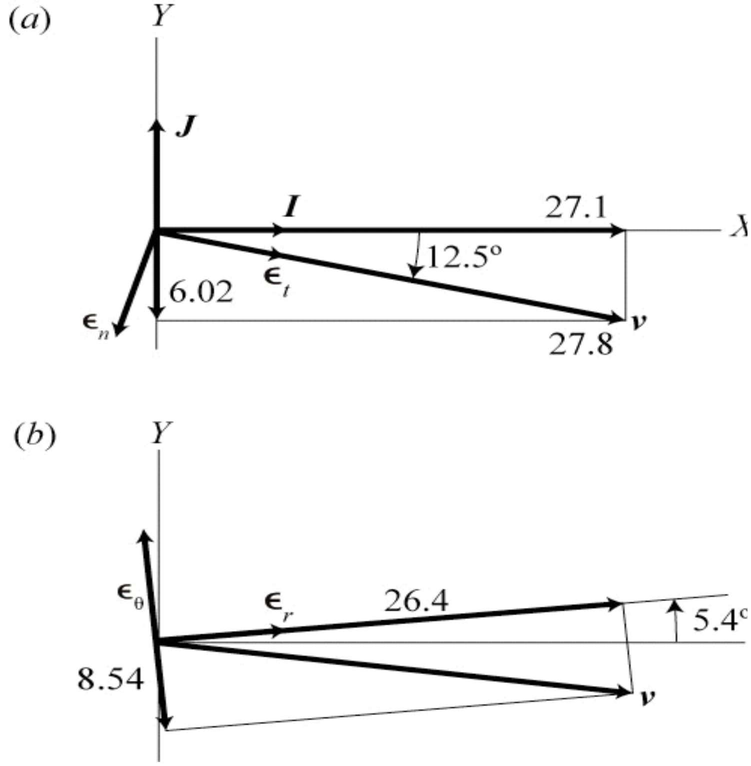

Figure 2.20 Velocity components (m/sec) at $X = 750\,m$, (a) Path and Cartesian components, (b) Polar components

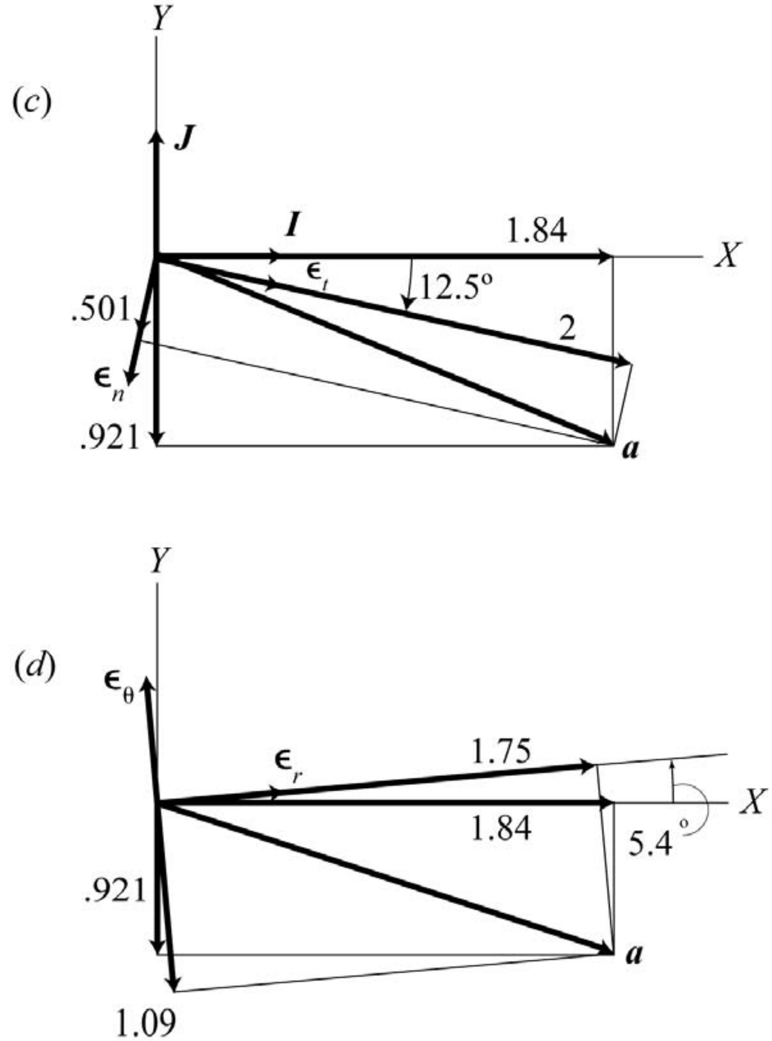

Figure 2.20 Acceleration components (m/sec^2) for $X = 750\,m$, (c) Path and Cartesian components, and (d) Polar and Cartesian components