Lecture 18. PLANAR KINEMATICS OF RIGID BODIES, GOVERNING EQUATIONS

Planar kinematics of rigid bodies involve no new equations. The particle kinematics results of Chapter 2 will be used.

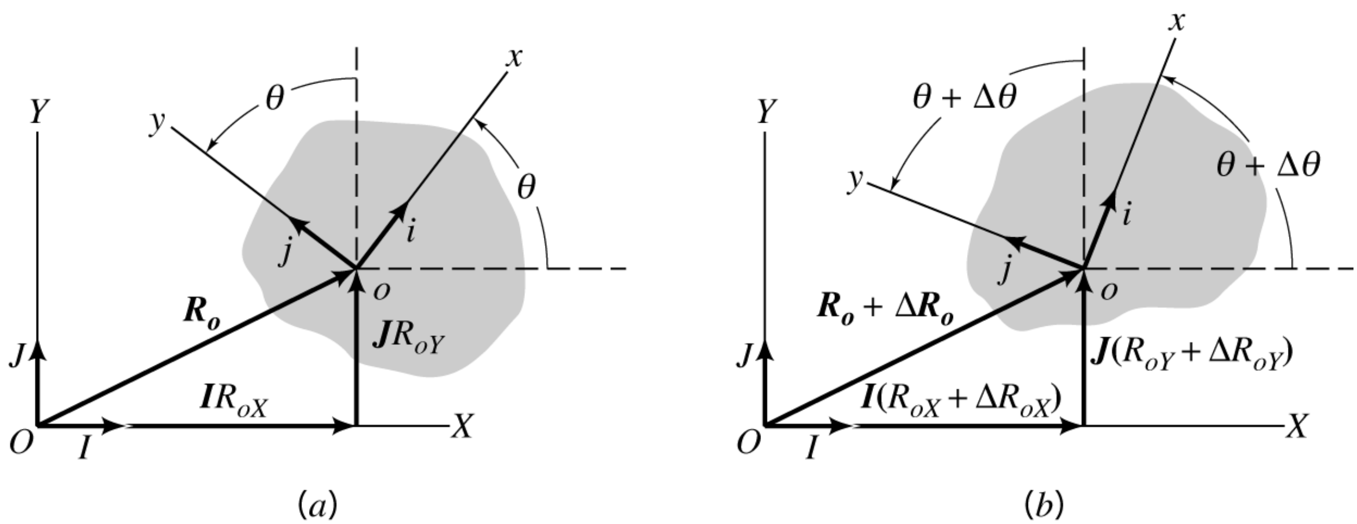


Figure 4.1 Planar motion of a rigid body moving in the plane of the page. Point *o*'s position in the body is defined in the *X*, *Y* coordinate system by $\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$. The orientation of the body with respect to the *X*, *Y* coordinate system is defined by θ. (a). The body at time *t* and orientation θ. (b). The body at a slightly later time $t + \Delta t$ with a new position $\boldsymbol{R_o} + \boldsymbol{\Delta R_o}$ and new orientation $\theta + \Delta\theta$.

The x, y, z coordinate system is fixed to the body; the X, Y, Z system is fixed to "ground."

$\omega = d\theta/dt$ = angular speed of rigid body and x, y, z coordinate system relative to ground or the X, Y, Z system.

Right-hand rule, angular velocity vector of the rigid body and the x, y coordinate system is

$$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} = \boldsymbol{K}\dot{\theta}$$

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} = \boldsymbol{K}\dot{\theta}$ = angular velocity of x, y, z relative to X, Y, Z.

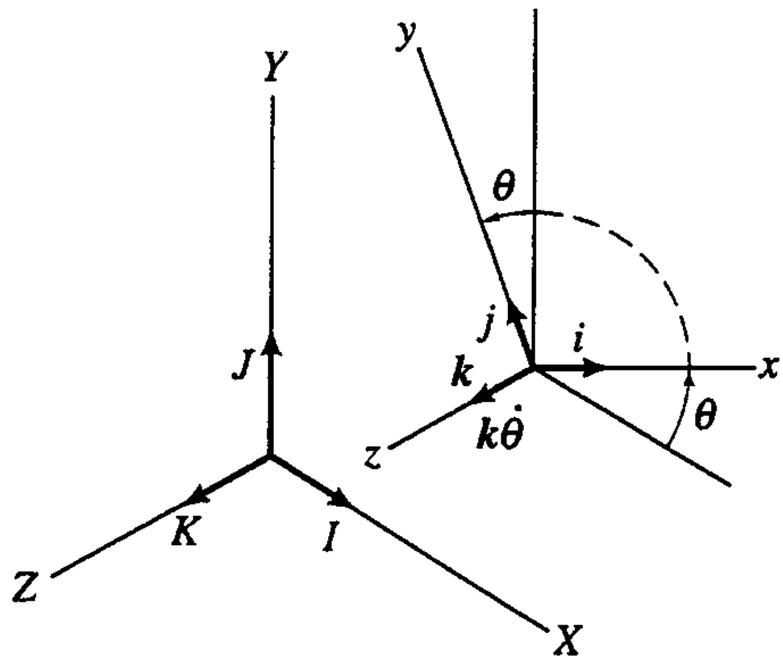


Figure 4.2 Rigid body moving in the X, Y plane with its angular velocity vector $\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ aligned with the z and Z axes.

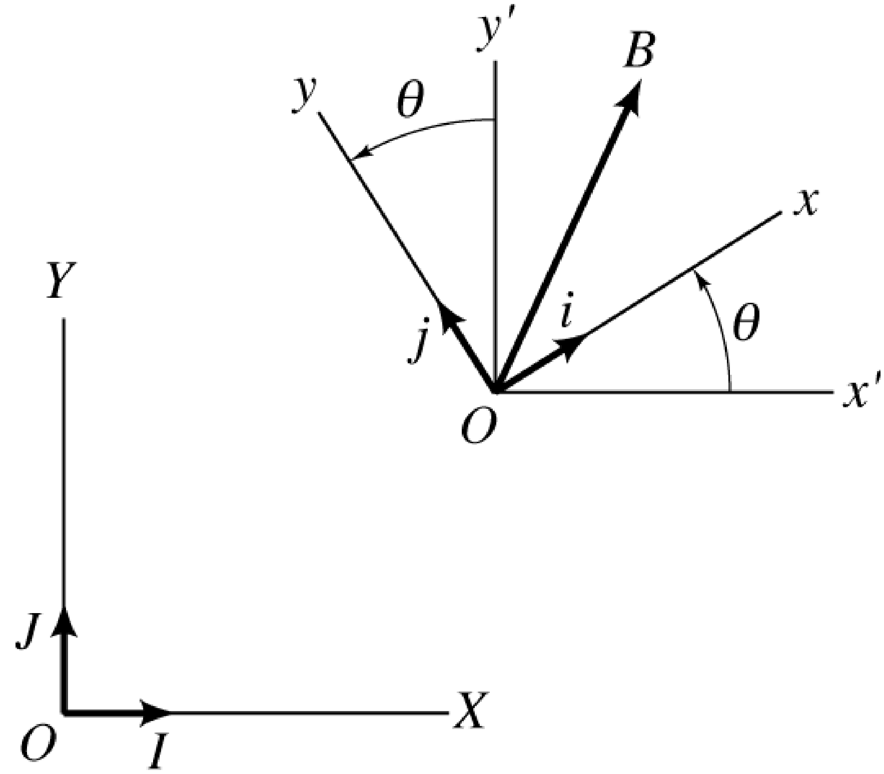


Figure 2.21

X, *Y* and *x*, *y* coordinate systems.

Components of ***B***:

$$\boldsymbol{B} = \boldsymbol{I}\ B_X + \boldsymbol{J}\ B_Y$$
$$\boldsymbol{B} = \boldsymbol{i}\ B_x + \boldsymbol{j}\ B_y\ . \qquad \textbf{(2.41)}$$

Coordinate Transformation for components

$$\begin{Bmatrix} B_x \\ B_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix}\ , \qquad \textbf{(2.42a)}$$

or

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_x \\ B_y \end{Bmatrix} . \qquad \textbf{(2.42b)}$$

Unit vector definitions:

$$\boldsymbol{i} = \boldsymbol{I} \cos\theta + \boldsymbol{J} \sin\theta$$

$$\boldsymbol{j} = -\boldsymbol{I} \sin\theta + \boldsymbol{J} \cos\theta .$$

Derivatives of unit vectors with respect to *X*, *Y* coordinate system:

$$\dot{\boldsymbol{i}} = \frac{d\boldsymbol{i}}{dt}\Big|_{XYZ} = -\boldsymbol{I} \sin\theta\, \dot{\theta} + \boldsymbol{J} \cos\theta\, \dot{\theta} = \boldsymbol{j}\, \dot{\theta} = \boldsymbol{\omega} \times \boldsymbol{i}$$

$$\dot{\boldsymbol{j}} = \frac{d\boldsymbol{j}}{dt}\Big|_{XYZ} = -\boldsymbol{I} \cos\theta\, \dot{\theta} - \boldsymbol{J} \sin\theta\, \dot{\theta} = -\boldsymbol{i}\, \dot{\theta} = \boldsymbol{\omega} \times \boldsymbol{j} .$$

Differentiating $\boldsymbol{B} = \boldsymbol{i}\, B_x + \boldsymbol{j}\, B_y$ with respect to the *X*, *Y* system,

$$\dot{\boldsymbol{B}} = \frac{dB}{dt}\Big|_{XYZ} = \boldsymbol{i}\, \dot{B}_x + \boldsymbol{j}\, \dot{B}_y + B_x \overset{\boldsymbol{\omega} \times \boldsymbol{i}}{\frac{d\boldsymbol{i}}{dt}\Big|_{XYZ}} + B_y \overset{\boldsymbol{\omega} \times \boldsymbol{j}}{\frac{d\boldsymbol{j}}{dt}\Big|_{XYZ}} .$$

or

$$\dot{\boldsymbol{B}} = \boldsymbol{i}\dot{B}_x + \boldsymbol{j}\dot{B}_y + \boldsymbol{\omega} \times (\boldsymbol{i}B_x + \boldsymbol{j}B_y)$$

$$\frac{d\boldsymbol{B}}{dt}\Big|_{XYZ} = \frac{d\boldsymbol{B}}{dt}\Big|_{xyz} + \boldsymbol{\omega} \times \boldsymbol{B}$$

$$\dot{\boldsymbol{B}} = \hat{\dot{\boldsymbol{B}}} + \boldsymbol{\omega} \times \boldsymbol{B} \ .$$

Derivatives with respect to coordinate systems:

$$\dot{\boldsymbol{B}} \triangleq \frac{d\boldsymbol{B}}{dt}\Big|_{XY} = \boldsymbol{I}\ \dot{B}_X + \boldsymbol{J}\ \dot{B}_Y$$

$$\hat{\dot{\boldsymbol{B}}} \triangleq \frac{d\boldsymbol{B}}{dt}\Big|_{x,y} = \boldsymbol{i}\ \dot{B}_x + \boldsymbol{j}\ \dot{B}_y \ .$$

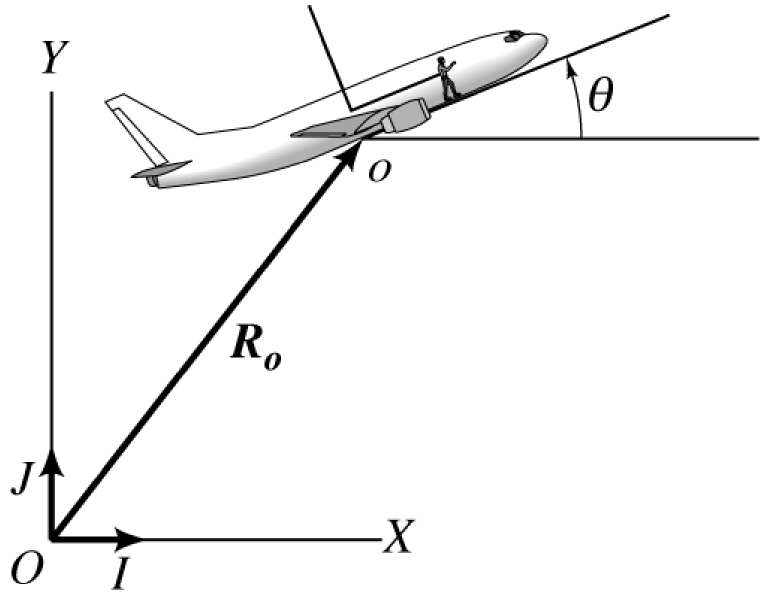


Figure 2.23 An airplane passenger moving down the aisle, as the airplane moves with respect to ground and pitches upwards relative to ground.

VELOCITY AND ACCELERATION RELATIONSHIPS IN TWO COORDINATE SYSTEMS

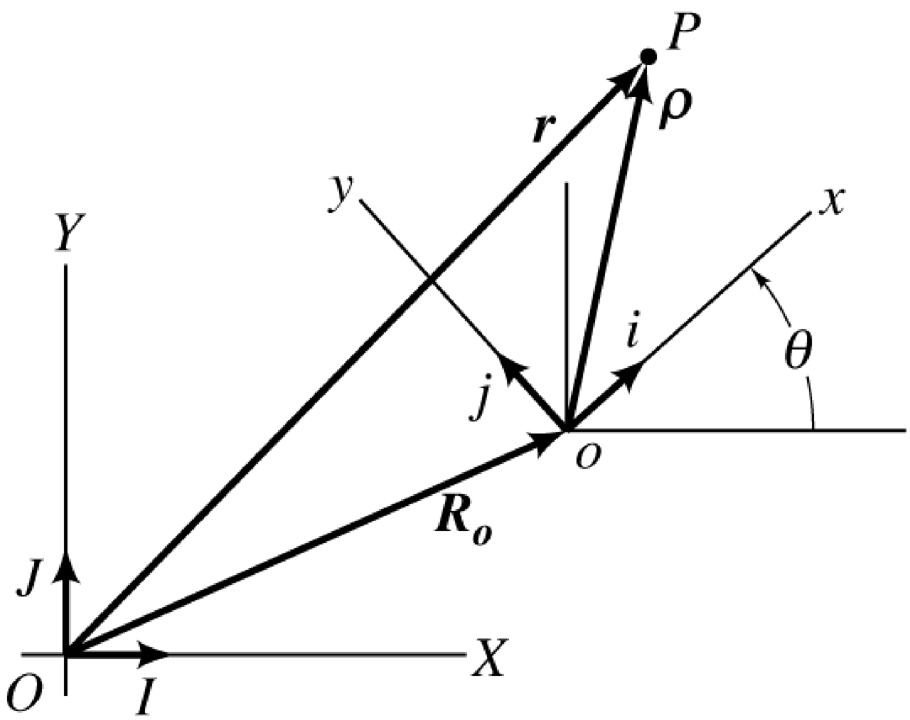


Figure 4.5 Two-coordinate arrangement for general planar kinematics

The point P is located in the X,Y system by

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \; . \tag{2.59}$$

P is located in the x,y,z system by $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y$

Velocity Equations

Taking the time derivative of Eq.(2.59) with respect to the X, Y system yields:

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{X,Y} = \left.\frac{d\boldsymbol{R_o}}{dt}\right|_{X,Y} + \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{X,Y} \tag{2.60}$$

$$= \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\rho}} \; .$$

Applying $\dot{\boldsymbol{B}} = \hat{\dot{\boldsymbol{B}}} + \boldsymbol{\omega} \times \boldsymbol{B}$ to obtain $\dot{\boldsymbol{\rho}} = \hat{\dot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \boldsymbol{\rho}$, nets

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{X,Y} = \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{x,y} + \boldsymbol{\omega} \times \boldsymbol{\rho} \; , \tag{2.62a}$$

or

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \hat{\dot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \; . \tag{2.62b}$$

Acceleration Equations
Differentiating Eq.(2.62b),

$$\ddot{\boldsymbol{r}} = \left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{X,Y} = \ddot{\boldsymbol{R}}_o + \left.\frac{d\hat{\dot{\boldsymbol{\rho}}}}{dt}\right|_{X,Y} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{X,Y} . \quad \textbf{(2.63)}$$

Applying $\dot{\boldsymbol{B}} = \hat{\dot{\boldsymbol{B}}} + \boldsymbol{\omega} \times \boldsymbol{B}$,

$$\left.\frac{d\hat{\dot{\boldsymbol{\rho}}}}{dt}\right|_{X,Y} = \left.\frac{d\hat{\dot{\boldsymbol{\rho}}}}{dt}\right|_{x,y} + \boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} = \hat{\ddot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} . \quad \textbf{(2.64)}$$

nets

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_o + \hat{\ddot{\boldsymbol{\rho}}} + 2\boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} + (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) , \quad \textbf{(2.65)}$$

Comparisons to Polar-Coordinate Definitions

Parallels between the present vector results and earlier polar-coordinate definitions for velocity and acceleration become more apparent if we require that $\boldsymbol{R}_o = 0$ and that $\boldsymbol{\rho}$ lies along the x axis; i.e.,

$$\boldsymbol{\rho} = \boldsymbol{i}x . \quad \textbf{(2.66)}$$

Hence,

$$\hat{\dot{\boldsymbol{\rho}}} = \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{x,y} = \boldsymbol{i}\,\dot{x}$$

$$\hat{\ddot{\boldsymbol{\rho}}} = \left.\frac{d^2\boldsymbol{\rho}}{dt^2}\right|_{x,y} = \boldsymbol{i}\,\ddot{x}$$

(2.67)

$$\boldsymbol{\omega}\times\boldsymbol{\rho} = \boldsymbol{k}\dot{\theta}\times\boldsymbol{i}x = \boldsymbol{j}\,x\dot{\theta}$$

$$\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} = \boldsymbol{k}\ddot{\theta}\times\boldsymbol{i}x = \boldsymbol{j}\,x\ddot{\theta}$$

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) = \boldsymbol{k}\dot{\theta}\times\boldsymbol{j}x\dot{\theta} = -\boldsymbol{i}\,x\dot{\theta}^2$$

$$\boldsymbol{\omega}\times\hat{\dot{\boldsymbol{\rho}}} = \boldsymbol{k}\dot{\theta}\times\boldsymbol{i}\dot{x} = \boldsymbol{j}\,\dot{x}\dot{\theta}$$

Substitution into Eq.(2.62) (with $\dot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$) gives

$$\dot{\boldsymbol{r}} = \boldsymbol{i}\dot{x} + \boldsymbol{j}x\dot{\theta}\ .$$

Comparing this result to

$$\dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{\varepsilon}_{\boldsymbol{r}} + r\dot{\theta}\,\boldsymbol{\varepsilon}_{\boldsymbol{\theta}} = v_r\,\boldsymbol{\varepsilon}_{\boldsymbol{r}} + v_{\theta}\,\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}\ . \qquad \textbf{(2.27)}$$

shows the following parallel in physical terms:

$$\boldsymbol{\varepsilon_r}\ \dot{\boldsymbol{r}} = \boldsymbol{i}\ \dot{x} = \hat{\dot{\boldsymbol{\rho}}}$$

$$\boldsymbol{\varepsilon_\theta}\ r\dot{\theta} = \boldsymbol{j}\ \dot{\theta}\ x = \boldsymbol{\omega} \times \boldsymbol{\rho}$$

For comparison of the acceleration terms, substituting from Eq.(2.67) into Eq.(2.65) gives (with $\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$) :

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \boldsymbol{i}\ddot{x} + 2\boldsymbol{j}\dot{\theta}\dot{x} + \boldsymbol{j}\ddot{\theta}x - \boldsymbol{i}\dot{\theta}^2 x \\ &= \boldsymbol{i}\,(\ddot{x} - x\dot{\theta}^2) + \boldsymbol{j}(x\ddot{\theta} + 2\,\dot{x}\dot{\theta}) \end{aligned} \qquad \textbf{(2.68)}$$

By comparison to the polar-coordinate definition,

$$\begin{aligned} \ddot{\boldsymbol{r}} &= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{\varepsilon_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\varepsilon_\theta} \\ &= a_r\,\boldsymbol{\varepsilon_r} + a_\theta\,\boldsymbol{\varepsilon_\theta}\ . \end{aligned} \qquad \textbf{(2.30)}$$

the following physical equivalence of terms is established:

$$\boldsymbol{\varepsilon_r}\ddot{r} = \boldsymbol{i}\,\ddot{x} = \hat{\ddot{\boldsymbol{\rho}}}$$

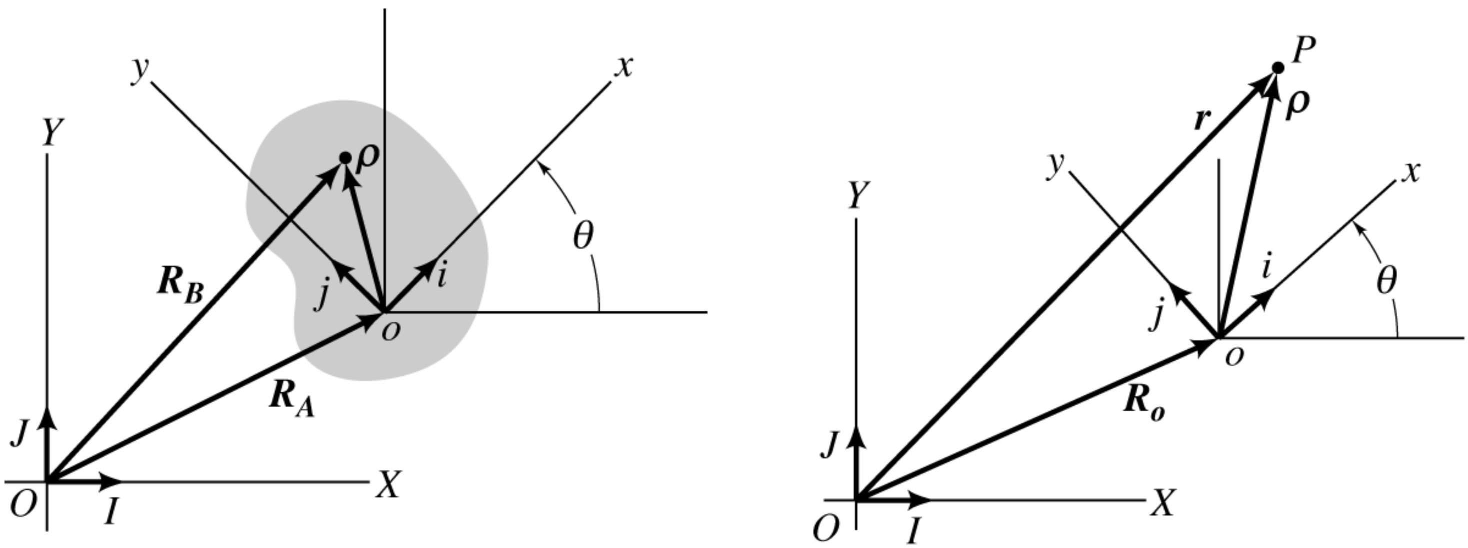
$$2\,\boldsymbol{\varepsilon_\theta}\dot{r}\dot{\theta} = 2\boldsymbol{j}\dot{x}\dot{\theta} = 2\,\boldsymbol{\omega}\times\hat{\dot{\boldsymbol{\rho}}}\ \ (\textit{Coriolis acceleration term})$$

$$\boldsymbol{\varepsilon_\theta} r\ddot{\theta} = \boldsymbol{j}\,x\ddot{\theta} = \dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}$$

$$-\boldsymbol{\varepsilon_r} r\dot{\theta}^2 = -\boldsymbol{i}\,x\dot{\theta}^2 = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})(\textit{Centrifugal acceleration term})$$

Hence, Eq.(2.65) merely presents old physical terms in a new vector format.

VELOCITY AND ACCELERATION RELATIONSHIPS FOR TWO POINTS IN A RIGID BODY



Velocity Equations

The x, y, z system is **<u>fixed in a rigid body</u>**.

$$\boldsymbol{R_o} \Rightarrow \boldsymbol{r_A} \ , \ \boldsymbol{r} \Rightarrow \boldsymbol{r_B}$$

$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$ = position vector locating a point **<u>in the rigid body</u>**. Hence,

$$\hat{\dot{\boldsymbol{\rho}}} = \frac{d\boldsymbol{\rho}}{dt}\Big|_{x,y,z} = 0 \ , \ \hat{\ddot{\boldsymbol{\rho}}} = \frac{d^2\boldsymbol{\rho}}{dt^2}\Big|_{x,y,z} = 0 \ .$$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_o + \hat{\dot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \Rightarrow \dot{\boldsymbol{r}}_B = \dot{\boldsymbol{r}}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AB} ,$$

and

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_o + \hat{\ddot{\boldsymbol{\rho}}} + 2\boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} + (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) ,$$

becomes

$$\ddot{\boldsymbol{r}}_B = \ddot{\boldsymbol{r}}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AB})$$

Alternatively,

$$\boxed{\begin{aligned} \boldsymbol{v}_B &= \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AB} \\ \boldsymbol{a}_B &= \boldsymbol{a}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AB}) \end{aligned}}$$

Example Problem 4.1

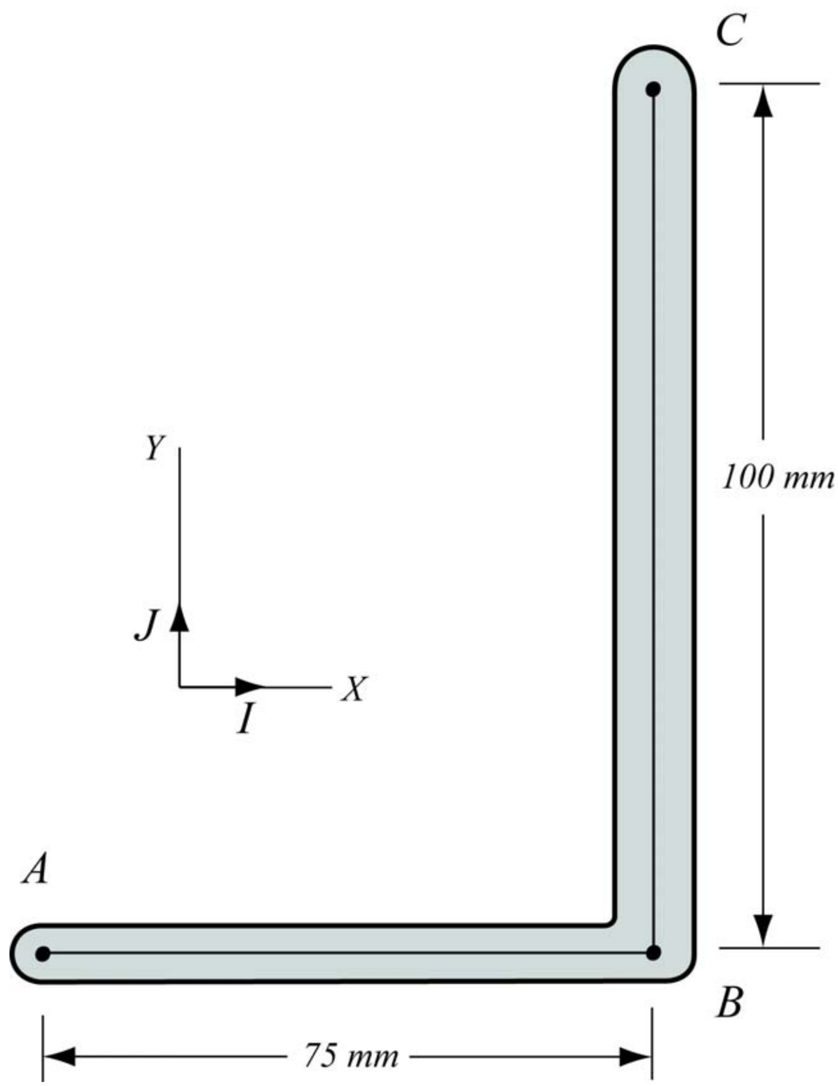


Figure XP4.1a Bar with points *A*, *B*, and *C*

Given:

$$\boldsymbol{v}_A = 5.\boldsymbol{I}\, mm/\sec \ , \ \boldsymbol{a}_A = .5\boldsymbol{I}\ mm/\sec^2 \qquad \textbf{(i)}$$

$$\boldsymbol{\omega} = 0.1\boldsymbol{K} rad/\sec \ , \ \dot{\boldsymbol{\omega}} = .02\boldsymbol{K}\ rad/\sec^2 \qquad \textbf{(ii)}$$

Tasks:

a. Determine the velocity and acceleration vectors for points *C* and *B*.

b. Draw the velocity and acceleration vectors for points *A*, *B*, and *C*.

Solution: Applying the first of Eqs.(4.3) gives

$$\boldsymbol{v}_{\boldsymbol{B}} = \boldsymbol{v}_{\boldsymbol{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\boldsymbol{AB}} = 5\boldsymbol{I} + .1\boldsymbol{K} \times 75\boldsymbol{I} = 5\boldsymbol{I} + 7.5\boldsymbol{J}\ \ mm/\text{sec}$$

$$\begin{aligned}\boldsymbol{v}_{\boldsymbol{C}} &= \boldsymbol{v}_{\boldsymbol{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\boldsymbol{AC}} = 5\boldsymbol{I} + .1\boldsymbol{K} \times (75\boldsymbol{I} + 100\boldsymbol{J}) \\ &= (5 - 10)\boldsymbol{I} + 7.5\boldsymbol{J}\ \ mm/\text{sec} \\ &= -5\boldsymbol{I} + 7.5\boldsymbol{J} mm/\text{sec}\ \ .\end{aligned}$$

The acceleration vectors of points *B* is obtained from the second of Eqs.(4.3) as

$$\begin{aligned}\boldsymbol{a}_{\boldsymbol{B}} &= \boldsymbol{a}_{\boldsymbol{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\boldsymbol{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\boldsymbol{AB}}) \\ &= .5\boldsymbol{I} + .02\boldsymbol{K} \times 75.\boldsymbol{I} + 0.1\boldsymbol{K} \times [0.1\boldsymbol{K} \times 75.\boldsymbol{I}] = .5\boldsymbol{I} + 1.5\boldsymbol{J} - .75I \\ &= -.25I + 1.5\boldsymbol{J}\ \ mm/\text{sec}^2\ \ .\end{aligned}$$

Similarly for point *C*

$$\begin{aligned}\boldsymbol{a}_{\boldsymbol{C}} &= \boldsymbol{a}_{\boldsymbol{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\boldsymbol{AC}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\boldsymbol{AC}}) \\ &= .5\boldsymbol{I} + .02\boldsymbol{K} \times (75.\boldsymbol{I} + 100\boldsymbol{J}) + 0.1\boldsymbol{K} \times [0.1\boldsymbol{K} \times (75.\boldsymbol{I} + 100\boldsymbol{J})] \\ &= .5I + (1.5\boldsymbol{J} - 2.\boldsymbol{I}) - (.75\boldsymbol{I} + 1.\boldsymbol{J}) = -2.25\boldsymbol{I} + .5\boldsymbol{J}\ \ mm/\text{sec}^2\ \ .\end{aligned}$$

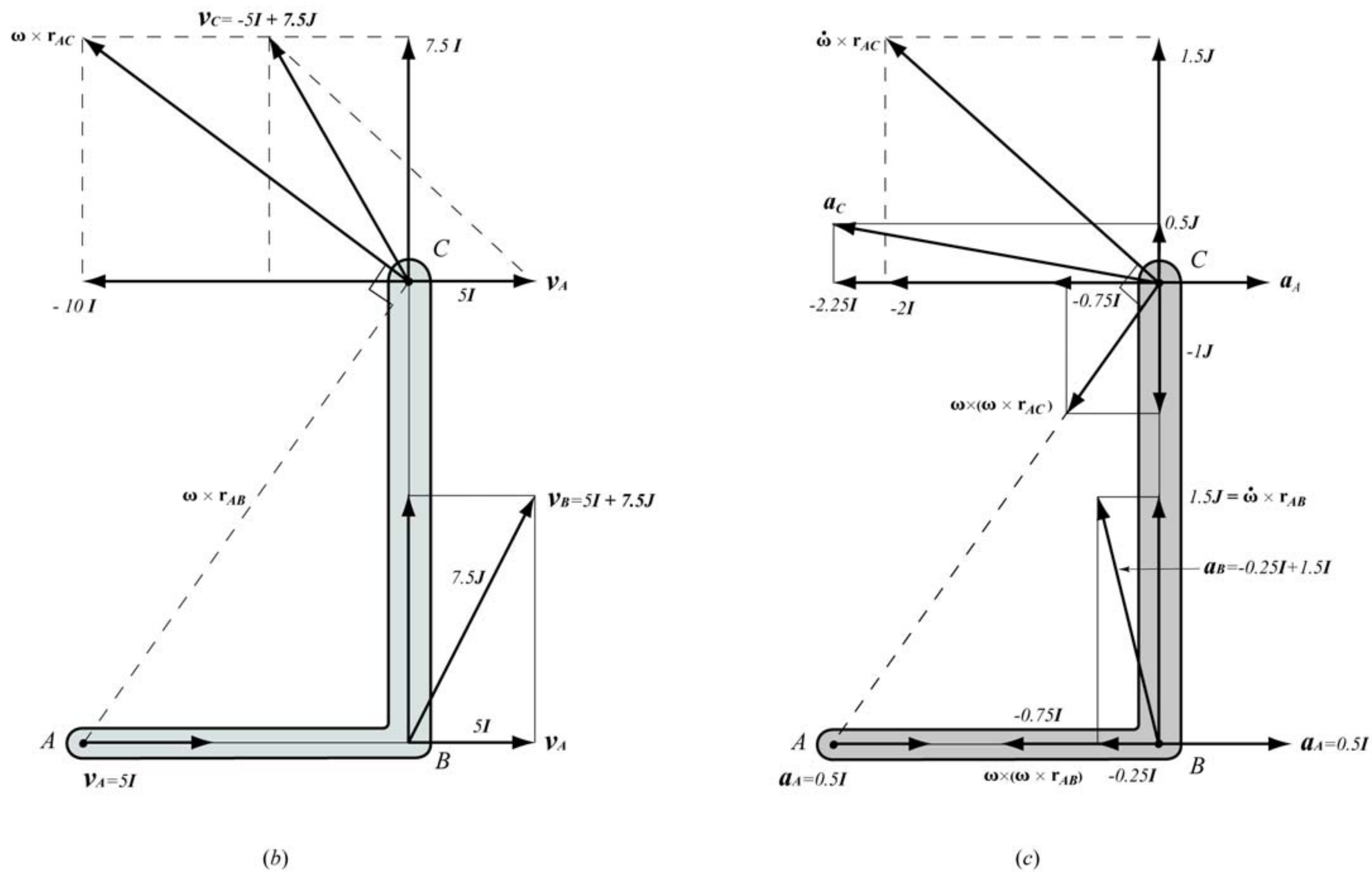
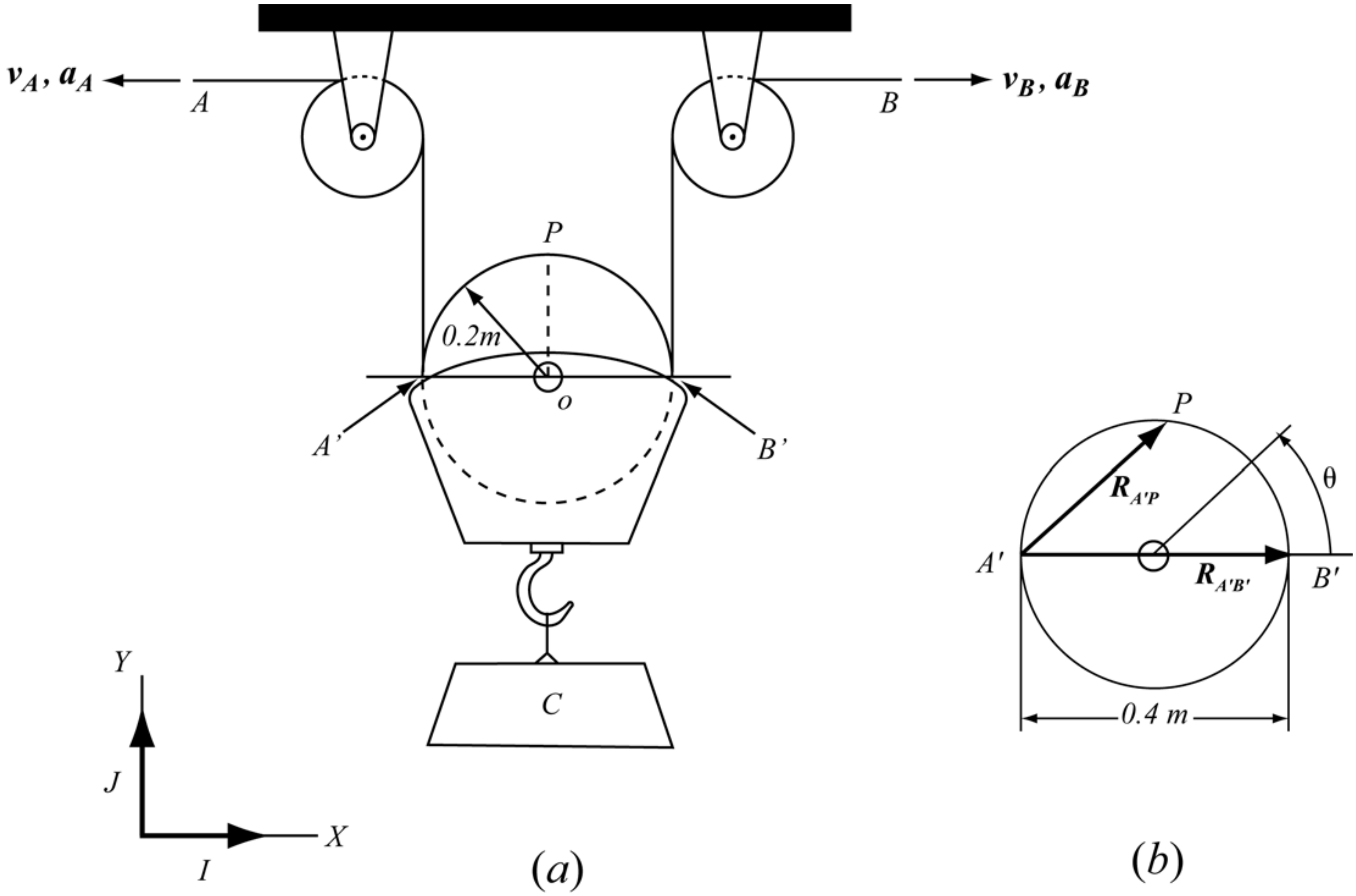


Figure XP4.1 (b) Velocity (mm/sec) and (c) Acceleration ($mm/\sec^2$)

Example Problem 4.2



Example Problem 4.2 Point *A*, the end of the left cable has a velocity of .6 m/sec and an acceleration of .13 m/sec^2 . Point *B*, the end of the right cable has a velocity of 1.2 m/sec and an acceleration of -.13m/sec^2. The central pulley has a radius of 0.4 *m*. Point *o* is at the center of the pulley, and point *P* is at the top of the pulley

Tasks: Determine the velocity and acceleration of points *o* and *P*.

Solution: We are going to use Eqs.(4.3), the velocity and acceleration vectors for two points on a rigid body to work through this example. The rigid body is the central pulley illustrated in figure 4.2b. Since the cable is inextensible, points A' and B' on the central pulley have the same velocities and *vertical* acceleration magnitudes as A and B, respectively. Specifically,

$$\boldsymbol{v}_{A'} = .6\boldsymbol{J}\ m/\sec, \quad \boldsymbol{v}_{B'} = 1.2\ \boldsymbol{J}\ m/\sec$$

$$\boldsymbol{a}_{A'Y} = 0.13\boldsymbol{J}\ m/\sec^2\ ,\ \boldsymbol{a}_{B'Y} = -.13\boldsymbol{J}\ m/\sec^2\ . \tag{i}$$

Velocity results. We start this development knowing $\boldsymbol{v}_{A'}, \boldsymbol{v}_{B'}$, and needing $\boldsymbol{v}_o, \boldsymbol{v}_P$. We could use the velocity relationship to find these unknowns providing that we knew $\boldsymbol{\omega} = \boldsymbol{K}\omega = \boldsymbol{K}\dot{\theta}$. We can determine $\boldsymbol{\omega}$ by applying $\boldsymbol{v}_B' = \boldsymbol{v}_A' + \boldsymbol{\omega} \times \boldsymbol{r}_{A'B'}$ for points A' and B' on the pulley. Substituting for $\boldsymbol{v}_{A'}, \boldsymbol{v}_{B'}$ from Eq.(i) and $\boldsymbol{r}_{A'B'} = 0.4\boldsymbol{I}\ m$ gives

$$1.2\boldsymbol{J}\frac{m}{\sec} = 0.6\boldsymbol{J}\frac{m}{\sec} + \boldsymbol{K}\dot{\theta}\frac{rad}{\sec} \times .4\boldsymbol{I}m$$

$$\dot{\theta} = \frac{.6}{.4}\frac{rad}{\sec} = 1.5\frac{rad}{\sec} \tag{ii}$$

Proceeding with this result for $\boldsymbol{\omega}$ gives:

$$\boldsymbol{v_o} = \boldsymbol{v_{A'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{A'o}}) = .6\boldsymbol{J}\frac{m}{\sec} + (1.5\boldsymbol{K}\frac{rad}{\sec} \times .2\boldsymbol{I}m) = .9\boldsymbol{J}\frac{m}{\sec}$$

$$\boldsymbol{v_P} = \boldsymbol{v_{A'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{A'P}}) = .6\boldsymbol{J}\frac{m}{\sec} + [1.5\boldsymbol{K}\frac{rad}{\sec} \times (.2\boldsymbol{I} + .2\boldsymbol{J})m]$$
$$= (.6\boldsymbol{J} + .3\boldsymbol{J} - .3\boldsymbol{I})\frac{m}{\sec} = (.9\boldsymbol{J} - .3\boldsymbol{I})\frac{m}{\sec}$$

As expected, $\boldsymbol{v_o}'s$ velocity is vertically upwards. Because of the pulley's rotation, point *P* has a velocity component in the -*X* direction. Note that we could have just as easily used the equations, $\boldsymbol{v_o} = \boldsymbol{v_{B'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{B'o}})$ and $\boldsymbol{v_P} = \boldsymbol{v_{B'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{B'P}})$ to get these results since we know $\boldsymbol{v_{B'}}$.

Acceleration Development. Recall in the sentence above Eq.(i) that the ***vertical*** acceleration of points A', B' have the same magnitudes, respectively, as $\boldsymbol{a_{A'}}, \boldsymbol{a_{B'}}$. The acceleration vectors $\boldsymbol{a_{A'}}, \boldsymbol{a_{B'}}$ have horizontal components due to the pulley's rotation; hence, $\boldsymbol{a_{A'}} \neq .13\boldsymbol{J}m/\sec^2$, and $\boldsymbol{a_{B'}} \neq -.13\boldsymbol{J}m/\sec^2$. We can verify this statement starting with $\boldsymbol{a_o} = a_o\boldsymbol{J}$, the acceleration of point *o*, even though a_o is unknown. Applying the acceleration equation from Eqs.(4.3) gives:

$$a_A' = a_O + \dot{\omega} \times r_{OA'} + \omega \times (\omega \times r_{OA'}) = Ja_o + (K\ddot{\theta} \times -.2I) + [K\dot{\theta} \times (K\dot{\theta} \times -.2I)]$$
$$= Ja_o - J.2\ddot{\theta} + I.2\dot{\theta}^2 = J(a_O - .2\ddot{\theta}) + I.2\dot{\theta}^2 \ m/\sec^2$$

$$a_B' = a_O + \dot{\omega} \times r_{OB'} + \omega \times (\omega \times r_{OB'}) = Ja_o + (K\ddot{\theta} \times .2I) + [K\dot{\theta} \times (K\dot{\theta} \times .2I)]$$
$$= Ja_o + J.2\ddot{\theta} - I.2\dot{\theta}^2 = J(a_O + .2\ddot{\theta}) - I.2\dot{\theta}^2 \ m/\sec^2 \ . \tag{iv}$$

Note particularly the horizontal components of $\boldsymbol{a}_{A'}, \boldsymbol{a}_{B'}$ arising from the centripetal acceleration term $r\dot{\theta}^2$ induced by the pulley rotation. With that result firmly in mind, we can proceed to solve for $\boldsymbol{a}_o, \boldsymbol{a}_P$.

The first step is solving for $\dot{\boldsymbol{\omega}}$. Applying $\boldsymbol{a}_{B'} = \boldsymbol{a}_{a'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{A'B'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{A'B'})$ gives

$$(-.2\dot{\theta}^2 I - .13J) = (.2\dot{\theta}^2 I + .13J) + (K\ddot{\theta} \times .4I) + [K\dot{\theta} \times (K\dot{\theta} \times .4I)]$$
$$= .2\dot{\theta}^2 I + .13J + .4\ddot{\theta}J - .4\dot{\theta}^2 I \ m/\sec^2$$

Taking the ***I*** and ***J*** components separately gives:

$$I: \ -.2\dot{\theta}^2 = .2\dot{\theta}^2 - .4\dot{\theta}^2 \qquad J: -.13 = .13 + .4\ddot{\theta}$$

$$\therefore \ \ddot{\theta} = -\frac{.26}{.4}\frac{rad}{\sec^2} = -.65\frac{rad}{\sec^2} \ .$$

The *X* component result gave nothing; the *Y* component allowed us to solve for $\ddot{\theta}$. At this point, we are in a position to directly solve for $\boldsymbol{a}_o, \boldsymbol{a}_P$ as:

$$\boldsymbol{a_o} = \boldsymbol{a_{A'}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{A'o}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{A'o}})$$
$$= (.2\dot{\theta}^2 \boldsymbol{I} + .13\boldsymbol{J}) + (\boldsymbol{K}\ddot{\theta} \times .2\boldsymbol{I}) + [\boldsymbol{K}\dot{\theta} \times (\boldsymbol{K}\dot{\theta} \times .2\boldsymbol{I})]$$
$$= .2\dot{\theta}^2 \boldsymbol{I} + .13\boldsymbol{J} + .2\ddot{\theta}\boldsymbol{J} - .2\dot{\theta}^2 \boldsymbol{I} = (.13 + .2 \times -.65)\boldsymbol{J} = -1.17\boldsymbol{J}\ m/\sec^2$$

$$\boldsymbol{a_P} = \boldsymbol{a_{A'}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{A'P}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{A'P}})$$
$$= (.2\dot{\theta}^2 \boldsymbol{I} + .13\boldsymbol{J}) + \boldsymbol{K}\ddot{\theta} \times (.2\boldsymbol{I} + .2\boldsymbol{J}) + \boldsymbol{K}\dot{\theta} \times [\boldsymbol{K}\dot{\theta} \times (.2\boldsymbol{I} + .2\boldsymbol{J})]$$
$$= .2\dot{\theta}^2 \boldsymbol{I} + .13\boldsymbol{J} + .2\ddot{\theta}(\boldsymbol{J} - \boldsymbol{I}) - .2\dot{\theta}^2(\boldsymbol{I} + \boldsymbol{J}) = -.2\ddot{\theta}\boldsymbol{I} + (.13 + .2\ddot{\theta} - .2\dot{\theta}^2)\boldsymbol{J}$$
$$= (-.2 \times -.65)\boldsymbol{I} + [.13 + (.2 \times -.65) - .2 \times 1.5^2]\boldsymbol{J} = -1.3\boldsymbol{I} - 1.5\boldsymbol{J}\ m/\sec^2\ .$$

As expected, point *o*'s acceleration is vertical. Also, *P*'s vertical acceleration is entirely due to the $r\dot{\theta}^2$ term. This last step could have proceeded equally well from B' instead of A', since we also know $\boldsymbol{a_{B'}}$.

The customary development in this type of problem uses the following sequential steps:

(1). Starting with known velocities at two points, in this case A' and B', calculate $\boldsymbol{\omega}$. Then using a known velocity and $\boldsymbol{\omega}$ calculate any additional required velocities.

(2). Starting with known accelerations at two points, calculate $\dot{\boldsymbol{\omega}}$. Then using one of the known accelerations plus $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$, calculate other required accelerations.

Lecture 19. ROLLING WITHOUT SLIPPING

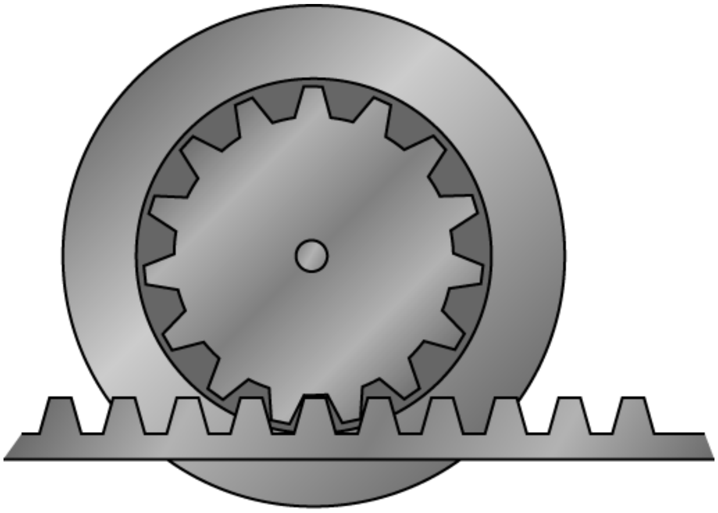


Figure 4.8 Gear rolling in geared horizontal guides.

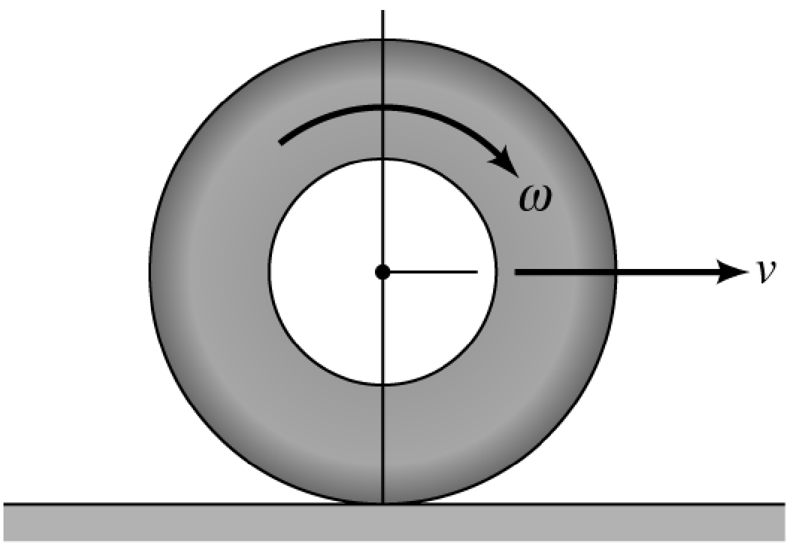


Figure 4.9 Wheel rolling on a horizontal surface.

The derivation and understanding of velocity and acceleration relationships for a wheel that is rolling without slipping is the fundamental objective of this lecture.

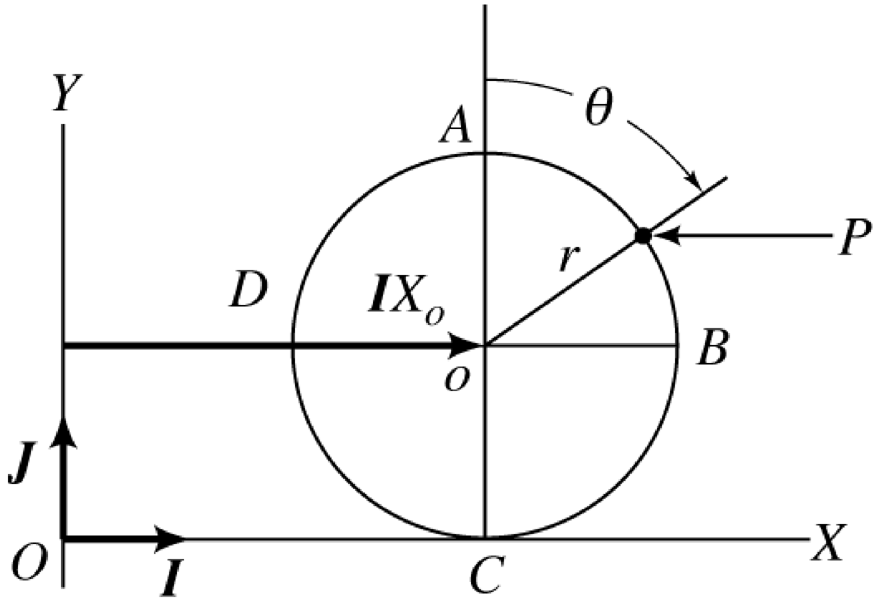


Figure 4.10 Wheel *rolling without slipping* on a horizontal surface.

Geometric Development

The wheel in figure 4.10 advances to the right as θ increases. The question is: *Without slipping, how is the rotation angle* θ *related to the displacement of the wheel center* X_o *?* Note first that the contact point between the wheel, denoted as *C*, advances to the right precisely the same distance as point *o*. If the wheel starts with θ and X_o at zero, and rolls forward through one rotation *without slipping*, both the new contact point and the now displaced point *o* will have moved to the right a distance equal to the circumference of the wheel; i.e., $X_o = 2\pi r$. It may help you to think of the wheel as a paint roller and imagine the length of the paint strip that would be laid out on the plane during one rotation. Comparable to the result for a full rotation, *without slipping* the geometric constraint relating X_o and θ is

$$X_o = r\theta \ . \tag{4.4}$$

Differentiating with respect to time gives :

$$\dot{X}_o = r\dot{\theta} \ , \quad \ddot{X}_o = r\ddot{\theta} \ . \tag{4.5}$$

These are the desired kinematic constraint equations for a wheel that is rolling without slipping.

We want to define the trajectory of the point P on the wheel located by θ. The coordinates of the displacement vector locating P are defined by

$$\begin{aligned} X &= X_o + r\sin\theta = r\theta + r\sin\theta \\ Y &= r\cos\theta \ + \ r \ . \end{aligned} \tag{4.6}$$

The trajectory followed by point P as the wheel rolls to the right is the cycloid traced out in figure 4.11. The cycloid was obtained for $r = 1$ by varying θ through two cycles of rotation to generate coordinates $X(\theta)$ and $Y(\theta)$ and then plotting Y versus X .

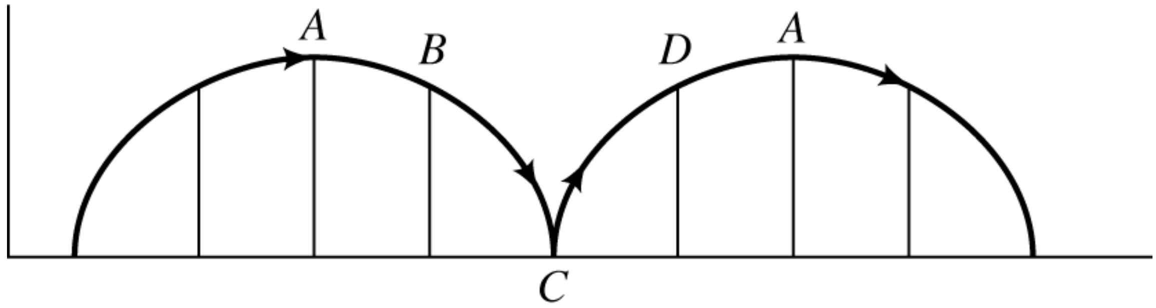


Figure 4.11 Cycloidal path traced out by a point on a wheel that is rolling without slipping. The letters *A* through *D* on the wheel indicate locations occupied by point *P* on the cycloid.

Referring to the locations *A* through *D* of figures 4.10 and 4.11, point *P* starts at *A*, reaches *B* after the wheel has rotated $\pi/2$ radians, reaches the contact location after π radians, reaches *D* after $3\pi/2$ radians and then returns to *A* after a full rotation.

Differentiating the components of the position vector locating *P* (defined by Eq.(4.6)) to obtain components of the velocity vector for point *P* with respect to the *X,Y* coordinate system gives:

$$\begin{aligned} \dot{X} &= r\dot{\theta} + r\dot{\theta}\cos\theta \\ \dot{Y} &= -r\dot{\theta}\sin\theta \ . \end{aligned} \tag{4.7}$$

Note that $\dot{X}$ and $\dot{Y}$ are components of the velocity vector of a point *P* on the wheel, located by the angle θ. The corresponding acceleration components are:

$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$

$$\ddot{Y} = -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \quad . \tag{4.8}$$

Eqs.(4.7) and (4.8) can be used to evaluate the instantaneous velocity and acceleration components of any point on the wheel by specifying an appropriate value for θ. Values for θ = 0, π/2, π, and 3π/2 correspond, respectively, to locations *A*, *B*, *C*, and *D* in figure 4.10.

Position A (*top of the wheel;* θ = 0)*:*

$$\dot{X} = r\dot{\theta} + r\dot{\theta}(1) = 2r\dot{\theta} = 2\dot{X}_o$$

$$\dot{Y} = -r\dot{\theta}(0) = 0$$

$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}(1) - r\dot{\theta}^2(0) = 2r\ddot{\theta} = 2\ddot{X}_o$$

$$\ddot{Y} = -r\ddot{\theta}(0) - r\dot{\theta}^2(1) = -r\dot{\theta}^2 \quad . \tag{4.9}$$

Position B (*right-hand side of the wheel;* θ = π/2) *:*

$$\begin{aligned}
\dot{X} &= r\dot{\theta} + r\dot{\theta}(0) = r\dot{\theta} = \dot{X}_o \\
\dot{Y} &= -r\dot{\theta}(1) = -r\dot{\theta} \\
\ddot{X} &= r\ddot{\theta} + r\ddot{\theta}(0) - r\dot{\theta}^2(1) = r\ddot{\theta} - r\dot{\theta}^2 = \ddot{X}_o - r\dot{\theta}^2 \\
\ddot{Y} &= -r\ddot{\theta}(1) - r\dot{\theta}^2(0) = -r\ddot{\theta} \;.
\end{aligned} \tag{4.10}$$

Position C (*bottom of the wheel at the contact location;* θ = π) *:*

$$\begin{aligned}
\dot{X} &= r\dot{\theta} + r\dot{\theta}(-1) = 0 \\
\dot{Y} &= -r\dot{\theta}(0) = 0 \\
\ddot{X} &= r\ddot{\theta} + r\ddot{\theta}(-1) - r\dot{\theta}^2(0) = 0 \\
\ddot{Y} &= -r\ddot{\theta}(0) - r\dot{\theta}^2(-1) = r\dot{\theta}^2 \;.
\end{aligned} \tag{4.11}$$

Position D (*left-hand side of the wheel;* θ = 3π/2)*:*

$$\begin{aligned}
\dot{X} &= r\dot{\theta} + r\dot{\theta}(0) = r\dot{\theta} = \dot{X}_o \\
\dot{Y} &= -r\dot{\theta}(-1) = r\dot{\theta} \\
\ddot{X} &= r\ddot{\theta} + r\ddot{\theta}(0) - r\dot{\theta}^2(-1) = r\ddot{\theta} + r\dot{\theta}^2 = \ddot{X}_o + r\dot{\theta}^2 \\
\ddot{Y} &= -r\ddot{\theta}(-1) - r\dot{\theta}^2(0) = r\ddot{\theta} \;.
\end{aligned} \tag{4.12}$$

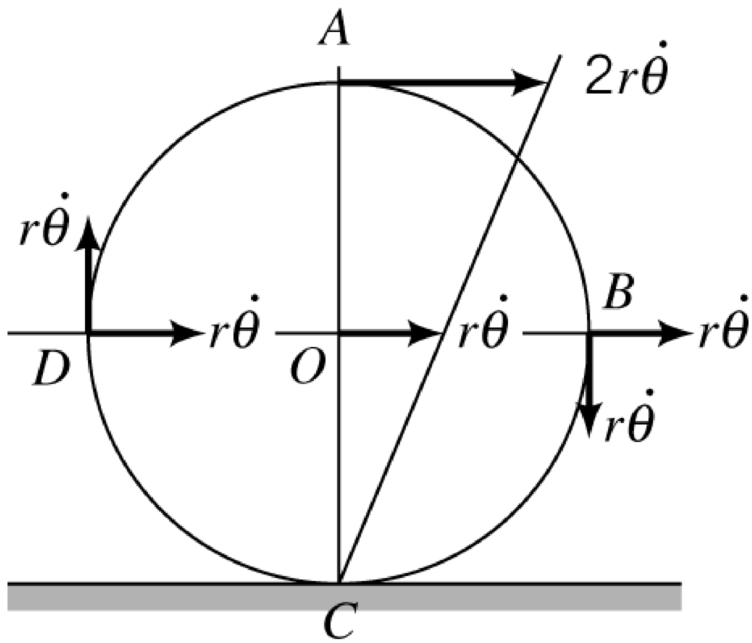


Figure 4.12 Velocity vectors for points on the wheel at locations *A* through *D* and *o*.

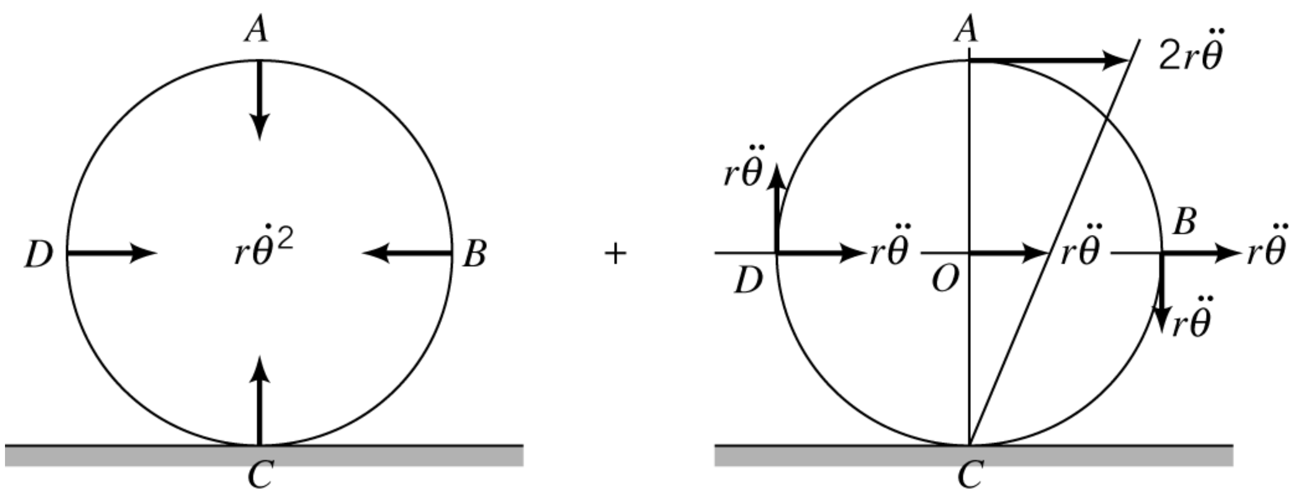


Figure 4.13 Acceleration vectors for points on the wheel at locations *A* through *D*.

The point P in contact with the ground has zero velocity *at the instant of contact*. Constant-velocity components over a *finite* time period are required to give zero acceleration. *Note carefully that a point on the wheel at the contact location has a vertical acceleration of* $r\dot{\theta}^2$.

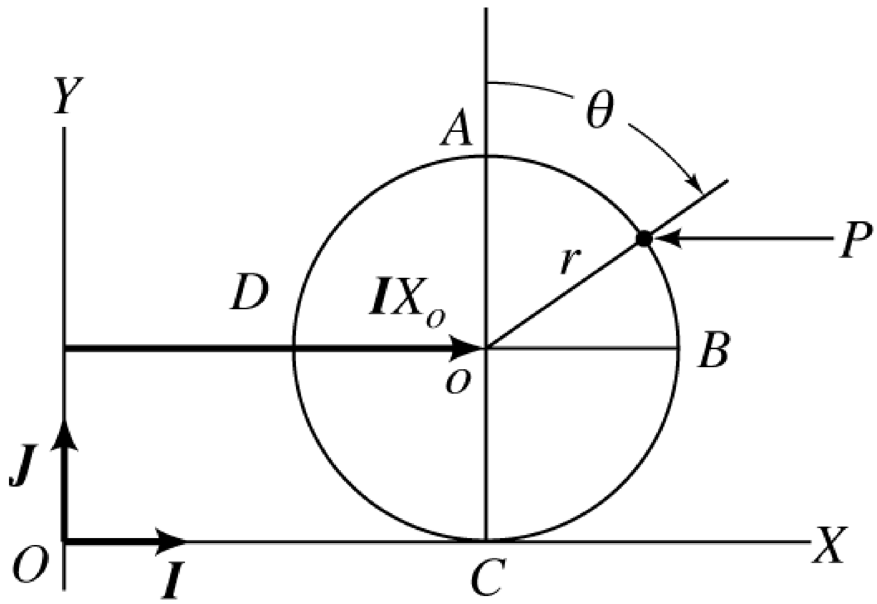
Geometric approach :

a. State (write out) the geometric X and Y component equations.

b. Differentiate the displacement component equations to obtain velocity component equations.

c. Differentiate the velocity component equations to obtain acceleration component equations.

Vector Development of Velocity Relationships



$\boldsymbol{v_o} = \boldsymbol{I}\dot{X}_o$ for point o. Using the right-hand-rule convention for defining angular velocity vectors, the wheel's angular velocity vector is $\boldsymbol{\omega} = \boldsymbol{K}(-\dot{\theta}) = -\boldsymbol{K}\dot{\theta}$. (If the wheel were rolling to the left, $\dot{\theta}$ would be negative, and the angular velocity vector would be $\boldsymbol{\omega} = +\boldsymbol{K}\,|\,\dot{\theta}\,|$.) C is a point on the wheel at the instantaneous contact location between the wheel and the ground, and has a velocity of zero; i.e., $\boldsymbol{v_C} = \boldsymbol{I}\,0 + \boldsymbol{J}\,0$.

Vector Velocity and Acceleration Relationships

$$\boxed{\begin{aligned} \boldsymbol{v_B} &= \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{AB}} \\ \boldsymbol{a_B} &= \boldsymbol{a_A} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{AB}}) \ . \end{aligned}} \qquad \textbf{(4.3)}$$

Applying the first of Eqs.(4.3) to points o and C gives

$$\boldsymbol{v_C} = \boldsymbol{v_o} + \boldsymbol{\omega} \times \boldsymbol{r_{oC}} \ .$$

Setting $\boldsymbol{v_C}$ to zero and substituting: $\boldsymbol{v_o} = \boldsymbol{I}\dot{X}_o$, $\boldsymbol{\omega} = -\boldsymbol{K}\dot{\theta}$, $\boldsymbol{r_{oC}} = -\boldsymbol{J}r$, gives

$$0 = \boldsymbol{I}\dot{X}_o + (-\boldsymbol{K}\dot{\theta} \times -\boldsymbol{J}r) \Rightarrow 0 = \boldsymbol{I}(\dot{X}_o - r\dot{\theta}) \ .$$

Hence, the rolling-without-slipping kinematic result for velocity is (again)

$$\dot{X}_o = r\dot{\theta} \ . \qquad \textbf{(4.5a)}$$

Predictably, the vector approach has given us the rolling-without-slipping kinematic condition for velocity.

Find the velocity vectors for points *A* through *D* of figure 4.10. Starting with point *A*, and applying the velocity relationship from Eqs.(4.3) to points *A* and *C* on the wheel gives

$$\boldsymbol{v}_A = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{r}_{CA} \ .$$

Substituting: $\boldsymbol{v}_C = \boldsymbol{0}$, $\boldsymbol{\omega} = -\boldsymbol{K}\dot{\theta}$, and $\boldsymbol{r}_{CA} = \boldsymbol{J}\,2r$ yields

$$\boldsymbol{v}_A = 0 - \boldsymbol{K}\dot{\theta} \times \boldsymbol{J}2r \ = \ \boldsymbol{I}2r\dot{\theta} \ .$$

The velocity of a point on the wheel at location *B* can be found by applying Eqs.(4.3) as:

$$\boldsymbol{v}_B = \boldsymbol{v}_o + \boldsymbol{\omega} \times \boldsymbol{r}_{oB}$$

$$\boldsymbol{v}_B = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{r}_{CB} \ .$$

The first equation defines $\boldsymbol{v}_B$ by starting from a known velocity at point *o* ; the second equation starts from a known velocity at point *C*. The vectors $\boldsymbol{v}_o$, $\boldsymbol{v}_C$, and $\boldsymbol{\omega}$ have already been identified. The required new vectors are $\boldsymbol{r}_{oB} = \boldsymbol{I}\,r$ and $\boldsymbol{r}_{CB} = \boldsymbol{I}r + \boldsymbol{J}r$.
Substitution gives:

$$\boldsymbol{v}_B = \boldsymbol{I}r\dot{\theta} - \boldsymbol{K}\dot{\theta} \times \boldsymbol{I}r \ = \boldsymbol{I}r\dot{\theta} - \boldsymbol{J}r\dot{\theta}$$

$$\boldsymbol{v}_B = 0 - \boldsymbol{K}\dot{\theta} \times (\boldsymbol{I}r + \boldsymbol{J}r) \ = \boldsymbol{I}r\dot{\theta} - \boldsymbol{J}r\dot{\theta} \ .$$

The velocity of point *D* can be obtained by any of the following:

$$\boldsymbol{v}_D = \boldsymbol{v}_o + \boldsymbol{\omega} \times \boldsymbol{r}_{oD}$$

$$\boldsymbol{v}_D = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AD}$$

$$\boldsymbol{v}_D = \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{r}_{BD}$$

$$\boldsymbol{v}_D = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{r}_{CD} \ .$$

We know the velocity vectors for point *o*, *A*, *B*, and *C*, and can write expressions for the vectors $\boldsymbol{r}_{oD}$, $\boldsymbol{r}_{AD}$, $\boldsymbol{r}_{BD}$, and $\boldsymbol{r}_{CD}$. Applying (arbitrarily) the last equation with $\boldsymbol{r}_{CD} = -\boldsymbol{I}\mathrm{r} + \boldsymbol{J}\mathrm{r}$ gives

$$\boldsymbol{v}_D = 0 - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{I}r + \boldsymbol{J}r) = \boldsymbol{I}r\dot{\theta} + \boldsymbol{J}r\dot{\theta} \ .$$

Results From Vector Developments of Acceleration

In this subsection, we will use Eqs.(4.3), relating the acceleration vectors of two points on a rigid body to: (i) derive the rolling-without-slipping acceleration result of Eq.(4.5) ($\ddot{X}_o = r\ddot{\theta}$) and (ii) determine the acceleration vectors of points *A* through *D*. Starting with points *o* and *C*, we can apply the acceleration result of Eq.(4.3) as

$$\boldsymbol{a}_C = \boldsymbol{a}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{oC} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{oC}) \ .$$

Substituting $\boldsymbol{a_o} = \boldsymbol{I}\ddot{X}_o$, $\dot{\boldsymbol{\omega}} = -\boldsymbol{K}\ddot{\theta}$, and our earlier result $\boldsymbol{r_{oC}} = -r\boldsymbol{J}$ gives:

$$\boldsymbol{a_C} = (\boldsymbol{I}a_{CX} + \boldsymbol{J}a_{CY}) = \boldsymbol{I}\ddot{X}_o + (-\boldsymbol{K}\ddot{\theta} \times -\boldsymbol{J}r) - \boldsymbol{K}\dot{\theta} \times (\boldsymbol{K}\dot{\theta} \times -\boldsymbol{J}r) \ ,$$

$$\boldsymbol{I}a_{CX} + \boldsymbol{J}a_{CY} = \boldsymbol{I}(\ddot{X}_o - r\ddot{\theta}) + \boldsymbol{J}r\dot{\theta}^2 \ .$$

Equating the $\boldsymbol{I}$ and $\boldsymbol{J}$ components gives:

$$\begin{aligned} &\boldsymbol{I} \ : \ \ a_{CX} = (\ddot{X}_o - r\ddot{\theta}) = 0 \ \Rightarrow \ \ \ddot{X}_o = r\ddot{\theta} \\ &\boldsymbol{J} \ : \ \ a_{CY} = r\dot{\theta}^2 \ . \end{aligned} \tag{4.13}$$

The accelerations of points A, B and D can also be obtained via Eqs.(4.3), starting from any point on the wheel where the acceleration vector is known. Choosing point o (arbitrarily) gives

$$\boldsymbol{a_A} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oA}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oA}})$$

$$\boldsymbol{a_B} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oB}})$$

$$\boldsymbol{a_D} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oD}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oD}}) \ .$$

Substituting for the variables on the right-hand side of these equations gives

$$\boldsymbol{a}_{\boldsymbol{A}} = \boldsymbol{I} r\ddot{\theta} + (-\boldsymbol{K}\ddot{\theta} \times \boldsymbol{J} r) - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{K}\dot{\theta} \times \boldsymbol{J} r) = \boldsymbol{I} 2r\ddot{\theta} - \boldsymbol{J} r\dot{\theta}^2$$

$$\boldsymbol{a}_{\boldsymbol{B}} = \boldsymbol{I} r\ddot{\theta} + (-\boldsymbol{K}\ddot{\theta} \times \boldsymbol{I} r) - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{K}\dot{\theta} \times \boldsymbol{I} r) = \boldsymbol{I}(r\ddot{\theta} - r\dot{\theta}^2) - \boldsymbol{J} r\ddot{\theta}$$

$$\boldsymbol{a}_{\boldsymbol{D}} = \boldsymbol{I} r\ddot{\theta} + (-\boldsymbol{K}\ddot{\theta} \times -\boldsymbol{I} r) - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{K}\dot{\theta} \times -\boldsymbol{I} r)$$
$$= \boldsymbol{I}(r\ddot{\theta} + r\dot{\theta}^2) + \boldsymbol{J} r\ddot{\theta}.$$

Example Problem 4.3

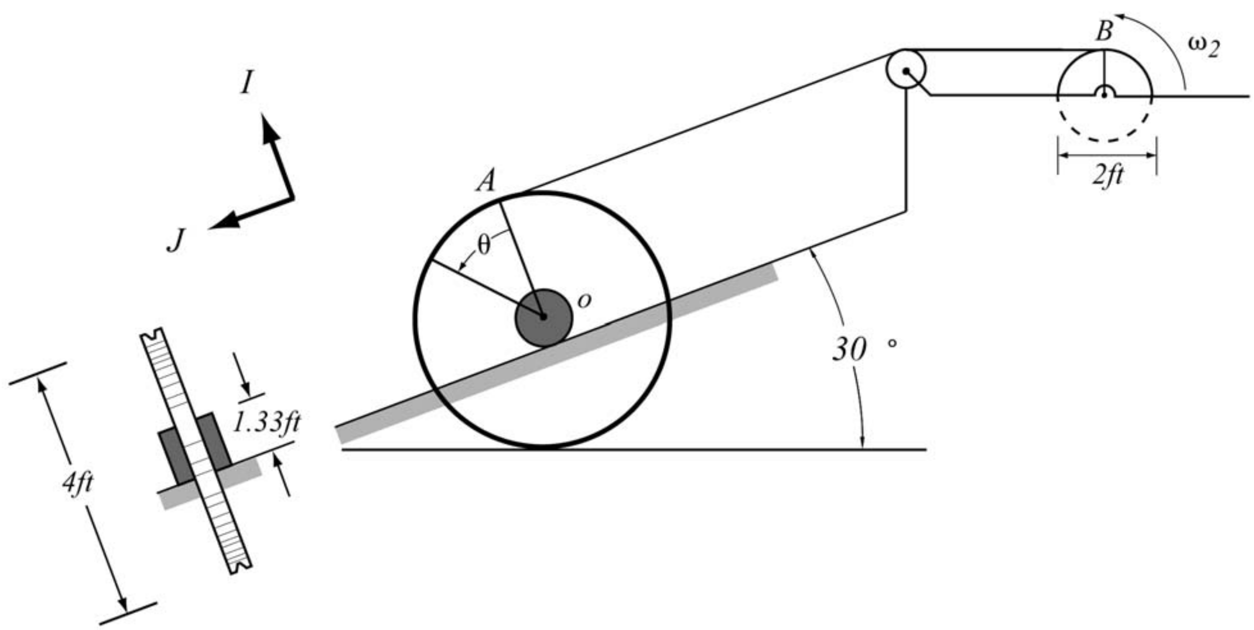


Figure XP4.3 Rolling-wheel assembly

The lower wheel assembly is rolling without slipping on a plane that is inclined at $30°$ to the horizontal. It is connected to the top spool via an inextensible cable that is playing out cable. The center of the lower spool and its contact point are denoted, respectively by o and C. Point A denotes the top of the spool. At the instant of interest, the acceleration of the center of the lower spool is $\boldsymbol{a}_{\boldsymbol{o}} = 6.44\boldsymbol{J}\,ft/\sec^2$, and its velocity is $\boldsymbol{v}_{\boldsymbol{o}} = 2.6\boldsymbol{J}\,ft/\sec$.

Tasks: For the instant considered, determine the velocity and acceleration of points C and A. Determine the top spool's angular velocity and acceleration.

Solution.
Rolling without slipping $\Rightarrow$ $\boldsymbol{v}_{\boldsymbol{C}} = 0$, plus from Eqs.(4.5)

$$\dot{X}_o = 2.6\,ft/\sec = r\dot{\theta} = (1.331/2)\,ft \times \dot{\theta}\,rad/\sec \;\Rightarrow\; \dot{\theta} = 3.91\,rad/\sec$$

$$\therefore\; \boldsymbol{\omega}_{\boldsymbol{1}} = 3.91\boldsymbol{K}\,rad/\sec\;.$$

With $\boldsymbol{\omega}_{\boldsymbol{1}}$ defined, starting from either o or C, $\boldsymbol{v}_{\boldsymbol{A}}$ is

$$\begin{aligned}\boldsymbol{v}_{\boldsymbol{A}} &= \boldsymbol{v}_{\boldsymbol{C}} + \boldsymbol{\omega}_{\boldsymbol{1}} \times \boldsymbol{r}_{\boldsymbol{CA}} = 0 + 3.91\boldsymbol{K}\,rad/\sec \times (.665 + 2)\boldsymbol{I}\,ft \\ &= 10.42\boldsymbol{J}\,ft/\sec\end{aligned}$$

$$\begin{aligned}\boldsymbol{v}_{\boldsymbol{A}} &= \boldsymbol{v}_{\boldsymbol{o}} + \boldsymbol{\omega}_{\boldsymbol{1}} \times \boldsymbol{r}_{\boldsymbol{oA}} = 2.6\boldsymbol{J}\,ft/\sec + 3.91\boldsymbol{K}\,rad/\sec \times 2\boldsymbol{I}\,ft \\ &= 10.42\boldsymbol{J}\,ft/\sec\;.\end{aligned}$$

The velocity of the lower spool at the cable contact point is $v_A = 10.42\,ft/\sec$. Since the cable is inextensible, the cable contact point on the upper spool has the same velocity, and

$v_B = v_A = 10.42\,ft/\sec = \omega_2 \times 1\,ft \;\Rightarrow\; \omega_2 = 10.42\,rad/\sec$, and

$\boldsymbol{\omega_2} = 10.42\boldsymbol{K}\;\; rad/\sec$.

From Eq.(4.11),

$\boldsymbol{a_C} = \boldsymbol{I} r\dot{\theta}^2 = .665\,ft \times (3.91\,rad/\sec)^2 = 10.16\boldsymbol{I}\,ft/\sec^2$.

The rolling-without-slipping conditions of Eqs.(4.5) relates *o*'s acceleration to the spool's angular acceleration as

$\ddot{X}_o = 6.44\,ft/\sec^2 = r\ddot{\theta} \;\Rightarrow\; \ddot{\theta} = 6.44\,(ft/\sec^2)/.665\,ft = 9.68\,rad/\sec^2$

$\therefore\; \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} = 9.68\boldsymbol{K}\,rad/\sec^2$.

Starting at *o*, $\boldsymbol{a_A}$ is

$$
\begin{aligned}
\boldsymbol{a_A} &= \boldsymbol{a_o} + \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} \times \boldsymbol{r_{oA}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{oA}}) \\
&= 6.44\,\frac{ft}{\sec^2}\boldsymbol{J} + (9.68\,\frac{rad}{\sec^2}\boldsymbol{K} \times 2ft\boldsymbol{I}) \\
&\quad + [3.91\,\frac{rad}{\sec}\boldsymbol{K} \times (3.91\,\frac{rad}{\sec}\boldsymbol{K} \times 2ft\boldsymbol{I})] \\
&= (6.44 + 19.36)\boldsymbol{J} - 30.6\boldsymbol{I} = -30.6\boldsymbol{I} + 25.8\boldsymbol{J} \quad ft/\sec^2 \;.
\end{aligned}
$$

Starting at *C*, $\boldsymbol{a_A}$ is

$$\boldsymbol{a_A} = \boldsymbol{a_C} + \dot{\boldsymbol{\omega}}_{\mathbf{1}} \times \boldsymbol{r_{CA}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{CA}})$$

$$= 10.16\,\boldsymbol{I}\,\frac{ft}{\sec^2} + (9.68\,\frac{rad}{\sec^2}\boldsymbol{K} \times 2.665\,ft\,\boldsymbol{I})$$

$$+ [3.91\,\frac{rad}{\sec}\boldsymbol{K} \times (3.91\,\frac{rad}{\sec}\boldsymbol{K} \times 2.665\,ft\,\boldsymbol{I})]$$

$$= 25.8\boldsymbol{J} + (10.16 - 40.74)\boldsymbol{I} = -30.6\boldsymbol{I} + 25.8\boldsymbol{J} \quad ft/\sec^2 \;.$$

Solving for $\dot{\boldsymbol{\omega}}_{\mathbf{2}}$: The acceleration of the wheel assembly at A is $\boldsymbol{a_A} = 25.8\boldsymbol{J} - 30.6\boldsymbol{I}\ ft/\sec^2$. The acceleration of spool 2 at its contact point B is $\boldsymbol{a_B} = 1\,\dot{\omega}_2\boldsymbol{J} - 1\,\omega_2^2\boldsymbol{I}\ ft/\sec^2$. Because the cable is inextensible, the $\boldsymbol{J}$ components of these accelerations (along the cable) must be equal. Hence,

$$\dot{\omega}_2 = (25.8\,ft/\sec^2)/1\,ft = 25.8\,rad/\sec^2, \text{ and } \dot{\boldsymbol{\omega}}_{\mathbf{2}} = 25.8\boldsymbol{K}\ rad/\sec^2 .$$

Example Problem 4.4 A cylinder of radius $r_2 = 1\,m$ is rolling without slipping on a horizontal surface. At the instant of interest, the surface is moving to the left with $\boldsymbol{v}_s = -\boldsymbol{I}\,1.25\,m/\sec$, and has an acceleration to the right of $\boldsymbol{a}_s = \boldsymbol{I}\,1.0\,m/\sec^2$. An inextensible cord is wrapped around an inner cylinder of radius $r_1 = 0.5\,m$ and anchored to a wall at *D*. *Determine the angular velocity and acceleration of the cylinder and the velocity and acceleration of points A, o, B, and C.*

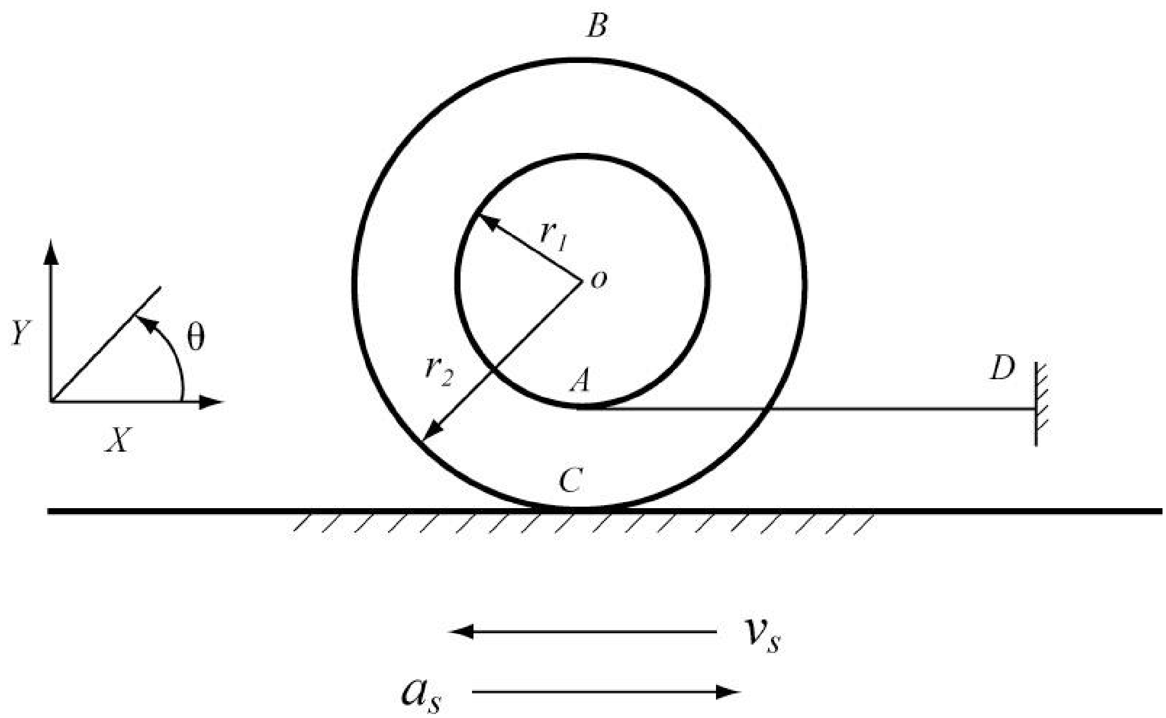


Figure XP4.4 Cylinder rolling without slipping on a horizontal (moving surface) while being restrained by a cord from *D*

Solution. First find $\dot{\theta}$. Point *C*, the contact point between the outer surface of the cylinder and the moving surface, has the velocity $\boldsymbol{v_C} = \boldsymbol{I} v_s$. Point *A*, the contact point of the inner cylinder with the cord, has zero velocity, $\boldsymbol{v_A} = 0$. The cylinder can be visualized as rolling without slipping (to the right) on the cord line *A*-*D*. We could use the vector equation $\boldsymbol{v_I} = \boldsymbol{v_J} + \boldsymbol{\omega} \times \boldsymbol{r_{JI}}$ to state multiple correct equations between the velocities of points *A*, *B*, *o*, and *C*, and most of these equations would not be helpful in determining $\dot{\theta}$. We will use *A* and *C* because we know their velocities, and we do not know the velocities of the remaining points. Since points *C* and *A* are on the cylinder (rigid body),

$$\boldsymbol{v_C} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{AC}} \ \ or,$$

$$\begin{aligned} -\boldsymbol{I}(1.25\, m/\sec) &= 0 + \boldsymbol{K}\dot{\theta}(rad/\sec) \times -0.5\, m\boldsymbol{J} \\ &= \boldsymbol{I} 0.5\dot{\theta}\,(m/\sec) \end{aligned} \qquad \textbf{(i)}$$

$$\therefore \ \dot{\theta} = -2.5\, rad/\sec \ , \ \boldsymbol{\omega} = -\boldsymbol{K} 2.5\, rad/\sec$$

Hence, the cylinder is rotating in a clockwise direction.

We can determine the velocity of point *o* at the center of the cylinder using either $\boldsymbol{v_o} = \boldsymbol{v_C} + \boldsymbol{\omega} \times \boldsymbol{r_{Co}}$ or $\boldsymbol{v_o} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{Ao}}$, because we know $\boldsymbol{v_A} = 0$ and $\boldsymbol{v_C} = -\boldsymbol{I}\, 1.25\, m/\sec$. Proceeding from point *A*,

$$\boldsymbol{v_o} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{Ao}} = 0 - \boldsymbol{K}2.5\,(rad/\sec) \times \boldsymbol{J}0.5\,(m)$$
$$= \boldsymbol{I}1.25\,m/\sec \ .$$

Hence, point o at the center of the cylinder moves horizontally to the right. Similarly,

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{Ab}} = 0 - \boldsymbol{K}2.5\,(rad/\sec) \times \boldsymbol{J}1.5\,(m)$$
$$= \boldsymbol{I}3.75 \ \ m/\sec \ .$$

We have now completed the velocity analysis determining $\dot{\theta}, \boldsymbol{\omega}, \boldsymbol{v_o}, \boldsymbol{v_B}, \boldsymbol{v_A}$.

For the acceleration analysis, our first objective is the solution for $\ddot{\theta}$. Point A has a zero horizontal acceleration component; i.e., $\boldsymbol{a_A} = \boldsymbol{J} a_{AY}$. From the nonslipping condition, point C has the horizontal acceleration component, $a_{CX} = 1.0\,m/\sec^2$; hence, $\boldsymbol{a_C} = \boldsymbol{I}1. + \boldsymbol{J} a_{CY} \ m/\sec^2$.

Hence,

$$\boldsymbol{a_C} = \boldsymbol{a_A} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{AC}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{AC}}) \ \ or,$$

$$\boldsymbol{I}1\,m/\sec^2 + \boldsymbol{J} a_{CY} = \boldsymbol{J} a_{AY} + [\boldsymbol{K}\ddot{\theta}\,(rad/\sec^2) \times -.5\,m\boldsymbol{J}]$$
$$+ \{\boldsymbol{K}\dot{\theta}\,(rad/\sec) \times [\boldsymbol{K}\dot{\theta}\,(rad/\sec) \times -0.5\,m\boldsymbol{J}]\}$$

$$= [\boldsymbol{I}0.5\,\ddot{\theta} + \boldsymbol{J}(a_{AY} + 0.5\,\dot{\theta}^2)]\,(m/\sec^2) \ .$$

Taking the $\boldsymbol{I}$ and $\boldsymbol{J}$ components separately gives:

$$\boldsymbol{I}:\ 1m\sec^2 = 0.5\ddot{\theta}\, m/\sec^2$$
$$\Rightarrow\ \ddot{\theta} = 2\,rad/\sec^2\ \Rightarrow\ \dot{\boldsymbol{\omega}} = \boldsymbol{K}2\ \ rad/\sec^2$$

(ii)

$$\boldsymbol{J}:\ a_{CY}(m/\sec^2) = [a_{AY} + 0.5\dot{\theta}^2]\,m/\sec^2$$
$$= [a_{AY} + 0.5\times(-2.5)^2]\,m/\sec^2\ = (a_{AY} + 3.125)\,m/\sec^2$$

The $\boldsymbol{I}$ component result is immediately useful, determining $\ddot{\theta}$.

The $\boldsymbol{J}$ component result is not helpful, since it only provides a relationship between the two unknowns a_{AY} and a_{CY}. We still need to calculate these components. Point *o* has no vertical motion; i.e, $\boldsymbol{a_o} = \boldsymbol{I}a_{oX}$. Hence,

$$\boldsymbol{a_C} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}}\times\boldsymbol{r_{oC}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r_{oC}})\ \ or,$$

$$\boldsymbol{I}1.m/\sec^2 + \boldsymbol{J}a_{CY} = \boldsymbol{I}a_{oX} + [\boldsymbol{K}\ddot{\theta}(rad/\sec^2)\times -1.m\boldsymbol{J}]$$
$$+ \{\boldsymbol{K}\dot{\theta}(rad/\sec)\times[\boldsymbol{K}\dot{\theta}(rad/\sec)\times -1.m\boldsymbol{J}]\}$$

$$= \boldsymbol{I}(a_{oX} + \ddot{\theta}) + \boldsymbol{J}\dot{\theta}^2\,(m/\sec^2)\ .$$

The $\boldsymbol{I}$ and $\boldsymbol{J}$ components give:

$$\boldsymbol{I}:\ 1m\sec^2 = (a_{oX} + \ddot{\theta})\, m/\sec^2$$

$$\therefore \quad a_{oX} = (1 - 2)\, m/\sec^2 = -1\, m/\sec^2 \qquad \textbf{(iii)}$$

$$\boldsymbol{J}:\ a_{CY}(m/\sec^2) = (\dot{\theta}^2)\, m/\sec^2 = (2.5)^2\, m/\sec^2$$
$$= 6.25\, m/\sec^2\ .$$

We substituted $\ddot{\theta} = 2\, rad/\sec^2$ and $\dot{\theta} = -2.5\, rad/\sec$ into the ***I*** and ***J*** component equations, respectively. You can verify that we could have also used $\boldsymbol{a_A} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oA}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oA}})$ successfully to determine a_{AY}.

From Eq.(ii), $a_{CY} = a_{AY} + 3.125\,(m/\sec^2)$; hence, using the second of Eq. (ii), $a_{AY} = 6.25 - 3.125 = 3.125\,(m/\sec^2)$. Note that $a_A = r_{oA}\dot{\theta}^2 = 0.5 \times (2.5)^2 = 3.125$. In this example, point *A* corresponds to the "contact point C". It is in contact with cord line *AD*, and its only acceleration is vertical due to the centrifugal acceleration term.

We now have

$$\boldsymbol{a_A} = \boldsymbol{J}3.125\ (m/\sec^2)$$

$$\boldsymbol{a_o} = -\boldsymbol{I}1.(m/\sec^2)$$

$$\boldsymbol{a_C} = -\boldsymbol{I}1. + \boldsymbol{J}6.25\ (m/\sec^2)$$

The acceleration of point B can be obtained starting from points A, o, or C , since we have the acceleration of all these points. Starting from o,

$$\boldsymbol{a}_B = \boldsymbol{a}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{oB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{oB})$$

$$\begin{aligned} &= \ -\boldsymbol{I}1\,(m/\sec^2) \\ &+ [\boldsymbol{K}2\,(rad\sec^2) \times \boldsymbol{J}1\,m] + \\ &- \boldsymbol{K}2.5\,(rad/\sec) \times [-\boldsymbol{K}2.5\,(rad/\sec) \times \boldsymbol{J}1.m] \end{aligned}$$

$$= \boldsymbol{I}(-1.-2.) - \boldsymbol{J}6.25 \quad m/\sec^2 = -\boldsymbol{I}3. - \boldsymbol{J}6.25 \quad m/\sec^2 \ .$$

Note in reviewing this example the key to the solution is: (i) first find $\dot{\theta}$, and (ii) then find $\ddot{\theta}$. We used a velocity relation between points A and C to find $\dot{\theta}$ and an acceleration relation between the same two points to find $\ddot{\theta}$. These points work because *we know the X components* of $\boldsymbol{v}_C, \boldsymbol{a}_C, \boldsymbol{v}_A, \boldsymbol{a}_A$. We can write valid equations relating the velocity and acceleration for any two of the points; however, only the combination of points A and C will produce directly useful results in calculating $\dot{\theta}$ and $\ddot{\theta}$.

Also, note that to determine the Y components of $\boldsymbol{a}_A, \boldsymbol{a}_B, \boldsymbol{a}_C$, we needed to use an acceleration relationship involving $\boldsymbol{a}_0$, the acceleration of center of the cylinder. Valid acceleration relationships can certainly be stated between the points A, B, and C However, the results are not helpful; e.g., we obtained $a_{CY} = a_{AY} + 3.125 \ m/\sec^2$ in Eq.(ii), which involves two unknowns a_{CY} and a_{AY}. Point o "works" because we know that its vertical acceleration is zero.

NOTE: THE GEOMETRIC APPROACH WORKS WELL ON MECHANISMS. THE VECTOR APPROACH GENERALLY WORKS BETTER IN ROLLING-WITHOUT-SLIPPING PROBLEMS WHEN YOU NEED TO FIND THE ACCELERATION OF A POINT OR THE ANGULAR VELOCITY OR ACCELERATION OF A WHEEL.

Lecture 20. PLANAR KINEMATIC-PROBLEM EXAMPLES

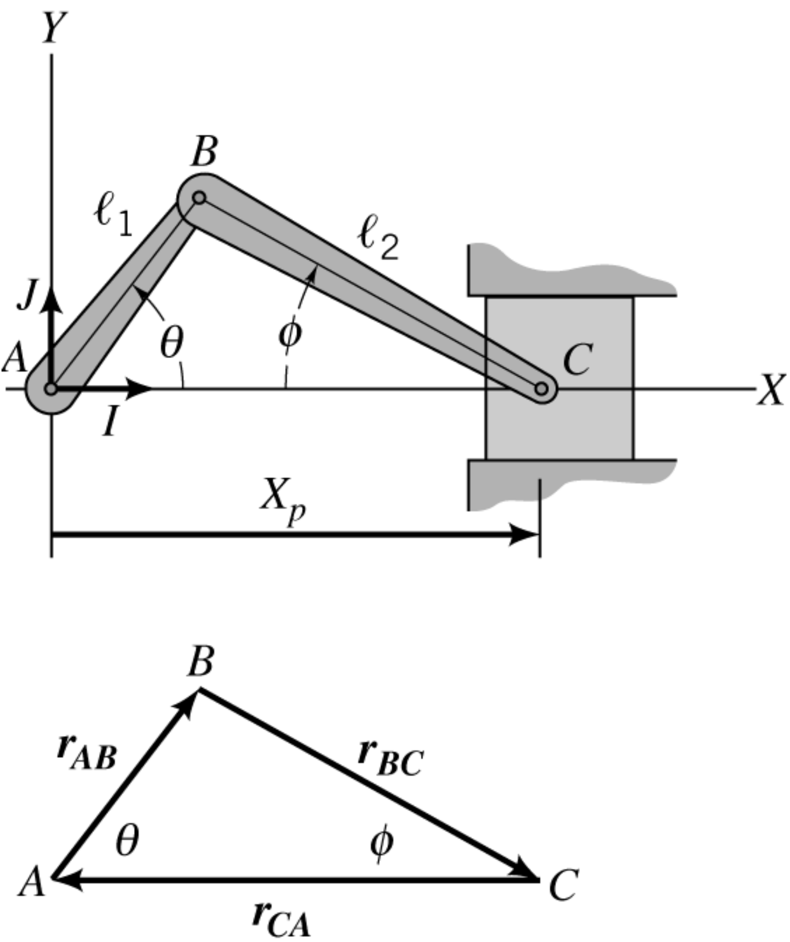


Figure 4.15 Slider-crank mechanism.

TASK: *For a given constant rotation rate* $\dot{\theta} = \omega$, *find the velocity* $\dot{X}_P, \dot{\varphi}$ *and acceleration* $\ddot{X}_P, \ddot{\varphi}$ *terms of the piston for one cycle of* θ.

Geometric Approach: There are three variables (θ, φ, and X_p) but only one degree of freedom. The following (constraint) relationships may be obtained by inspection:

$$X_P = l_1 \cos\theta + l_2 \cos\varphi \quad (X components)$$
$$l_1 \sin\theta = l_2 \sin\varphi \qquad (Y components) \ . \tag{4.16}$$

With θ as the input (known) variable, these equations can be easily solved for the output variables X_P, φ. The vector diagram in figure 4.15 shows the position vectors $\boldsymbol{r}_{AB}$, $\boldsymbol{r}_{BC}$, and $\boldsymbol{r}_{CA}$. For these vectors,

$$\boldsymbol{r}_{AB} + \boldsymbol{r}_{BC} + \boldsymbol{r}_{CA} = 0 \ . \tag{4.17}$$

Substituting,

$$\boldsymbol{r}_{AB} = \boldsymbol{I} l_1 \cos\theta + \boldsymbol{J} l_1 \sin\theta$$
$$\boldsymbol{r}_{BC} = \boldsymbol{I} l_2 \cos\varphi - \boldsymbol{J} l_2 \sin\varphi$$
$$\boldsymbol{r}_{CA} = -\boldsymbol{I} X_P \ ,$$

gives :

$$\boldsymbol{I}: \quad l_1 \cos\theta + l_2 \cos\varphi - X_p = 0$$
$$\boldsymbol{J}: \quad l_1 \sin\theta - l_2 \sin\varphi = 0 \ .$$

Differentiating Eq.(4.16) w.r.t. time gives:

$$\dot{X}_P + l_2 \sin\varphi\,\dot{\varphi} = -\,l_1 \sin\theta\,\dot{\theta} = -\,l_1\,\omega\sin\theta$$

$$l_2 \cos\varphi\,\dot{\varphi} = l_1 \cos\theta\,\dot{\theta} = l_1\,\omega\cos\theta \quad . \tag{4.18a}$$

Differentiating again gives:

$$\begin{aligned}\ddot{X}_P + l_2 \sin\varphi\,\ddot{\varphi} &= -\,l_1 \sin\theta\,\ddot{\theta} - l_1 \cos\theta\,\dot{\theta}^2 - l_2 \cos\varphi\,\dot{\varphi}^2 \\ &= -\,l_1\,\omega^2 \cos\theta - l_2 \cos\varphi\,\dot{\varphi}^2\end{aligned}$$

$$\begin{aligned}l_2 \cos\varphi\,\ddot{\varphi} &= l_1 \cos\theta\,\ddot{\theta} - l_1 \sin\theta\,\dot{\theta}^2 + l_2 \sin\varphi\,\dot{\varphi}^2 \\ &= -\,l_1\,\omega^2 \sin\theta + l_2 \sin\varphi\,\dot{\varphi}^2 \quad .\end{aligned} \tag{4.18b}$$

Matrix equations of unknowns

$$\begin{bmatrix} 1 & \sin\varphi \\ 0 & \cos\varphi \end{bmatrix} \begin{Bmatrix} \dot{X}_P \\ l_2\dot{\varphi} \end{Bmatrix} = l_1\omega \begin{Bmatrix} -\sin\theta \\ \cos\theta \end{Bmatrix} . \tag{4.19a}$$

$$\begin{aligned}\begin{bmatrix} 1 & \sin\varphi \\ 0 & \cos\varphi \end{bmatrix} \begin{Bmatrix} \ddot{X}_P \\ l_2\ddot{\varphi} \end{Bmatrix} &= -\,l_1\omega^2 \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \\ &\quad + \, l_2\dot{\varphi}^2 \begin{Bmatrix} -\cos\varphi \\ \sin\varphi \end{Bmatrix} .\end{aligned} \tag{4.19b}$$

The engineering-analysis tasks are accomplished by the following steps:

1. Vary θ over the range of $[\ 0, 2\pi\ ]$, yielding discrete values θ_i .

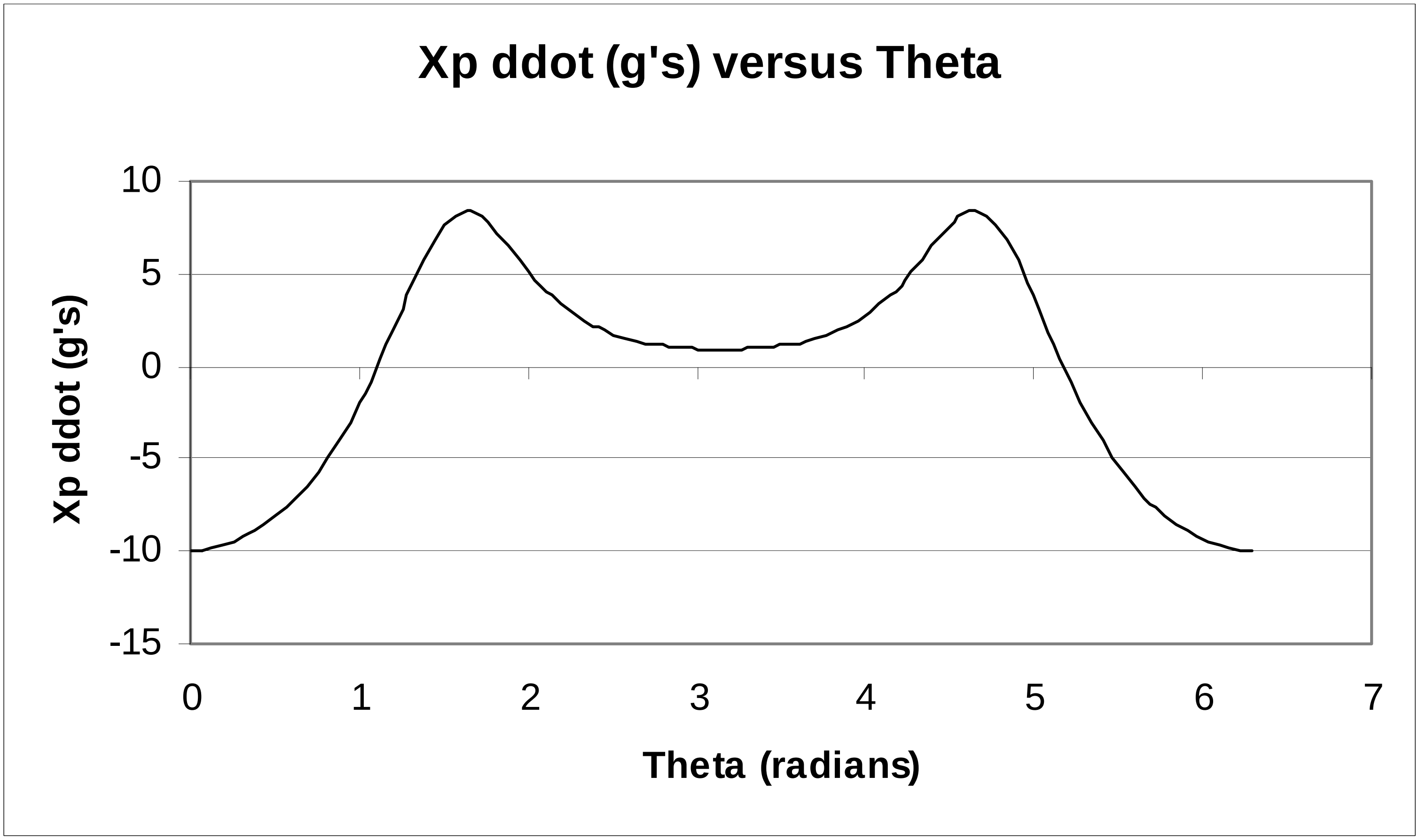
2. For each θ_i value, solve Eq.(4.16) to determine corresponding values for φ_i .

3. Use Eqs.(4.18) with known values for θ_i and φ_i to determine $\dot{\varphi}_i$.

4. Use Eqs.(4.19) with known values for θ_i , φ_i , and $\dot{\varphi}_i$ to determine $\ddot{X}_{Pi}$.

5. Plot $\ddot{X}_{Pi}$ versus θ_i.

Spread-sheet solution for $\ddot{X}_p$ for one θ cycle with $l_1 = 250\,mm, l_2 = 300\,mm, \omega = 14.6\,rad/\sec$.



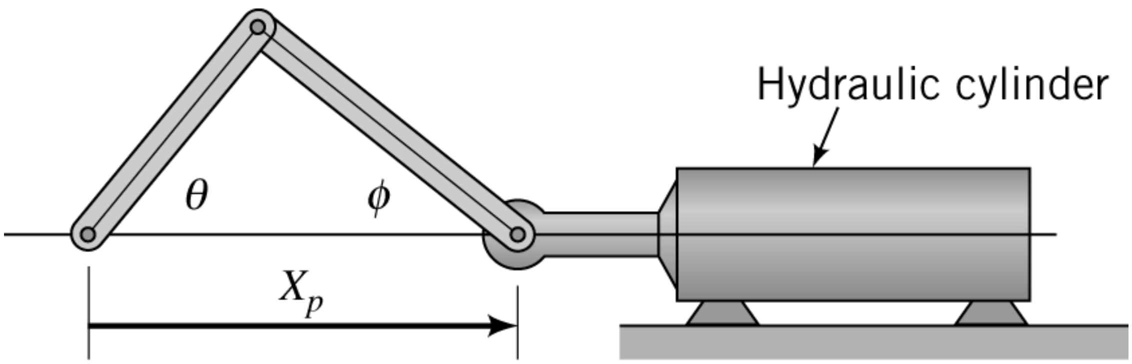


Figure 4.18 Slider crank mechanism with displacement input from a hydraulic cylinder.

For $X_P(t)$ as the input, with (θ and φ), ($\dot{\theta}$ and $\dot{\varphi}$), and ($\ddot{\theta}$ and $\ddot{\varphi}$) as the desired output coordinates. The equations for the coordinates are:

$$l_1\cos\theta + l_2\cos\varphi = X_P \;, \quad l_1\sin\theta - l_2\sin\varphi = 0 \;. \qquad \textbf{(4.20a)}$$

From Eq.(4.20a), the velocity relationships are:

$$l_2\sin\varphi\,\dot{\varphi} + l_1\sin\theta\,\dot{\theta} = -\dot{X}_P \;, \quad l_2\cos\varphi\,\dot{\varphi} - l_1\cos\theta\,\dot{\theta} = 0 \;. \qquad \textbf{(4.20b)}$$

From Eq.(4.20b), the required acceleration component equations are:

$$\begin{aligned} l_2\sin\varphi\,\ddot{\varphi} + l_1\sin\theta\,\ddot{\theta} &= -\ddot{X}_P - l_1\cos\theta\,\dot{\theta}^2 - l_2\cos\varphi\,\dot{\varphi}^2 \\ l_2\cos\varphi\,\ddot{\varphi} - l_1\cos\theta\,\ddot{\theta} &= -l_1\sin\theta\,\dot{\theta}^2 + l_2\sin\varphi\,\dot{\varphi}^2 \;. \end{aligned} \qquad \textbf{(4.20c)}$$

The problem solution is obtained for specified values of $X_P(t)$, $\dot{X}_P(t)$, $\ddot{X}_P(t)$ by proceeding *sequentially* through Eqs.(4.20a), (4.20b), and (4.20c). Note that Eqs.(4.20a) defining θ and φ are nonlinear, while Eqs.(4.20b) for $\dot{\varphi}$ and $\dot{\theta}$, and Eqs.(4.20c) for $\ddot{\varphi}$ and $\ddot{\theta}$ are linear.

The essential first step in developing kinematic equations for planar mechanisms via geometric relationships is drawing a picture of the mechanism in a *general* orientation, yielding equations that can be subsequently differentiated.

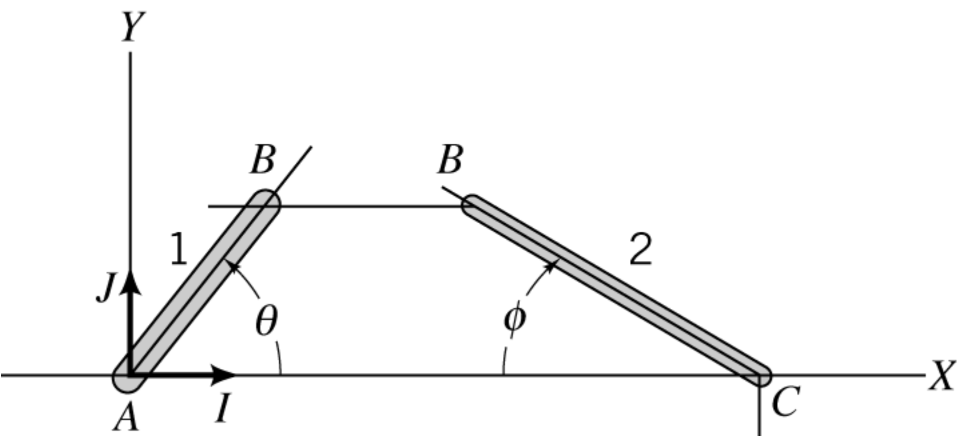


Figure 4.19 Disassembled view of the slider-crank mechanism for vector analysis.

Vector Approach for Velocity and Acceleration Results

Applying the velocity result of Eq.(4.3) separately to links 1 and 2, gives:

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}} \ , \quad \boldsymbol{v_C} = \boldsymbol{v_B} + \boldsymbol{\omega_2} \times \boldsymbol{r_{BC}} \ .$$

Equating the two answers that these equations provide for $\boldsymbol{v_B}$,

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}} = \boldsymbol{v_C} - \boldsymbol{\omega_2} \times \boldsymbol{r_{BC}} \ .$$

Since point A is fixed in the X ,Y system, $\boldsymbol{v_A} = 0$. Similarly, given that point C can only move horizontally, $\boldsymbol{v_C} = \boldsymbol{I} \, \dot{X}_P$. The vector $\boldsymbol{\omega_1}$ is the angular velocity of link 1 with respect to the X,Y system. Using the right-hand rule,

$$\boldsymbol{\omega_1} = \boldsymbol{K}\dot{\theta} \ , \quad \boldsymbol{\omega_2} = -\boldsymbol{K}\dot{\varphi}$$

The position vectors $\boldsymbol{r_{AB}}$ and $\boldsymbol{r_{BC}}$ are defined by

$$\boldsymbol{r_{AB}} = l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\cos\theta) \ , \quad \boldsymbol{r_{BC}} = l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$

Substitution gives

$$0 + \boldsymbol{K}\dot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) = \boldsymbol{I}\dot{X}_p - (-\boldsymbol{K}\dot{\varphi}) \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi) \ .$$

Carrying out the cross products and gathering terms,

$$\boldsymbol{I}: \quad -l_1\dot{\theta}\sin\theta = \dot{X}_p + l_2\sin\varphi\,\dot{\varphi}$$

$$\boldsymbol{J}: \quad l_1\dot{\theta}\cos\theta = l_2\dot{\varphi}\cos\varphi \ .$$

To find the acceleration relationships, applying the second of Eqs.(4.3) to figure 4.17 :

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_{AB} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_{AB})$$

$$\boldsymbol{a}_C = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{BC} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{BC}) \ .$$

Equating the separate definitions for $\boldsymbol{a}_B$ gives

$$\begin{aligned}\boldsymbol{a}_B &= \boldsymbol{a}_A + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_{AB} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_{AB}) \\ &= \boldsymbol{a}_C - \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{BC} - \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{BC}) \ .\end{aligned}$$

Since point A is fixed, $\boldsymbol{a}_A = 0$. Also, since point C is constrained to move in the horizontal plane, $\boldsymbol{a}_C = \boldsymbol{I}\,\ddot{X}_P$. The remaining undefined variables are $\dot{\boldsymbol{\omega}}_1 = \boldsymbol{K}\ddot{\theta}$, the angular acceleration of link 1 with respect to the X, Y system, and $\dot{\boldsymbol{\omega}}_2 = -\boldsymbol{K}\ddot{\varphi}$, the angular acceleration of link 2 with respect to the X, Y system. Substituting gives

$$0 + \boldsymbol{K}\ddot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) + \boldsymbol{K}\dot{\theta} \times [\boldsymbol{K}\dot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)]$$

$$= \boldsymbol{I}\ddot{X}_P - (-\boldsymbol{K}\ddot{\varphi}) \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$

$$-(-\boldsymbol{K}\dot{\varphi}) \times [-\boldsymbol{K}\dot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)] \ .$$

Completing the cross products and algebra gives the following component equations:

$$\boldsymbol{I}: \quad -l_1\ddot{\theta}\sin\theta - l_1\dot{\theta}^2\cos\theta = \ddot{X}_P + l_2\ddot{\varphi}\sin\varphi + l_2\dot{\varphi}^2\cos\varphi$$

$$\boldsymbol{J}: \quad l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta = l_2\ddot{\varphi}\cos\varphi - l_2\dot{\varphi}^2\sin\varphi \ ,$$

4.5b A Four-Bar-Linkage Example

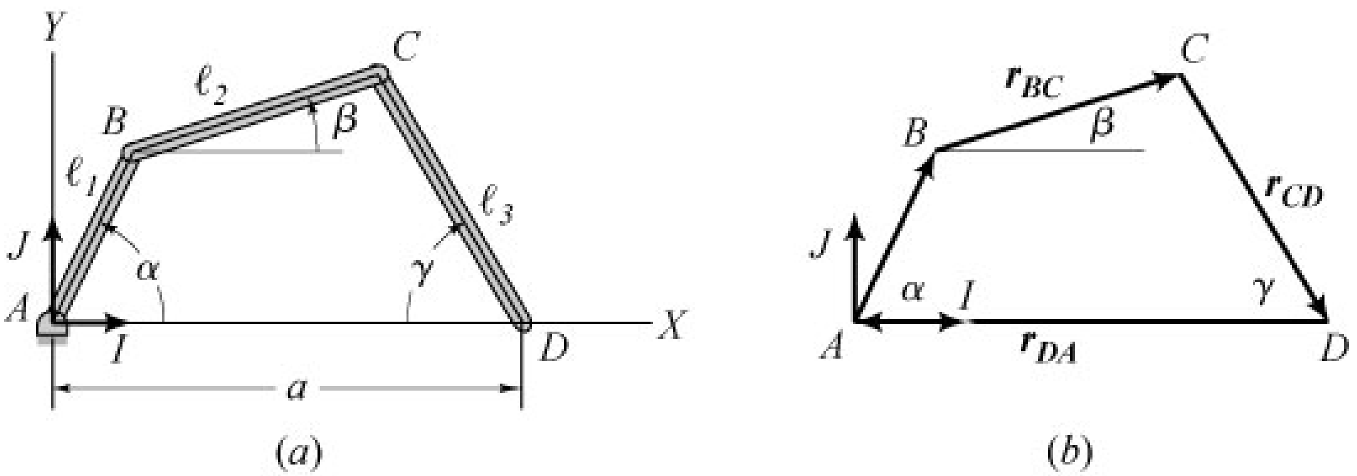


Figure 4.19 (a) Four-bar linkage, (b) Vector diagram for linkage

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Consider the following engineering-analysis task: *For a constant rotation rate* , $\dot{\alpha} = \omega_o$ *determine the angular velocities* $\dot{\beta}$, $\dot{\gamma}$ *and angular accelerations* $\ddot{\beta}$, $\ddot{\gamma}$ *for one rotation of* α.

Geometric Approach

Inspecting figure 4.19a yields:

$$\begin{aligned} X: &\quad l_1\cos\alpha + l_2\cos\beta + l_3\cos\gamma = a \\ Y: &\quad l_1\sin\alpha + l_2\sin\beta - l_3\sin\gamma = 0 \ . \end{aligned} \qquad \textbf{(4.21)}$$

Figure 4.19b shows a closed-loop vector representation that can be formally used to obtain Eqs.(4.21). The results from figure 4.19b can be stated, $\boldsymbol{r}_{AB} + \boldsymbol{r}_{BC} + \boldsymbol{r}_{CD} + \boldsymbol{r}_{DA} = 0$. Substituting:

$$\boldsymbol{r}_{AB} = l_1(\boldsymbol{I}\cos\alpha + \boldsymbol{J}\sin\alpha)\ ,\ \boldsymbol{r}_{BC} = l_2(\boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta)$$
$$\boldsymbol{r}_{CD} = l_3(\boldsymbol{I}\cos\gamma - \boldsymbol{J}\sin\gamma)\ ,\ \boldsymbol{r}_{DA} = -\boldsymbol{I}a$$

gives the same result as Eqs.(4.21).

Restating Eqs.(4.21) as:

$$\begin{aligned} l_2\cos\beta + l_3\cos\gamma &= a - l_1\cos\alpha \\ l_2\sin\beta - l_3\sin\gamma &= -l_1\sin\alpha\ . \end{aligned} \tag{4.22}$$

shows α as the input coordinate and β and γ as output coordinates. Differentiating with respect to time gives:

$$\begin{aligned} -l_2\sin\beta\,\dot\beta - l_3\sin\gamma\,\dot\gamma &= l_1\sin\alpha\,\dot\alpha = l_1\omega\sin\alpha \\ l_2\cos\beta\,\dot\beta - l_3\cos\gamma\,\dot\gamma &= -l_1\cos\alpha\,\dot\alpha = -l_1\omega\cos\alpha\ . \end{aligned} \tag{4.23}$$

In matrix format,

$$\begin{bmatrix} \sin\beta & \sin\gamma \\ -\cos\beta & \cos\gamma \end{bmatrix} \begin{Bmatrix} l_2\dot\beta \\ l_3\dot\gamma \end{Bmatrix} = l_1\omega \begin{Bmatrix} -\sin\alpha \\ \cos\alpha \end{Bmatrix}\ . \tag{4.23}$$

Using Cramers rule to solve these equations gives

$$l_2 \dot{\beta} = \frac{-l_1 \omega \sin(\alpha + \gamma)}{\sin(\beta + \gamma)}$$

$$l_3 \dot{\gamma} = \frac{l_1 \omega \sin(\beta - \alpha)}{\sin(\beta + \gamma)} \quad , \qquad \textbf{(4.24)}$$

The solution is undefined for $\beta + \gamma = \pi, 0$

Differentiating Eqs.(4.23) gives:

$$-l_2 \sin\beta \ddot{\beta} - l_3 \sin\gamma \ddot{\gamma} = l_1 \sin\alpha \ddot{\alpha} + l_1 \cos\alpha \dot{\alpha}^2 + l_2 \cos\beta \dot{\beta}^2 + l_3 \cos\gamma \dot{\gamma}^2$$

$$l_2 \cos\beta \ddot{\beta} - l_3 \cos\gamma \ddot{\gamma} = -l_1 \cos\alpha \ddot{\alpha} + l_1 \sin\alpha \dot{\alpha}^2 + l_2 \sin\beta \dot{\beta}^2 - l_3 \sin\gamma \dot{\gamma}^2$$

Setting $\ddot{\alpha}$=0, and $\dot{\alpha} = \omega_o$ reduces them to:

$$-l_2 \sin\beta \ddot{\beta} - l_3 \sin\gamma \ddot{\gamma} = l_1 \cos\alpha \omega_o^2 + l_2 \cos\beta \dot{\beta}^2 + l_3 \cos\gamma \dot{\gamma}^2$$

$$l_2 \cos\beta \ddot{\beta} - l_3 \cos\gamma \ddot{\gamma} = -l_1 \sin\alpha \omega_o^2 + l_2 \sin\beta \dot{\beta}^2 - l_3 \sin\gamma \dot{\gamma}^2 \quad ,$$

or, in matrix format,

$$
\begin{bmatrix} -\sin\beta & -\sin\gamma \\ \cos\beta & -\cos\gamma \end{bmatrix} \begin{Bmatrix} l_2\ddot{\beta} \\ l_3\ddot{\gamma} \end{Bmatrix} =
$$

$$
\begin{Bmatrix} l_1\omega^2\cos\alpha \;+\; l_2\dot{\beta}^2\cos\beta \;+\; l_3\dot{\gamma}^2\cos\gamma \\ -l_1\omega^2\sin\alpha \;+\; l_2\dot{\beta}^2\sin\beta \;-\; l_3\dot{\gamma}^2\sin\gamma \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} .
$$

Using Cramer's rule, the solution is

$$
l_2\,\ddot{\beta} \;=\; \frac{-g_1\cos\gamma + g_2\sin\gamma}{\sin(\beta+\gamma)}
$$

$$
\textbf{(4.26)}
$$

$$
l_3\,\ddot{\gamma} \;=\; \frac{-g_1\cos\beta - g_2\sin\beta}{\sin(\beta+\gamma)} \;.
$$

The solution is undefined for $\beta+\gamma=\pi$.

The engineering-analysis task is accomplished by executing the following sequential steps:

1. Vary α over the range [0, 2π], yielding discrete values α_i.

2. For each α_i value, solve Eq.(4.22) to determine corresponding values for β_i and γ_i.

3. Enter Eqs.(4.23a) with known values for α_i, β_i and γ_i. to determine $\dot{\beta}_i$ and $\dot{\gamma}_i$.

4. Enter Eqs.(4.26a) with known values for α_i, β_i, γ_i, $\dot{\beta}_i$ and $\dot{\gamma}_i$ to determine $\ddot{\beta}_i$ and $\ddot{\gamma}_i$.

5. Plot $\dot{\beta}_i$, $\dot{\gamma}_i$, $\ddot{\beta}_i$ and $\ddot{\gamma}_i$ versus α_I.

Eqs.(4.22) can be solved analytically, starting with the restatement

$$l_2 \cos\beta = (a - l_1 \cos\alpha) - l_3 \cos\gamma = h_1 - l_3 \cos\gamma$$

$$l_2 \sin\beta = -l_1 \sin\alpha + l_3 \sin\gamma = h_2 + l_3 \sin\gamma \ .$$

$h_1 = a - l_1 \cos\alpha$ and $h_2 = -l_1 \sin\alpha$ are defined in terms of α and are known quantities. Squaring both of these equations and adding them together gives

$$l_2^2(\cos^2\beta + \sin^2\beta) = h_1^2 - 2h_1 l_3 \cos\gamma + l_3^2 \cos^2\gamma + h_2^2$$
$$+ 2h_2 l_3 \sin\gamma + l_3^2 \sin^2\gamma$$

$$\therefore \ l_2^2 = h_1^2 + h_2^2 + l_3^2 - 2h_1 l_3 \cos\gamma + 2h_2 l_3 \sin\gamma .$$

Rearranging gives

$$2h_1 l_3 \cos\gamma = h_1^2 + h_2^2 + l_3^2 - l_2^2 + 2h_2 l_3 \sin\gamma = 2d + 2h_2 l_3 \sin\gamma$$

$$\therefore\ h_1^2 l_3^2 \cos^2\gamma = h_1^2 l_3^2 (1 - \sin^2\gamma) = d^2 + 2dh_2 l_3 \sin\gamma + h_2^2 l_3^2 \sin^2\gamma\ ,$$

where $2d = h_1^2 + h_2^2 + l_3^2 - l_2^2$. Restating this result gives

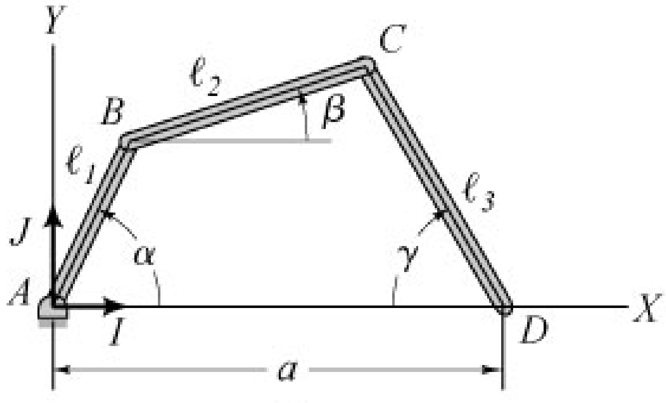
$$\sin^2\gamma + \frac{2dh_2}{l_3(h_1^2 + h_2^2)}\sin\gamma + \frac{(d^2 - h_1^2 l_3^2)}{l_3^2(h_1^2 + h_2^2)} = 0$$

The equation $\sin^2\gamma + B\sin\gamma + C = 0$ has the two roots:

$$\sin\gamma = -\frac{B}{2} + \frac{1}{2}\sqrt{B^2 - 4C}$$

$$\sin\gamma = -\frac{B}{2} - \frac{1}{2}\sqrt{B^2 - 4C}\ . \qquad \textbf{(4.27)}$$

Depending on values for *B* and *C*, this equation can have one real root, two real roots, or two complex roots. Two real roots implies two distinct solutions, and this possibility is illustrated by figure 4.20 below where the same α value gives an orientation that differs from figure 4.19a.



(a)

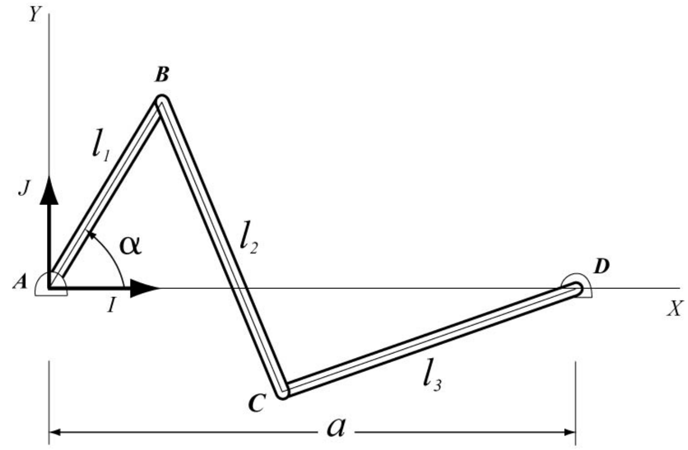


Figure 4.20 Alternate configuration for the linkage of figure 4.19a

The one real-root solution corresponding to $B^2 = 4C$ defines an extreme "locked" position for the mechanism, as illustrated in figure 4.21. Note that this position corresponds to $\beta_1 = -\gamma_1$ netting $\sin(\beta_1 + \gamma_1) = 0$, which also caused the angular velocities and angular accelerations to be undefined in Eqs.(4.23) and (4.25), respectively.

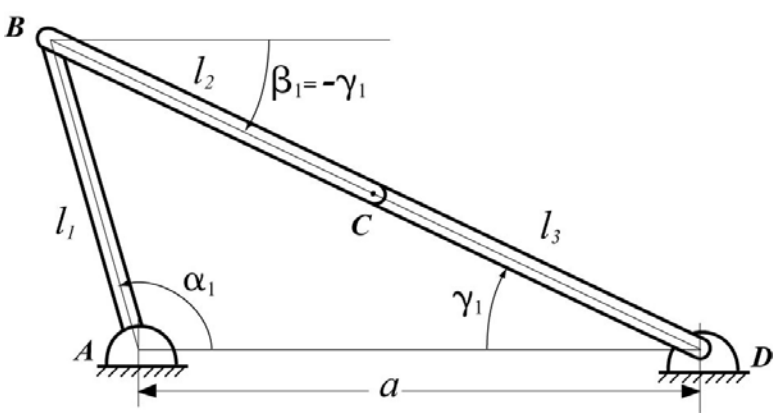


Figure 4.21 Locked position for the linkage of figure 4.19a with $l_1 = l_3 = 1.45\, l_1, a = 2.28\, l_1$

We can solve for the limiting α value in figure 4.21 by substituting $\beta_1 = -\gamma_1$ into Eq.(4.22) to get

$$l_2 \cos\gamma_1 + l_3 \cos\gamma_1 = (l_2 + l_3)\cos\gamma_1 = a - l_1 \cos\alpha_1$$

$$-l_2 \sin\gamma_1 - l_3 \sin\gamma_1 = -(l_2 + l_3)\sin\gamma_1 = -l_1 \sin\alpha_1 \ .$$

Squaring both equations and adding them together gives

$$a^2\, 2 a l_1 \cos\alpha_1 + l_1^2 = (l_2 + l_3)^2$$

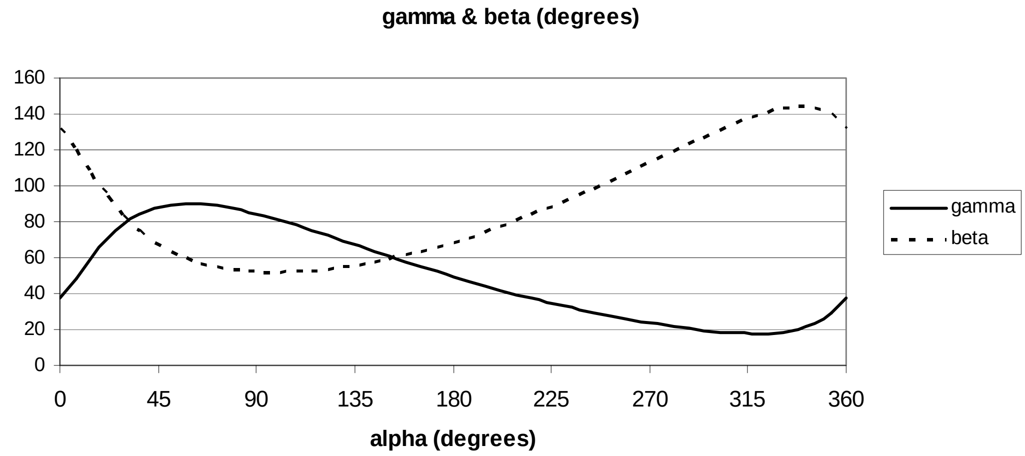
$$\therefore \ \cos\alpha_1 = \frac{a^2 + l_1^2 - (l_2 + l_3)^2}{2 a l_1} \ . \qquad \textbf{(4.28)}$$

If the parameters l_1, l_2, l_3, a are such that a solution exists for α_1, then a locked position can occur. For figure 4.21, the limiting value for α_1 corresponds to

$$\alpha_1 = \pi \ \Rightarrow \ (a + l_1)^2 = (l_2 + l_3)^2 \ \Rightarrow \ a + l_1 = l_2 + l_3 \ .$$

For $l_2 + l_3 > a + l_1$ there is no limiting rotation angle α_1, and the left hand link can rotate freely through 360 degrees.

Figure 4.22 illustrates a solution for $\beta, \gamma, \dot\beta, \dot\gamma$ with $l_1 = 0.35\,m;\ l_2 = .816\,m;\ l_3 = 1\,m; a = 0.6\,m$ and $\omega = 3 rad/\sec$ for α_i over [0, 2π]. The solution illustrated corresponds to the first solution (positive square root) in Eq.(4.27).



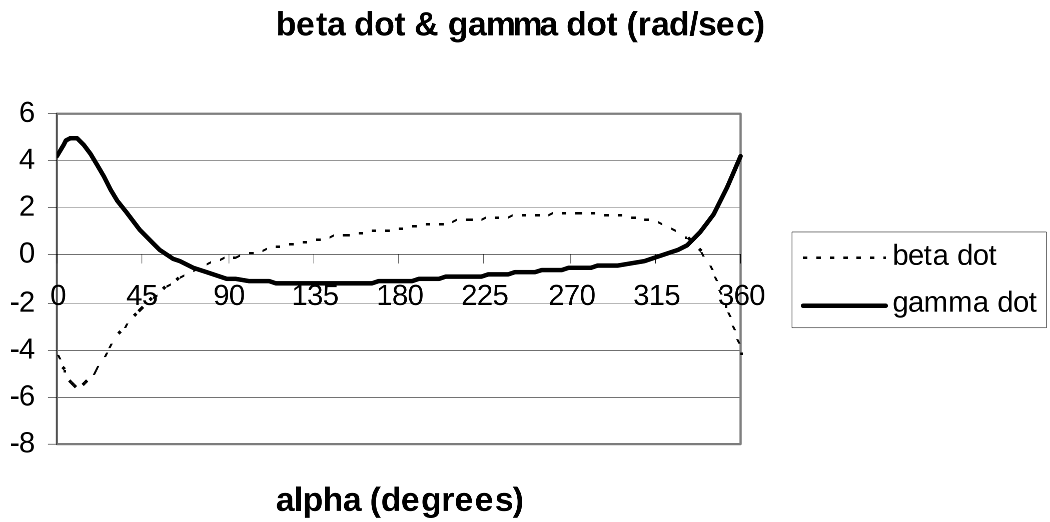


Figure 4.22 Numerical solution for $\beta, \gamma, \dot{\beta}, \dot{\gamma}$ versus α for
$l_1 = 0.35\,m;\ l_2 = .816\,m;\ l_3 = 1\,m;\ a = 0.6\,m$

Caution is advisable in using Eq.(4.27) to solve for γ_i to make sure that the solution is in the correct quadrant.

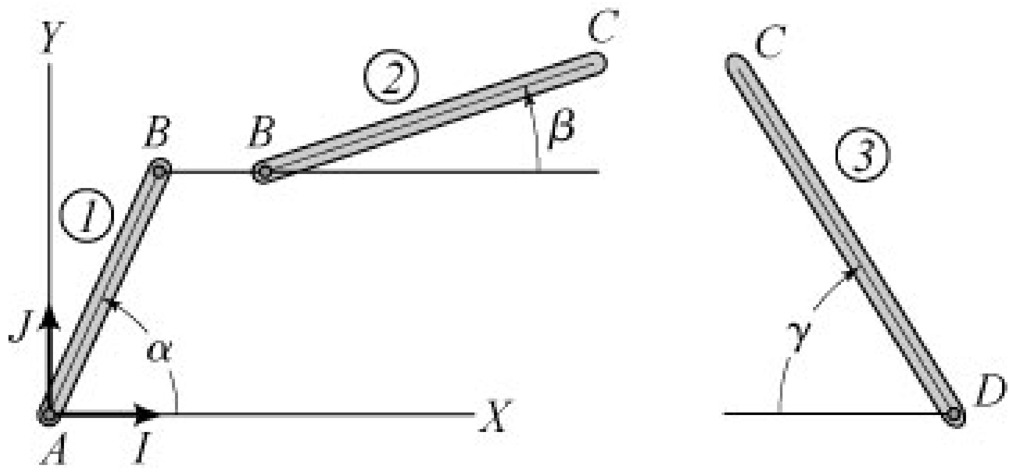


Figure 4.21 Disassembled view of the three-bar linkage of figure 4.19 for vector analysis.

Starting on the left with link 1, and looking from point *A* to *B* gives

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \omega_1 \times \boldsymbol{r_{AB}} \ .$$

Next, for link 3, looking from point *D* back to point *C* we can write

$$\boldsymbol{v_C} = \boldsymbol{v_D} + \omega_3 \times \boldsymbol{r_{DC}} \ .$$

Finally, for link 2, looking from point *B* to point *C* gives

$$\boldsymbol{v}_{\boldsymbol{C}} = \boldsymbol{v}_{\boldsymbol{B}} + \omega_2 \times \boldsymbol{r}_{\boldsymbol{BC}} \quad .$$

Substituting into this last equation for $\boldsymbol{v}_{\boldsymbol{B}}$ and $\boldsymbol{v}_{\boldsymbol{C}}$, and observing that $\boldsymbol{v}_{\boldsymbol{A}} = \boldsymbol{v}_{\boldsymbol{D}} = 0$ gives

$$\omega_3 \times \boldsymbol{r}_{\boldsymbol{DC}} = \omega_1 \times \boldsymbol{r}_{\boldsymbol{AB}} + \omega_2 \times \boldsymbol{r}_{\boldsymbol{BC}} \quad .$$

Eqs.(4.22) define the position vectors of this equation. Using the right-hand rule, the angular velocity vectors are defined as $\boldsymbol{\omega}_{\boldsymbol{1}} = \boldsymbol{K}\dot{\alpha}$, $\boldsymbol{\omega}_{\boldsymbol{2}} = \boldsymbol{K}\dot{\beta}$, and $\boldsymbol{\omega}_{\boldsymbol{3}} = -\boldsymbol{K}\dot{\gamma}$. Substituting gives

$$\begin{aligned} &-\boldsymbol{K}\dot{\gamma} \times l_3(-\boldsymbol{I}\cos\gamma + \boldsymbol{J}\sin\gamma) \\ &\quad = \boldsymbol{K}\dot{\alpha} \times l_1(\boldsymbol{I}\cos\alpha + \boldsymbol{J}\sin\alpha) + \boldsymbol{K}\dot{\beta} \times l_2(\boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta) \quad . \end{aligned}$$

Carrying out the cross products gives:

$$\boldsymbol{I} : \quad l_3\dot{\gamma}\sin\gamma = -l_1\dot{\alpha}\sin\alpha - l_2\dot{\beta}\sin\beta$$

$$\boldsymbol{J} : \quad l_3\dot{\gamma}\cos\gamma = l_1\dot{\alpha}\cos\alpha + l_2\dot{\beta}\cos\beta \quad .$$

The acceleration relationships are obtained via the same logic from

$$\begin{aligned} \boldsymbol{a}_{\boldsymbol{B}} &= \boldsymbol{a}_{\boldsymbol{A}} + \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} \times \boldsymbol{r}_{\boldsymbol{AB}} + \boldsymbol{\omega}_{\boldsymbol{1}} \times (\boldsymbol{\omega}_{\boldsymbol{1}} \times \boldsymbol{r}_{\boldsymbol{AB}}) \\ \boldsymbol{a}_{\boldsymbol{C}} &= \boldsymbol{a}_{\boldsymbol{D}} + \dot{\boldsymbol{\omega}}_{\boldsymbol{3}} \times \boldsymbol{r}_{\boldsymbol{DC}} + \boldsymbol{\omega}_{\boldsymbol{3}} \times (\boldsymbol{\omega}_{\boldsymbol{3}} \times \boldsymbol{r}_{\boldsymbol{DC}}) \\ \boldsymbol{a}_{\boldsymbol{C}} &= \boldsymbol{a}_{\boldsymbol{B}} + \dot{\boldsymbol{\omega}}_{\boldsymbol{2}} \times \boldsymbol{r}_{\boldsymbol{BC}} + \boldsymbol{\omega}_{\boldsymbol{2}} \times (\boldsymbol{\omega}_{\boldsymbol{2}} \times \boldsymbol{r}_{\boldsymbol{BC}}) \quad . \end{aligned}$$

Substituting from the first and second equations for $\boldsymbol{a_B}$ and $\boldsymbol{a_C}$ into the last gives

$$\boldsymbol{a_D} + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{r_{DC}} + \boldsymbol{\omega_3} \times (\boldsymbol{\omega_3} \times \boldsymbol{r_{BC}})$$

$$= \boldsymbol{a_A} + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r_{AB}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}) + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r_{BC}} + \boldsymbol{\omega_2} \times (\boldsymbol{\omega_2} \times \boldsymbol{r_{AC}}) \ .$$

Noting that $\boldsymbol{a_D}$ and $\boldsymbol{a_A}$ are zero, and substituting $\dot{\boldsymbol{\omega}}_1 = \boldsymbol{K}\ddot{\alpha}$, $\dot{\boldsymbol{\omega}}_2 = \boldsymbol{K}\ddot{\beta}$, and $\dot{\boldsymbol{\omega}}_3 = -\boldsymbol{K}\ddot{\gamma}$ into this equation gives

$$-\boldsymbol{K}\ddot{\gamma} \times l_3(-\boldsymbol{I}\cos\gamma + \boldsymbol{J}\sin\gamma) - \boldsymbol{K}\dot{\gamma} \times [-\boldsymbol{K}\dot{\gamma} \times l_3(-\boldsymbol{I}\cos\gamma + \boldsymbol{J}\sin\gamma)]$$

$$= \boldsymbol{K}\ddot{\alpha} \times l_1(\boldsymbol{I}\cos\alpha + \boldsymbol{J}\sin\alpha) + \boldsymbol{K}\dot{\alpha} \times [\boldsymbol{K}\dot{\alpha} \times l_1(\boldsymbol{I}\cos\alpha + \boldsymbol{J}\sin\alpha)]$$
$$+ \boldsymbol{K}\ddot{\beta} \times l_2(\boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta) + \boldsymbol{K}\dot{\beta} \times [\boldsymbol{K}\dot{\beta} \times l_2(\boldsymbol{I}\cos\beta + \boldsymbol{J}\sin\beta)] \ .$$

Carrying out the cross products and gathering terms,

$$\boldsymbol{I} \ : \quad l_3\ddot{\gamma}\sin\gamma + l_3\dot{\gamma}^2\cos\gamma = -l_1\ddot{\alpha}\sin\alpha - l_1\dot{\alpha}^2\cos\beta$$
$$-l_2\ddot{\beta}\sin\beta - l_2\dot{\beta}^2\cos\beta$$

$$\boldsymbol{J} \ : \quad l_3\ddot{\gamma}\cos\gamma - l_3\dot{\gamma}^2\sin\gamma = l_1\ddot{\alpha}\cos\alpha - l_1\dot{\alpha}^2\sin\alpha$$
$$+l_2\ddot{\beta}\cos\beta - l_2\dot{\beta}^2\sin\beta.$$

If general governing equations are required, the geometric relationships of Eq.(4.23) *must be developed.*

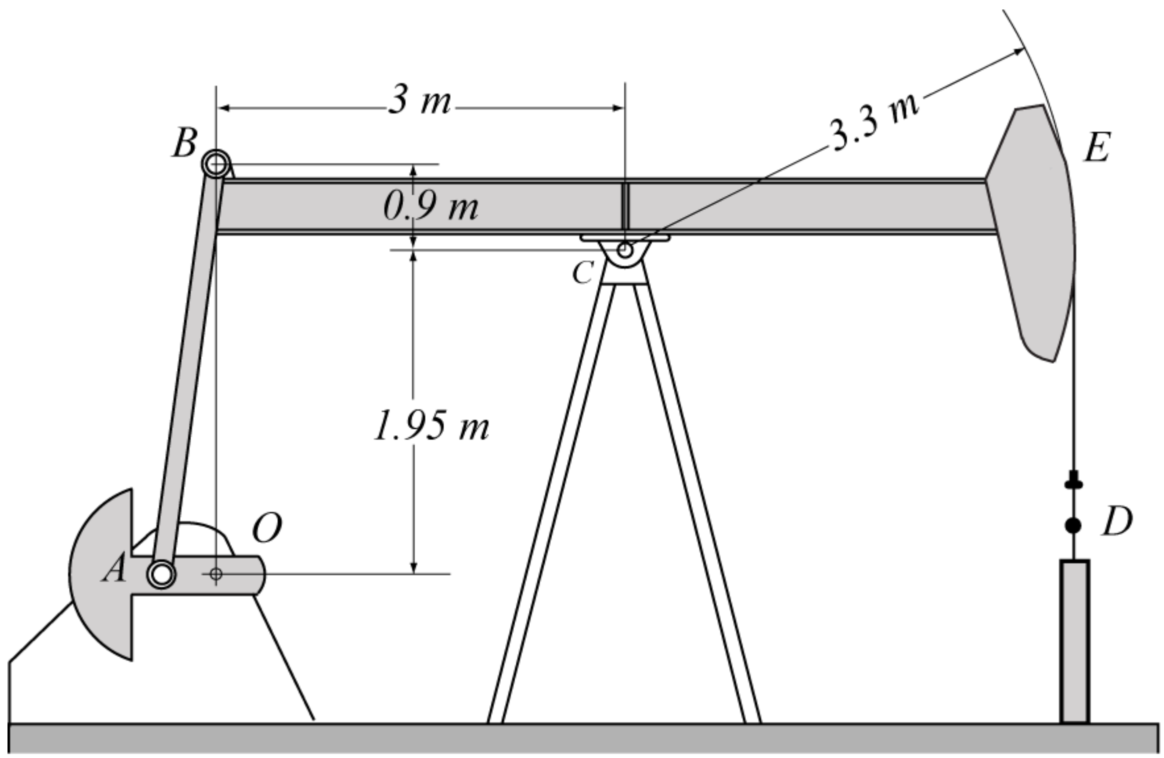


Figure XP5.4a Oil well pumping rig, adapted from Meriam and Kraige (1992)

Example Problem 4.5 Figure XP4.5a illustrates an oil pumping rig that is typically used for shallow oil wells. An electric motor drives the rotating arm *OA* at a constant, clock-wise angular velocity $\omega = 20\, rpm$. A cable attaches the pumping rod at *D* to the end of the rocking arm *BE*. Rotation of the driving link produces a vertical oscillation that drives a positive-displacement pump at the bottom of the well.

Tasks:

a. Draw the rig in a general position and select coordinates to define the bars' general position. State the kinematic constraint equations defining the angular positions of bars *AB* and *BCE* in terms of bar *OA*'s angular position.

b. Outline a solution procedure to determine the orientations of bars *AB* and *BCE* in terms of bar *OA*'s angular position.

c. Derive general expressions for the angular velocities of bars *AB* and *BCE* in terms of bar *OA*'s angular position and angular velocity. Solve for the unknown angular velocities.

d. Derive general expressions for the angular accelerations of bars *AB* and *BCE* in terms of bar *OA*'s angular position, velocity, and acceleration. Solve for the unknown angular accelerations.

e. Derive general expressions for the change in vertical position and vertical acceleration of point *D* as a function of bar *OA*'s angular position.

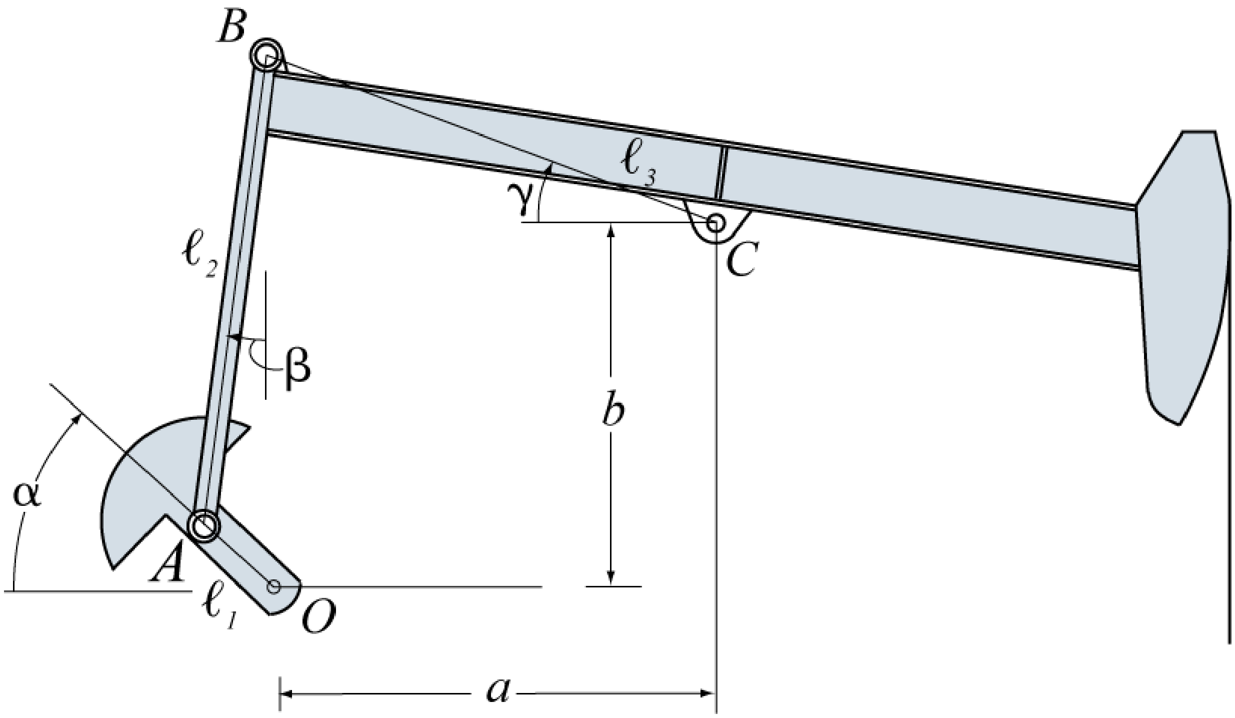


Figure XP5.4b Pumping rig in a general position with coordinates

Solution. The sketch of figure XP4.5b shows the angles α, β, γ defining the angular positions of bars *OA*, *AB* and *BC*, respectively. α is the (known) input variable, while β and γ are the (unknown) output variables. The length l_3 extends from *B* to *C*. Stating the components of the bars in the *X* and *Y* directions gives:

$$-l_1 \cos\alpha + l_2 \sin\beta + l_3 \cos\gamma = a$$
$$\Rightarrow\ l_2 \sin\beta + l_3 \cos\gamma = a + l_1 \cos\alpha = h_1(\alpha)$$

(i)

$$l_1 \sin\alpha + l_2 \cos\beta - l_3 \sin\gamma = b$$
$$\Rightarrow\ l_2 \cos\beta - l_3 \sin\gamma = b - l_1 \sin\alpha = h_2(\alpha),$$

and concludes *Task a*.

As a first step in solving for (β,γ), we state the equations as
$l_2 \sin\beta = h_1 - l_3 \cos\gamma$, $l_2 \cos\beta = h_2 + l_3 \sin\gamma$

Squaring these equations and adding them together gives:

$$l_2^2(\sin^2\beta + \cos^2\beta) = l_2^2$$
$$= h_1^2 - 2h_1 l_3 \cos\gamma + l_3^2\cos^2\gamma + h_2^2 + 2h_2 l_3 \sin\gamma + l_3^2 \sin^2\gamma$$

$$\therefore\ 2h_1 l_3 \cos\gamma = h_1^2 + h_2^2 + l_3^2 - l_2^2 + 2h_2 l_3 \sin\gamma = 2d + 2h_2 l_3 \sin\gamma \ ,$$

where $2d = h_1^2 + h_2^2 + l_3^2 - l_2^2$. Substituting $\cos\gamma = \sqrt{1-\sin^2\gamma}$ nets

$$h_1^2 l_3^2 (1 - \sin^2\gamma)$$
$$= d^2 + 2dh_2 l_3 \sin\gamma + h_2^2 l_3^2 \sin^2\gamma$$

$$\therefore\ \sin^2\gamma + \frac{2dh_2}{l_3(h_1^2 + h_2^2)}\sin\gamma + \frac{d^2 - h_1^2 l_3^2}{l_3^2(h_1^2 + h_2^2)} = 0\ . \qquad \textbf{(ii)}$$

For a specified value of α, solving this quadratic equation gives $\sin\gamma \Rightarrow \gamma = \sin^{-1}\gamma$, and back substitution into Eq.(i) nets β. These steps concludes *Task b*, and figure XP4.5b illustrates the results for the lengths of figure XP4.5a.

Proceeding to *Task c*, we can differentiate Eq.(i) with respect to time to obtain:

$$l_2 \cos\beta\,\dot{\beta} - l_3 \sin\gamma\,\dot{\gamma} = -l_1 \sin\alpha\,\dot{\alpha} = -l_1 \omega \sin\alpha$$

$$-l_2 \sin\beta\,\dot{\beta} - l_3 \cos\gamma\,\dot{\gamma} = -l_1 \cos\alpha\,\dot{\alpha} = -l_1 \omega \cos\alpha .$$

(iii)

In matrix format, these equations become

$$\begin{bmatrix} \cos\beta & -\sin\gamma \\ \sin\beta & \cos\gamma \end{bmatrix} \begin{Bmatrix} l_2 \dot{\beta} \\ l_3 \dot{\gamma} \end{Bmatrix} = l_1 \omega \begin{Bmatrix} -\sin\alpha \\ \cos\alpha \end{Bmatrix} .$$

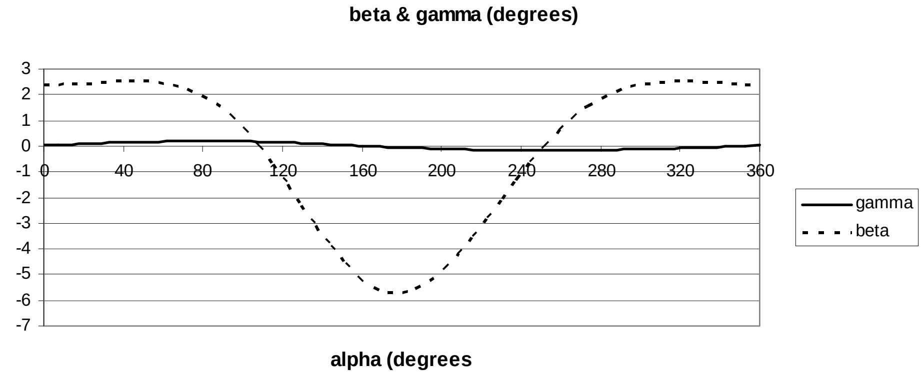
Using Cramer's rule (Appendix A), their solution can be stated:

$$l_2 \dot{\beta} = \frac{l_1 \omega}{\cos\gamma \cos\beta + \sin\gamma \cos\beta} \begin{vmatrix} -\sin\alpha & -\sin\gamma \\ \cos\alpha & \cos\gamma \end{vmatrix} = \frac{l_1 \omega \sin(\gamma - \alpha)}{\cos(\beta - \gamma)}$$

$$l_3 \dot{\gamma} = \frac{l_1 \omega}{\cos\gamma \cos\beta + \sin\gamma \cos\beta} \begin{vmatrix} \cos\beta & -\sin\alpha \\ \sin\beta & \cos\alpha \end{vmatrix} = \frac{l_1 \omega \cos(\alpha - \beta)}{\cos(\beta - \gamma)} ,$$

(iv)

concluding *Task c*. Figure XP 4.5c illustrates $\dot{\gamma}, \dot{\beta}$ versus α



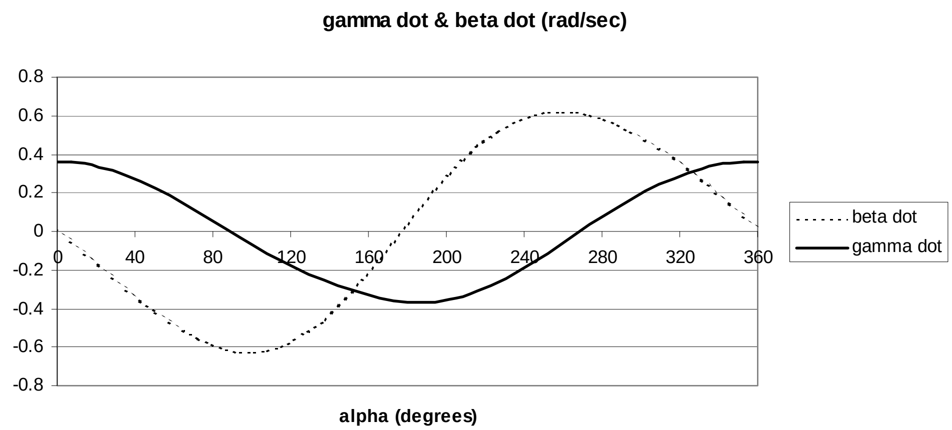


Figure XP 4.5c γ, β and $\dot{\gamma}, \dot{\beta}$ versus α

Moving to *Task d*, we can differentiate Eq.(iii) with respect to time to obtain:

$$l_2 \cos\beta\,\ddot{\beta} - l_3 \sin\gamma\,\ddot{\gamma}$$
$$= -l_1 \omega^2 \cos\alpha + l_2 \sin\beta\,\dot{\beta}^2 + l_3 \cos\gamma\,\dot{\gamma}^2 = g_1$$

$$-l_2 \sin\beta\,\ddot{\beta} - l_3 \cos\gamma\,\ddot{\gamma}$$
$$= l_1 \omega^2 \sin\alpha + l_2 \cos\beta\,\dot{\beta}^2 - l_3 \sin\gamma\,\dot{\gamma}^2 = -g_2.$$

In matrix format, these equations become

$$\begin{bmatrix} \cos\beta & -\sin\gamma \\ \sin\beta & \cos\gamma \end{bmatrix} \begin{Bmatrix} l_2\ddot{\beta} \\ l_3\ddot{\gamma} \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}.$$

The solution can be stated:

$$l_2\,\ddot{\beta} = \frac{1}{\cos\gamma\cos\beta + \sin\gamma\cos\beta} \begin{vmatrix} g_1 & -\sin\gamma \\ g_2 & \cos\gamma \end{vmatrix} = \frac{g_1\cos\gamma + g_2\sin\gamma}{\cos(\beta-\gamma)}$$

$$l_3\,\ddot{\gamma} = \frac{1}{\cos\gamma\cos\beta + \sin\gamma\cos\beta} \begin{vmatrix} \cos\beta & g_1 \\ \sin\beta & g_2 \end{vmatrix} = \frac{g_2\cos\beta - g_1\sin\beta}{\cos(\beta-\gamma)}\,, \tag{vi}$$

concluding *Task d*.

In regard to *Task e*, as long as the circular arc at the end of the rocker arm is long enough, the tangent point of the cable with the circular-faced end of the rocker arm will be at a horizontal line running through *C*. Hence, the change in the horizontal position of point *D* is the amount of cable rolled off the arc due to a change in the rocker arm γ, i.e., $\delta Y = -3.3\,\delta\gamma$. Similarly, the vertical acceleration of the sucker rod at *D* is the circumferential acceleration of a point on the arc, i.e, $a_\theta = r\ddot{\theta} \Rightarrow \ddot{Y} = 3.3\,\ddot{\gamma}$. Figure XP4.5 d illustrates $\delta Y, \ddot{Y}$ as a function of alpha. The distance traveled by the pump rod in one cyle is .633 - (-.633) = 1.27 m. The peak positive acceleration is 1.17 g and the minimum is -0.63 g. Note that $|-0.63\,g| < 1\,g$, indicating from a rigid-body viewpoint that the cable will remain in tension during its downward motion.

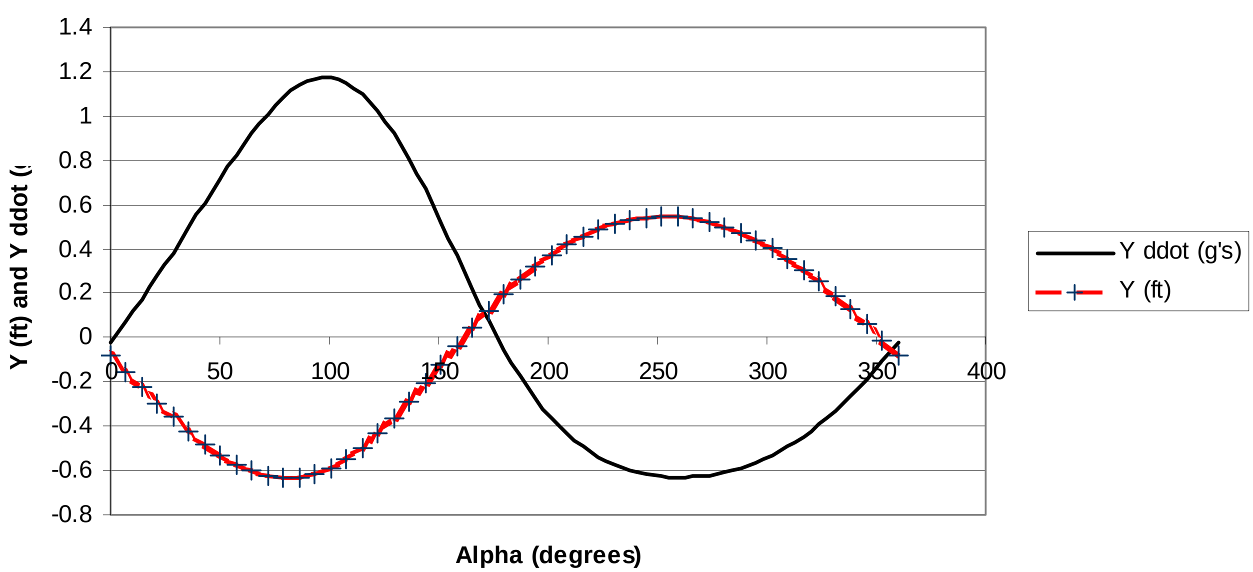


Figure XP4.5d Vertical acceleration and change in position of the pumping rod

Lecture 21. MORE PLANAR KINEMATIC EXAMPLES

4.5c Another Slider-Crank Mechanism

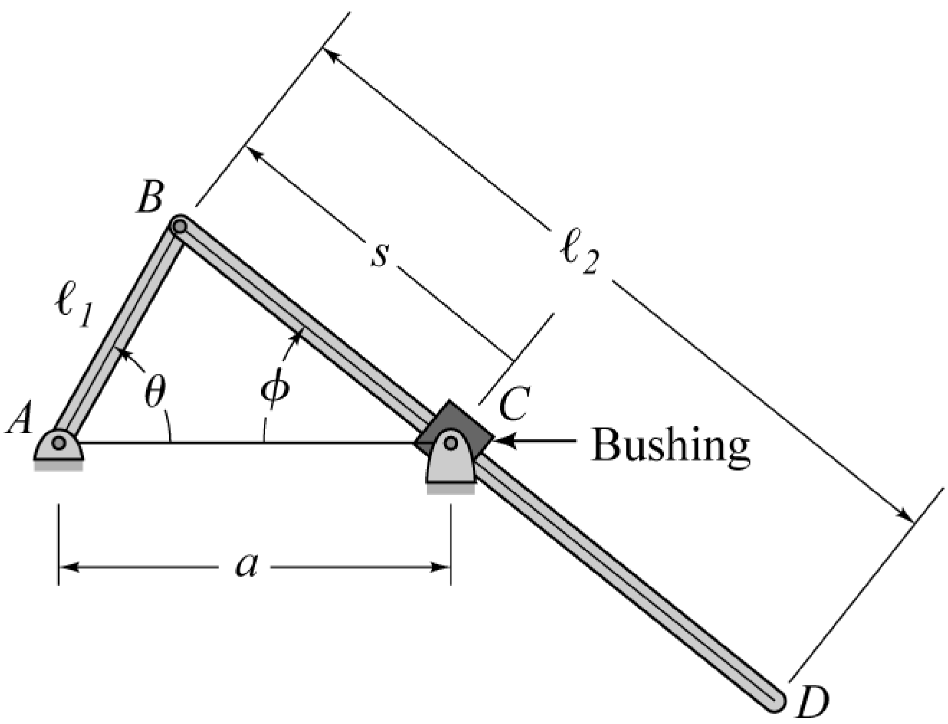


Figure 4.24 Alternative slider-crank mechanism.

Engineering-analysis task: *For* $\dot{\theta} = \omega$ = *constant, determine* φ *and S and their first and second derivatives for one cycle of* θ.

Geometric Approach. From figure 4.22:

$$X: \quad l_1 \cos\theta + S\cos\varphi = a$$

$$Y: \quad l_1 \sin\theta = S\sin\varphi \quad . \tag{4.25}$$

Reordering these equations to

$$S\cos\varphi = a - l_1\cos\theta$$

$$S\sin\varphi = l_1\sin\theta \quad , \qquad \textbf{(4.27a)}$$

emphasizes that θ is the input coordinate, with φ and S the output coordinates. These equations are nonlinear but can be solved for φ and S in terms of θ, via

$$S^2(\cos^2\varphi + \sin^2\varphi) = (a - l_1\cos\theta)^2 + (l_1\sin\theta^2)$$

$$\therefore \quad S^2 = a^2 - 2al_1\cos\theta + l_1^2 \quad , \qquad \textbf{(4.27b)}$$

and

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{l_1\sin\theta}{a - l_1\cos\theta} \quad . \qquad \textbf{(4.27c)}$$

Differentiating Eqs.(4.25) nets:

$$\dot{S}\cos\varphi - S\sin\varphi\,\dot{\varphi} = l_1\sin\theta\,\dot{\theta}$$

$$\dot{S}\sin\varphi + S\cos\varphi\,\dot{\varphi} = l_1\cos\theta\,\dot{\theta} \quad , \qquad \textbf{(4.28a)}$$

or

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} \dot{S} \\ S\dot{\varphi} \end{Bmatrix} = l_1\ \omega \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix} . \qquad \textbf{(4.28b)}$$

Differentiating Eq.(4.28a) w.r.t. time nets:

$$\begin{aligned} \ddot{S}\cos\varphi - S\sin\varphi\,\ddot{\varphi} - S\cos\varphi\,\dot{\varphi}^2 &= l_1\sin\theta\,\ddot{\theta} + 2\dot{S}\dot{\varphi}\sin\varphi \\ &\quad + l_1\cos\theta\,\dot{\theta}^2 \\ \ddot{S}\sin\varphi + S\cos\varphi\,\ddot{\varphi} - S\sin\varphi\,\dot{\varphi}^2 &= l_1\cos\theta\,\ddot{\theta} - 2\dot{S}\dot{\varphi}\cos\varphi \\ &\quad - l_1\sin\theta\,\dot{\theta}^2 . \end{aligned} \qquad \textbf{(4.29)}$$

Substituting $\dot{\theta} = \omega$, $\ddot{\theta} = 0$,and rearranging gives:

$$\begin{aligned} \ddot{S}\cos\varphi - S\ddot{\varphi}\sin\varphi &= l_1\omega^2\cos\theta + 2\dot{S}\dot{\varphi}\sin\varphi + S\cos\varphi\,\dot{\varphi}^2 \\ \ddot{S}\sin\varphi + S\ddot{\varphi}\cos\varphi &= -l_1\omega^2\sin\theta - 2\dot{S}\dot{\varphi}\cos\varphi + S\sin\varphi\,\dot{\varphi}^2 . \end{aligned} \qquad \textbf{(4.30a)}$$

The matrix equation for the unknown $\ddot{S}$ and $S\ddot{\varphi}$ is

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} \ddot{S} \\ S\ddot{\varphi} \end{Bmatrix}$$

$$= \begin{Bmatrix} l_1\omega^2\cos\theta & + & 2\dot{S}\,\dot{\varphi}\,\sin\varphi & + & S\,\dot{\varphi}^2\cos\varphi \\ -l_1\omega^2\sin\theta & - & 2\dot{S}\,\dot{\varphi}\,\cos\varphi & + & S\,\dot{\varphi}^2\sin\varphi \end{Bmatrix} . \qquad \textbf{(4.30b)}$$

The engineering-analysis task is accomplished by executing the following sequential steps:

1. Vary θ over the range [0, 2π], yielding discrete values θ_i .

2. For each θ_i value, solve Eq.(4.25) to determine corresponding values for φ_i and S_i.

3. Enter Eqs.(4.28) with known values for θ_i , φ_i and S_i. to determine $\dot{\varphi}_i$ and $\dot{S}_i$.

4 Enter Eqs.(4.30) with known values for θ_i , φ_i , S_i , $\dot{\varphi}_i$ and $\dot{S}_i$ to determine $\ddot{\varphi}_i$ and $\ddot{S}_i$.

5. Plot $\dot{\varphi}_i$, $\dot{\theta}_i$, $\ddot{\varphi}_i$ and $\ddot{S}_i$ versus θ_i .

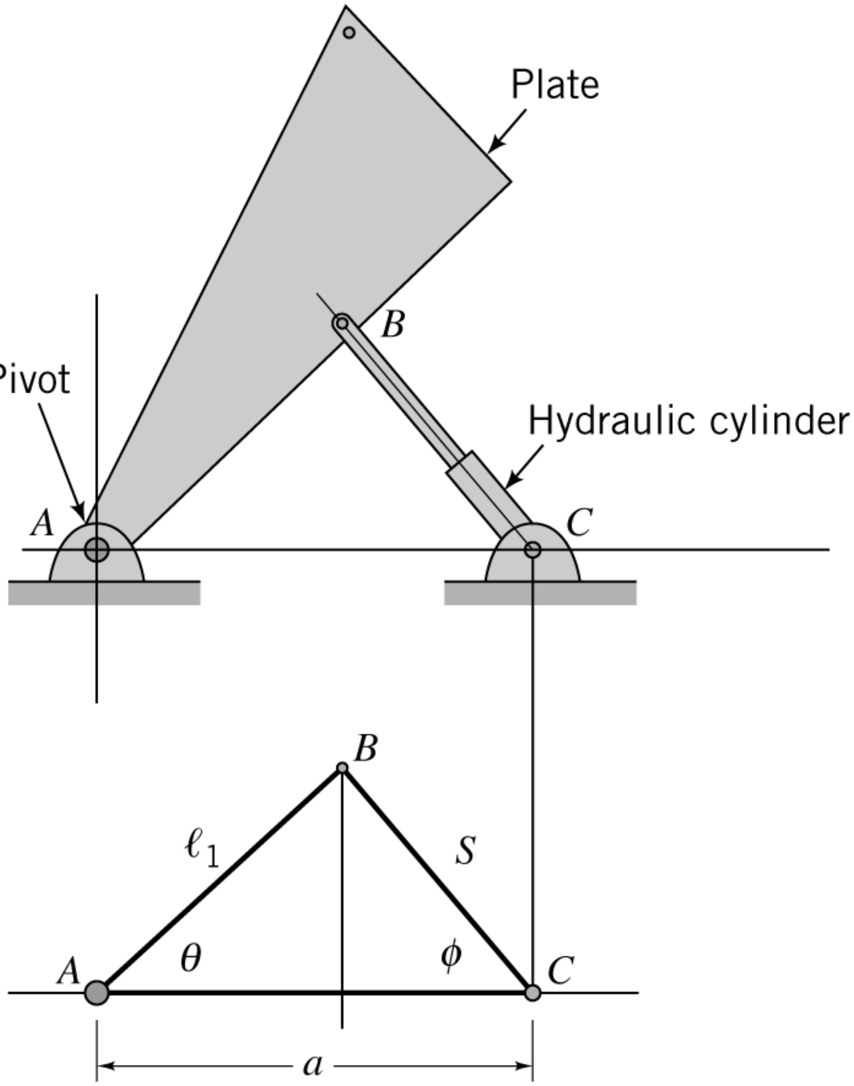


Figure 4.24 Alternate version of the mechanism in figure 4.22 with *S* as the input.

For S as the input, Eqs.(4.28a) are reordered as:

$$S\dot{\varphi}\sin\varphi + l_1\dot{\theta}\sin\theta = \dot{S}\cos\varphi$$

$$S\dot{\varphi}\cos\varphi - l_1\dot{\theta}\cos\theta = -\dot{S}\sin\varphi \ ,$$

to define $\dot{\varphi}$ and $\dot{\theta}$. Rearranging Eqs.(4.29) defines $\ddot{\varphi}$ and $\ddot{\theta}$ via:

$$-S\sin\varphi\,\ddot{\varphi} - l_1\sin\theta\,\ddot{\theta} = -\ddot{S}\cos\varphi + S\cos\varphi\,\dot{\varphi}^2 + 2\dot{S}\dot{\varphi}\sin\varphi + l_1\cos\theta\,\dot{\theta}^2$$

$$S\cos\varphi\,\ddot{\varphi} - l_1\cos\theta\,\ddot{\theta} = -\ddot{S}\sin\varphi + S\sin\varphi\,\dot{\varphi}^2 - 2\dot{S}\dot{\varphi}\cos\varphi - l_1\sin\theta\,\dot{\theta}^2 \quad .$$

The basic geometry of figures 4.22 and 4.24 tends to show up regularly in planar mechanisms.

Vector, Two-Coordinate-System Approach for Velocity and Acceleration Relationships

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_o + \hat{\dot{\rho}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \qquad \textbf{(4.1)}$$

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_o + \hat{\ddot{\boldsymbol{\rho}}} + 2\boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

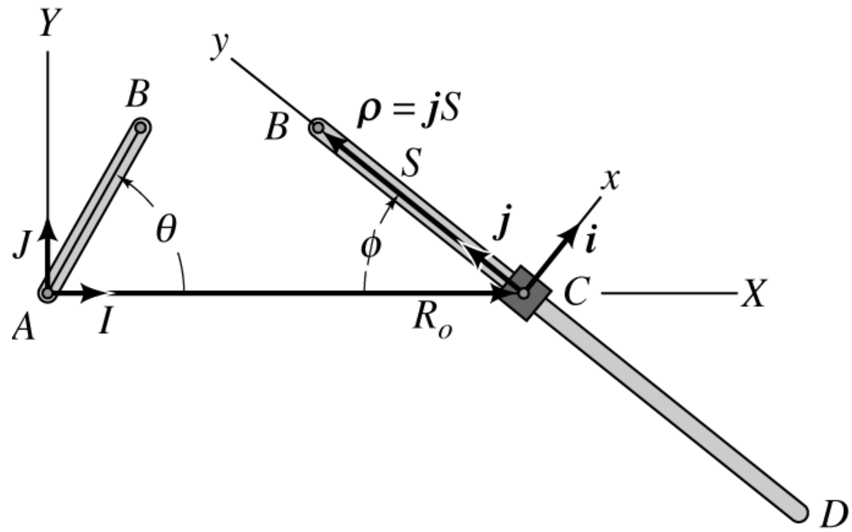


Figure 4.26
Two-coordinate arrangement for rod *BD* of the slider-crank mechanism in figure 4.22.

The vector $\boldsymbol{\omega}$ is defined as the angular velocity of the *x, y* system relative to the *X,Y* system. From figure 4.26 , using the right-hand-screw convention,

$$\boldsymbol{\omega} = -\boldsymbol{K}\dot{\varphi} = -\boldsymbol{k}\dot{\varphi} \ .$$

Given that $\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt}\big|_{X,Y}$, we obtain by direct differentiation

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{K}\ddot{\varphi} = -\boldsymbol{k}\ddot{\varphi} \ .$$

From figure 4.26,

$$\boldsymbol{\rho} = \boldsymbol{j}S \ .$$

Differentiating this vector holding $\boldsymbol{j}$ constant gives

$$\dot{\hat{\boldsymbol{\rho}}} = \boldsymbol{j}\dot{S} \ .$$

Differentiating again gives

$$\ddot{\hat{\boldsymbol{\rho}}} = \boldsymbol{j}\ddot{S} \ .$$

Substituting these results into the definitions provided by Eqs.(4.1) gives:

$$\dot{\boldsymbol{r}}_{\boldsymbol{B}} = \boldsymbol{v}_{\boldsymbol{B}} = 0 + \boldsymbol{j}\dot{S} - \boldsymbol{k}\dot{\varphi} \times \boldsymbol{j}S$$

$$\ddot{\boldsymbol{r}}_{\boldsymbol{B}} = \boldsymbol{a}_{\boldsymbol{B}} = 0 + \boldsymbol{j}\ddot{S} + 2(-\boldsymbol{k}\dot{\varphi}) \times \boldsymbol{j}\dot{S} - \boldsymbol{k}\ddot{\varphi} \times \boldsymbol{j}S - \boldsymbol{k}\dot{\varphi} \times (-\boldsymbol{k}\dot{\varphi} \times \boldsymbol{j}S)$$

Carrying through the cross products and completing the algebra nets:

$$\begin{aligned} \boldsymbol{v}_{\boldsymbol{B}} &= \boldsymbol{i}S\dot{\varphi} + \boldsymbol{j}\dot{S} \\ \boldsymbol{a}_{\boldsymbol{B}} &= \boldsymbol{i}(S\ddot{\varphi} + 2\dot{S}\dot{\varphi}) + \boldsymbol{j}(\ddot{S} - S\dot{\varphi}^2) \end{aligned} \qquad \textbf{(4.31)}$$

By comparison to figure 4.25, the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ of the *x,y* system coincide with the unit vectors $\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}$ and $\boldsymbol{\varepsilon}_{\mathbf{r2}}$ used in the polar-coordinate solution for $\boldsymbol{v}_{\boldsymbol{B}}$ and $\boldsymbol{a}_{\boldsymbol{B}}$.

Returning to figure 4.25, we can apply Eq.(4.3) to state the velocities and accelerations of points *A* and *B* as:

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}$$
$$\boldsymbol{a_B} = \boldsymbol{a_A} + \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} \times \boldsymbol{r_{AB}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}) \ .$$

Since point A is fixed, $\boldsymbol{v_A} = \boldsymbol{a_A} = 0$. From the right-hand-rule convention, $\boldsymbol{\omega_1} = \boldsymbol{K}\dot{\theta}$, $\dot{\boldsymbol{\omega}}_{\boldsymbol{1}} = \boldsymbol{K}\ddot{\theta}$. From figure 4.25, $\boldsymbol{r_{AB}} = l_1 (\, \boldsymbol{I} \cos\theta + \boldsymbol{J} \sin\theta \,)$. Substituting, we obtain

$$\boldsymbol{v_B} = 0 + \boldsymbol{K}\dot{\theta} \times l_1 (\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$$

$$\begin{aligned}\boldsymbol{a_B} &= 0 + \boldsymbol{K}\ddot{\theta}_1 \times l_1 (\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \\ &+ \boldsymbol{K}\dot{\theta} \times [\boldsymbol{K}\dot{\theta} \times l_1 (\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)].\end{aligned}$$

Carrying out the cross products and algebra gives:

$$\begin{aligned}\boldsymbol{v_B} &= l_1 \dot{\theta} (-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) \\ \boldsymbol{a_B} &= l_1 \ddot{\theta} (-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) - l_1 \dot{\theta}^2 (\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \ .\end{aligned} \qquad \textbf{(4.32)}$$

The results in Eqs.(4.32) are given in terms of $\boldsymbol{I}$ and $\boldsymbol{J}$ unit vectors, versus $\boldsymbol{i}$ and $\boldsymbol{j}$ for Eq.(4.31).

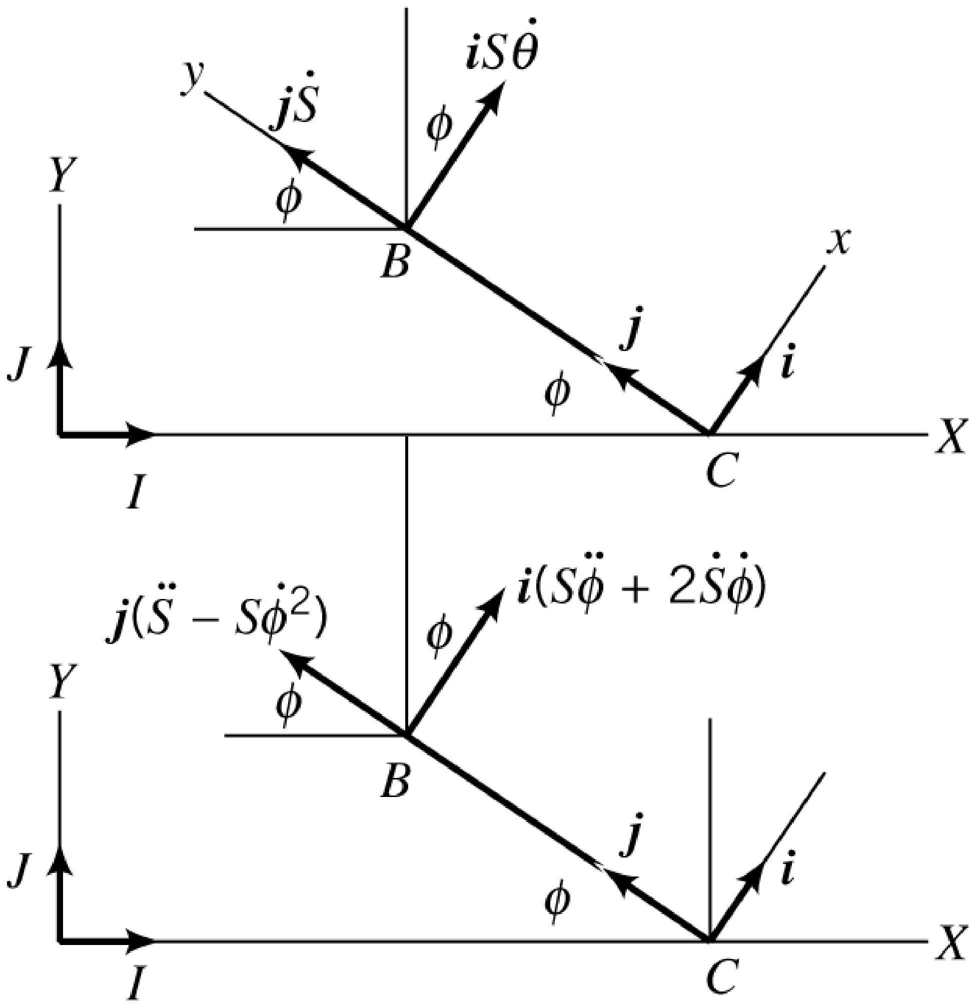


Figure 4.27 Velocity and acceleration definitions for the velocity point B in the x, y system.

From figure 4.27:

$$\boldsymbol{v_B} = S\dot{\varphi}(\boldsymbol{I}\sin\varphi + \boldsymbol{J}\cos\varphi) + \dot{S}(-\boldsymbol{I}\cos\varphi + \boldsymbol{J}\sin\varphi)$$

$$\boldsymbol{a_B} = (S\ddot{\varphi} - S\dot{\varphi}^2)(\boldsymbol{I}\sin\varphi + \boldsymbol{J}\cos\varphi) + (\ddot{S} + 2\dot{S}\dot{\varphi})(-\boldsymbol{I}\cos\varphi + \boldsymbol{J}\sin\varphi) \quad \textbf{(4.33)}$$

Equating these definition with the result from Eqs.(4.32) gives:

$$I: \quad -l_1\dot{\theta}\sin\theta = -\dot{S}\cos\varphi - S\dot{\varphi}\sin\varphi$$

$$J: \quad l_1\dot{\theta}\cos\theta = \dot{S}\sin\varphi + S\dot{\varphi}\cos\varphi \quad ,$$

and

$$I: \; -l_1\ddot{\theta}\sin\theta - l_1\dot{\theta}^2\cos\theta = -(\ddot{S} - S\dot{\varphi}^2)\cos\varphi + (S\ddot{\varphi} + 2\dot{S}\dot{\varphi})\sin\varphi$$

$$J: \quad l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta = (\ddot{S} - S\dot{\varphi}^2)\sin\varphi + (S\ddot{\varphi} + 2\dot{S}\dot{\varphi})\cos\varphi \; ,$$

which repeats our earlier results.

Solution for the Velocity and Acceleration of Point D

The simplest approach (given that we now know $\varphi, \dot{\varphi}$, and $\ddot{\varphi}$) is the direct vector formulation. Applying Eqs.(4.3) to points *A* and *B* gives:

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}$$

$$\boldsymbol{a_B} = \boldsymbol{a_A} + \dot{\boldsymbol{\omega}}_\mathbf{1} \times \boldsymbol{r_{AB}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}) \; .$$

We have already worked through these equations, obtaining solutions for $\boldsymbol{v_B}$ and $\boldsymbol{a_B}$ in Eqs.(4.32). We can also apply Eqs.(4.3) to points *B* and *D*, since they are points on a rigid body (unlike point C), obtaining:

$$\boldsymbol{v}_D = \boldsymbol{v}_B + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{BD}$$

$$\boldsymbol{a}_D = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{BD} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{BD}) \ .$$

Substituting from Eq.(4.34) for $\boldsymbol{v}_B$ and $\boldsymbol{a}_B$ plus $\boldsymbol{\omega}_2 = -\boldsymbol{K}\dot{\varphi}$, $\dot{\boldsymbol{\omega}}_2 = -\boldsymbol{K}\ddot{\varphi}$, and $\boldsymbol{r}_{BD} = l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$ into these equation gives:

$$\boldsymbol{v}_D = l_1\dot{\theta}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) - \boldsymbol{K}\dot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$

$$\boldsymbol{a}_D = l_1\ddot{\theta}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) - l_1\dot{\theta}^2(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$$
$$-\boldsymbol{K}\ddot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi) - \boldsymbol{K}\dot{\varphi} \times [-\boldsymbol{K}\dot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)] \ .$$

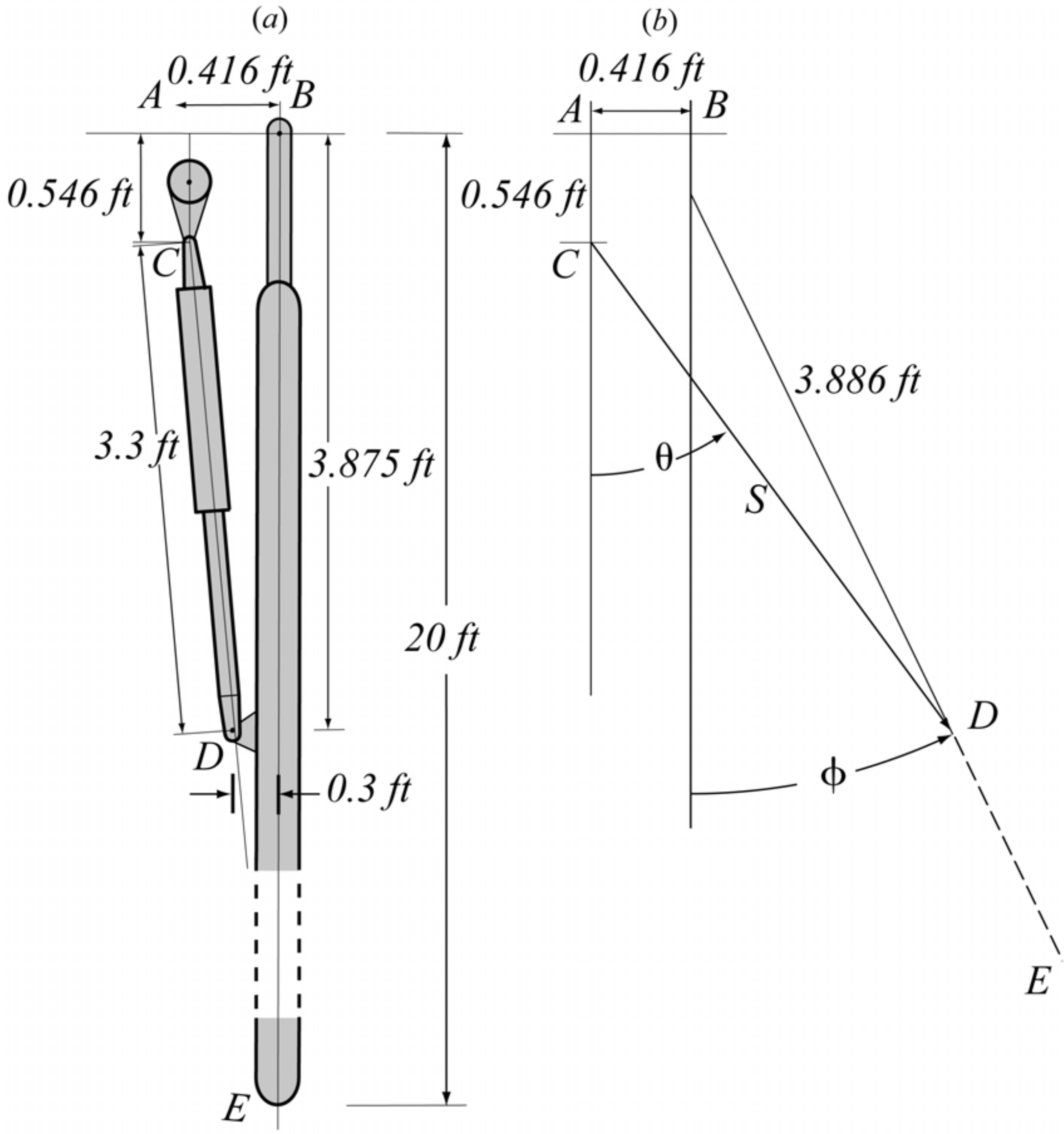
Carrying out the cross products and gathering terms yields:

$$\boldsymbol{v}_D = -\boldsymbol{I}(l_1\dot{\theta}\sin\theta + l_2\dot{\varphi}\sin\varphi) + \boldsymbol{J}(l_1\dot{\theta}\,co\theta - l_2\dot{\varphi}\cos\varphi)$$

$$\boldsymbol{a}_D = \boldsymbol{I}(-l_1\ddot{\theta}\sin\theta - l_1\dot{\theta}^2\cos\theta - l_2\ddot{\varphi}\sin\varphi - l_2\dot{\varphi}^2\cos\varphi)$$
$$+\boldsymbol{J}(l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta - l_2\ddot{\varphi}\cos\varphi + l_2\dot{\varphi}^2\sin\varphi) \ .$$

These are general equations for $\boldsymbol{v}_D$ and $\boldsymbol{a}_D$. Substituting $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$ completes the present effort, with $\varphi, \dot{\varphi}$, and $\ddot{\varphi}$ defined, respectively, by Eqs.(4.28), (4.29), and (4.30).

Lesson: The “best” method for finding the velocity and acceleration of a specific point is frequently not the “best” method for finding kinematic relationships.



Example Problem 4.6 Figure XP4.6a provides a top view of a power-gate actuator. An electric motor drives a lead screw mounted in the arm connecting points *C* and *D*. Lengthening arm *CD* closes the gate; shortening it opens the gate. During closing action, arm *CD* extends from a length of 3.3 ft to 4.3 ft in about 17 seconds to proceed from a fully open to fully closed positions. The gate reaches its steady extension rate quickly at the outset and decelerates rapidly when the gate nears the closed position.

Tasks:

a. Draw the gate actuator in a general position and derive governing equations that define the orientations of bars *CD* and *BC* as a function of the length of arm CD.

b. Assume that arm *CD* extends at a constant rate (gate is closing) and determine a relationship for the angular velocities of arms *CD* and *BE*.

c. Continuing to assume that bar *CD* extends at a constant rate, determine a relationship for the angular accelerations of arms *CD* and *BE*.

Solution From figure XP4.6b:

$$\begin{aligned}&\textit{horizontal}\ :\\ &S\sin\theta = .416 + 3.886\sin\varphi\ ;\ 3.886 = \sqrt{3.875^2 + .3^2}\\ &\textit{vertical}\ :\\ &S\cos\theta = 3.886\cos\varphi - .546\end{aligned} \qquad \textbf{(i)}$$

S is the input and θ, φ are the unknown output variables. Differentiating Eq.(i) with respect to time gives:

$$\dot{S}\sin\theta + S\cos\theta\,\dot{\theta} = 4.92\cos\varphi\,\dot{\varphi}\ ,\ \dot{S}\cos\theta - S\sin\theta\,\dot{\theta} = -4.92\sin\varphi\,\dot{\varphi} \qquad \textbf{(ii)}$$

Rearranging Eq.(ii) and putting them in matrix format gives

$$\begin{bmatrix} 4.92\cos\varphi & -S\cos\theta \\ -4.92\sin\varphi & S\sin\theta \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \end{Bmatrix} = \dot{S} \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix}. \qquad \textbf{(iii)}$$

Differentiating Eq.(ii) gives:

$$\ddot{S}\sin\theta + 2\dot{S}\dot{\theta}\cos\theta + S\cos\theta\,\dot{\theta} - S\sin\theta\,\dot{\theta}^2$$
$$= 3.886\cos\varphi\,\ddot{\varphi} - 3.886\cos\varphi\,\dot{\varphi}^2$$

$$\ddot{S}\cos\theta - 2\dot{S}\dot{\theta}\sin\theta - S\sin\theta\,\ddot{\theta} - S\cos\theta\,\dot{\theta}^2$$
$$= -3.886\sin\varphi\,\ddot{\varphi} - 3.886\cos\varphi\,\dot{\varphi}^2$$

Rearranging the equations gives:

$$3.886\cos\varphi\,\ddot{\varphi} + S\cos\theta\,\ddot{\theta}$$
$$= \ddot{S}\sin\theta + 2\dot{S}\dot{\theta}\cos\theta - S\sin\theta\,\dot{\theta}^2 + 3.886\cos\varphi\,\dot{\varphi}^2 = g_1$$

$$-3.886\sin\varphi\,\ddot{\varphi} + S\sin\theta\,\ddot{\theta}$$
$$= \ddot{S}\cos\theta - 2\dot{S}\dot{\theta}\sin\theta - S\cos\theta\,\dot{\theta}^2 + 3.886\cos\varphi\,\dot{\varphi}^2 = g_2$$

(iv)

In matrix format, Eq.(iv) becomes

$$\begin{bmatrix} 3.886\cos\varphi & -S\cos\theta \\ 3.886\sin\varphi & S\sin\theta \end{bmatrix} \begin{Bmatrix} \ddot{\varphi} \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}. \qquad \textbf{(v)}$$

Figure XP4.6c illustrates the solution for θ, φ, $\dot{\theta}, \dot{\varphi}$, and $\ddot{\theta}, \ddot{\varphi}$ versus *S*.

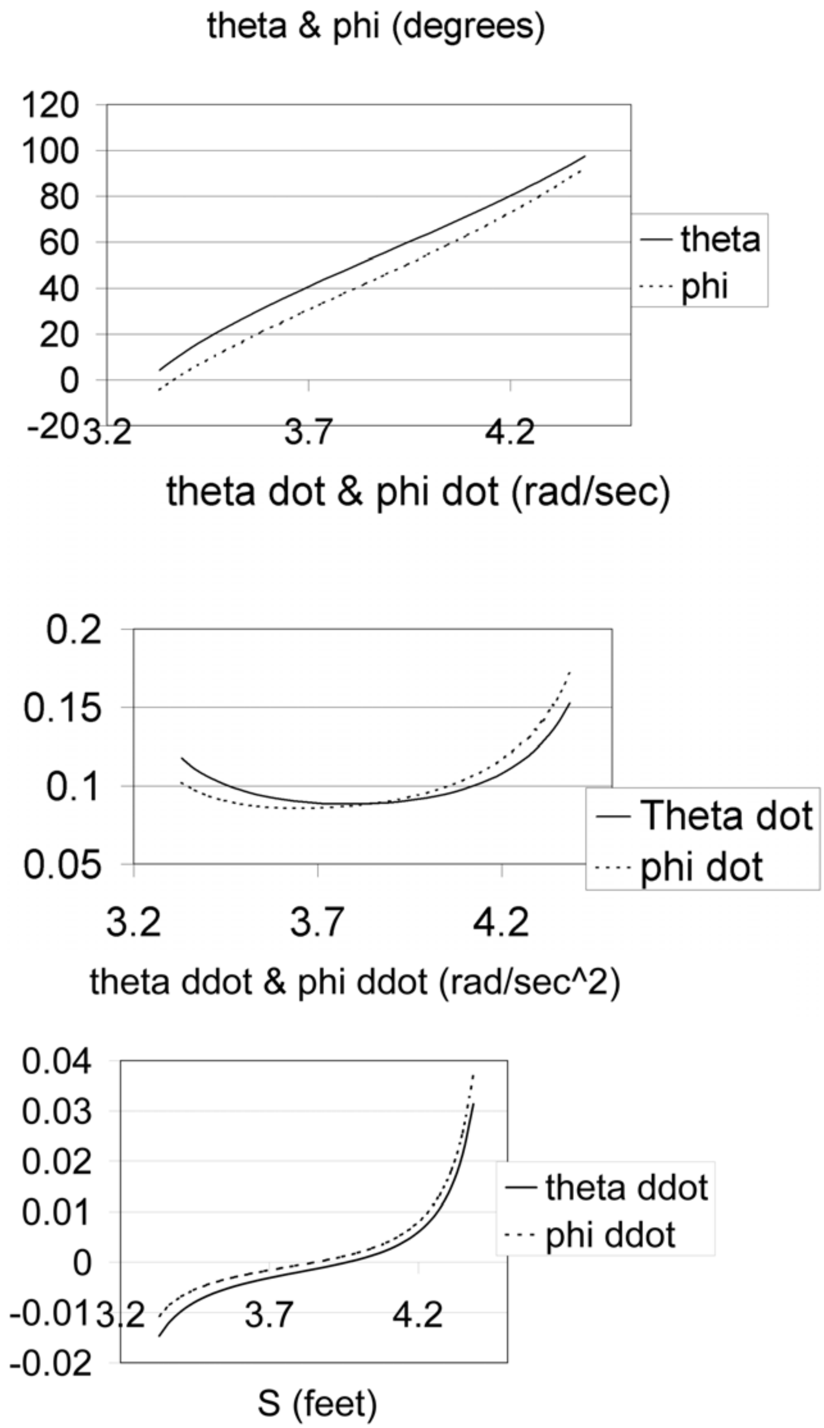


Figure XP4.6c Angular positions, velocity, and accelerations versus *S*

Lecture 22. PLANAR KINEMATIC PROBLEM EXAMPLES

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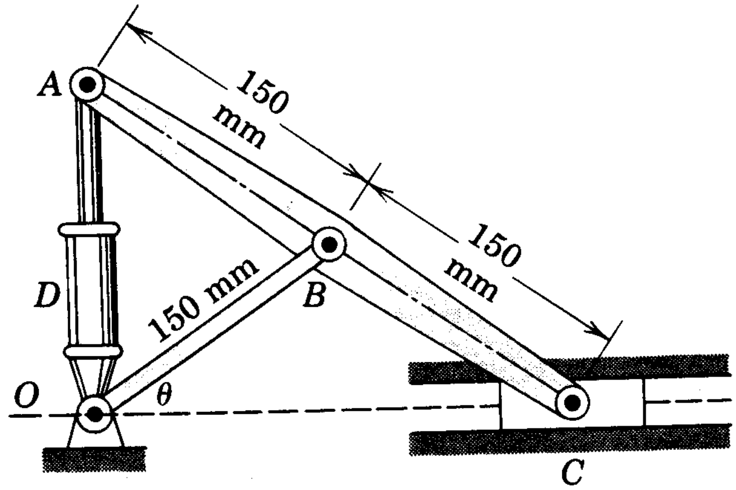
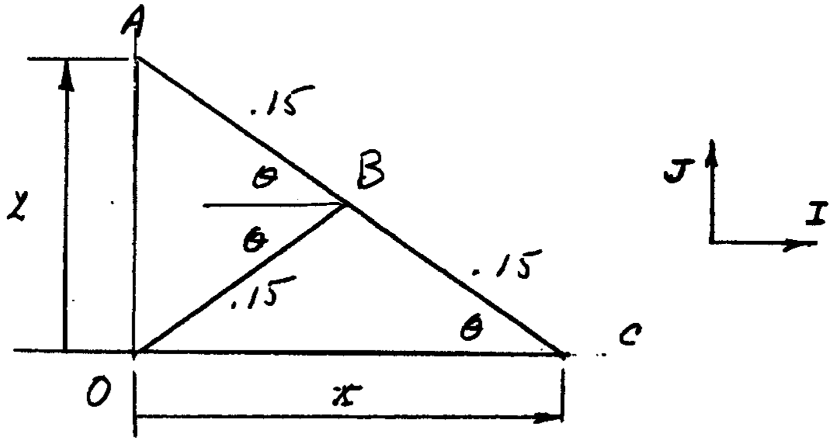


The hydraulic cylinder is causing the distance *OA* to increase at the <u>constant</u> rate of 50 *mm*/*sec*. Calculate the velocity and acceleration of the pin at *C* when $\theta = 50^o$.

GEOMETRY



Geometric constraint equations:

$$X = 2(.15\,m\ \cos\theta) = .3\,m\ \cos\theta$$

$$Y = 2(.15\,m\ \sin\theta) = .3\,m \sin\theta\ .$$

Differentiating w.r.t. time

$$\dot{X} = -.3\ \sin\theta\ \dot{\theta}\ m/\sec\ ,\quad \dot{Y} = .3\ \cos\theta\,\dot{\theta}\ m/\sec\ . \qquad \text{(i)}$$

Setting $\dot{Y} = 50\ mm/\sec = .05\,m/\sec$ and solving for $\dot{\theta}$ at $\theta = 50°$ gives

$$\dot{\theta} = \frac{.05\ m/\sec}{.3\ m\ \cos(50°)} = .259\ \frac{rad}{\sec}$$

Solving for $\dot{X}$,

$$\dot{X} = -0.3\sin(50°)\times .259 = -.0596\ \frac{m}{\sec} \Rightarrow \boldsymbol{V_C} = -\boldsymbol{I}.0596\ \frac{m}{\sec}$$

Differentiating Eq.(i) w.r.t time gives

$$\begin{aligned} \ddot{X} &= -.3\ \sin\theta\ \ddot{\theta} - .3\ \cos\theta\ \dot{\theta}^2\ m/\sec^2 \\ \ddot{Y} &= .3\ \cos\theta\,\ddot{\theta} - .3\ \sin\theta\,\dot{\theta}^2\ m/\sec^2 \end{aligned} \qquad \text{(ii)}$$

Substituting $\dot{\theta} = .259\,rad/\sec$ and $\ddot{Y} = 0$ nets

$$\ddot{Y} = 0 = .3 \cos 50^o \ddot{\theta} - .3 \sin 50^o (.259^2)$$

$$\therefore \ddot{\theta} = .259^2 \frac{\sin 50^o}{\cos 50^o} = 0.08 \frac{rad}{\sec^2} . \qquad \text{(iii)}$$

Substituting $\dot{\theta} = .259\, rad/\sec$ and $\ddot{\theta} = .08\, rad/\sec^2$ gives

$$\ddot{X} = -.3 \times \sin 50° \times 0.08 - 0.3 \times \cos 50 degr \times (.259)^2$$

$$= -0.0267\, m/\sec^2 .$$

Hence, $\boldsymbol{a}_C = -\boldsymbol{I} 0.0267\, m/\sec^2$.

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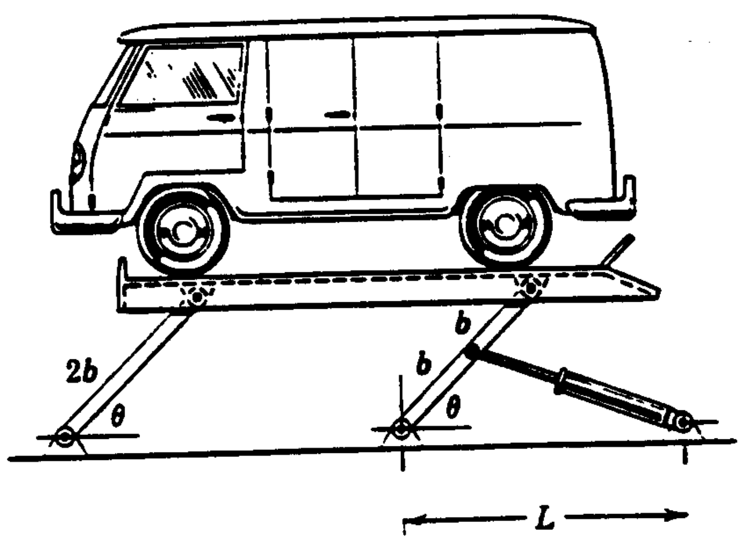
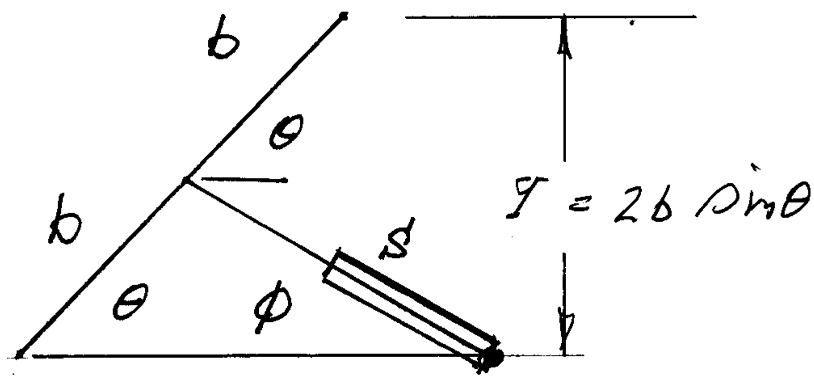


For the time of interest, the hydraulic cylinder is extending at the rate $\dot{S}$. Determine relationships for the vertical velocity of the lift.

GEOMETRY



Geometric constraint equations:

$$Y:\ b\ \sin\theta\ =\ S\ \sin\ \varphi$$

$$X:\ b\ \cos\theta\ +\ S\ \cos\ \varphi\ =\ L\ .$$

Differentiating the geometric constraint equations w.r.t. time gives:

$$\begin{aligned} b\ \cos\theta\ \dot{\theta} &= \dot{S}\ \sin\varphi\ +\ S\ \cos\varphi\ \dot{\varphi} \\ -\ b\ \sin\theta\ \dot{\theta}\ +\ \dot{S}\ \cos\varphi\ -S\ \sin\varphi\ \dot{\varphi} &= 0\ . \end{aligned} \qquad \textbf{(iv)}$$

In Matrix format,

$$\begin{bmatrix} \cos\theta & -\cos\varphi \\ \sin\theta & \sin\varphi \end{bmatrix} \begin{Bmatrix} b\ \dot{\theta} \\ S\ \dot{\varphi} \end{Bmatrix} = \begin{Bmatrix} \dot{S}\ \sin\ \varphi \\ \dot{S}\ \cos\ \varphi \end{Bmatrix} .$$

Solution from Cramer's Rule

$$D = \cos\theta \sin\varphi + \sin\theta \cos\varphi = \sin(\theta + \varphi)$$

$$D\, b\, \dot{\theta} = \begin{vmatrix} \dot{S}\sin\varphi & -\cos\varphi \\ \dot{S}\cos\varphi & \sin\varphi \end{vmatrix} = \dot{S}\,(\sin^2\varphi + \cos^2\varphi) = \dot{S}$$

$$\therefore\ b\,\dot{\theta} = \frac{\dot{S}}{\sin(\theta + \varphi)}\ .$$

Substituting to define $\dot{Y}$

$$\dot{Y} = 2\, b \cos\theta\, \dot{\theta} = \frac{2\cos\theta\, \dot{S}}{\sin(\theta + \varphi)}$$

If we needed $\ddot{Y}$, we could differentiate this equation w.r.t. time .

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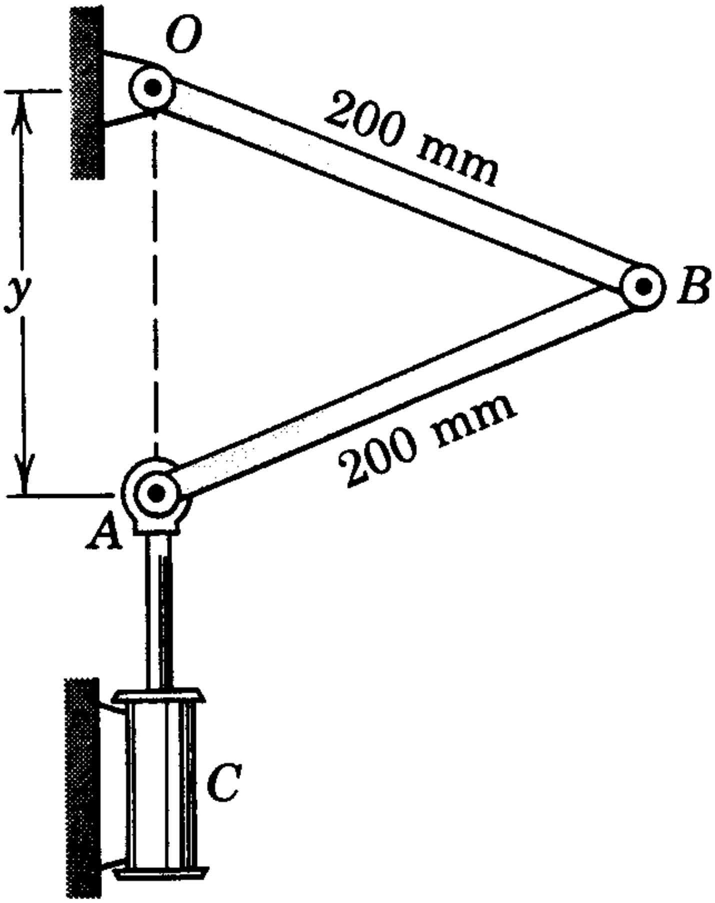
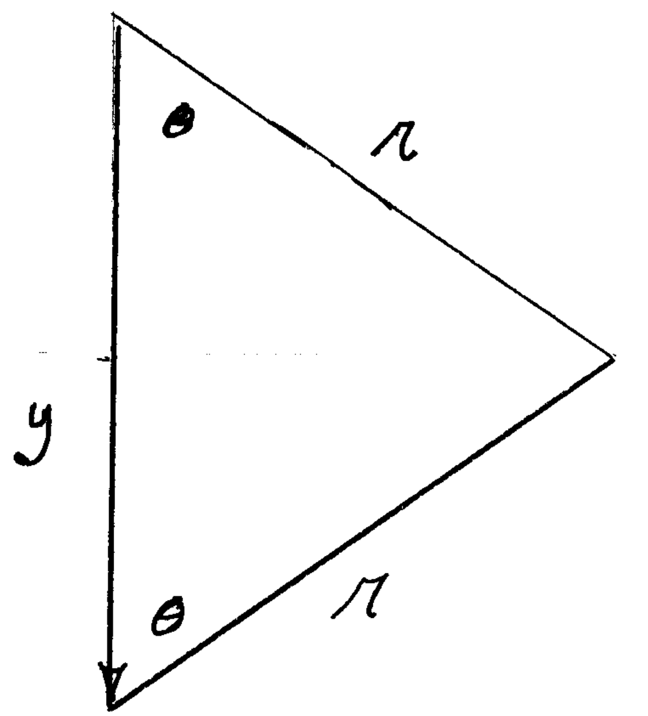


For the instant when $y = 200\,mm$, the hydraulic cylinder at C provides the vertical motion defined by $\dot{y} = 400 mm/\sec$ and $\ddot{y} = -100 mm/\sec$. Solve for the angular velocity and angular acceleration of link AB.



GEOMETRY

Kinematic constraint equation

$$y = 2r\cos\theta$$

Differentiating w.r.t. time gives

$$\dot{y} = -2r\sin\theta\,\dot{\theta}$$

$$\ddot{y} = -2r\cos\theta\,\dot{\theta}^2 - 2r\sin\theta\,\ddot{\theta} \qquad \textbf{(v)}$$

For y = 200 mm, θ = 60°

$\dot{y}$ = 400 mm / sec = -2(200 mm)sin (60°) $\dot{\theta}$

$$\therefore \boxed{\dot{\theta} = -1.155 \ \frac{rad}{\text{sec}}}$$

$\ddot{y}$ = -100 = -2(200) cos (60°) $(1.155)^2$ -2(200) sin(60°) $\ddot{\theta}$

$$\therefore \boxed{\ddot{\theta} = -.481 \ \frac{rad}{\text{sec}}}$$

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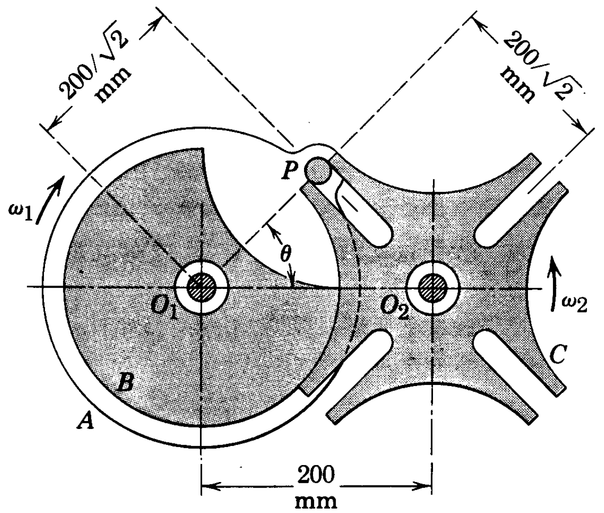
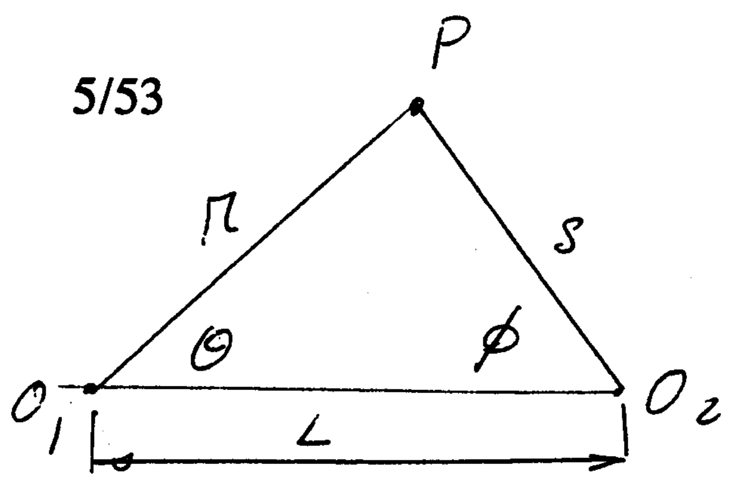


The Geneva wheel produces intermittent motion of the right-hand side wheel due to a constant rotation rate of the left-hand (drive) wheel. For $\theta = 45^o$, $\omega_1 = 2\, rad/\sec$, determine ω_2 and $\dot{\omega}_2$.



Geometry

$$\left.\begin{aligned} r \sin\theta &= S \sin\varphi \\ r\cos\theta + S\cos\varphi &= L \end{aligned}\right\} \Rightarrow \begin{aligned} S &= \sqrt{r^2 + L^2 - 2Lr\cos\theta} \\ \tan\varphi &= \frac{r\sin\theta}{L - r\cos\theta} \end{aligned}$$

Differentiating w.r.t. time gives

$$r\cos\theta\ \dot{\theta} = \dot{S}\ \sin\varphi + S\ \cos\varphi\ \dot{\varphi}$$
$$-r\sin\theta\ \dot{\theta} + \dot{S}\ \cos\varphi - S\ \sin\varphi\ \dot{\varphi} = 0\ .$$

$\dot{\theta}$ is given; $\dot{S}$ and $S\dot{\varphi}$ are the unknowns.

Matrix equation of unknowns,

$$\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} S\dot{\varphi} \\ \dot{S} \end{Bmatrix} = r\ \dot{\theta} \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} .$$

Solving using Cramer's rule, $D = \cos^2\varphi + \sin^2\varphi = 1$, and

$$DS\dot{\varphi} = r\dot{\theta} \begin{vmatrix} \cos\theta & \sin\varphi \\ \sin\theta & \cos\varphi \end{vmatrix} = r\dot{\theta}(\cos\theta\ \cos\varphi - \sin\theta\ \sin\varphi\ )$$

$$\dot{\varphi} = r\dot{\theta}\ \cos(\theta + \varphi)/S,$$

and $\boldsymbol{\omega_2} = -\boldsymbol{K}\dot{\varphi}$. Solving for $\dot{S}$,

$$D\dot{S} = r\dot{\theta} \begin{vmatrix} \cos\varphi & \cos\theta \\ -\sin\varphi & \sin\theta \end{vmatrix} = r\dot{\theta}\ (\ \cos\varphi\ \sin\theta + \cos\theta\ \sin\varphi\ )$$

$$\dot{S} = r\dot{\theta}\ \sin(\varphi + \theta\ )\ .$$

Differentiating $S\dot{\varphi}$ yields:

$$\dot{S}\dot{\varphi} + S\ddot{\varphi} = r\ddot{\theta}\cos(\varphi+\theta) - r\dot{\theta}\sin(\varphi+\theta)(\dot{\varphi}+\dot{\theta})$$
$$= -r\dot{\theta}\sin(\varphi+\theta)(\dot{\varphi}+\dot{\theta}),$$

since $\ddot{\theta} = 0$. Hence

$$\ddot{\varphi} = -[\dot{S}\dot{\varphi} + r(\dot{\varphi}\ \dot{\theta} + \dot{\theta}^2)\ \sin(\varphi+\theta)\ ]\ /\ S$$

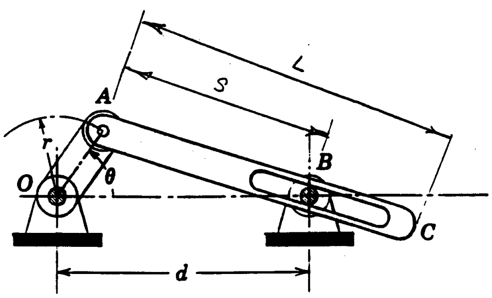
and $\boldsymbol{\omega_2} = -\boldsymbol{K}\ddot{\varphi}$.

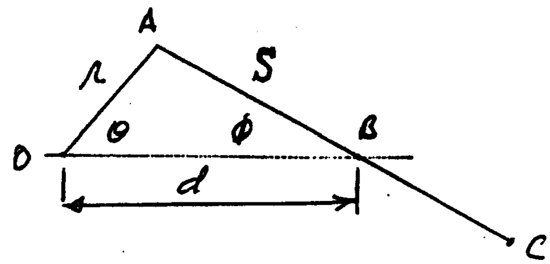
Differentiating $\dot{S}$ gives

$$\ddot{S} = r\,\ddot{\theta}\,\sin(\varphi+\theta) + r\,\dot{\theta}\,\cos(\varphi+\theta)(\dot{\varphi}+\dot{\theta})$$
$$= r(\dot{\theta}\,\dot{\varphi} + \dot{\theta}^2)\cos(\varphi+\theta)\,.$$

Lecture 23. PLANAR KINEMATIC EXAMPLE PROBLEMS

Crank *OA* is rotating with constant angular velocity $\omega_o = \dot{\theta}$. Derive general expressions for the angular velocity and angular acceleration of link *AC*. Also, derive general expressions for $\dot{s}$ and $\ddot{s}$. What is the velocity and acceleration of point *C*?





Geometric-constraint equations:

$$
\begin{aligned}
r \sin \theta &= S \sin \varphi \\
r \cos \theta + S \cos \varphi = d \;\Rightarrow\; r \cos \theta &= d - S \cos \varphi \,.
\end{aligned}
\tag{1}
$$

Differentiating w.r.t. time gives:

$$
\begin{aligned}
r \cos \theta \, \dot{\theta} &= \dot{S} \sin \varphi + S \cos \varphi \, \dot{\varphi} \\
-r \sin \theta \, \dot{\theta} &= -\dot{S} \cos \varphi + S \sin \varphi \, \dot{\varphi} \,.
\end{aligned}
\tag{2}
$$

Matrix equation for unknowns

$$\begin{bmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix} \begin{Bmatrix} \dot{S} \\ S\dot{\varphi} \end{Bmatrix} = r\,\dot{\theta} \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}$$

Solving using Cramer's rule, $D = -\sin^2\varphi - \cos^2\varphi = -1$

$$D\dot{S} = r\,\dot{\theta} \begin{vmatrix} \cos\theta & \cos\varphi \\ \sin\theta & -\sin\varphi \end{vmatrix} \begin{matrix} = r\,\dot{\theta}\,(-\sin\varphi\,\cos\theta - \sin\theta\,\cos\varphi) \\ = r\dot{\theta}\,[-\sin(\varphi+\theta)]\,. \end{matrix}$$

$$\therefore\ \dot{S} = r\,\dot{\theta}\,\sin(\varphi+\theta)$$

$$DS\dot{\varphi} = r\dot{\theta} \begin{vmatrix} \sin\varphi & \cos\theta \\ \cos\varphi & \sin\theta \end{vmatrix} \begin{matrix} = r\dot{\theta}\,(\sin\varphi\,\cos\theta - \cos\,\varphi\cos\varphi) \\ = r\dot{\theta}\,[-\cos(\varphi+\theta)] \end{matrix}$$

$$\therefore\ S\dot{\varphi} = r\dot{\theta}\cos(\varphi+\theta)$$

Differentiating $\dot{S}$ w.r.t. time with $\ddot{\theta}=0$ gives

$$\ddot{S} = r\ddot{\theta}\sin(\varphi+\theta) + r\dot{\theta}\,\cos(\varphi+\theta)(\dot{\varphi}+\dot{\theta}) = r(\dot{\theta}\dot{\varphi}+\dot{\theta}^2)\cos(\varphi+\theta)$$

Differentiating $S\dot{\varphi}$ w.r.t. time with $\ddot{\theta} = 0$ gives

$$\dot{S}\dot{\varphi} + S\ddot{\varphi} = r\ddot{\theta}\cos(\varphi+\theta) - r\dot{\theta}\sin(\varphi+\theta)(\dot{\varphi}+\dot{\theta})$$
$$= -r(\dot{\theta}^2 + \dot{\theta}\dot{\varphi})\sin(\varphi+\theta)$$

Alternatively, differentiate Eq. (2) to obtain

$$r\ddot{\theta} - r\sin\theta\,\dot{\theta}^2 = \ddot{S}\sin\varphi + \dot{S}\dot{\varphi}\cos\varphi + \dot{S}\dot{\varphi}\cos\varphi + \dot{S}\ddot{\varphi}\cos\varphi - S\dot{\varphi}^2\sin\varphi$$

$$r\ddot{\theta} + r\cos\theta\,\dot{\theta}^2 = \ddot{S}\cos\varphi - \dot{S}\dot{\varphi}\sin\varphi - \dot{S}\dot{\varphi}\sin\varphi - S\ddot{\varphi}\sin\varphi - S\dot{\varphi}^2\cos\varphi$$

Matrix equations

$$\begin{bmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix} \begin{Bmatrix} \ddot{S} \\ S\ddot{\varphi} \end{Bmatrix} = \begin{Bmatrix} -r\sin\theta\,\dot{\theta}^2 - 2\dot{S}\dot{\varphi}\cos\varphi + S\dot{\varphi}^2\sin\varphi \\ r\cos\theta\,\dot{\theta}^2 + 2\dot{S}\dot{\varphi}\sin\varphi + S\dot{\varphi}^2\cos\varphi \end{Bmatrix}$$

Vector-approach solution to find the velocity and acceleration of C.

Vector relations for points A and O:

$$\boldsymbol{v_A} = \boldsymbol{v_O} + \boldsymbol{\omega_{OA}} \times \boldsymbol{r_{OA}} = 0 + \boldsymbol{K}\dot{\boldsymbol{\theta}} \times r\,(\boldsymbol{J}\cos\theta + \boldsymbol{I}\sin\theta)$$
$$= r\dot{\theta}(\,-\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$$

$$\boldsymbol{a_A} = \boldsymbol{a_O} + \dot{\boldsymbol{\omega}}_{\boldsymbol{OA}} \times \boldsymbol{r_{OA}} + \boldsymbol{\omega_{0A}} \times (\boldsymbol{\omega_{OA}} \times \boldsymbol{r_{OA}})$$
$$= 0 + 0 + \boldsymbol{K}\dot{\boldsymbol{\theta}} \times r\dot{\theta}(\boldsymbol{J}\cos\theta + \boldsymbol{I}\sin\theta) = -r\dot{\theta}^2(\boldsymbol{J}\cos\theta + \boldsymbol{I}\sin\theta)$$

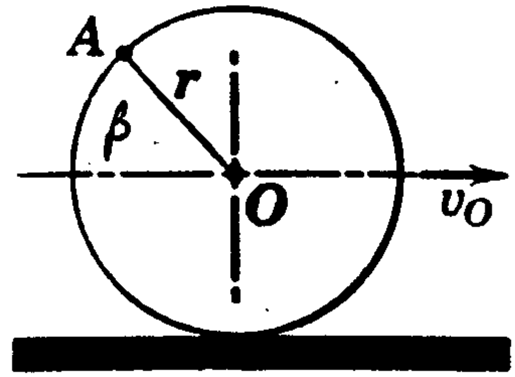
Vector relations Vector relations for points A and C:

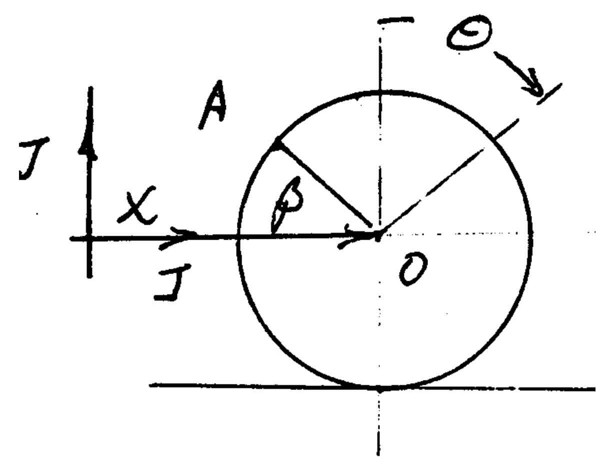
$$\boldsymbol{v_C} = \boldsymbol{v_A} + \boldsymbol{\omega_{AC}} \times \boldsymbol{r_{AC}} = \boldsymbol{v_A} - \boldsymbol{K}\dot{\varphi} \times L(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$
$$= r\dot{\theta}(\boldsymbol{J}\cos\theta - \boldsymbol{I}\sin\theta) - L\dot{\varphi}(\boldsymbol{J}\cos\varphi + \boldsymbol{I}\sin\varphi)$$

$$\boldsymbol{a_C} = \boldsymbol{a_A} + \dot{\boldsymbol{\omega}}_{\boldsymbol{CA}} \times \boldsymbol{r_{CA}} + \boldsymbol{\omega_{CA}} \times (\boldsymbol{\omega} \times \boldsymbol{r_{CA}})$$
$$= \boldsymbol{a_A} - \boldsymbol{K}\ddot{\varphi} \times L(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi$$
$$-\boldsymbol{K}\dot{\varphi} \times -L\dot{\varphi}(\boldsymbol{J}\cos\varphi + \boldsymbol{I}\sin\varphi)$$

$$= r\dot{\theta}^2(\,-\boldsymbol{I}\cos\theta - \boldsymbol{J}\sin\theta) + L\ddot{\varphi}(\,-\boldsymbol{J}\cos\varphi - \boldsymbol{I}\,\sin\varphi)$$
$$+ L\dot{\varphi}^2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$

2.The wheel rolls to the right without slipping, and its center O has a constant velocity v_o. Determine the velocity v and acceleration a of a point A on the rim of the wheel in terms of the angle β measured clockwise from the horizontal. Note, β is not the rotation angle of the wheel. It simply locates point A.





Rolling without slipping implies:

$$x = r\,\theta\ ,\ \dot{x} = v_0 = r\,\dot{\theta}\ ,\ \ddot{x} = r\,\ddot{\theta} = a_O = 0$$

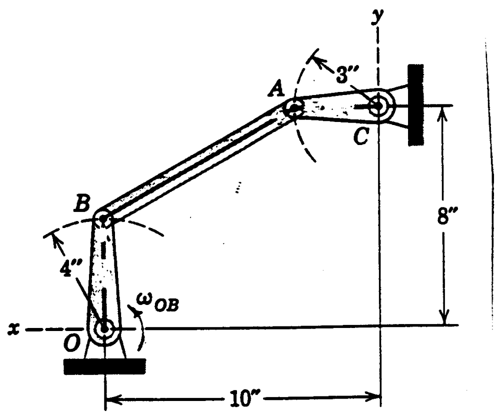
The velocity equation gives

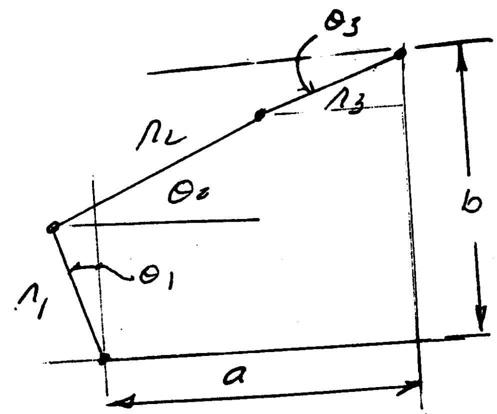
$$DS\dot{\varphi} = r\dot{\theta}\begin{vmatrix} \sin\varphi & \cos\theta \\ \cos\varphi & \sin\theta \end{vmatrix} \begin{aligned} &= r\dot{\theta}\ (\sin\varphi\ \cos\theta\ -\ \cos\ \varphi\cos\varphi) \\ &= r\dot{\theta}[-\cos(\varphi+\theta)] \end{aligned}$$

The acceleration equation gives

$$
\begin{aligned}
\boldsymbol{a}_A &= \boldsymbol{a}_O + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{0A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{0A}) \\
&= 0 - \boldsymbol{K}\ddot{\theta} \times r(-\boldsymbol{I}\cos\beta + \boldsymbol{j}\sin\beta) - \boldsymbol{K}\dot{\theta} \times r\dot{\theta}(\boldsymbol{J}\cos\beta + \boldsymbol{I}\sin\beta) \\
&= r\ddot{\theta}(\boldsymbol{J}\cos\beta + \boldsymbol{I}\sin\beta) + r\dot{\theta}^2(\boldsymbol{I}\cos\beta - \boldsymbol{J}\sin\beta)\ .
\end{aligned}
$$

3. Link *OB* has a constant clockwise angular velocity ω_{OB}. For general positions of the assembly, derive general expressions for the angular velocities and angular accelerations of links *BA* and *AC*.





Kinematic-constraint equations:

$$r_1 \cos\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 = b$$

$$-r_1 \sin\theta_1 + r_2 \cos\theta_2 + r_3 \cos\theta_3 = a \ ,$$

or restated with θ_1 as the input variable

$$\begin{aligned} r_2 \sin\theta_2 + r_3 \sin\theta_3 &= b - r_1 \cos\theta_1 \\ r_2 \cos\theta_2 + r_3 \cos\theta_3 &= a_1 + r_1 \sin\theta_1 \end{aligned} \tag{1}$$

Differentiating w.r.t. time gives

$$\begin{aligned} r_2 \cos\theta_2 \, \dot{\theta}_2 + r_3 \cos\theta_3 \, \dot{\theta}_3 &= r_1 \sin\theta_1 \, \dot{\theta}_1 \\ -r_2 \sin\theta_2 \, \dot{\theta}_2 - r_3 \sin\theta_3 \, \dot{\theta}_3 &= r_1 \cos\theta_1 \, \dot{\theta}_1 \end{aligned} \tag{2}$$

In matrix format,

$$\begin{bmatrix} \cos\theta_2 & \cos\theta_3 \\ -\sin\theta_2 & -\sin\theta_3 \end{bmatrix} \begin{Bmatrix} r_2 \, \dot{\theta}_2 \\ r_3 \, \dot{\theta}_3 \end{Bmatrix} = r_1 \dot{\theta}_1 \begin{Bmatrix} \sin\theta_1 \\ \cos\theta_1 \end{Bmatrix}$$

Differentiate Eq.(2) w.r.t. time to obtain:

$$r_2 \cos\theta_2 \, \ddot{\theta}_2 + r_3 \cos\theta_3 \, \ddot{\theta}_3 - r_2 \sin\theta_2 \, \dot{\theta}_2^2 - r_3 \sin\theta_3 \, \dot{\theta}_3^2 =$$

$$r_1 \sin\theta_1 \, \ddot{\theta}_1 + r_1 \cos\theta_1 \, \dot{\theta}_1^2 = r_1 \cos\theta_1 \, \dot{\theta}_1^2$$

$$-r_2 \sin \theta_2 \, \ddot{\theta}_2 - r_3 \sin \theta_3 \, \ddot{\theta}_3 - r_2 \cos\theta_2 \, \dot{\theta}_2^2 - r_3 \cos\theta_3 \, \dot{\theta}_3^2 =$$
$$r_1 \cos\theta_1 \, \ddot{\theta}_1 - r_1 \sin\theta_1 \, \dot{\theta}_1^2 = - r_1 \sin\theta_1 \, \dot{\theta}_1^2$$

In matrix format,

$$\begin{bmatrix} \cos\theta_2 & \cos\theta_3 \\ -\sin\theta_2 & -\sin\theta_3 \end{bmatrix} \begin{Bmatrix} r_2\ \ddot{\theta}_2 \\ r_3\ \ddot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

where

$$f_1 = r_1\ \cos\theta_1\ \dot{\theta}_1^2 + r_2\ \sin\theta_2\ \dot{\theta}_2^2 + r_3\ \sin\theta_3\ \dot{\theta}_3^2$$

$$f_2 = -r_1\ \sin\theta_1\ \dot{\theta}_1^2 + r_2\ \cos\theta_2\ \dot{\theta}_2^2 + r_3\ \cos\theta_3\ \dot{\theta}_3^2$$

Solve using Cramer's rule.