Lecture 18. PLANAR KINEMATICS OF RIGID BODIES, GOVERNING EQUATIONS

Planar kinematics of rigid bodies involve no new equations. The particle kinematics results of Chapter 2 will be used.

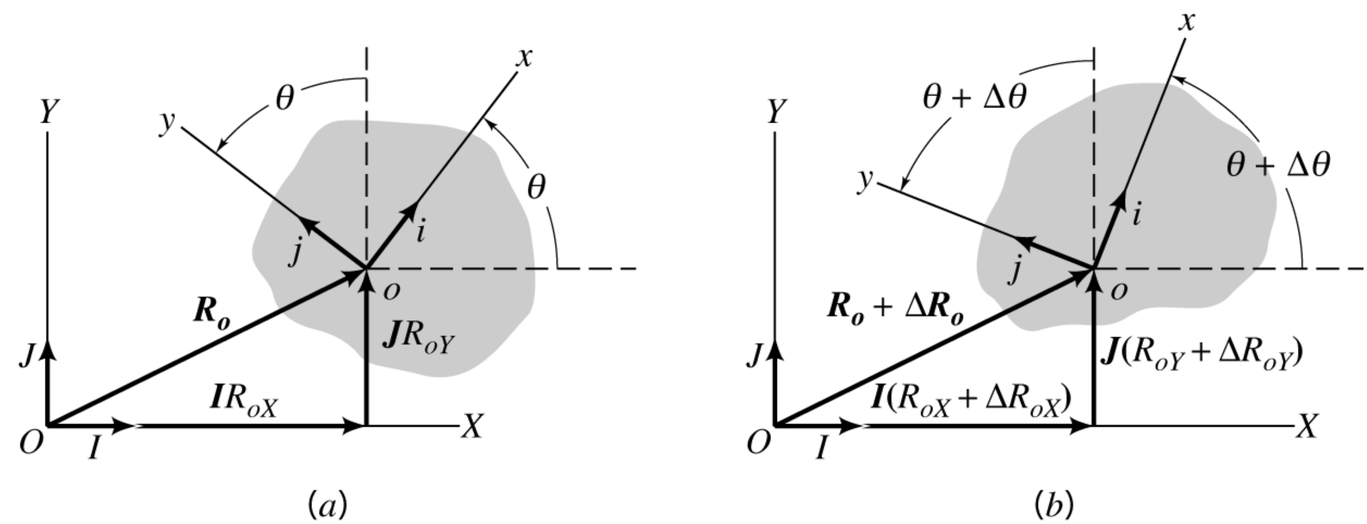


Figure 4.1 Planar motion of a rigid body moving in the plane of the page. Point *o*'s position in the body is defined in the *X*, *Y* coordinate system by $\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$. The orientation of the body with respect to the *X*, *Y* coordinate system is defined by θ. (a). The body at time *t* and orientation θ. (b). The body at a slightly later time $t + \Delta t$ with a new position $\boldsymbol{R_o} + \Delta\boldsymbol{R_o}$ and new orientation $\theta + \Delta\theta$.

The x, y, z coordinate system is fixed to the body; the X, Y, Z system is fixed to "ground."

$\omega = d\theta/dt$ = angular speed of rigid body and x, y, z coordinate system relative to ground or the X, Y, Z system.

Right-hand rule, angular velocity vector of the rigid body and the x, y coordinate system is

$$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} = \boldsymbol{K}\dot{\theta}$$

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} = \boldsymbol{K}\dot{\theta}$ = angular velocity of x, y, z relative to X, Y, Z.

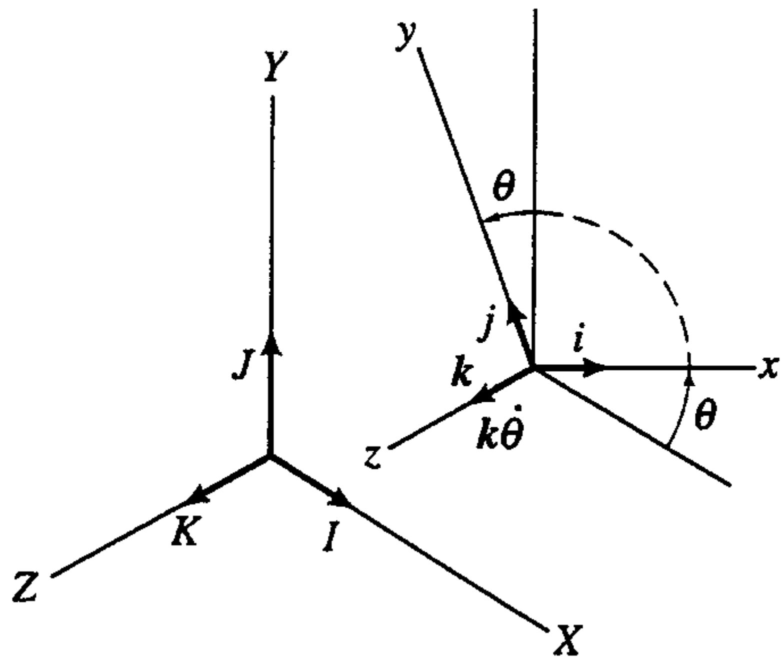


Figure 4.2 Rigid body moving in the X, Y plane with its angular velocity vector $\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ aligned with the z and Z axes.

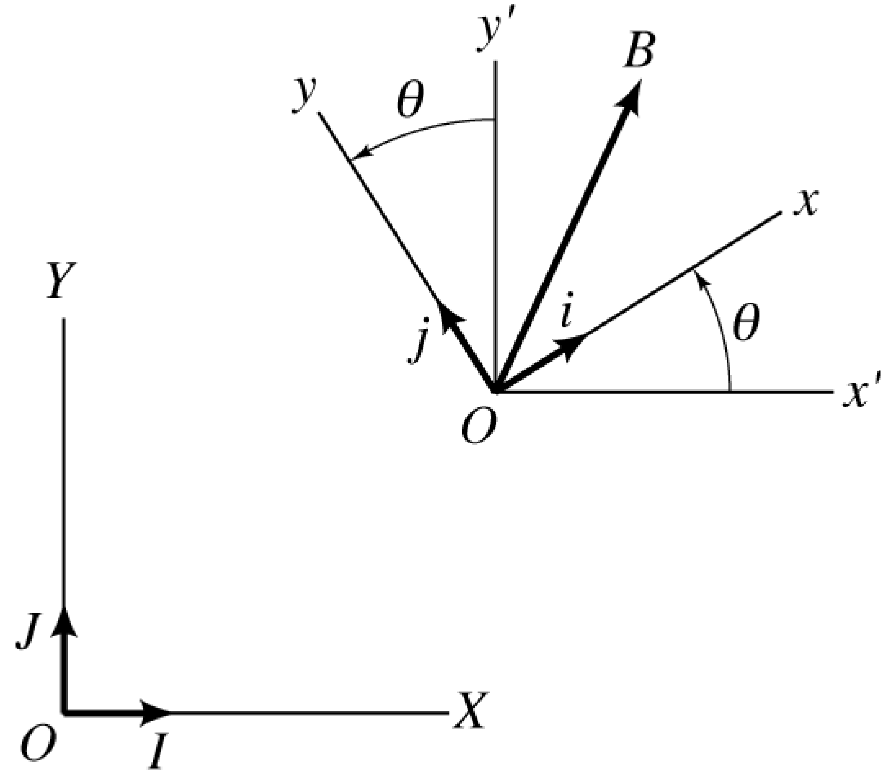


Figure 2.21

X, Y and x, y coordinate systems.

Components of $\boldsymbol{B}$:

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{I}\, B_X + \boldsymbol{J}\, B_Y \\ \boldsymbol{B} &= \boldsymbol{i}\, B_x + \boldsymbol{j}\, B_y \ . \end{aligned} \qquad \textbf{(2.41)}$$

Coordinate Transformation for components

$$\begin{Bmatrix} B_x \\ B_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} \ , \qquad \textbf{(2.42a)}$$

or

$$\begin{Bmatrix} B_X \\ B_Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} B_x \\ B_y \end{Bmatrix} . \qquad \textbf{(2.42b)}$$

Unit vector definitions:

$$\boldsymbol{i} = \boldsymbol{I} \cos\ \theta + \boldsymbol{J} \sin\ \theta$$

$$\boldsymbol{j} = -\boldsymbol{I} \sin\ \theta + \boldsymbol{J} \cos\ \theta\ .$$

Derivatives of unit vectors with respect to X, Y coordinate system:

$$\dot{\boldsymbol{i}} = \frac{d\boldsymbol{i}}{dt}\Big|_{XYZ} = -\boldsymbol{I} \sin\theta\ \dot{\theta} + \boldsymbol{J} \cos\theta\ \dot{\theta} = \boldsymbol{j}\ \dot{\theta} = \boldsymbol{\omega} \times \boldsymbol{i}$$

$$\dot{\boldsymbol{j}} = \frac{d\boldsymbol{j}}{dt}\Big|_{XYZ} = -\boldsymbol{I} \cos\theta\ \dot{\theta} - \boldsymbol{J} \sin\theta\ \dot{\theta} = -\boldsymbol{i}\ \dot{\theta} = \boldsymbol{\omega} \times \boldsymbol{j}\ .$$

Differentiating $\boldsymbol{B} = \boldsymbol{i}\ B_x + \boldsymbol{j}\ B_y$ with respect to the X, Y system,

$$\dot{\boldsymbol{B}} = \frac{dB}{dt}\Big|_{XYZ} = \boldsymbol{i}\ \dot{B}_x + \boldsymbol{j}\ \dot{B}_y + B_x \overbrace{\frac{d\boldsymbol{i}}{dt}\Big|_{XYZ}}^{\boldsymbol{\omega} \times \boldsymbol{i}} + B_y \overbrace{\frac{d\boldsymbol{j}}{dt}\Big|_{XYZ}}^{\boldsymbol{\omega} \times \boldsymbol{j}}\ .$$

or

$$\dot{\boldsymbol{B}} = \boldsymbol{i}\dot{B}_x + \boldsymbol{j}\dot{B}_y + \boldsymbol{\omega} \times (\boldsymbol{i}B_x + \boldsymbol{j}B_y)$$

$$\frac{d\boldsymbol{B}}{dt}\Big|_{XYZ} = \frac{d\boldsymbol{B}}{dt}\Big|_{xyz} + \boldsymbol{\omega} \times \boldsymbol{B}$$

$$\dot{\boldsymbol{B}} = \hat{\dot{\boldsymbol{B}}} + \boldsymbol{\omega} \times \boldsymbol{B} \ .$$

Derivatives with respect to coordinate systems:

$$\dot{\boldsymbol{B}} \triangleq \frac{d\boldsymbol{B}}{dt}\Big|_{XY} = \boldsymbol{I}\ \dot{B}_X + \boldsymbol{J}\ \dot{B}_Y$$

$$\hat{\dot{\boldsymbol{B}}} \triangleq \frac{d\boldsymbol{B}}{dt}\Big|_{x,y} = \boldsymbol{i}\ \dot{B}_x + \boldsymbol{j}\ \dot{B}_y \ .$$

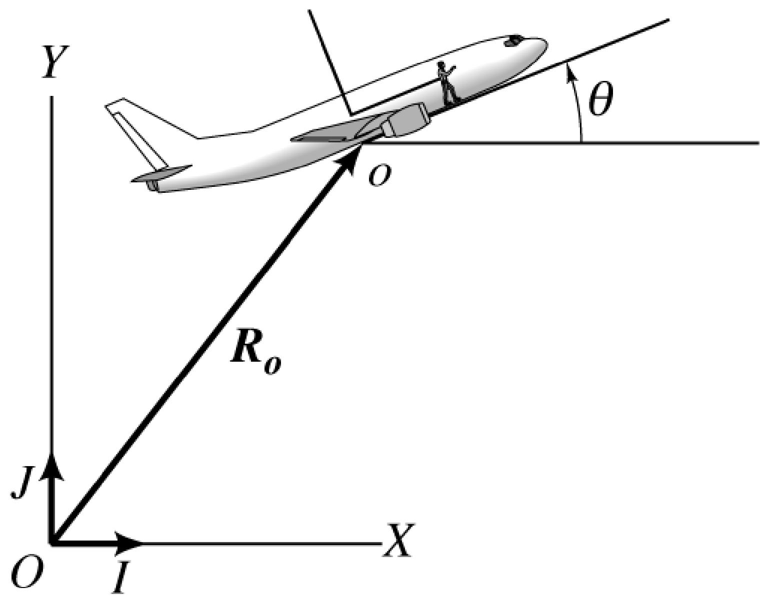


Figure 2.23 An airplane passenger moving down the aisle, as the airplane moves with respect to ground and pitches upwards relative to ground.

VELOCITY AND ACCELERATION RELATIONSHIPS IN TWO COORDINATE SYSTEMS

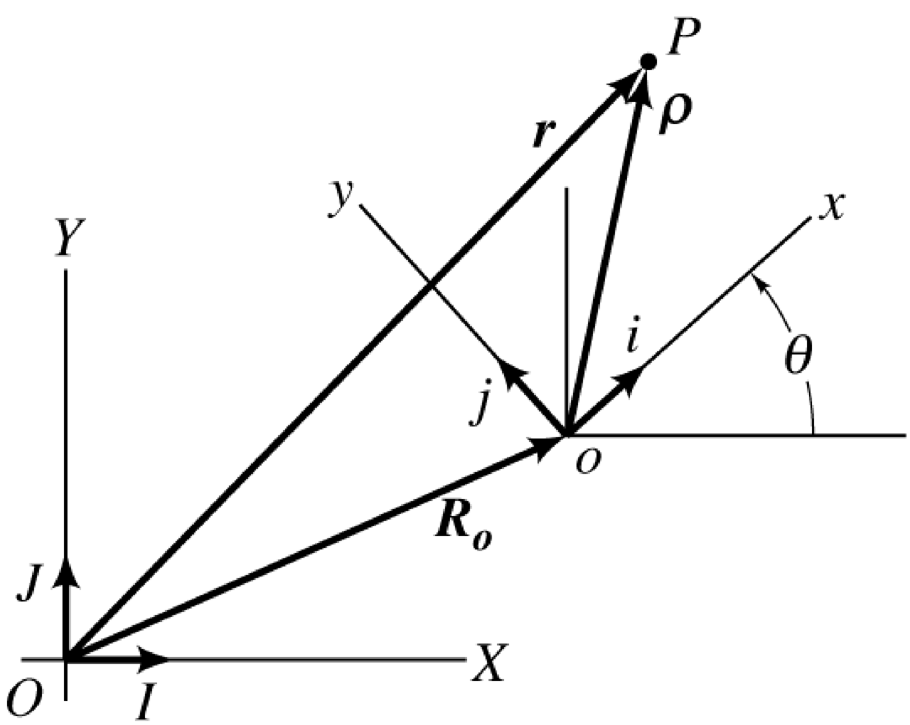


Figure 4.5 Two-coordinate arrangement for general planar kinematics

The point P is located in the X,Y system by

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \ . \tag{2.59}$$

P is located in the x,y,z system by $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y$

Velocity Equations

Taking the time derivative of Eq.(2.59) with respect to the X, Y system yields:

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{X,Y} = \left.\frac{d\boldsymbol{R_o}}{dt}\right|_{X,Y} + \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{X,Y} \tag{2.60}$$

$$= \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\rho}} \ .$$

Applying $\dot{\boldsymbol{B}} = \hat{\dot{\boldsymbol{B}}} + \boldsymbol{\omega} \times \boldsymbol{B}$ to obtain $\dot{\boldsymbol{\rho}} = \hat{\dot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \boldsymbol{\rho}$, nets

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{X,Y} = \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{x,y} + \boldsymbol{\omega} \times \boldsymbol{\rho} \ , \tag{2.62a}$$

or

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \hat{\dot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \ . \tag{2.62b}$$

Acceleration Equations

Differentiating Eq.(2.62b),

$$\ddot{\boldsymbol{r}} = \left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{X,Y} = \ddot{\boldsymbol{R}}_o + \left.\frac{d\hat{\dot{\boldsymbol{\rho}}}}{dt}\right|_{X,Y} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{X,Y} . \qquad \textbf{(2.63)}$$

Applying $\dot{\boldsymbol{B}} = \hat{\dot{\boldsymbol{B}}} + \boldsymbol{\omega} \times \boldsymbol{B}$,

$$\left.\frac{d\hat{\dot{\boldsymbol{\rho}}}}{dt}\right|_{X,Y} = \left.\frac{d\hat{\dot{\boldsymbol{\rho}}}}{dt}\right|_{x,y} + \boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} = \hat{\ddot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} . \qquad \textbf{(2.64)}$$

nets

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_o + \hat{\ddot{\boldsymbol{\rho}}} + 2\boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} + (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) , \qquad \textbf{(2.65)}$$

Comparisons to Polar-Coordinate Definitions

Parallels between the present vector results and earlier polar-coordinate definitions for velocity and acceleration become more apparent if we require that $\boldsymbol{R}_o = 0$ and that $\boldsymbol{\rho}$ lies along the x axis; i.e.,

$$\boldsymbol{\rho} = \boldsymbol{i}x . \qquad \textbf{(2.66)}$$

Hence,

$$\hat{\dot{\boldsymbol{\rho}}} = \left.\frac{d\boldsymbol{\rho}}{dt}\right|_{x,y} = \boldsymbol{i}\,\dot{x}$$

$$\hat{\ddot{\boldsymbol{\rho}}} = \left.\frac{d^2\boldsymbol{\rho}}{dt^2}\right|_{x,y} = \boldsymbol{i}\,\ddot{x}$$

(2.67)

$$\boldsymbol{\omega}\times\boldsymbol{\rho} = \boldsymbol{k}\dot{\theta}\times\boldsymbol{i}x = \boldsymbol{j}\,x\dot{\theta}$$

$$\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} = \boldsymbol{k}\ddot{\theta}\times\boldsymbol{i}x = \boldsymbol{j}\,x\ddot{\theta}$$

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) = \boldsymbol{k}\dot{\theta}\times\boldsymbol{j}x\dot{\theta} = -\boldsymbol{i}\,x\dot{\theta}^2$$

$$\boldsymbol{\omega}\times\hat{\dot{\boldsymbol{\rho}}} = \boldsymbol{k}\dot{\theta}\times\boldsymbol{i}\dot{x} = \boldsymbol{j}\,\dot{x}\dot{\theta}$$

Substitution into Eq.(2.62) (with $\dot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$) gives

$$\dot{\boldsymbol{r}} = \boldsymbol{i}\dot{x} + \boldsymbol{j}x\dot{\theta}\ .$$

Comparing this result to

$$\dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{\varepsilon}_{\boldsymbol{r}} + r\dot{\theta}\,\boldsymbol{\varepsilon}_{\boldsymbol{\theta}} = v_r\,\boldsymbol{\varepsilon}_{\boldsymbol{r}} + v_\theta\,\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}\ . \qquad \textbf{(2.27)}$$

shows the following parallel in physical terms:

$$\boldsymbol{\varepsilon}_r\, \dot{r} = \boldsymbol{i}\, \dot{x} = \hat{\dot{\boldsymbol{\rho}}}$$

$$\boldsymbol{\varepsilon}_\theta\, r\dot{\theta} = \boldsymbol{j}\, \dot{\theta}\, x = \boldsymbol{\omega} \times \boldsymbol{\rho}$$

For comparison of the acceleration terms, substituting from Eq.(2.67) into Eq.(2.65) gives (with $\ddot{\boldsymbol{R}}_o = 0$) :

$$\ddot{\boldsymbol{r}} = \boldsymbol{i}\ddot{x} + 2\boldsymbol{j}\dot{\theta}\dot{x} + \boldsymbol{j}\ddot{\theta}x - \boldsymbol{i}\dot{\theta}^2 x$$

$$= \boldsymbol{i}\,(\ddot{x} - x\dot{\theta}^2) + \boldsymbol{j}(x\ddot{\theta} + 2\,\dot{x}\dot{\theta}) \qquad \textbf{(2.68)}$$

By comparison to the polar-coordinate definition,

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{\varepsilon}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\varepsilon}_\theta$$

$$= a_r\,\boldsymbol{\varepsilon}_r + a_\theta\,\boldsymbol{\varepsilon}_\theta\ . \qquad \textbf{(2.30)}$$

the following physical equivalence of terms is established:

$$\boldsymbol{\varepsilon}_r\ddot{r} = \boldsymbol{i}\,\ddot{x} = \hat{\ddot{\boldsymbol{\rho}}}$$

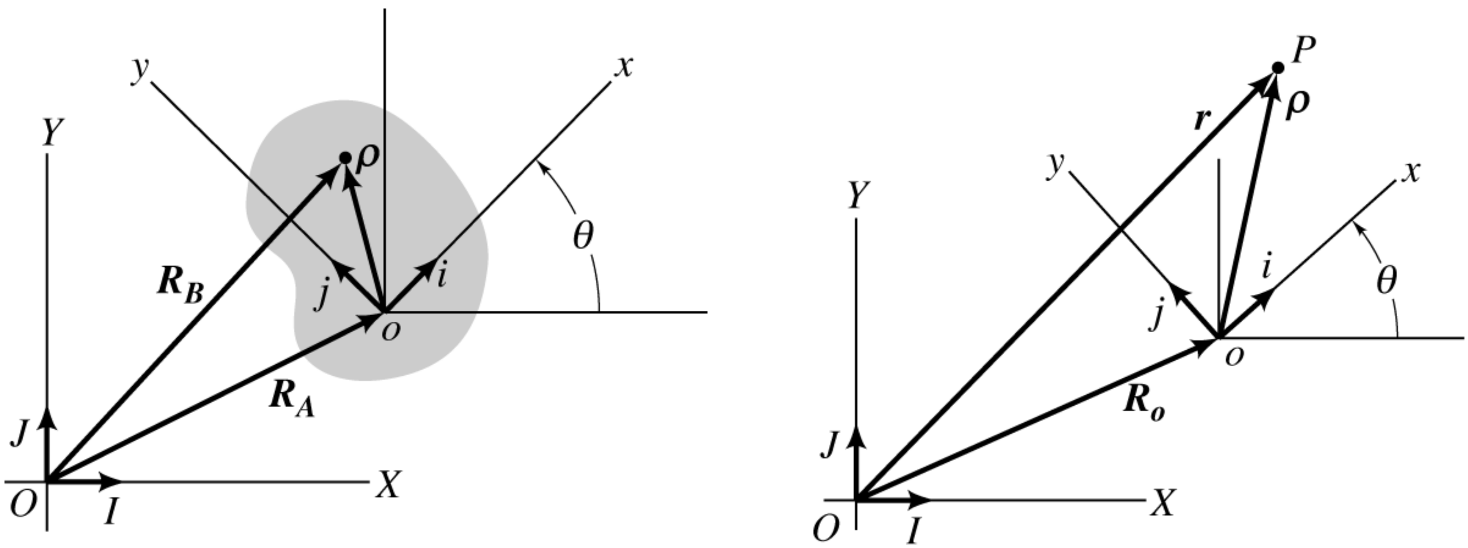
$$2\,\boldsymbol{\varepsilon}_\theta\dot{r}\dot{\theta} = 2\boldsymbol{j}\dot{x}\dot{\theta} = 2\,\boldsymbol{\omega}\times\hat{\dot{\boldsymbol{\rho}}}\ \ (\textit{Coriolis acceleration term})$$

$$\boldsymbol{\varepsilon}_\theta r\ddot{\theta} = \boldsymbol{j}\,x\ddot{\theta} = \dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}$$

$$-\boldsymbol{\varepsilon}_r r\dot{\theta}^2 = -\boldsymbol{i}\,x\dot{\theta}^2 = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})(\textit{Centrifugal acceleration term})$$

Hence, Eq.(2.65) merely presents old physical terms in a new vector format.

VELOCITY AND ACCELERATION RELATIONSHIPS FOR TWO POINTS IN A RIGID BODY



Velocity Equations

The x, y, z system is **fixed in a rigid body**.

$$\boldsymbol{R_o} \Rightarrow \boldsymbol{r_A} \ , \ \boldsymbol{r} \Rightarrow \boldsymbol{r_B}$$

$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$ = position vector locating a point **in the rigid body**. Hence,

$$\hat{\dot{\boldsymbol{\rho}}} = \frac{d\boldsymbol{\rho}}{dt}\Big|_{x,y,z} = 0 \ , \ \hat{\ddot{\boldsymbol{\rho}}} = \frac{d^2\boldsymbol{\rho}}{dt^2}\Big|_{x,y,z} = 0 \ .$$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_o + \hat{\dot{\boldsymbol{\rho}}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \Rightarrow \dot{\boldsymbol{r}}_B = \dot{\boldsymbol{r}}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AB} ,$$

and

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_o + \hat{\ddot{\boldsymbol{\rho}}} + 2\boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} + (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) ,$$

becomes

$$\ddot{\boldsymbol{r}}_B = \ddot{\boldsymbol{r}}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AB})$$

Alternatively,

$$\boxed{\begin{aligned} \boldsymbol{v}_B &= \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AB} \\ \boldsymbol{a}_B &= \boldsymbol{a}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AB}) \end{aligned}}$$

Example Problem 4.1

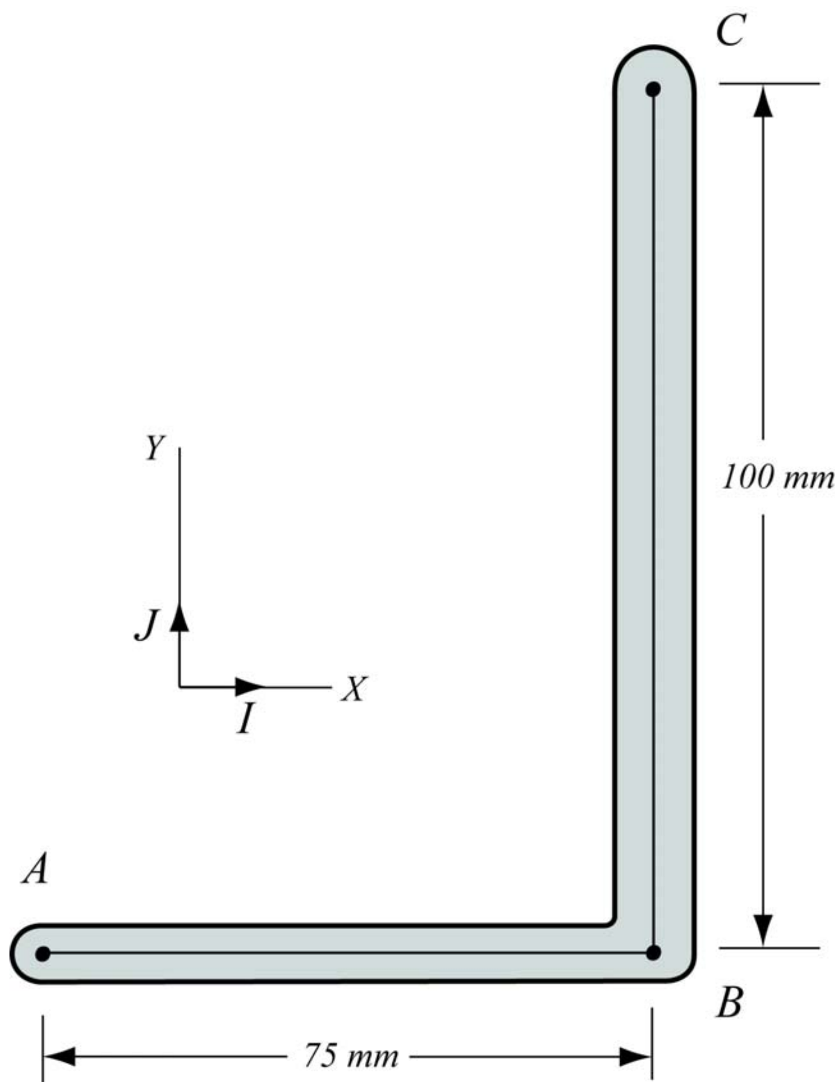


Figure XP4.1a Bar with points *A*, *B*, and *C*

Given:

$$\boldsymbol{v}_A = 5.\boldsymbol{I}\, mm/\sec \;,\; \boldsymbol{a}_A = .5\boldsymbol{I}\;\; mm/\sec^2 \qquad \textbf{(i)}$$

$$\boldsymbol{\omega} = 0.1\boldsymbol{K} rad/\sec \;,\; \dot{\boldsymbol{\omega}} = .02\boldsymbol{K}\;\; rad/\sec^2 \qquad \textbf{(ii)}$$

Tasks:

a. Determine the velocity and acceleration vectors for points *C* and *B*.

b. Draw the velocity and acceleration vectors for points *A*, *B*, and *C*.

Solution: Applying the first of Eqs.(4.3) gives

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AB} = 5\boldsymbol{I} + .1\boldsymbol{K} \times 75\boldsymbol{I} = 5\boldsymbol{I} + 7.5\boldsymbol{J}\ \ mm/\text{sec}$$

$$\begin{aligned}\boldsymbol{v}_C &= \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AC} = 5\boldsymbol{I} + .1\boldsymbol{K} \times (75\boldsymbol{I} + 100\boldsymbol{J}) \\ &= (5 - 10)\boldsymbol{I} + 7.5\boldsymbol{J}\ \ mm/\text{sec} \\ &= -5\boldsymbol{I} + 7.5\boldsymbol{J} mm/\text{sec}\ \ .\end{aligned}$$

The acceleration vectors of points *B* is obtained from the second of Eqs.(4.3) as

$$\begin{aligned}\boldsymbol{a}_B &= \boldsymbol{a}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AB}) \\ &= .5\boldsymbol{I} + .02\boldsymbol{K} \times 75.\boldsymbol{I} + 0.1\boldsymbol{K} \times [0.1\boldsymbol{K} \times 75.\boldsymbol{I}] = .5\boldsymbol{I} + 1.5\boldsymbol{J} - .75I \\ &= -.25I + 1.5\boldsymbol{J}\ \ mm/\text{sec}^2\ \ .\end{aligned}$$

Similarly for point *C*

$$\begin{aligned}\boldsymbol{a}_C &= \boldsymbol{a}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{AC} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AC}) \\ &= .5\boldsymbol{I} + .02\boldsymbol{K} \times (75.\boldsymbol{I} + 100\boldsymbol{J}) + 0.1\boldsymbol{K} \times [0.1\boldsymbol{K} \times (75.\boldsymbol{I} + 100\boldsymbol{J})] \\ &= .5I + (1.5\boldsymbol{J} - 2.\boldsymbol{I}) - (.75\boldsymbol{I} + 1.\boldsymbol{J}) = -2.25\boldsymbol{I} + .5\boldsymbol{J}\ \ mm/\text{sec}^2\ \ .\end{aligned}$$

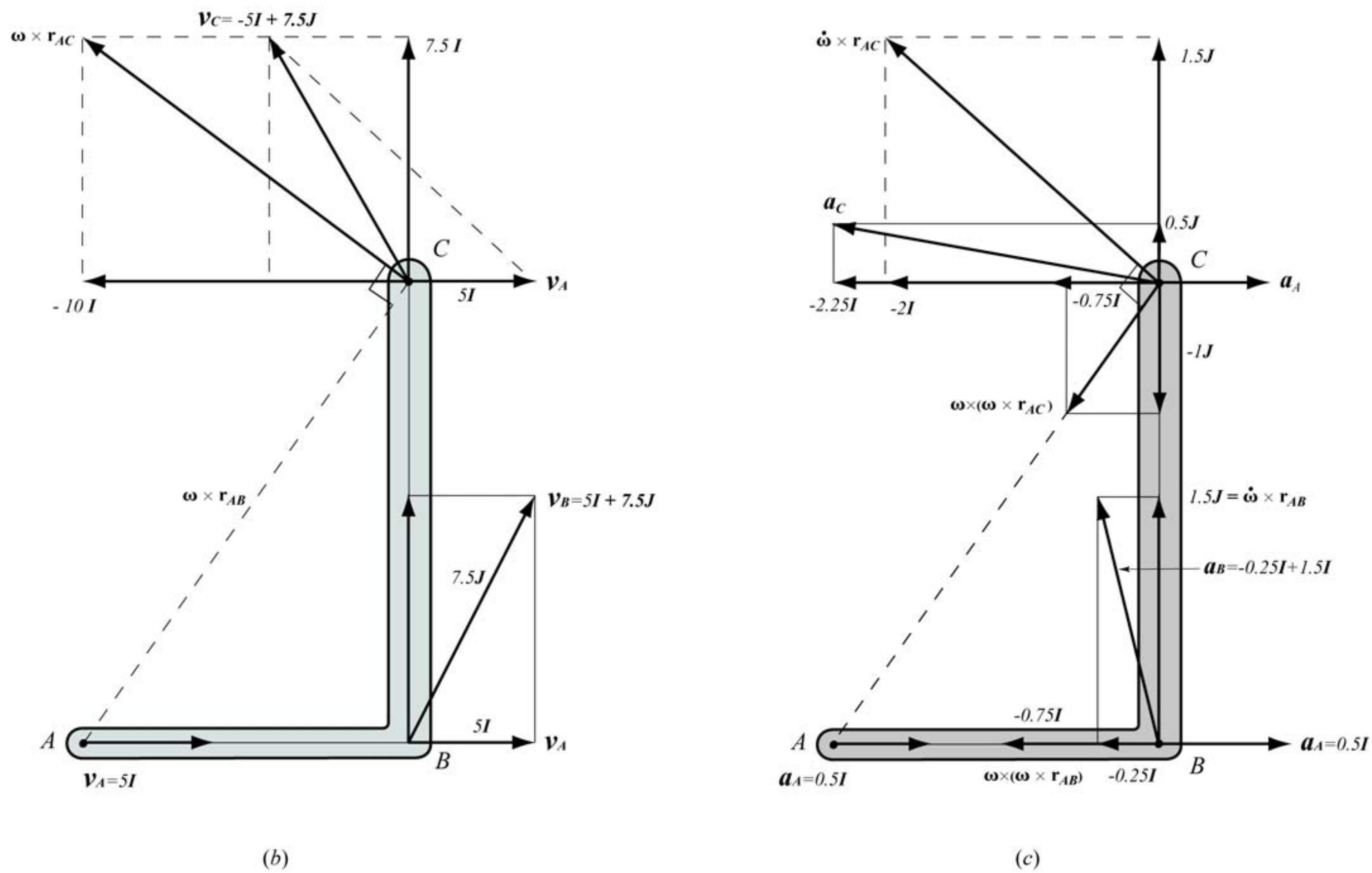
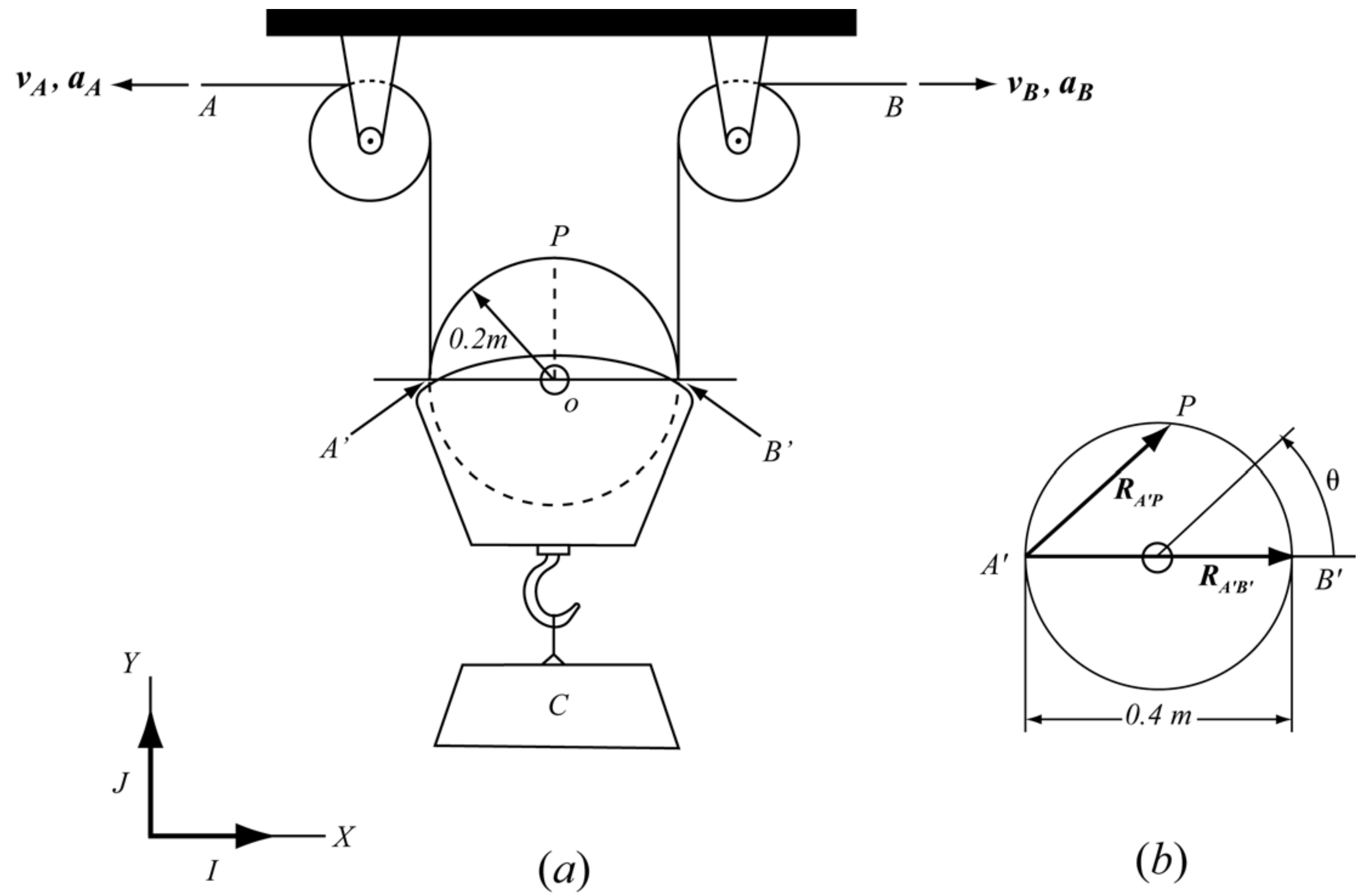


Figure XP4.1 (b) Velocity (mm/sec) and (c) Acceleration ($mm/\sec^2$)

Example Problem 4.2



Example Problem 4.2 Point *A*, the end of the left cable has a velocity of .6 m/sec and an acceleration of .13 m/sec^2 . Point *B*, the end of the right cable has a velocity of 1.2 m/sec and an acceleration of -.13m/sec^2. The central pulley has a radius of 0.4 *m*. Point *o* is at the center of the pulley, and point *P* is at the top of the pulley

Tasks: Determine the velocity and acceleration of points *o* and *P*.

Solution: We are going to use Eqs.(4.3), the velocity and acceleration vectors for two points on a rigid body to work through this example. The rigid body is the central pulley illustrated in figure 4.2b. Since the cable is inextensible, points A' and B' on the central pulley have the same velocities and *vertical* acceleration magnitudes as A and B, respectively. Specifically,

$$\boldsymbol{v}_{A'} = .6\boldsymbol{J}\ m/\sec, \quad \boldsymbol{v}_{B'} = 1.2\ \boldsymbol{J}\ m/\sec$$

$$\boldsymbol{a}_{A'Y} = 0.13\boldsymbol{J}\ m/\sec^2\ ,\ \boldsymbol{a}_{B'Y} = -.13\boldsymbol{J}\ m/\sec^2\ . \qquad \textbf{(i)}$$

Velocity results. We start this development knowing $\boldsymbol{v}_{A'}, \boldsymbol{v}_{B'}$, and needing $\boldsymbol{v}_o, \boldsymbol{v}_P$. We could use the velocity relationship to find these unknowns providing that we knew $\boldsymbol{\omega} = \boldsymbol{K}\omega = \boldsymbol{K}\dot{\theta}$. We can determine $\boldsymbol{\omega}$ by applying $\boldsymbol{v}_B' = \boldsymbol{v}_A' + \boldsymbol{\omega} \times \boldsymbol{r}_{A'B'}$ for points A' and B' on the pulley. Substituting for $\boldsymbol{v}_{A'}, \boldsymbol{v}_{B'}$ from Eq.(i) and $\boldsymbol{r}_{A'B'} = 0.4\boldsymbol{I}\ m$ gives

$$1.2\boldsymbol{J}\frac{m}{\sec} = 0.6\boldsymbol{J}\frac{m}{\sec} + \boldsymbol{K}\dot{\theta}\frac{rad}{\sec} \times .4\boldsymbol{I}m$$

$$\dot{\theta} = \frac{.6}{.4}\frac{rad}{\sec} = 1.5\frac{rad}{\sec} \qquad \textbf{(ii)}$$

Proceeding with this result for $\boldsymbol{\omega}$ gives:

$$\boldsymbol{v_o} = \boldsymbol{v_{A'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{A'o}}) = .6\boldsymbol{J}\frac{m}{\sec} + (1.5\boldsymbol{K}\frac{rad}{\sec} \times .2\boldsymbol{I}m) = .9\boldsymbol{J}\frac{m}{\sec}$$

$$\boldsymbol{v_P} = \boldsymbol{v_{A'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{A'P}}) = .6\boldsymbol{J}\frac{m}{\sec} + [1.5\boldsymbol{K}\frac{rad}{\sec} \times (.2\boldsymbol{I} + .2\boldsymbol{J})m]$$

$$= (.6\boldsymbol{J} + .3\boldsymbol{J} - .3\boldsymbol{I})\frac{m}{\sec} = (.9\boldsymbol{J} - .3\boldsymbol{I})\frac{m}{\sec}$$

As expected, $\boldsymbol{v_o}'s$ velocity is vertically upwards. Because of the pulley's rotation, point *P* has a velocity component in the -*X* direction. Note that we could have just as easily used the equations, $\boldsymbol{v_o} = \boldsymbol{v_{B'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{B'o}})$ and $\boldsymbol{v_P} = \boldsymbol{v_{B'}} + (\boldsymbol{\omega} \times \boldsymbol{r_{B'P}})$ to get these results since we know $\boldsymbol{v_{B'}}$.

Acceleration Development. Recall in the sentence above Eq.(i) that the ***vertical*** acceleration of points A', B' have the same magnitudes, respectively, as $\boldsymbol{a_{A'}}, \boldsymbol{a_{B'}}$. The acceleration vectors $\boldsymbol{a_{A'}}, \boldsymbol{a_{B'}}$ have horizontal components due to the pulley's rotation; hence, $\boldsymbol{a_{A'}} \neq .13\boldsymbol{J}m/\sec^2$, and $\boldsymbol{a_{B'}} \neq -.13\boldsymbol{J}m/\sec^2$. We can verify this statement starting with $\boldsymbol{a_o} = a_o\boldsymbol{J}$, the acceleration of point *o*, even though a_o is unknown. Applying the acceleration equation from Eqs.(4.3) gives:

$$\boldsymbol{a}_A^{\prime}=\boldsymbol{a}_O+\dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{OA^{\prime}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{OA^{\prime}})=\boldsymbol{J}a_o+(\boldsymbol{K}\ddot{\theta}\times-.2\boldsymbol{I})+[\boldsymbol{K}\dot{\theta}\times(\boldsymbol{K}\dot{\theta}\times-.2\,I)]$$
$$=\boldsymbol{J}a_o-\boldsymbol{J}.2\ddot{\theta}+\boldsymbol{I}.2\dot{\theta}^2=\boldsymbol{J}(a_O-.2\ddot{\theta})+\boldsymbol{I}.2\dot{\theta}^2\ \ m/\sec^2$$

$$\boldsymbol{a}_B^{\prime}=\boldsymbol{a}_O+\dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{OB^{\prime}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{OB^{\prime}})=\boldsymbol{J}a_o+(\boldsymbol{K}\ddot{\theta}\times.2\boldsymbol{I})+[\boldsymbol{K}\dot{\theta}\times(\boldsymbol{K}\dot{\theta}\times.2\,I)]$$
$$=\boldsymbol{J}a_o+\boldsymbol{J}\,.2\ddot{\theta}-\boldsymbol{I}.2\dot{\theta}^2=\boldsymbol{J}(a_O+.2\ddot{\theta})-\boldsymbol{I}.2\dot{\theta}^2\ \ m/\sec^2\ .$$

(iv)

Note particularly the horizontal components of $\boldsymbol{a}_{A^{\prime}},\boldsymbol{a}_{B^{\prime}}$ arising from the centripetal acceleration term $r\dot{\theta}^2$ induced by the pulley rotation. With that result firmly in mind, we can proceed to solve for $\boldsymbol{a}_o,\boldsymbol{a}_P$.

The first step is solving for $\dot{\boldsymbol{\omega}}$. Applying $\boldsymbol{a}_{B^{\prime}}=\boldsymbol{a}_{a^{\prime}}+\dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{A^{\prime}B^{\prime}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{A^{\prime}B^{\prime}})$ gives

$$(-.2\dot{\theta}^2\boldsymbol{I}-.13\boldsymbol{J})=(.2\dot{\theta}^2\boldsymbol{I}+.13\boldsymbol{J})+(\boldsymbol{K}\ddot{\theta}\times.4\boldsymbol{I})+[\boldsymbol{K}\dot{\theta}\times(\boldsymbol{K}\dot{\theta}\times.4\,I)]$$
$$=.2\dot{\theta}^2\boldsymbol{I}+.13\boldsymbol{J}+.4\ddot{\theta}\boldsymbol{J}-.4\dot{\theta}^2\boldsymbol{I}\ \ m/\sec^2$$

Taking the ***I*** and ***J*** components separately gives:

$$\boldsymbol{I}:\ -.2\dot{\theta}^2=.2\dot{\theta}^2-.4\dot{\theta}^2\qquad \boldsymbol{J}:-.13=.13+.4\ddot{\theta}$$

$$\therefore\ \ddot{\theta}=-\frac{.26}{.4}\frac{rad}{\sec^2}=-.65\frac{rad}{\sec^2}\ .$$

The *X* component result gave nothing; the *Y* component allowed us to solve for $\ddot{\theta}$. At this point, we are in a position to directly solve for $\boldsymbol{a}_o,\boldsymbol{a}_P$ as:

$$\boldsymbol{a_o} = \boldsymbol{a_{A'}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{A'o}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{A'o}})$$
$$= (.2\dot{\theta}^2\boldsymbol{I} + .13\boldsymbol{J}) + (\boldsymbol{K}\ddot{\theta} \times .2\boldsymbol{I}) + [\boldsymbol{K}\dot{\theta} \times (\boldsymbol{K}\dot{\theta} \times .2\boldsymbol{I})]$$
$$= .2\dot{\theta}^2\boldsymbol{I} + .13\boldsymbol{J} + .2\ddot{\theta}\boldsymbol{J} - .2\dot{\theta}^2\boldsymbol{I} = (.13 + .2 \times -.65)\boldsymbol{J} = -1.17\boldsymbol{J}\ m/\sec^2$$

$$\boldsymbol{a_P} = \boldsymbol{a_{A'}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{A'P}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{A'P}})$$
$$= (.2\dot{\theta}^2\boldsymbol{I} + .13\boldsymbol{J}) + \boldsymbol{K}\ddot{\theta} \times (.2\boldsymbol{I} + .2\boldsymbol{J}) + \boldsymbol{K}\dot{\theta} \times [\boldsymbol{K}\dot{\theta} \times (.2\boldsymbol{I} + .2\boldsymbol{J})]$$
$$= .2\dot{\theta}^2\boldsymbol{I} + .13\boldsymbol{J} + .2\ddot{\theta}(\boldsymbol{J} - \boldsymbol{I}) - .2\dot{\theta}^2(\boldsymbol{I} + \boldsymbol{J}) = -.2\ddot{\theta}\boldsymbol{I} + (.13 + .2\ddot{\theta} - .2\dot{\theta}^2)\boldsymbol{J}$$
$$= (-.2 \times -.65)\boldsymbol{I} + [.13 + (.2 \times -.65) - .2 \times 1.5^2]\boldsymbol{J} = -1.3\boldsymbol{I} - 1.5\boldsymbol{J}\ m/\sec^2\ .$$

As expected, point *o*'s acceleration is vertical. Also, *P*'s vertical acceleration is entirely due to the $r\dot{\theta}^2$ term. This last step could have proceeded equally well from B' instead of A', since we also know $\boldsymbol{a_{B'}}$.

The customary development in this type of problem uses the following sequential steps:

(1). Starting with known velocities at two points, in this case A' and B', calculate $\boldsymbol{\omega}$. Then using a known velocity and $\boldsymbol{\omega}$ calculate any additional required velocities.

(2). Starting with known accelerations at two points, calculate $\dot{\boldsymbol{\omega}}$. Then using one of the known accelerations plus $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$, calculate other required accelerations.