# **Lecture 19.** ROLLING WITHOUT SLIPPING

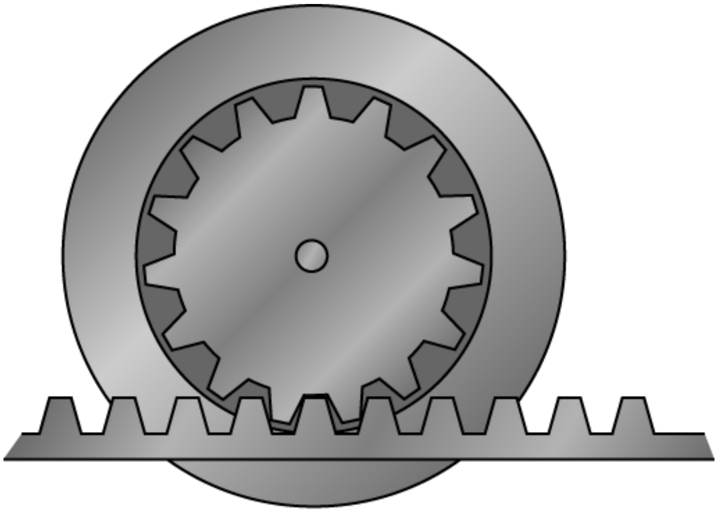


**Figure 4.8** Gear rolling in geared horizontal guides.

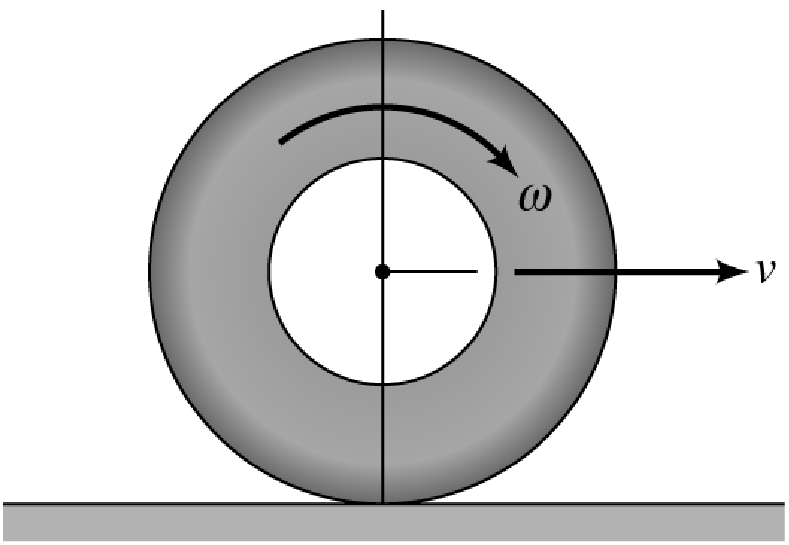



**Figure 4.9** Wheel rolling on a horizontal surface.

The derivation and understanding of velocity and acceleration relationships for a wheel that is rolling without slipping is the fundamental objective of this lecture.

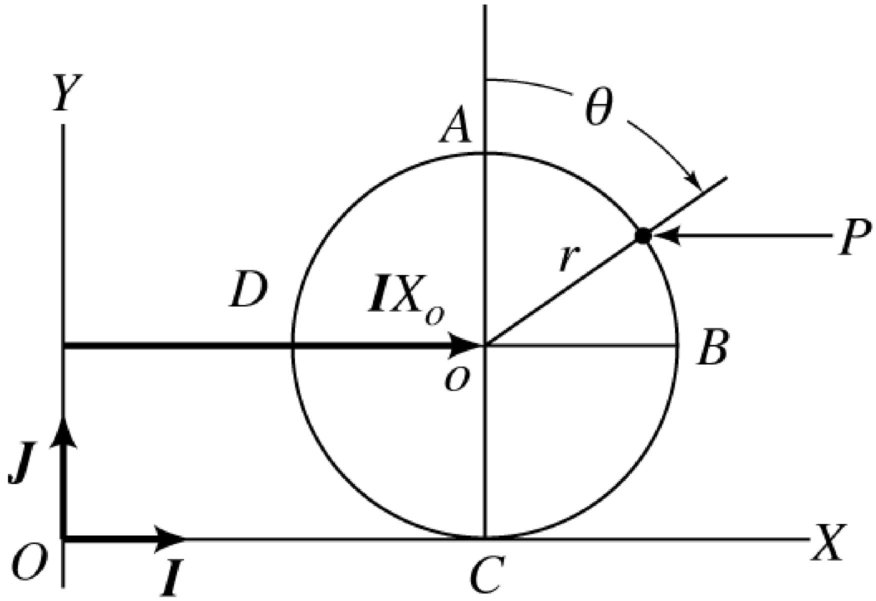



**Figure 4.10** Wheel *rolling without slipping* on a horizontal surface.

***Geometric Development***

The wheel in figure 4.10 advances to the right as θ increases. The question is: *Without slipping, how is the rotation angle* θ *related to the displacement of the wheel center* $X_o$ *?* Note first that the contact point between the wheel, denoted as *C*, advances to the right precisely the same distance as point *o*. If the wheel starts with θ and $X_o$ at zero, and rolls forward through one rotation *without slipping*, both the new contact point and the now displaced point *o* will have moved to the right a distance equal to the circumference of the wheel; i.e., $X_o = 2\pi r$. It may help you to think of the wheel as a paint roller and imagine the length of the paint strip that would be laid out on the plane during one rotation. Comparable to the result for a full rotation, *without slipping* the geometric constraint relating $X_o$ and θ is

$$X_o = r\theta \ . \tag{4.4}$$

Differentiating with respect to time gives :

$$\dot{X}_o = r\dot{\theta} \ , \quad \ddot{X}_o = r\ddot{\theta} \ . \tag{4.5}$$

These are the desired kinematic constraint equations for a wheel that is rolling without slipping.

We want to define the trajectory of the point $P$ on the wheel located by $\theta$. The coordinates of the displacement vector locating $P$ are defined by

$$\begin{aligned} X &= X_o + r\sin\theta = r\theta + r\sin\theta \\ Y &= r\cos\theta + r \ . \end{aligned} \tag{4.6}$$

The trajectory followed by point $P$ as the wheel rolls to the right is the cycloid traced out in figure 4.11. The cycloid was obtained for $r = 1$ by varying $\theta$ through two cycles of rotation to generate coordinates $X(\theta)$ and $Y(\theta)$ and then plotting $Y$ versus $X$ .

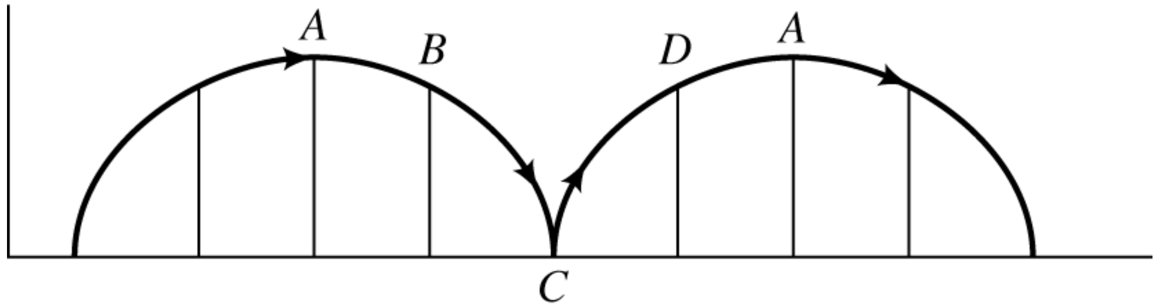



**Figure 4.11** Cycloidal path traced out by a point on a wheel that is rolling without slipping. The letters *A* through *D* on the wheel indicate locations occupied by point *P* on the cycloid.

Referring to the locations *A* through *D* of figures 4.10 and 4.11, point *P* starts at *A*, reaches *B* after the wheel has rotated $\pi/2$ radians, reaches the contact location after $\pi$ radians, reaches *D* after $3\pi/2$ radians and then returns to *A* after a full rotation.

Differentiating the components of the position vector locating *P* (defined by Eq.(4.6)) to obtain components of the velocity vector for point *P* with respect to the *X*,*Y* coordinate system gives:

$$\dot{X} = r\dot{\theta} + r\dot{\theta}\cos\theta$$
$$\dot{Y} = -r\dot{\theta}\sin\theta \quad . \qquad \mathbf{(4.7)}$$

Note that $\dot{X}$ and $\dot{Y}$ are components of the velocity vector of a point *P* on the wheel, located by the angle $\theta$. The corresponding acceleration components are:

$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$
$$\ddot{Y} = -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \quad . \tag{4.8}$$

Eqs.(4.7) and (4.8) can be used to evaluate the instantaneous velocity and acceleration components of any point on the wheel by specifying an appropriate value for θ. Values for $\theta = 0, \pi/2, \pi$, and $3\pi/2$ correspond, respectively, to locations *A*, *B*, *C*, and *D* in figure 4.10.

*Position A* ( *top of the wheel;* θ = 0 )*:*

$$\dot{X} = r\dot{\theta} + r\dot{\theta}(1) = 2r\dot{\theta} = 2\dot{X}_o$$
$$\dot{Y} = -r\dot{\theta}(0) = 0$$
$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}(1) - r\dot{\theta}^2(0) = 2r\ddot{\theta} = 2\ddot{X}_o$$
$$\ddot{Y} = -r\ddot{\theta}(0) - r\dot{\theta}^2(1) = -r\dot{\theta}^2 \quad . \tag{4.9}$$

*Position B* (*right-hand side of the wheel;* $\theta = \pi/2$ ) *:*

$$\dot{X} = r\dot{\theta} + r\dot{\theta}(0) = r\dot{\theta} = \dot{X}_o$$
$$\dot{Y} = -r\dot{\theta}(1) = -r\dot{\theta}$$
$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}(0) - r\dot{\theta}^2(1) = r\ddot{\theta} - r\dot{\theta}^2 = \ddot{X}_o - r\dot{\theta}^2 \tag{4.10}$$
$$\ddot{Y} = -r\ddot{\theta}(1) - r\dot{\theta}^2(0) = -r\ddot{\theta} \quad .$$

*Position C* (*bottom of the wheel at the contact location;* $\theta = \pi$) *:*

$$\dot{X} = r\dot{\theta} + r\dot{\theta}(-1) = 0$$
$$\dot{Y} = -r\dot{\theta}(0) = 0$$
$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}(-1) - r\dot{\theta}^2(0) = 0 \tag{4.11}$$
$$\ddot{Y} = -r\ddot{\theta}(0) - r\dot{\theta}^2(-1) = r\dot{\theta}^2 \quad .$$

*Position D* (*left-hand side of the wheel;* $\theta = 3\pi/2$)*:*

$$\dot{X} = r\dot{\theta} + r\dot{\theta}(0) = r\dot{\theta} = \dot{X}_o$$
$$\dot{Y} = -r\dot{\theta}(-1) = r\dot{\theta}$$
$$\ddot{X} = r\ddot{\theta} + r\ddot{\theta}(0) - r\dot{\theta}^2(-1) = r\ddot{\theta} + r\dot{\theta}^2 = \ddot{X}_o + r\dot{\theta}^2 \tag{4.12}$$
$$\ddot{Y} = -r\ddot{\theta}(-1) - r\dot{\theta}^2(0) = r\ddot{\theta} \quad .$$

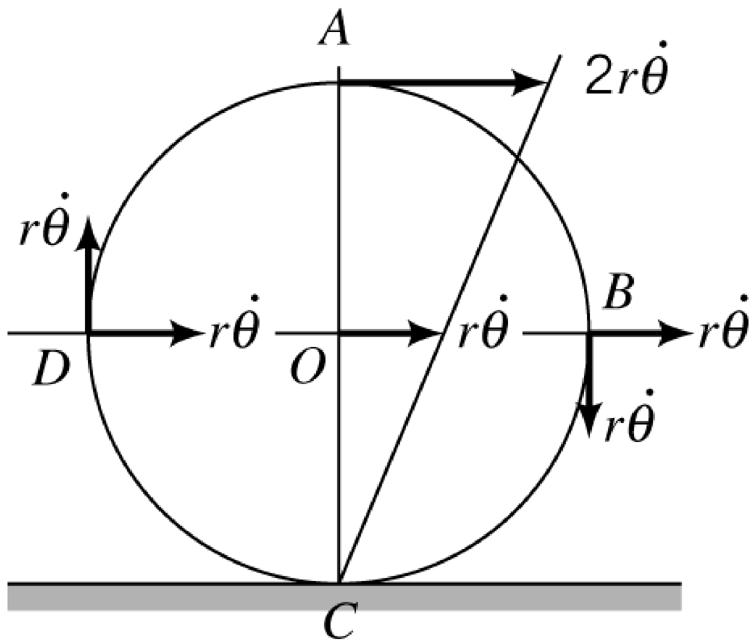



**Figure 4.12** Velocity vectors for points on the wheel at locations *A* through *D* and *o*.

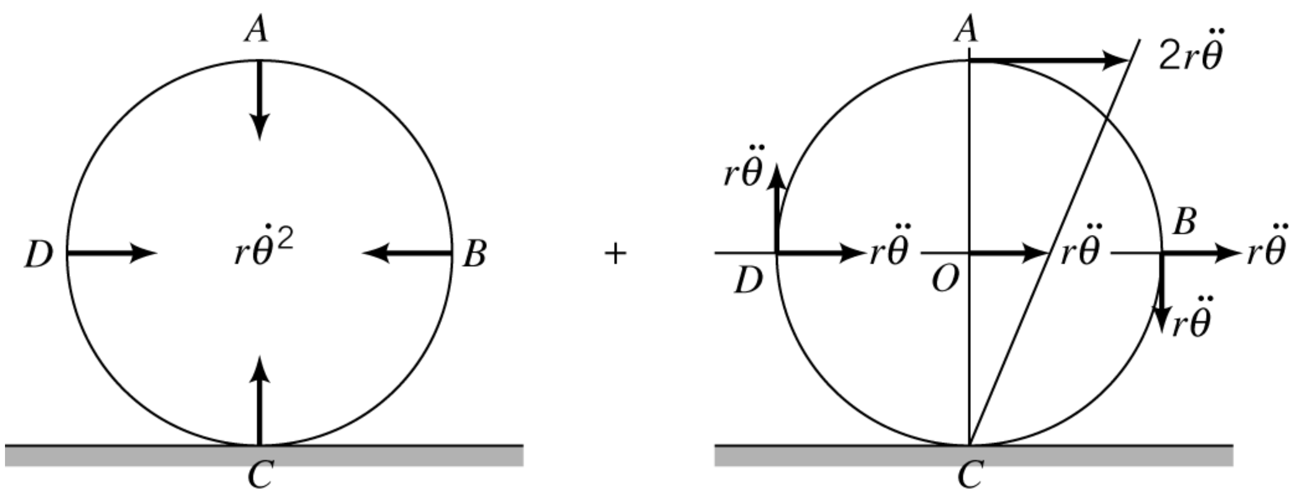



**Figure 4.13** Acceleration vectors for points on the wheel at locations *A* through *D*.

The point $P$ in contact with the ground has zero velocity *at the instant of contact*. Constant-velocity components over a *finite* time period are required to give zero acceleration. *Note carefully that a point on the wheel at the contact location has a vertical acceleration of* $r\dot{\theta}^2$.

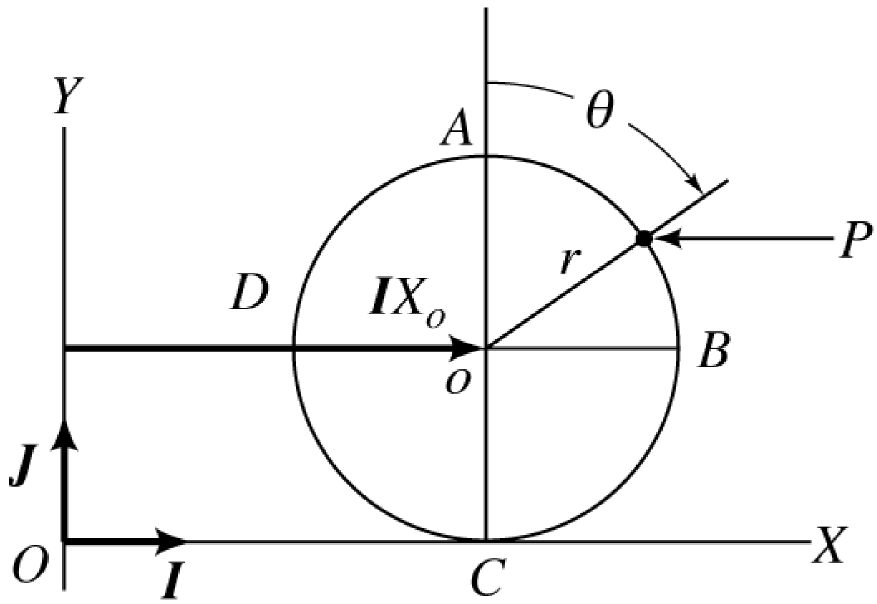
**Geometric approach** :

*a.* State (write out) the geometric $X$ and $Y$ component equations.

*b.* Differentiate the displacement component equations to obtain velocity component equations.

*c.* Differentiate the velocity component equations to obtain acceleration component equations.

# Vector Development of Velocity Relationships




$\boldsymbol{v_o} = \boldsymbol{I}\dot{X}_o$ for point $o$. Using the right-hand-rule convention for defining angular velocity vectors, the wheel's angular velocity vector is $\boldsymbol{\omega} = \boldsymbol{K}(-\dot{\theta}) = -\boldsymbol{K}\dot{\theta}$. ( If the wheel were rolling to the left, $\dot{\theta}$ would be negative, and the angular velocity vector would be $\boldsymbol{\omega} = +\boldsymbol{K}\,|\,\dot{\theta}\,|$.) $C$ is a point on the wheel at the instantaneous contact location between the wheel and the ground, and has a velocity of zero; i.e., $\boldsymbol{v_C} = \boldsymbol{I}\,0 + \boldsymbol{J}\,0$.

## Vector Velocity and Acceleration Relationships

$$\boxed{\begin{aligned} \boldsymbol{v_B} &= \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{AB}} \\ \boldsymbol{a_B} &= \boldsymbol{a_A} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{AB}}) \ . \end{aligned}} \qquad \textbf{(4.3)}$$

Applying the first of Eqs.(4.3) to points *o* and *C* gives

$$\boldsymbol{v_C} = \boldsymbol{v_o} + \boldsymbol{\omega} \times \boldsymbol{r_{oC}} \, .$$

Setting $\boldsymbol{v_C}$ to zero and substituting: $\boldsymbol{v_o} = \boldsymbol{I}\dot{X}_o$, $\boldsymbol{\omega} = -\boldsymbol{K}\dot{\theta}$, $\boldsymbol{r_{oC}} = -\boldsymbol{J}r$, gives

$$0 = \boldsymbol{I}\dot{X}_o + ( -\boldsymbol{K}\dot{\theta} \times -\boldsymbol{J}r ) \ \Rightarrow \ 0 = \boldsymbol{I}(\dot{X}_o - r\dot{\theta}) \ .$$

Hence, the rolling-without-slipping kinematic result for velocity is (again)

$$\dot{X}_o = r\dot{\theta} \ . \qquad \textbf{(4.5a)}$$

Predictably, the vector approach has given us the rolling-without-slipping kinematic condition for velocity.

**Find the velocity vectors for points *A* through *D* of figure 4.10.** Starting with point *A*, and applying the velocity relationship from Eqs.(4.3) to points *A* and *C* on the wheel gives

$$\boldsymbol{v}_A = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{r}_{CA} \ .$$

Substituting: $\boldsymbol{v}_C = \boldsymbol{0}$, $\boldsymbol{\omega} = -\boldsymbol{K}\dot{\theta}$, and $\boldsymbol{r}_{CA} = \boldsymbol{J}\,2r$ yields

$$\boldsymbol{v}_A = 0 - \boldsymbol{K}\dot{\theta} \times \boldsymbol{J}2r \ = \ \boldsymbol{I}2r\dot{\theta} \ .$$

The velocity of a point on the wheel at location *B* can be found by applying Eqs.(4.3) as:

$$\boldsymbol{v}_B = \boldsymbol{v}_o + \boldsymbol{\omega} \times \boldsymbol{r}_{oB}$$

$$\boldsymbol{v}_B = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{r}_{CB} \ .$$

The first equation defines $\boldsymbol{v}_B$ by starting from a known velocity at point *o* ; the second equation starts from a known velocity at point *C*. The vectors $\boldsymbol{v}_o$ , $\boldsymbol{v}_C$, and $\boldsymbol{\omega}$ have already been identified. The required new vectors are $\boldsymbol{r}_{oB} = \boldsymbol{I}\,r$ and $\boldsymbol{r}_{CB} = \boldsymbol{I}r + \boldsymbol{J}r$.

Substitution gives:

$$\boldsymbol{v}_B = \boldsymbol{I}r\dot{\theta} - \boldsymbol{K}\dot{\theta} \times \boldsymbol{I}r \ = \boldsymbol{I}r\dot{\theta} - \boldsymbol{J}r\dot{\theta}$$

$$\boldsymbol{v}_B = 0 - \boldsymbol{K}\dot{\theta} \times (\boldsymbol{I}r + \boldsymbol{J}r) \ = \boldsymbol{I}r\dot{\theta} - \boldsymbol{J}r\dot{\theta} \ .$$

The velocity of point *D* can be obtained by any of the following:

$$\boldsymbol{v_D} = \boldsymbol{v_o} + \boldsymbol{\omega} \times \boldsymbol{r_{oD}}$$

$$\boldsymbol{v_D} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{AD}}$$

$$\boldsymbol{v_D} = \boldsymbol{v_B} + \boldsymbol{\omega} \times \boldsymbol{r_{BD}}$$

$$\boldsymbol{v_D} = \boldsymbol{v_C} + \boldsymbol{\omega} \times \boldsymbol{r_{CD}} \ .$$

We know the velocity vectors for point *o*, *A*, *B*, and *C*, and can write expressions for the vectors $\boldsymbol{r_{oD}}$, $\boldsymbol{r_{AD}}$, $\boldsymbol{r_{BD}}$, and $\boldsymbol{r_{CD}}$. Applying (arbitrarily) the last equation with $\boldsymbol{r_{CD}} = -\boldsymbol{I}r + \boldsymbol{J}r$ gives

$$\boldsymbol{v_D} = 0 - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{I}r + \boldsymbol{J}r) = \boldsymbol{I}r\dot{\theta} + \boldsymbol{J}r\dot{\theta} \ .$$

**Results From Vector Developments of Acceleration**

In this subsection, we will use Eqs.(4.3), relating the acceleration vectors of two points on a rigid body to: (i) derive the rolling-without-slipping acceleration result of Eq.(4.5) ( $\ddot{X}_o = r\ddot{\theta}$) and (ii) determine the acceleration vectors of points *A* through *D*. Starting with points *o* and *C*, we can apply the acceleration result of Eq.(4.3) as

$$\boldsymbol{a_C} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oC}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oC}}) \ .$$

Substituting $\boldsymbol{a_o} = \boldsymbol{I}\ddot{X}_o$ , $\dot{\boldsymbol{\omega}} = -\boldsymbol{K}\ddot{\theta}$, and our earlier result $\boldsymbol{r_{oC}} = -r\boldsymbol{J}$ gives:

$$\boldsymbol{a_C} = (\boldsymbol{I}a_{CX} + \boldsymbol{J}a_{CY}) = \boldsymbol{I}\ddot{X}_o + (-\boldsymbol{K}\ddot{\theta} \times -\boldsymbol{J}r) - \boldsymbol{K}\dot{\theta} \times (\boldsymbol{K}\dot{\theta} \times -\boldsymbol{J}r) \;,$$

$$\boldsymbol{I}a_{CX} + \boldsymbol{J}a_{CY} = \boldsymbol{I}(\ddot{X}_o - r\ddot{\theta}) + \boldsymbol{J}r\dot{\theta}^2 \;.$$

Equating the ***I*** and ***J*** components gives:

$$\begin{aligned} \boldsymbol{I} &: \quad a_{CX} = (\ddot{X}_o - r\ddot{\theta}) = 0 \;\Rightarrow\; \ddot{X}_o = r\ddot{\theta} \\ \boldsymbol{J} &: \quad a_{CY} = r\dot{\theta}^2 \;. \end{aligned} \tag{4.13}$$

The accelerations of points *A*, *B* and *D* can also be obtained via Eqs.(4.3), starting from any point on the wheel where the acceleration vector is known. Choosing point *o* (arbitrarily) gives

$$\begin{aligned} \boldsymbol{a_A} &= \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oA}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oA}}) \\ \boldsymbol{a_B} &= \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oB}}) \\ \boldsymbol{a_D} &= \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oD}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oD}}) \;. \end{aligned}$$

Substituting for the variables on the right-hand side of these equations gives

$$\boldsymbol{a_A} = \boldsymbol{I} r\ddot{\theta} + (-\boldsymbol{K}\ddot{\theta} \times \boldsymbol{J} r) - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{K}\dot{\theta} \times \boldsymbol{J} r) = \boldsymbol{I} 2 r\ddot{\theta} - \boldsymbol{J} r\dot{\theta}^2$$

$$\boldsymbol{a_B} = \boldsymbol{I} r\ddot{\theta} + (-\boldsymbol{K}\ddot{\theta} \times \boldsymbol{I} r) - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{K}\dot{\theta} \times \boldsymbol{I} r) = \boldsymbol{I}(r\ddot{\theta} - r\dot{\theta}^2) - \boldsymbol{J} r\ddot{\theta}$$

$$\boldsymbol{a_D} = \boldsymbol{I} r\ddot{\theta} + (-\boldsymbol{K}\ddot{\theta} \times -\boldsymbol{I} r) - \boldsymbol{K}\dot{\theta} \times (-\boldsymbol{K}\dot{\theta} \times -\boldsymbol{I} r)$$
$$= \boldsymbol{I}(r\ddot{\theta} + r\dot{\theta}^2) + \boldsymbol{J} r\ddot{\theta}.$$

## Example Problem 4.3

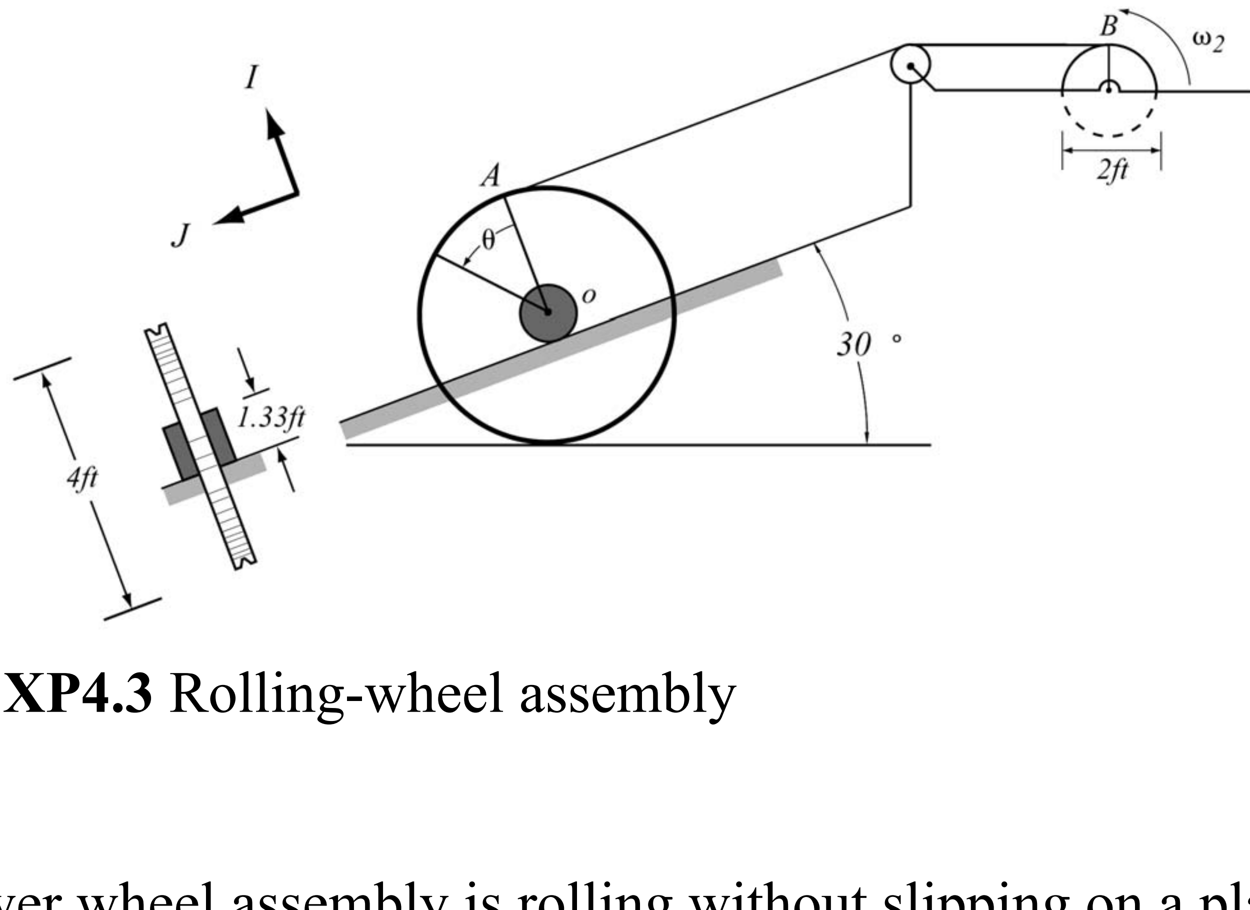



**Figure XP4.3** Rolling-wheel assembly

The lower wheel assembly is rolling without slipping on a plane that is inclined at $30°$ to the horizontal. It is connected to the top spool via an inextensible cable that is playing out cable. The center of the lower spool and its contact point are denoted, respectively by $o$ and $C$. Point $A$ denotes the top of the spool. At the instant of interest, the acceleration of the center of the lower spool is $\boldsymbol{a_o} = 6.44\boldsymbol{J}\, ft/\sec^2$, and its velocity is $\boldsymbol{v_o} = 2.6\boldsymbol{J}\, ft/\sec$.

**Tasks:** For the instant considered, determine the velocity and acceleration of points $C$ and $A$. Determine the top spool's angular velocity and acceleration.

**Solution.**
Rolling without slipping $\Rightarrow$ $\boldsymbol{v_C} = 0$, plus from Eqs.(4.5)

$$\dot{X}_o = 2.6 ft/\sec = r\dot{\theta} = (1.331/2)\, ft \times \dot{\theta}\, rad/\sec \;\Rightarrow\; \dot{\theta} = 3.91\, rad/\sec$$

$$\therefore\; \boldsymbol{\omega_1} = 3.91\boldsymbol{K}\, rad/\sec\ .$$

With $\boldsymbol{\omega_1}$ defined, starting from either $o$ or $C$, $\boldsymbol{v_A}$ is

$$\begin{aligned}\boldsymbol{v_A} &= \boldsymbol{v_C} + \boldsymbol{\omega_1} \times \boldsymbol{r_{CA}} = 0 + 3.91\boldsymbol{K}\, rad/\sec \times (.665 + 2)\boldsymbol{I} ft \\ &= 10.42\boldsymbol{J} ft/\sec\end{aligned}$$

$$\begin{aligned}\boldsymbol{v_A} &= \boldsymbol{v_O} + \boldsymbol{\omega_1} \times \boldsymbol{r_{oA}} = 2.6\boldsymbol{J} ft/\sec + 3.91\boldsymbol{K}\, rad/\sec \times 2\boldsymbol{I} ft \\ &= 10.42\boldsymbol{J} ft/\sec\ .\end{aligned}$$

The velocity of the lower spool at the cable contact point is $v_A = 10.42 ft/\sec$. Since the cable is inextensible, the cable contact point on the upper spool has the same velocity, and

$v_B = v_A = 10.42\, ft/\sec = \omega_2 \times 1\, ft \;\Rightarrow\; \omega_2 = 10.42\, rad/\sec$, and

$\boldsymbol{\omega_2} = 10.42\boldsymbol{K}\;\; rad/\sec$.

From Eq.(4.11),

$$\boldsymbol{a_C} = \boldsymbol{I} r\dot{\theta}^2 = .665\, ft \times (3.91\, rad/\sec)^2 = 10.16\boldsymbol{I}\, ft/\sec^2 .$$

The rolling-without-slipping conditions of Eqs.(4.5) relates *o*'s acceleration to the spool's angular acceleration as

$$\ddot{X}_o = 6.44\, ft/\sec^2 = r\ddot{\theta} \;\Rightarrow\; \ddot{\theta} = 6.44\,(ft/\sec^2)/\,.665\, ft = 9.68\, rad/\sec^2$$

$$\therefore\;\; \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} = 9.68\boldsymbol{K}\, rad/\sec^2 \;.$$

Starting at *o*, $\boldsymbol{a_A}$ is

$$\begin{aligned}
\boldsymbol{a_A} &= \boldsymbol{a_o} + \dot{\boldsymbol{\omega}}_{\boldsymbol{1}} \times \boldsymbol{r_{oA}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{oA}}) \\
&= 6.44\,\frac{ft}{\sec^2}\boldsymbol{J} + (9.68\,\frac{rad}{\sec^2}\boldsymbol{K} \times 2 ft \boldsymbol{I}) \\
&+ [3.91\,\frac{rad}{\sec}\boldsymbol{K} \times (3.91\,\frac{rad}{\sec}\boldsymbol{K} \times 2 ft \boldsymbol{I})] \\
&= (6.44 + 19.36)\boldsymbol{J} - 30.6\boldsymbol{I} = -30.6\boldsymbol{I} + 25.8\boldsymbol{J} \quad ft/\sec^2 \;.
\end{aligned}$$

Starting at *C*, $\boldsymbol{a_A}$ is

$$\begin{aligned}\boldsymbol{a_A} &= \boldsymbol{a_C} + \dot{\boldsymbol{\omega}}_{\mathbf{1}} \times \boldsymbol{r_{CA}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{CA}}) \\ &= 10.16\,\boldsymbol{I}\,\frac{ft}{\sec^2} + (9.68\,\frac{rad}{\sec^2}\boldsymbol{K} \times 2.665\,ft\,\boldsymbol{I}) \\ &+ [3.91\,\frac{rad}{\sec}\boldsymbol{K} \times (3.91\,\frac{rad}{\sec}\boldsymbol{K} \times 2.665\,ft\,\boldsymbol{I})] \\ &= 25.8\boldsymbol{J} + (10.16 - 40.74)\boldsymbol{I} = -30.6\boldsymbol{I} + 25.8\boldsymbol{J} \quad ft/\sec^2 \ .\end{aligned}$$

Solving for $\dot{\boldsymbol{\omega}}_{\mathbf{2}}$: The acceleration of the wheel assembly at $A$ is $\boldsymbol{a_A} = 25.8\boldsymbol{J} - 30.6\boldsymbol{I}\ ft/\sec^2$. The acceleration of spool 2 at its contact point $B$ is $\boldsymbol{a_B} = 1\,\dot{\omega}_2\boldsymbol{J} - 1\,\omega_2^2\boldsymbol{I}\ ft/\sec^2$. Because the cable is inextensible, the $\boldsymbol{J}$ components of these accelerations (along the cable) must be equal. Hence,

$$\dot{\omega}_2 = (25.8\,ft/\sec^2)/1\,ft = 25.8\,rad/\sec^2\text{, and } \dot{\boldsymbol{\omega}}_{\mathbf{2}} = 25.8\boldsymbol{K}\ rad/\sec^2 .$$

**Example Problem 4.4** A cylinder of radius $r_2 = 1\,m$ is rolling without slipping on a horizontal surface. At the instant of interest, the surface is moving to the left with $\boldsymbol{v_s} = -\boldsymbol{I}\,1.25\,m/\sec$, and has an acceleration to the right of $\boldsymbol{a_s} = \boldsymbol{I}\,1.0\,m/\sec^2$. An inextensible cord is wrapped around an inner cylinder of radius $r_1 = 0.5\,m$ and anchored to a wall at *D*. *Determine the angular velocity and acceleration of the cylinder and the velocity and acceleration of points A, o, B, and C.*

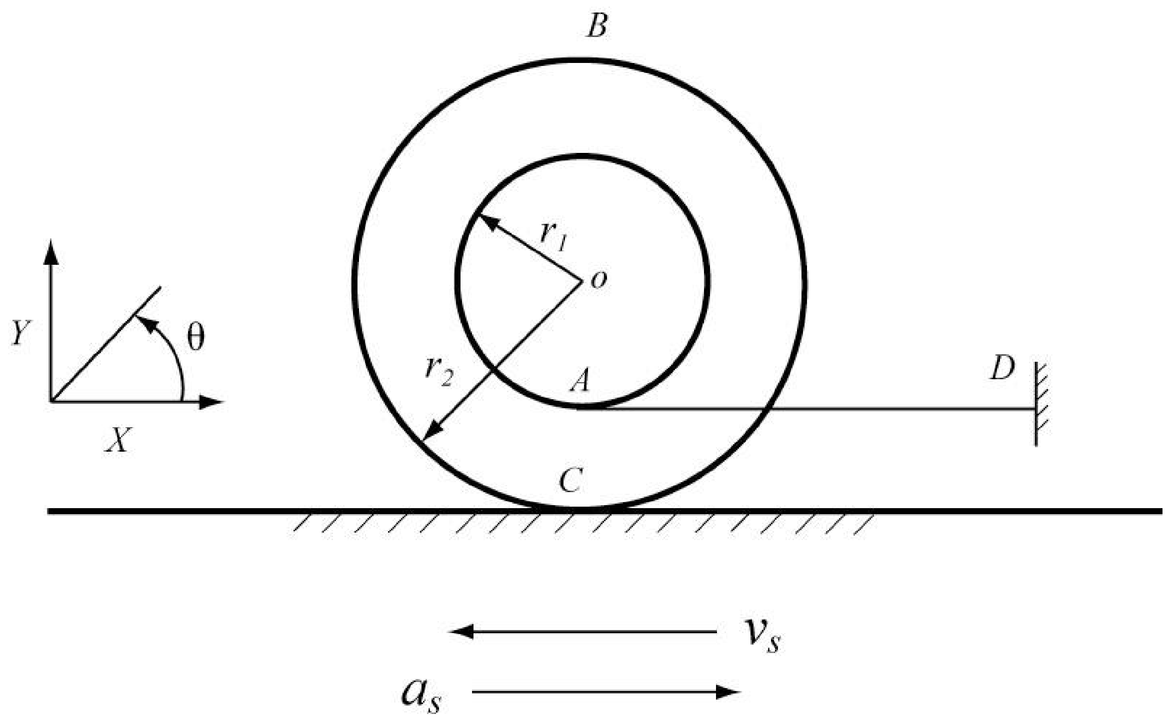



**Figure XP4.4** Cylinder rolling without slipping on a horizontal (moving surface) while being restrained by a cord from *D*

**Solution.** First find $\dot{\theta}$. Point *C*, the contact point between the outer surface of the cylinder and the moving surface, has the velocity $\boldsymbol{v_C} = \boldsymbol{I} v_s$. Point *A*, the contact point of the inner cylinder with the cord, has zero velocity, $\boldsymbol{v_A} = 0$. The cylinder can be visualized as rolling without slipping (to the right) on the cord line *A-D*. We could use the vector equation $\boldsymbol{v_I} = \boldsymbol{v_J} + \boldsymbol{\omega} \times \boldsymbol{r_{JI}}$ to state multiple correct equations between the velocities of points *A*, *B*, *o*, and *C*, and most of these equations would not be helpful in determining $\dot{\theta}$. We will use *A* and *C* because we know their velocities, and we do not know the velocities of the remaining points. Since points *C* and *A* are on the cylinder (rigid body),

$$\boldsymbol{v_C} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{AC}} \ \ or,$$

$$\begin{aligned} -\boldsymbol{I}(1.25\,m/\sec) &= 0 + \boldsymbol{K}\dot{\theta}(rad/\sec) \times -0.5\,m\boldsymbol{J} \\ &= \boldsymbol{I}0.5\dot{\theta}\,(m/\sec) \end{aligned} \qquad \textbf{(i)}$$

$$\therefore\ \dot{\theta} = -2.5\,rad/\sec\ ,\ \ \boldsymbol{\omega} = -\boldsymbol{K}2.5\,rad/\sec$$

Hence, the cylinder is rotating in a clockwise direction.

We can determine the velocity of point *o* at the center of the cylinder using either $\boldsymbol{v_o} = \boldsymbol{v_C} + \boldsymbol{\omega} \times \boldsymbol{r_{Co}}$ or $\boldsymbol{v_o} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{Ao}}$, because we know $\boldsymbol{v_A} = 0$ and $\boldsymbol{v_C} = -\boldsymbol{I}\,1.25\,m/\sec$. Proceeding from point *A*,

$$\boldsymbol{v_o} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{Ao}} = 0 - \boldsymbol{K}2.5\,(rad/\sec) \times \boldsymbol{J}0.5\,(m)$$
$$= \boldsymbol{I}1.25\,m/\sec\ .$$

Hence, point $o$ at the center of the cylinder moves horizontally to the right. Similarly,

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega} \times \boldsymbol{r_{Ab}} = 0 - \boldsymbol{K}2.5\,(rad/\sec) \times \boldsymbol{J}1.5\,(m)$$
$$= \boldsymbol{I}3.75\ \ m/\sec\ .$$

We have now completed the velocity analysis determining $\dot{\theta}, \boldsymbol{\omega}, \boldsymbol{v_o}, \boldsymbol{v_B}, \boldsymbol{v_A}$.

For the acceleration analysis, our first objective is the solution for $\ddot{\theta}$. Point $A$ has a zero horizontal acceleration component; i.e., $\boldsymbol{a_A} = \boldsymbol{J}a_{AY}$. From the nonslipping condition, point $C$ has the horizontal acceleration component, $a_{CX} = 1.0\,m/\sec^2$; hence, $\boldsymbol{a_C} = \boldsymbol{I}1. + \boldsymbol{J}a_{CY}\ m/\sec^2$.

Hence,

$$\boldsymbol{a_C} = \boldsymbol{a_A} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{AC}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{AC}})\ \ or,$$

$$\boldsymbol{I}1\,m/\sec^2 + \boldsymbol{J}a_{CY} = \boldsymbol{J}a_{AY} + [\boldsymbol{K}\ddot{\theta}\,(rad/\sec^2) \times -.5\,m\boldsymbol{J}]$$
$$+ \{\boldsymbol{K}\dot{\theta}\,(rad/\sec) \times [\boldsymbol{K}\dot{\theta}\,(rad/\sec) \times -0.5\,m\boldsymbol{J}]\}$$

$$= [\boldsymbol{I}0.5\,\ddot{\theta} + \boldsymbol{J}(a_{AY} + 0.5\,\dot{\theta}^2)]\,(m/\sec^2)\ .$$

Taking the $\boldsymbol{I}$ and $\boldsymbol{J}$ components separately gives:

$$\boldsymbol{I}:\ 1m\sec^2 = 0.5\ddot{\theta}\,m/\sec^2$$
$$\Rightarrow\ \ddot{\theta} = 2\,rad/\sec^2\ \Rightarrow\ \dot{\boldsymbol{\omega}} = \boldsymbol{K}2\ \ rad/\sec^2$$

**(ii)**

$$\boldsymbol{J}:\ a_{CY}(m/\sec^2) = [a_{AY} + 0.5\dot{\theta}^2]\,m/\sec^2$$
$$= [a_{AY} + 0.5\times(-2.5)^2]\,m/\sec^2\ = (a_{AY} + 3.125)\,m/\sec^2$$

The $\boldsymbol{I}$ component result is immediately useful, determining $\ddot{\theta}$.

The $\boldsymbol{J}$ component result is not helpful, since it only provides a relationship between the two unknowns $a_{AY}$ and $a_{CY}$. We still need to calculate these components. Point *o* has no vertical motion; i.e, $\boldsymbol{a_o} = \boldsymbol{I}a_{oX}$. Hence,

$$\boldsymbol{a_C} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}}\times\boldsymbol{r_{oC}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r_{oC}})\ \ or,$$

$$\boldsymbol{I}1.m/\sec^2 + \boldsymbol{J}a_{CY} = \boldsymbol{I}a_{oX} + [\boldsymbol{K}\ddot{\theta}(rad/\sec^2)\times -1.m\boldsymbol{J}]$$
$$+ \{\boldsymbol{K}\dot{\theta}(rad/\sec)\times[\boldsymbol{K}\dot{\theta}(rad/\sec)\times -1.m\boldsymbol{J}]\}$$

$$= \boldsymbol{I}(a_{oX} + \ddot{\theta}) + \boldsymbol{J}\dot{\theta}^2\,(m/\sec^2)\ .$$

The $\boldsymbol{I}$ and $\boldsymbol{J}$ components give:

$$\boldsymbol{I}: \; 1 m \sec^2 = (a_{oX} + \ddot{\theta}) m/\sec^2$$

$$\therefore \quad a_{oX} = (1 - 2) m/\sec^2 = -1\, m/\sec^2$$

**(iii)**

$$\boldsymbol{J}: \; a_{CY}(m/\sec^2) = (\dot{\theta}^2) m/\sec^2 = (2.5)^2 m/\sec^2$$
$$= 6.25\, m/\sec^2 \; .$$

We substituted $\ddot{\theta} = 2\, rad/\sec^2$ and $\dot{\theta} = -2.5\, rad/\sec$ into the $\boldsymbol{I}$ and $\boldsymbol{J}$ component equations, respectively. You can verify that we could have also used $\boldsymbol{a_A} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oA}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oA}})$ successfully to determine $a_{AY}$.

From Eq.(ii), $a_{CY} = a_{AY} + 3.125\,(m/\sec^2)$; hence, using the second of Eq. (ii), $a_{AY} = 6.25 - 3.125 = 3.125\,(m/\sec^2)$. Note that $a_A = r_{oA}\dot{\theta}^2 = 0.5 \times (2.5)^2 = 3.125$. In this example, point *A* corresponds to the "contact point C". It is in contact with cord line *AD*, and its only acceleration is vertical due to the centrifugal acceleration term.

We now have

$$\boldsymbol{a_A} = \boldsymbol{J}3.125\; (m/\sec^2)$$

$$\boldsymbol{a_o} = -\boldsymbol{I}1.\,(m/\sec^2)$$

$$\boldsymbol{a_C} = -\boldsymbol{I}1. + \boldsymbol{J}6.25\; (m/\sec^2)$$

The acceleration of point $B$ can be obtained starting from points $A$, $o$, or $C$ , since we have the acceleration of all these points. Starting from $o$,

$$\boldsymbol{a_B} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oB}})$$

$$\begin{aligned} &= \ -\boldsymbol{I}1\,(m/\sec^2) \\ &+ [\boldsymbol{K}2\,(rad\sec^2) \times \boldsymbol{J}1\,m] + \\ &-\boldsymbol{K}2.5\,(rad/\sec) \times [-\boldsymbol{K}2.5\,(rad/\sec) \times \boldsymbol{J}1.m] \end{aligned}$$

$$= \boldsymbol{I}(-1.-2.) - \boldsymbol{J}6.25 \quad m/\sec^2 = -\boldsymbol{I}3. - \boldsymbol{J}6.25 \quad m/\sec^2 \ .$$

Note in reviewing this example the key to the solution is: (i) first find $\dot{\theta}$, and (ii) then find $\ddot{\theta}$. We used a velocity relation between points $A$ and $C$ to find $\dot{\theta}$ and an acceleration relation between the same two points to find $\ddot{\theta}$. These points work because *we know the X components* of $\boldsymbol{v_C}, \boldsymbol{a_C}, \boldsymbol{v_A}, \boldsymbol{a_A}$. We can write valid equations relating the velocity and acceleration for any two of the points; however, only the combination of points $A$ and $C$ will produce directly useful results in calculating $\dot{\theta}$ and $\ddot{\theta}$.

Also, note that to determine the $Y$ components of $\boldsymbol{a_A}, \boldsymbol{a_B}, \boldsymbol{a_C}$, we needed to use an acceleration relationship involving $\boldsymbol{a_0}$, the acceleration of center of the cylinder. Valid acceleration relationships can certainly be stated between the points $A$, $B$, and $C$ However, the results are not helpful; e.g., we obtained $a_{CY} = a_{AY} + 3.125 \ m/\sec^2$ in Eq.(ii), which involves two unknowns $a_{CY}$ and $a_{AY}$. Point o "works" because we know that its vertical acceleration is zero.

**NOTE: THE GEOMETRIC APPROACH WORKS WELL ON MECHANISMS. THE VECTOR APPROACH GENERALLY WORKS BETTER IN ROLLING-WITHOUT-SLIPPING PROBLEMS WHEN YOU NEED TO FIND THE ACCELERATION OF A POINT OR THE ANGULAR VELOCITY OR ACCELERATION OF A WHEEL.**