Lecture 24. INERTIA PROPERTIES AND THE PARALLEL-AXIS FORMULA

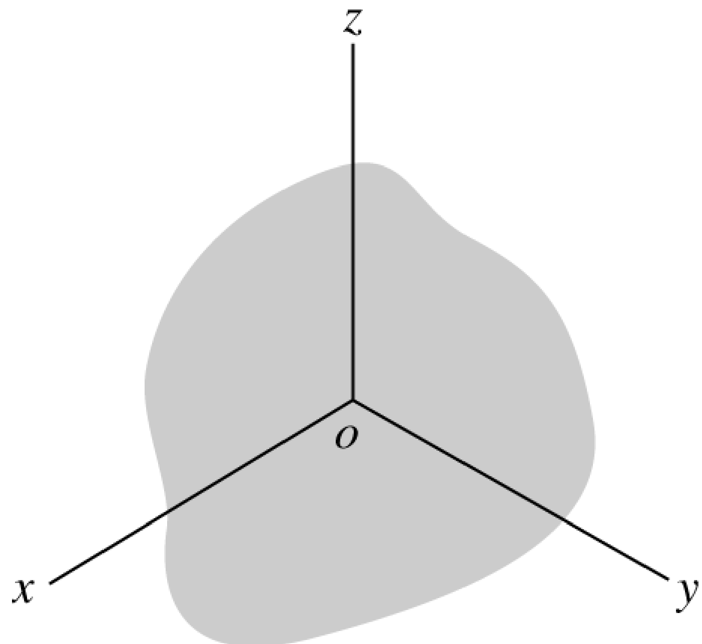


Figure 5.1 Rigid body with an imbedded x,y,z coordinate system.

The body's mass is defined by

$$m = \int_V \gamma(x,y,z)\,dx\,dy\,dz \quad , \tag{5.1}$$

where $\gamma(x,y,z)$ is the body's density at point x, y, z .

With the position vector of a point in the rigid body defined by

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \quad ,$$

the body's mass center is located in the x, y, z system by the vector $\boldsymbol{b}_{og}$, defined by

$$m\boldsymbol{b}_{og} = m(\boldsymbol{i}b_{ogx} + \boldsymbol{j}b_{ogy} + \boldsymbol{k}b_{ogz}) = \int_V \boldsymbol{\rho}\gamma\,dx\,dy\,dz = \int_m \boldsymbol{\rho}\,dm\ . \quad (5.2)$$

The mass moment of inertia about a z axis through o is defined by

$$I_{zzo} = \int_V (x^2 + y^2)\gamma\,dx\,dy\,dz = \int_m (x^2 + y^2)\,dm. \quad (5.3)$$

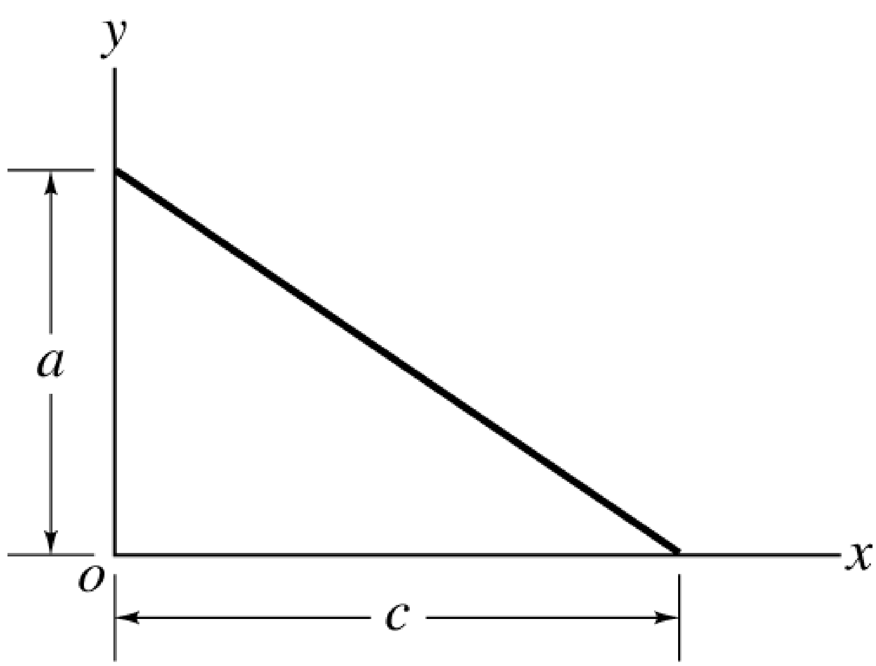


Figure 5.2 Triangular mass of unit depth and uniform mass per unit area $\bar{\gamma}$.

Applying Eq.(5.1) to the example of Figure 5.2 gives:

$$m = \bar{\gamma}\int_0^c y\,dx = \bar{\gamma}\int_0^c (a - \frac{a}{c}x)\,dx$$

$$= \bar{\gamma}\,|_0^c\,(ax - \frac{ax^2}{2c}) = \frac{ac\bar{\gamma}}{2} \quad .$$

Applying Eq.(5.2) to find the mass center gives

$$m\boldsymbol{b_{og}} = m(\boldsymbol{i}b_{ogx} + \boldsymbol{j}b_{ogy}) = \int_A \boldsymbol{\rho}\bar{\gamma}\,dx\,dy = \int_A (\boldsymbol{i}x + \boldsymbol{j}y)\bar{\gamma}\,dx\,dy \quad .$$

Hence,

$$m\,b_{ogx} = \bar{\gamma}\int_A x\,dx\,dy \quad , \quad m\,b_{ogy} = \bar{\gamma}\int_A y\,dx\,dy \quad ,$$

and:

$$\begin{aligned} m\,b_{ogx} &= \bar{\gamma}\int_0^c xy\,dx = \bar{\gamma}\int_0^c x(a - \frac{ax}{c})\,dx = \bar{\gamma}\,|^c{}_0(\frac{ax^2}{2} - \frac{ax^3}{3c}) \\ &= \bar{\gamma}\frac{ac^2}{6} = m\frac{c}{3} \\ m\,b_{ogy} &= \bar{\gamma}\int_0^c yx\,dy = \bar{\gamma}\int_0^a y(c - \frac{cy}{a})\,dy = \bar{\gamma}\,|^a{}_0(\frac{cy^2}{2} - \frac{cy^3}{3a}) \\ &= \gamma\frac{ca^2}{6} = m\frac{a}{3} \quad . \end{aligned} \tag{5.4}$$

The mass center is located in the *x, y* system by

$\boldsymbol{b}_{og} = \boldsymbol{i}c/3 + \boldsymbol{j}a/3$.

Proceeding from Eq.(5.3) for the moment-of-inertia definition about o is,

$$I_{zzo} = \int_A \bar{\gamma}(x^2 + y^2)\,dx\,dy = \bar{\gamma}\int_0^a \Big[\int_0^{c(1-\frac{y}{a})} (x^2 + y^2)dx\Big]dy \qquad (5.5)$$
$$= m(c^2 + a^2)/6 \ .$$

A considerable amount of work is hidden in getting across the last equality sign.

The radius of gyration is defined as the radius at which all of the mass could be concentrated to obtain the correct moment of inertia. For this example,

$$I_{zzo} = m k_g^2 \ \Rightarrow \ k_g = \sqrt{\frac{c^2 + a^2}{6}}$$

A particle has all of its mass concentrated at a point and has negligible dimensions of length, breadth, depth, etc. Rigid bodies have finite dimensions, yielding properties such as area, volume, and moment of inertia. Observe that continuing to reduce the dimensions of the triangular plate in figure 5.2 will cause the moment of inertia defined by Eq.(5.5) to rapidly approach zero, which is consistent with a particle.

The Parallel-Axis Formula

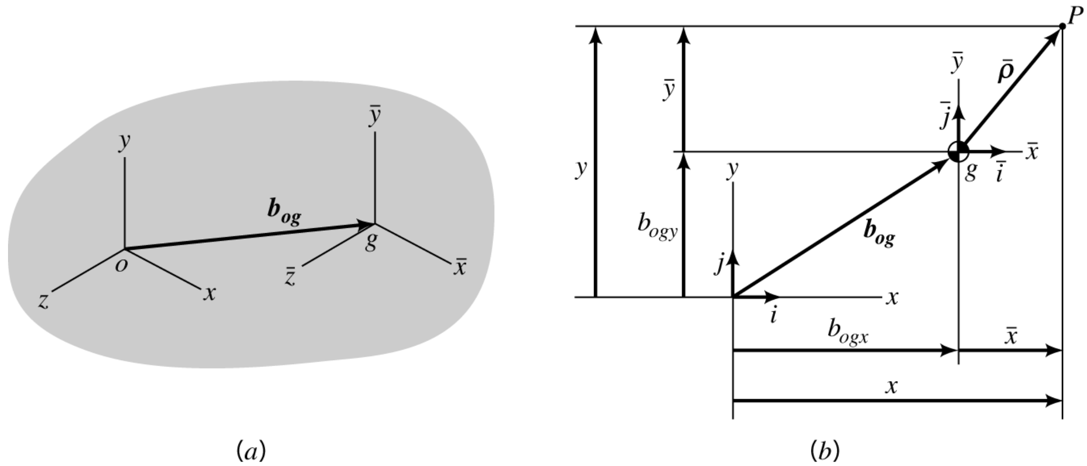


Figure 5.3 (a) Two coordinate systems fixed in a rigid body, (b) End view looking in along the z axis.

The *x, y, z* axes are parallel, respectively, to the $\bar{x}, \bar{y}, \bar{z}$ axes. The mass center of the body is located at the origin of the $\bar{x}, \bar{y}, \bar{z}$ coordinate system and is located in the *x, y, z* coordinate system by the vector $\boldsymbol{b_{og}} = \boldsymbol{i}\, b_{ogx} + \boldsymbol{j}\, b_{ogy} + \boldsymbol{k}\, b_{ogz}$. The question of interest is:

Suppose that we know the moment of inertia about the $\bar{z}$ axis, what is it about the z axis ?

Figure 5.3B provides an end view along the *z* axis of the $\bar{x}, \bar{y}$ and *x, y* axes. A point that is located in the $\bar{x}, \bar{y}, \bar{z}$ coordinate system by the vector $\bar{\boldsymbol{\rho}} = \bar{\boldsymbol{i}}\, \bar{x} + \bar{\boldsymbol{j}}\, \bar{y} + \bar{\boldsymbol{k}}\, \bar{z}$ is located in the *x, y, z* system by $\boldsymbol{\rho} = \bar{\boldsymbol{\rho}} + \boldsymbol{b_{og}}$, or $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z =$

$\boldsymbol{i}(b_{ogx}+\bar{x})+\boldsymbol{j}(b_{ogy}+\bar{y})+\boldsymbol{k}z$; hence,

$$x=b_{ogx}+\bar{x},\quad y=b_{ogy}+\bar{y}\ .$$

The moment of inertia about the z axis is defined to be

$$I_{zzo}=\int_m (x^2+y^2)\,dm\ .$$

Substituting for x and y gives

$$\begin{aligned}
I_{zzo} &= \int_m (b_{ogx}^2+2b_{ogx}\,\bar{x}+\bar{x}^2+b_{ogy}^2+2\,b_{ogy}\,\bar{y}+\bar{y}^2)\,dm \\
&= \int_m (\bar{x}^2+\bar{y}^2)\,dm + (b_{ogx}^2+b_{ogy}^2)\,m \\
&\quad + 2\,b_{ogx}\int_m \bar{x}\,dm + 2\,b_{ogy}\int_m \bar{y}\,dm
\end{aligned} \tag{5.6}$$

$$=I_{\bar{z}\bar{z}}+m\,|\,\boldsymbol{b_{og}}\,|^2+2b_{ogx}\int_m \bar{x}\,dm+2b_{ogy}\int_m \bar{y}\,dm\ .$$

Because the mass center is at the origin of the $\bar{x},\bar{y},\bar{z}$ coordinate system, the last two integrals in Eq.(5.6) are zero, and we obtain

$$I_{zz}=I_{\bar{z}\bar{z}}+m\,|\,\boldsymbol{b_{og}}\,|^2=I_g+m\,|\,\boldsymbol{b_{og}}\,|^2\ . \tag{5.7}$$

Note that this expression is only valid when the mass center of the body is at the origin of the $\bar{x},\bar{y},\bar{z}$ *coordinate system.*

Example 1

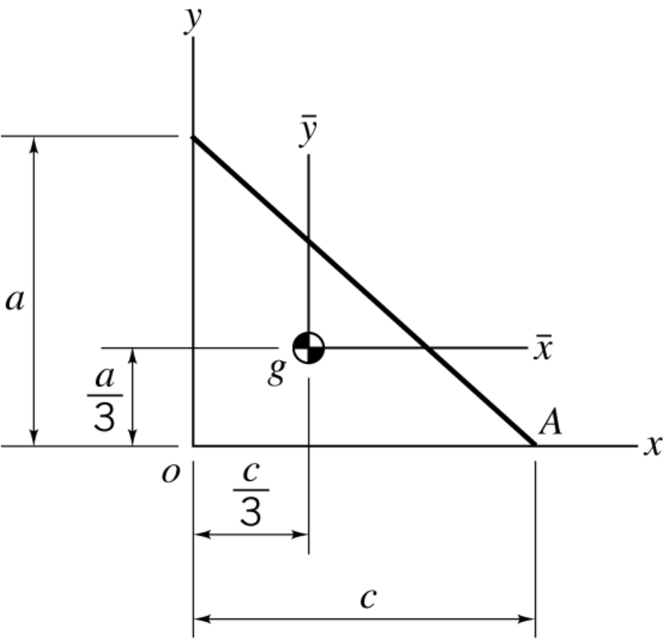


Figure 5.4 Two coordinate systems in the triangular plate of figure 5.2.

From Eq.(5.5)

$$I_{zzo} = I_o = m(c^2 + a^2)/6$$

Wanted: the moment of inertia about a *z* axis (perpendicular to the plate) through point *A* at the right-hand corner.

Procedure:

1. Use Eq.(5.7) to go from *o* to *g* and find I_g.

2. Use Eq.(5.7) to go from g to *A* and find I_A.

The vector from point *o* to *g* is $\boldsymbol{b_{og}} = \boldsymbol{i}c/3 + \boldsymbol{j}a/3$; hence, step 1 gives

$$I_g = I_o - m\,|\,\boldsymbol{b_{og}}\,|^2$$
$$= \frac{m}{6}(c^2 + a^2) - \frac{m}{9}(a^2 + c^2) = \frac{m}{18}(c^2 + a^2) \ .$$

The vector from point g to point A is $\boldsymbol{b_{gA}} = \boldsymbol{i}2c/3 - \boldsymbol{j}a/3$. Hence, step 2 gives

$$I_A = I_g + m\,|\boldsymbol{b_{gA}}|^2 = \frac{m}{18}(c^2 + a^2) + m(\frac{4c^2}{9} + \frac{a^2}{9})$$
$$= \frac{m}{18}(9c^2 + 3a^2) = \frac{mc^2}{2} + \frac{ma^2}{6} \ .$$

Note

$$I_A \neq I_o + m\,|\,\boldsymbol{b_{oA}}\,|^2$$

$$I_o + m\,|\,\boldsymbol{b_{oA}}\,|^2 = \frac{m}{6}(c^2 + a^2) + mc^2 = \frac{7mc^2}{6} + \frac{ma^2}{6} \ .$$

Example 2

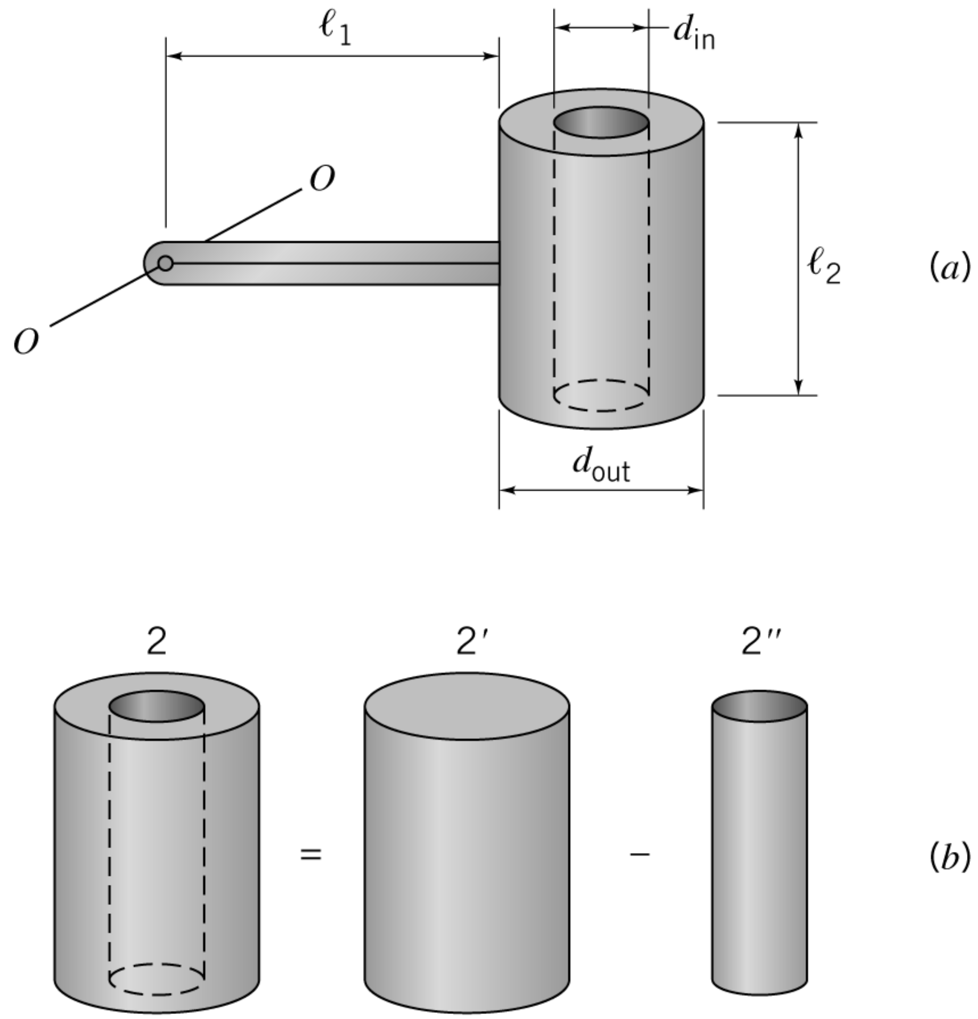


Figure XP5.1 (a) Assembly rotating about axis *o-o*, (b) Modeling approach for the hollow cylinder.

Figure XP5.1A illustrates a welded assembly consisting of a uniform bar with dimensions $l_1 = 25\,cm$, $d_1 = 2.5\,cm$ attached to a hollow cylinder with length $l_2 = 20\,cm$ and inner and outer radii $d_{out} = 150\,mm$; $d_{in} = 75\,mm$, respectively. The assembly is made from steel with density $\gamma = 7750\,kg/m^3$. The assembly rotates about the *o-o* axis.

The following engineering-analysis tasks apply:

> *a. Determine the moment of inertia of the assembly about the o-o axis.*
>
> *b. Determine the assembly's radius of gyration for rotation about the o-o axis.*
>
> *b. Determine the assembly's mass-center location.*

Solution Break the assembly into two pieces and analyze the bar and hollow cylinder separately.

Slender Bar: The moment of inertia for a slender bar about a transverse axis at its end is $I_{end} = ml^2/3$, and the moment of inertia for a transverse axis through the mass center is $I_g = ml^2/12$. These results are related to each other via the parallel-axis formula as

$$I_{end} = I_g + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{12} + \frac{ml^2}{4} = \frac{ml^2}{3}\,. \tag{i}$$

You should work at committing the bar's inertia-property definition to memory. The bar's mass is

$$m_1 = (\frac{\pi d_1^2}{4}) l_1 \gamma = \frac{3.14159 \times .025^2}{4} .25 \times 7750 \ = .951\, kg \ . \qquad \text{(ii)}$$

Hence, from Eq.(i),

$$I_{1o} = \frac{m_1 l_1^2}{3} = \frac{.951 \times .25^2}{3} = .0198\, kg m^2 \ . \qquad \text{(iii)}$$

Hollow Cylinder: figure XP5.1B shows that the hollow cylinder can be "constructed" by subtracting a solid cylinder (denoted $2''$) with the inner diameter d_{in} from a solid cylinder (denoted $2'$) with the outer radius d_{out}. Starting with the inner cylinder

$$m_{2''} = (\frac{\pi d_{in}^2}{4}) \times l_2 \times \gamma = (\frac{3.14159 \times .075^2}{4}) \times .2 \times 7550 = 6.671\, kg \, \text{(iv)}$$

From Appendix C, the moment of inertia for a transverse axis through the inner cylinder's mass center is

$$I_{2''g} = m_{2''} \times (\frac{r_{in}^2}{4} + \frac{l_2^2}{12}) = 6.671 \times (\frac{.0375^2}{4} + \frac{.2^2}{12}) = .0241\, kg m^2 . \text{(v)}$$

For the solid outer cylinder,

$$m_{2'} = (\frac{\pi d_{out}^2}{4}) \times l_2 \times \gamma = (\frac{3.14159 \times .15^2}{4}) \times .2 \times 7550 = 27.39\, kg , \text{(vi)}$$

and

$$I_{2'g} = m_{2'} \times \left(\frac{r_{out}^2}{4} + \frac{l_2^2}{12}\right) = 27.39 \times \left(\frac{.075^2}{4} + \frac{.2^2}{12}\right) = .1298\ kgm^2 \quad \text{(vii)}$$

From Eqs.(iv) through (vii),

$$m_2 = m_2' - m_2'' = 27.39 - 6.671 = 20.72\ kg$$

$$I_2 = I_2' - I_2'' = .1298 - .0246 = .1019\ kgm^2\ .$$

These values conclude the individual results for the hollow cylinder.

Combining the results for the bar and the hollow cylinder via the parallel-axis formula, the assembly moment of inertia is

$$I_o = I_{1o} + I_2 + m_2 \times b_{og}^2 = .0198 + .1019 + 20.72 \times .325^2 = 2.310\ kgm^2,$$

and concludes *Task a*. The radius of gyration k_g is obtained from $I_o = m k_g^2$ where *m* is the assembly moment of inertia; hence,

$$2.310 = (.951 + 20.72) k_g^2 = 21.67\ k_g^2 \Rightarrow \quad k_g = .326\ m\ , \qquad \text{(viii)}$$

and concludes *Task b*.

The assembly mass location is found from

$$md = (m_1 + m_2)d = 21.67\,d$$
$$= (m_1 d_1 + m_2 d_2) = .951 \times \frac{.25}{2} + 20.72 \times .325 = 6.853 \;\; ;$$

hence,

$$d = .316\,m \;\; .$$

Note that the mass center location defined by d is not related to the radius of gyration k_g .

Lecture 25. GOVERNING FORCE AND MOMENT EQUATIONS FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a *particle*.

Find: force and moment differential equations of motion for planar motion of a rigid body.

Force Equation

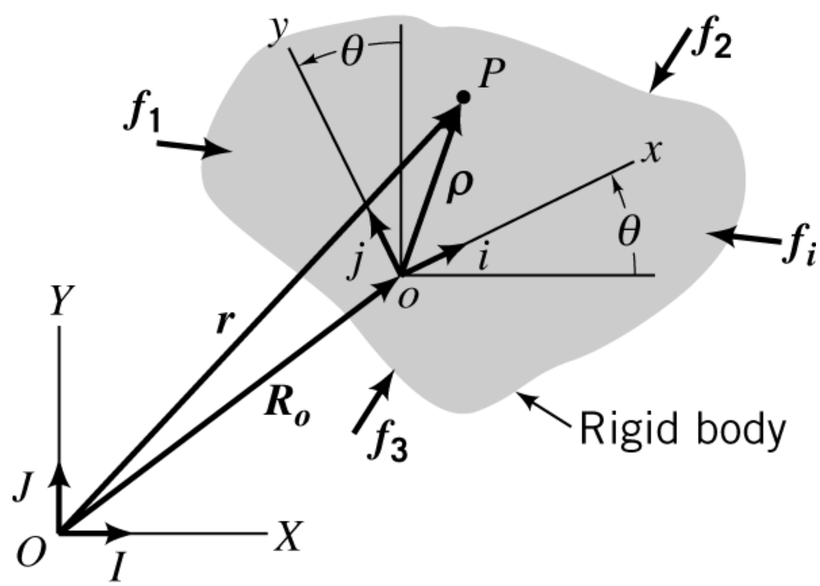


Figure 5.5 Rigid body acted on by external forces. The *x,y,z* coordinate system is fixed in the rigid body; the *X, Y, Z* system is an inertial coordinate system.

X, Y, Z inertial coordinate system

x, y, z coordinate system fixed in the rigid body.

θ defines the orientation of the rigid body (and the x, y, z coordinate system) with respect to the X, Y, Z system.

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ is the angular velocity of the rigid body and the x, y, z coordinate system, with respect to X, Y, Z coordinate system.

$\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$ locates the origin of the x, y, z system in the X, Y, Z system.

Point P in the rigid body is located in the X, Y, Z system by $\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y + \boldsymbol{K}r_Z$.

Point is located in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

Hence,

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \ .$$

Force Equation. Applying Newton's second law to the particle at P gives

$$\boldsymbol{f_P} = dm\,\ddot{\boldsymbol{r}} = dm\,\frac{d^2\boldsymbol{r}}{dt^2}\Big|_{\boldsymbol{X,Y,Z}}\ , \tag{5.8}$$

where:

$\boldsymbol{f_P}$ is the *resultant* force

$\ddot{\boldsymbol{r}}$ is the acceleration of the particle with respect to the inertial $X,\ Y,\ Z$ system.

$dm = \gamma\, dx\, dy\, dz$ where γ is the mass density of the rigid body.

The resultant force at P is

$$\boldsymbol{f_P} = \Sigma \boldsymbol{f_{external}} + \Sigma \boldsymbol{f_{internal}}$$

On the left hand side of Eq.(5.8), integrating over the mass of the body gives

$$\int_m \boldsymbol{f_p}\, dm = \Sigma \int_m \boldsymbol{f_{external}}\, dm + \Sigma \int_m \boldsymbol{f_{internal}}\, dm = \Sigma\ \boldsymbol{f_i} + 0\ \ ;$$

i.e., when integrated over the whole body, the internal forces cancel.

The integral expression of Eq.(5.8) is then

$$\sum \boldsymbol{f_i} = \int_V \ddot{\boldsymbol{r}}\,\gamma\, dx\, dy\, dz = \int_m \ddot{\boldsymbol{r}}\, dm \quad , \tag{5.9}$$

For the two points o and P in the rigid body

$$\boldsymbol{a_P} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oP}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oP}}) \ .$$

Since $\boldsymbol{r}$ and $\boldsymbol{R_o}$ locate points P and o, respectively, in the X, Y, Z system, and $\boldsymbol{\rho}$ is the vector from point o to P,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \ . \tag{5.10}$$

Since $dm = \gamma\, dx\, dy\, dz$, integration extends over the volume of the rigid body.

Since $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are constant with respect to the x, y, z integration variables they can be brought outside the integral sign yielding

$$\sum \boldsymbol{f_i} = m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \int_m \boldsymbol{\rho}\, dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\, dm) \ . \tag{5.11}$$

The mass center is located in the x, y, z system by $\boldsymbol{b_{og}}$, defined by

$$m\boldsymbol{b_{og}} = \int_m \boldsymbol{\rho}\, dm = m(\boldsymbol{i}\, b_{ogx} + \boldsymbol{j}\, b_{ogy} + \boldsymbol{k}\, b_{ogz}) \ . \tag{5.12}$$

Substituting from Eq.(5.12) into Eq.(5.11) gives

$$\sum \boldsymbol{f_i} = m[\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}})] \ . \tag{5.13}$$

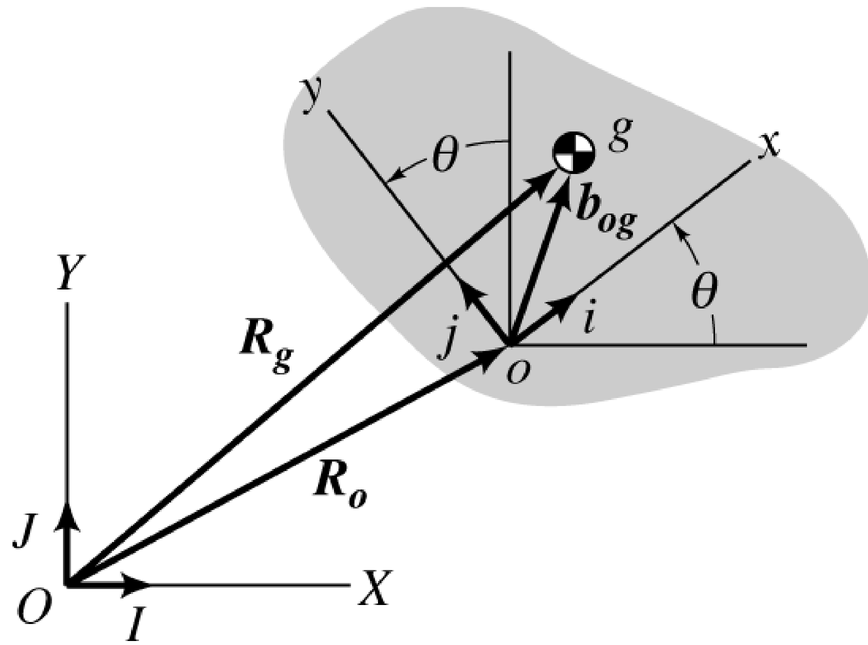


Figure 5.6 A rigid body with a mass center located in the body-fixed *x, y, z* coordinate system by the vector $\boldsymbol{b_{og}}$ and located in the inertial *X, Y, Z* system by $\boldsymbol{R_g}$.

Since *g* and *o* are fixed in the rigid body, their accelerations are related by

$$\boldsymbol{a_g} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{og}})$$

But $\boldsymbol{r_{og}} = \boldsymbol{b_{og}}$, and $\boldsymbol{a_o} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}}$; hence,

$$\ddot{\boldsymbol{R}}_{\boldsymbol{g}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}}) \ ,$$

and the force equation can be written (finally) as

$$\sum \boldsymbol{f_i} = m\,\ddot{\boldsymbol{R}}_{\boldsymbol{g}} \quad . \tag{5.14}$$

In words, Eq.(5.14) states that a rigid body can be treated like a particle, in that the summation of external forces acting on the rigid body equals the mass of the body times the acceleration of the mass center with respect to an inertial coordinate system.

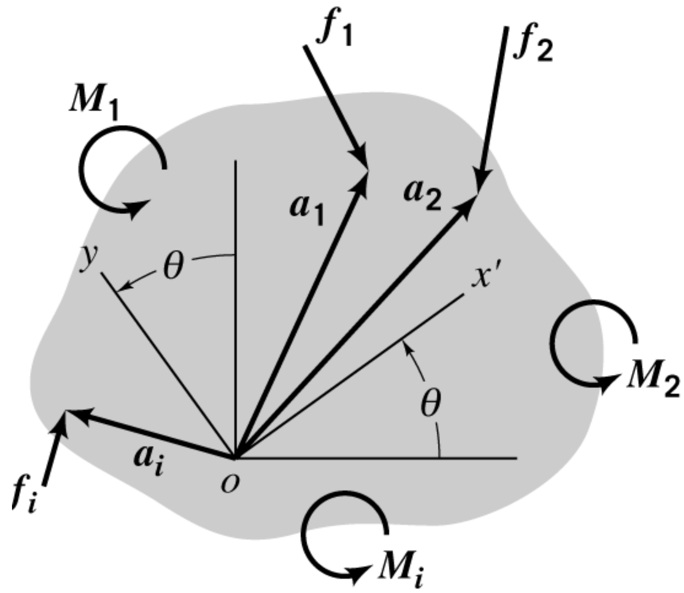
Cartesian component of Force equations:

$$\boxed{\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \qquad (5.15a)}$$

Polar version

$$\boxed{\sum f_{ir} = m(\ddot{r}_g - r_g\dot{\theta}^2) \ , \quad \sum f_{i\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) \ . \qquad (5.15b)}$$

Moment Equation



A rigid body acted on by several external forces $\boldsymbol{f_i}$ acting on the body at points located by the position vectors $\boldsymbol{a_i}$ and moments $\boldsymbol{M}_i$

In figure 5.5, the position vector $\boldsymbol{\rho}$ extends from o to a particle at point P. For moments about o, $\boldsymbol{\rho}$ is the moment arm, and the particle moment equation is

$$\boldsymbol{\rho} \times \boldsymbol{f_P} = \boldsymbol{\rho} \times dm\,\ddot{\boldsymbol{r}} \quad . \tag{5.16}$$

Integrating Eq.(5.16) over the mass of the rigid body yields

$$\sum (\boldsymbol{a_i} \times \boldsymbol{f_i}) + \sum \boldsymbol{M_i} = \boldsymbol{M_o} = \int_V \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, dm \tag{5.17}$$

The vector $\boldsymbol{M_o}$ on the left is the *resultant* external moment acting on the rigid body about point o, the origin of the x, y, z coordinate system.

Kinematics: Substituting from Eq.(5.10) gives

$$\boldsymbol{\rho}\times\ddot{\boldsymbol{r}} = (\boldsymbol{\rho}\times\ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}) + \boldsymbol{\rho}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})] \ .$$

The vector identity,

$$\boldsymbol{A}\times[\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{A})] = \boldsymbol{B}\times[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{A})] \ ,$$

gives

$$\boldsymbol{\rho}\times\ddot{\boldsymbol{r}} = (\boldsymbol{\rho}\times\ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}) + \boldsymbol{\omega}\times[\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})] \ . \tag{5.18}$$

Since $\ddot{\boldsymbol{R}}_o, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are not functions of the variables of integration, substitution from Eq.(5.18) into Eq.(5.17) gives

$$\begin{aligned} \boldsymbol{M}_o = {} & m(\boldsymbol{b}_{og}\times\ddot{\boldsymbol{R}}_o) + \int \boldsymbol{\rho}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho})\, dm \\ & + \boldsymbol{\omega}\times\int [\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})]\, dm \ . \end{aligned} \tag{5.19}$$

with $\boldsymbol{b}_{og}$ defined by Eq.(5.12).

To find component equations from Eq.(5.19)

$$\boldsymbol{b_{og}} \times m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = m\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_{ogx} & b_{ogy} & b_{ogz} \\ \ddot{R}_{ox} & \ddot{R}_{oy} & 0 \end{vmatrix}$$

$$\begin{aligned} & -\boldsymbol{i}\, m b_{ogz}\ddot{R}_{oy} \\ = & +\boldsymbol{j}\, m b_{ogz}\ddot{R}_{ox} \\ & +\boldsymbol{k} m(b_{ogx}\ \ddot{R}_{oy} - b_{ogy}\ \ddot{R}_{ox}) \end{aligned} \tag{5.20}$$

In carrying out the cross product, note that $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}$ is stated in terms of its components in the *x, y, z* coordinate system, versus the customary *X, Y, Z* system.

Defining the vectors in Eq.(5.19) in terms of their components gives

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \ , \ \boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} \ , \ \dot{\boldsymbol{\omega}} = \boldsymbol{k}\ddot{\theta} \ .$$

Hence,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\dot{\theta} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z) = \dot{\theta}(\boldsymbol{j}x - \boldsymbol{i}y)$$

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} = \ddot{\theta}\ (\boldsymbol{j}x - \boldsymbol{i}\ y) \ ,$$

and

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ -y & x & 0 \end{vmatrix} = \boldsymbol{k}\, \dot{\theta}\, (x^2 + y^2)\ . \quad (5.21)$$

Similarly,

$$\boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) = \boldsymbol{k}\, \ddot{\theta}\, (x^2 + y^2) \quad (5.22)$$

Substituting from Eqs.(5.20)-(5.22)into Eq.(5.19) gives the z component equation

$$\begin{aligned} \boldsymbol{k} M_{oz} &= \boldsymbol{k} m (\boldsymbol{b} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z \\ &\quad + \boldsymbol{k} \ddot{\theta} \int_m (x^2 + y^2)\, dm + \boldsymbol{k} \dot{\theta} \times \boldsymbol{k} \dot{\theta} \int_m (x^2 + y^2)\, dm\ . \end{aligned} \quad (5.23)$$

The last expression in this equation is zero because $\boldsymbol{k} \times \boldsymbol{k} = 0$. Since

$$I_o = \int_m (x^2 + y^2)\, dm\ ,$$

the moment Eq.(5.23) can be stated (finally) as

$$\sum M_{oz} = I_o \ddot{\theta} + m (\boldsymbol{b}_{\boldsymbol{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}}\ . \quad (5.24)$$

Summary of governing equations of motion for planar motion of a rigid body

Force-Equation Cartesian Components

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15}$$

Moment Equation

$$\sum M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b_{og}} \times \boldsymbol{\ddot{R}_o})_{\boldsymbol{z}} \ . \tag{5.24}$$

Reduced Forms for the Moment Equation

Moments taken about the mass center. If the point o about which moments are taken coincides with the mass center g, $\boldsymbol{b_{og}} = 0$, and Eq.(5.24) reduces to

$$M_{gz} = I_g \ddot{\theta} \ . \tag{5.25}$$

This equation is *only* correct for moments taken about the mass center of the rigid body.

Moments taken about a fixed point in inertial space. When point o is fixed in the (inertial) X, Y, Z coordinate system, $\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$, and the moment equation is

$$M_{oz} = I_o \ddot{\theta} \quad . \tag{5.26}$$

Fixed-Axis-Rotation Applications of the Force and Moment equations for Planar Motion of a Rigid Body

Rotor in Bearings

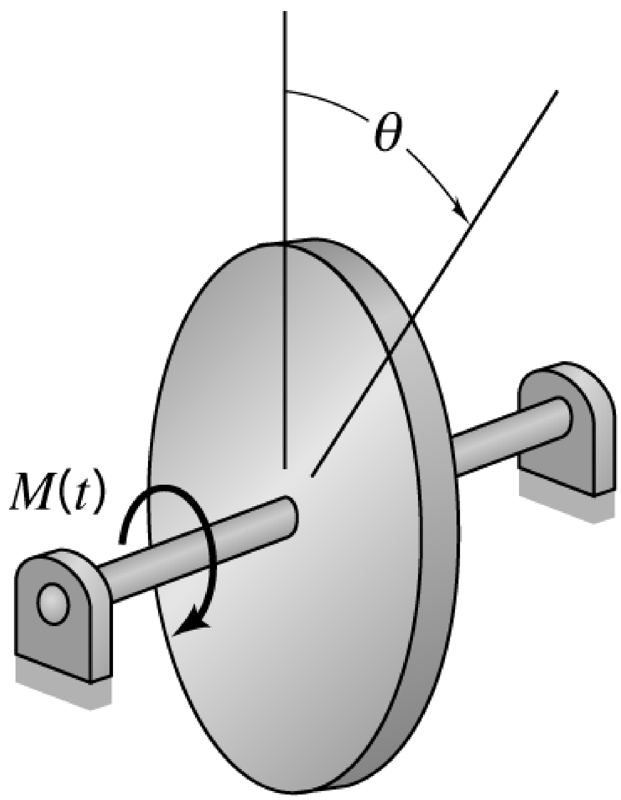


Figure 5.8 A disk mounted on a massless shaft, supported by two frictionless bearings, and acted on by the applied torque $M(t)$.

Derive the differential equation of motion for the rotor. The governing equation of motion for the present system is

$$M_{oz} = I_o\ddot{\theta} \Rightarrow \left(\frac{mr^2}{2}\right)\ddot{\theta} = M(t)\ .$$

The moment $M(t)$ is positive because it is acting in the same direction as $+\theta$. This is basically the same second-order differential equation obtained for a particle of mass m acted on by the force $f(t)$, namely, $m\ddot{x} = f(t)$, where x locates the particle in an inertial coordinate system.

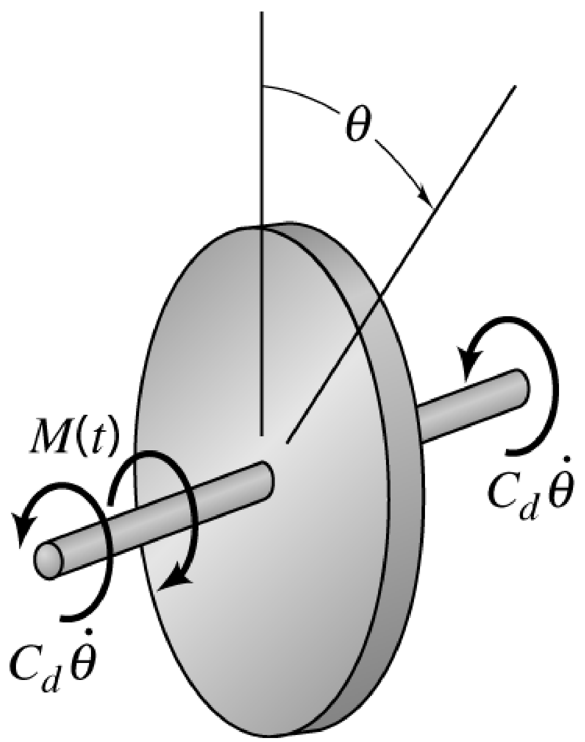


Figure 5.9 Free-body diagram for the rotor of figure 5.8 with a drag torque $C_d\dot{\theta}$ acting at each bearing.

The shaft is rotating in the $+\dot{\theta}$ direction; hence, the drag moment terms have negative signs because they are acting in -θ direction. The differential equation of motion to be obtained from the moment equation is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \frac{mr^2}{2}\ddot{\theta} = M_{oz}(t) - 2C_d\dot{\theta} \ , or$$

$$\frac{mr^2}{2}\ddot{\theta} + 2C_d\dot{\theta} = M_{oz}(t)$$

This equation has the same form as a particle of mass *m* acted on by the force *f(t)* and a linear dashpot with a damping coefficient *c* ; namely, $m\ddot{x} + c\dot{x} = f(t)$.

A-One Degree of Freedom Torsional Vibration Example

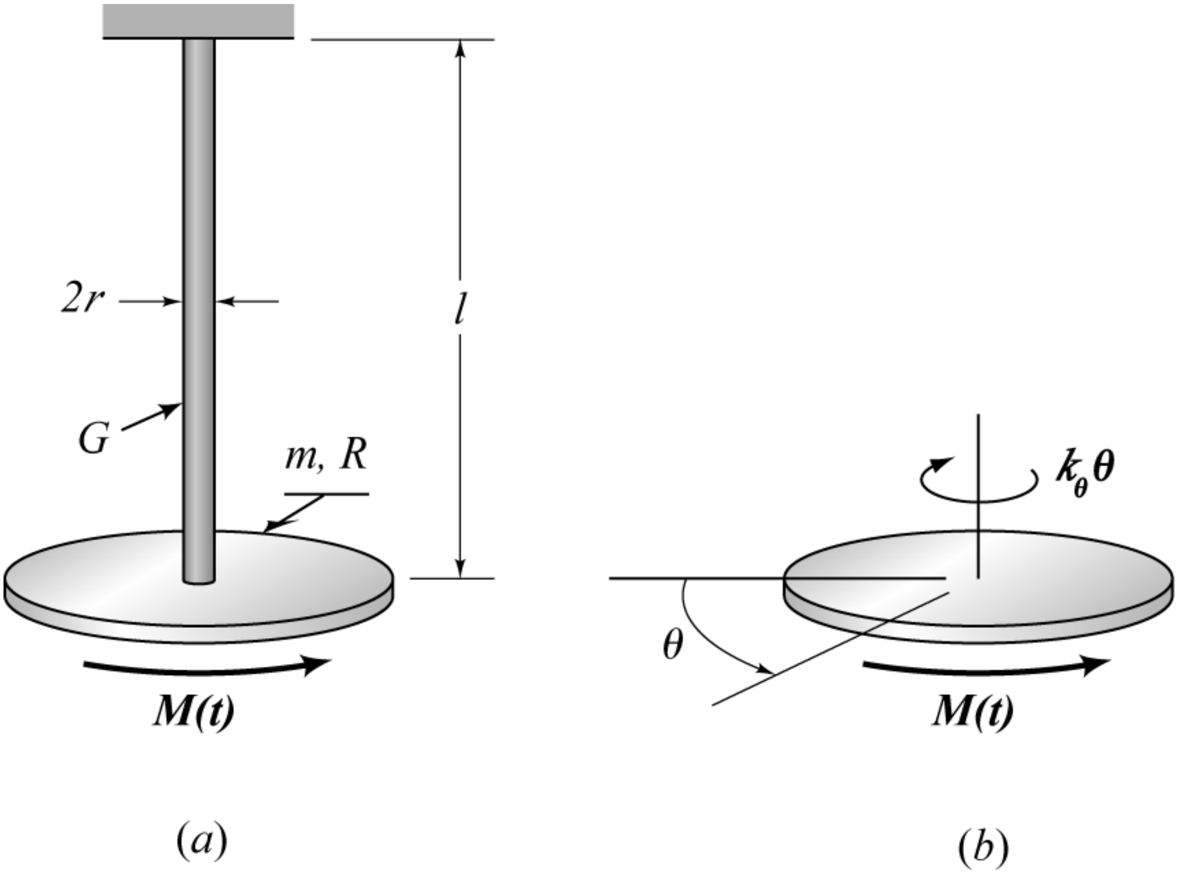


Figure 5.11 (a) Circular disk of mass *m* and radius *R*, supported by a slender rod of length *l*, radius *r*, and shear modulus *G*, (b) Free-body diagram for $\theta > 0$

Twisting the rod about its axis through an angle θ will create a reaction moment, related to θ by

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta .$$

$k_\theta = GJ/l$, where *G* is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. Recall that

the SI units for G is N/m^2; hence, k_θ has the units: *N-m/ radian*, i.e., moment per unit torsional rotation of the rod.

Derive the differential equation of motion for the disk. Applying Eq.(5.26) yields the moment equation

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \quad \frac{mR^2}{2}\ddot{\theta} = M(t) + M_\theta = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta \,,$$

The signs of the moments on the right hand side of this moment equation are positive or negative, depending on whether they are, respectively, in the $+\theta$ or $-\theta$ direction.

The differential equation of motion to be obtained from the moment equation is

$$\frac{mR^2}{2}\ddot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t) \,.$$

This result is analogous to the differential equation of motion for a particle of mass m, acted on by an external force $f(t)$, and supported by a linear spring with stiffness coefficient k ; viz., $m\ddot{x} + kx = f(t)$. For comparison, look at Eq.(3.13). This equation can be rewritten as

$$\ddot{\theta} + \omega_n^2 \theta = \frac{2M(t)}{mR^2} \,,$$

where the undamped natural frequency ω_n is defined by

$$\omega_n = \sqrt{\frac{\pi G r^4}{l m R^2}} \ .$$

Torsional Vibration Example with Viscous Damping

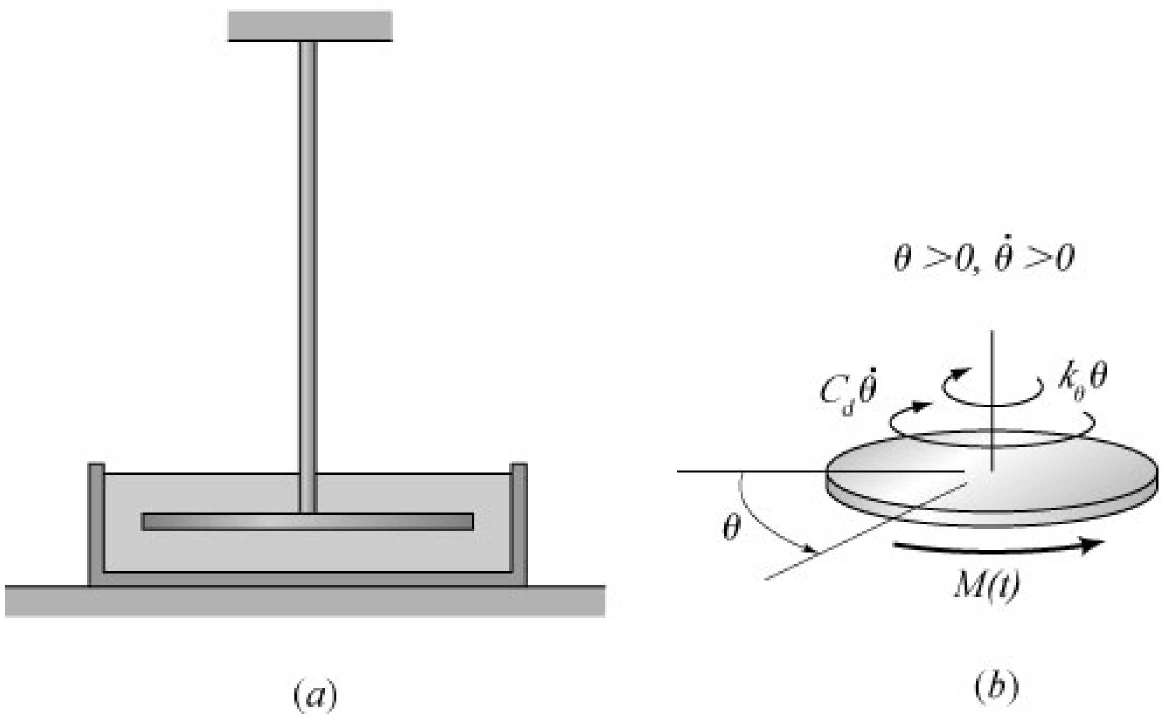


Figure 5.11 (a) The disk of figure 5.10 is now immersed in a viscous fluid, (b) Free-body diagram

Rotation of the disk at a finite rotational velocity $\dot{\theta}$ within the fluid causes the drag moment, $-C_d\dot{\theta}$, on the disk. The negative

sign for the drag term is chosen because it acts in the -θ direction. The complete moment equation is

$$\frac{mR^2}{2}\ddot{\theta} = \Sigma M_z = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta - C_d\dot{\theta},$$

with the governing differential equation

$$\frac{mR^2}{2}\ddot{\theta} + C_d\dot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t). \qquad (5.27)$$

This differential equation has the same form as a particle of mass *m* supported by a parallel arrangement of a spring with stiffness coefficient *k* and a linear damper with damping coefficient *c*; namely, $m\ddot{x} + c\dot{x} + kx = f(t)$.

Eq.(5.27) can be restated as

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2},$$

where ζ is the damping factor, defined by

$$2\zeta\omega_n = \frac{2C_d}{mR^2}, \quad \omega_n = \sqrt{\frac{\pi G r^4}{lmR^2}}.$$

The models developed from figures 5.10 and 5.11 show the same

damped and undamped vibration possibilities for rotational motion of a disk that we reviewed earlier for linear motion of a particle. The same possibilities exist to define damped and undamped natural frequencies, damping factors, etc.

An example involving kinematics between a disk and a particle

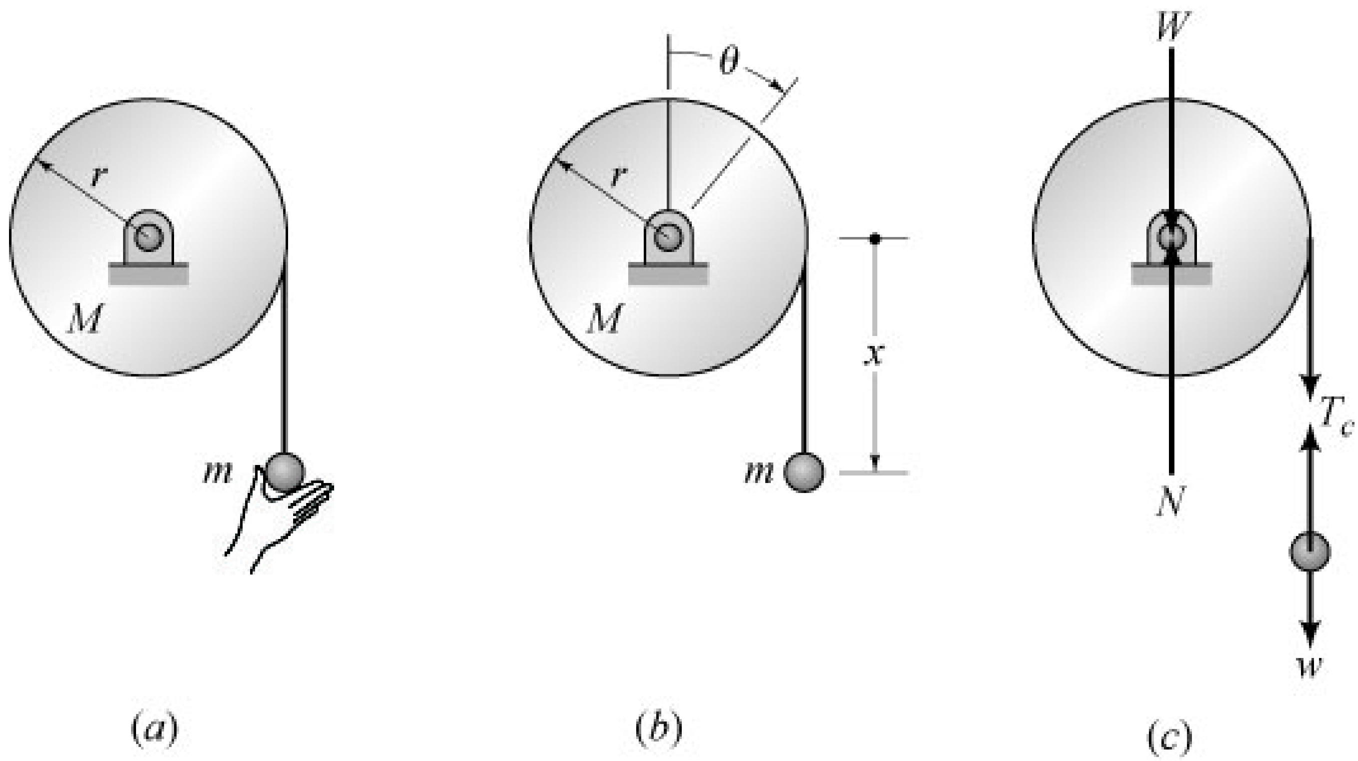


Figure 5.12 (a) Disk of mass M and radius r supported in frictionless bearings and connected to a particle of mass m by a light and inextensible cord, (b) Coordinates, (c) Free-body diagram.

Derive the differential equation of motion for the system.

Kinematics:

$$\delta x = r\delta\theta \ \Rightarrow \ \dot{x} = r\dot{\theta} \ , \ \ddot{x} = r\ddot{\theta}. \tag{5.28}$$

From the free-body diagram, the equation of motion for the disk is obtained by writing a moment equation about its axis of rotation. The equation of motion for mass *m* follows from $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a particle. The governing equations are:

$$\frac{Mr^2}{2}\ddot{\theta} = \Sigma M_{oz} = T_c r \ , \quad m\ddot{x} = \Sigma f = w - T_c, \tag{5.29}$$

where T_c is the tension in the cord. (The mass of the cord has been neglected in stating these equations.) In the first of Eq.(5.29), the moment term $T_c\, r$ is positive because it acts in the $+\theta$ direction. The sign of w is positive in the force equation because it acts in the $+\,x$ *direction* ; T_c has a negative sign because it is directed in the $-x$ direction.

Eqs.(5.29) provides two equations for the three unknowns: $\ddot{x}, \ddot{\theta}$, and T_c. Eliminating the tension T_c from Eqs.(5.29) gives

$$\frac{Mr^2}{2}\ddot{\theta} + rm\ddot{x} = wr. \tag{5.30}$$

Substituting from the last of Eq.(5.28) for $\ddot{x} = r\ddot{\theta}$ gives the final differential equation

$$(\frac{M}{2} + m)r^2\ddot{\theta} = wr.$$

Two driven pulleys connected by a belt

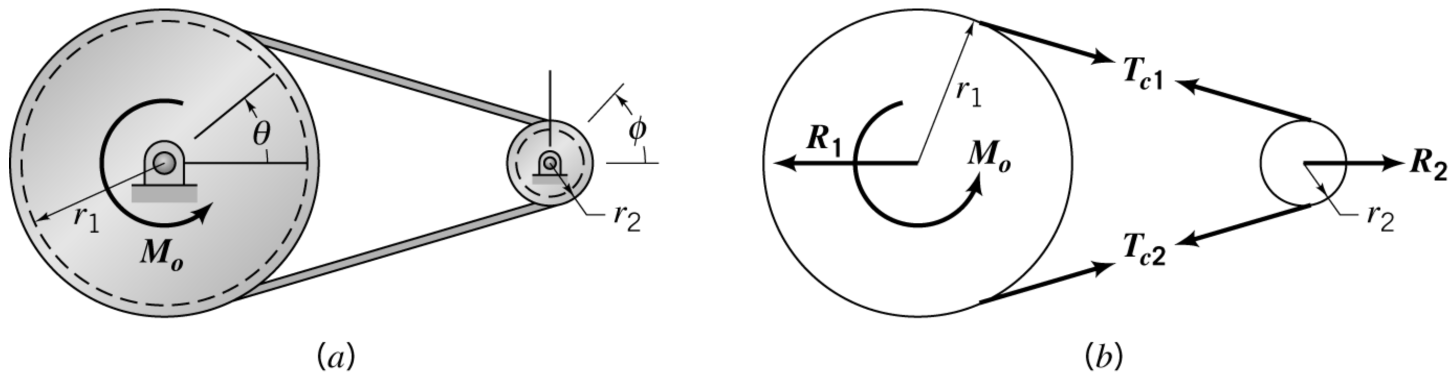


Figure 5.13 (a). Two disks connected by a belt. (b). Free-body diagram.

Figure 5.13A illustrates two pulleys that are connected to each other by a light and inextensible belt. The pulley at the left has mass m_1, radius of gyration k_{1g} about the pulley's axis of rotation, and is acted on by the counterclockwise moment M_o. The pulley at the right has mass m_2 and a radius of gyration k_{2g} about its axis of rotation. (The radius of gyration k_g defines the moment of inertia about the axis of rotation by $I = mk_g^2$.) The belt runs in a groove in pulley 1 with inner radius r_1. The inner radius of the belt groove for pulley 2 is r_2. The angle of rotation for pulleys 1 and 2 are, respectively, θ and φ.

Derive the governing differential equation of motion in terms of θ *and its derivatives.*

From the free-body diagram the fixed-axis rotation moment Eq.(5.26) gives:

$$
\begin{aligned}
I_1\ddot{\theta} &= \Sigma M_{1oz} = M_o(t) + r_1(T_{c2} - T_{c1}) \\
I_2\ddot{\varphi} &= \Sigma M_{2oz} = r_2(T_{c1} - T_{c2}) \ ,
\end{aligned}
\tag{5.31}
$$

where T_{c1} and T_{c2} are the tension components in the upper and lower belt segments. $M_o(t)$ has a positive sign because it is acting in the $+\theta$ direction; $r_1(T_{c1} - T_{c2})$ has a negative sign because it acts in the $-\theta$ direction. Similarly, $r_2(T_{c1} - T_{c2})$ has a positive sign in the second of Eq.(5.31) because it is acting in the $+\varphi$ direction.

The moments of inertia in Eq.(5.31) are defined in terms of their masses and radii of gyrations by

$$I_1 = m_1 k_1^2 \quad ; \ I_2 = m_2 k_2^2 \ .$$

Returning to Eq.(5.31), we can eliminate the tension terms in the two equations, obtaining

$$I_1\ddot{\theta} = M_o - \frac{r_1}{r_2} I_2\ddot{\varphi} \quad \Rightarrow \quad I_1\ddot{\theta} + \frac{r_1}{r_2} I_2\ddot{\varphi} = M_o(t) \ . \tag{5.32}$$

We now have one equation for the two unknowns $\ddot{\theta}$ and $\ddot{\varphi}$, and need an additional kinematic equation relating these two angular acceleration terms. Given that the belt connecting the pulleys is

inextensible (can not stretch) the velocity v of the belt leaving both pulleys must be equal; hence,

$$v = r_1\dot{\theta} = r_2\dot{\varphi} \;\Rightarrow\; r_1\ddot{\theta} = r_2\ddot{\varphi} \; .$$

Substituting this result back into Eq.(5.32) gives the desired final result

$$[I_1 + (\frac{r_1}{r_2})^2 I_2]\ddot{\theta} = I_{eff}\ddot{\theta} = M_o(t) \; .$$

Note that coupling the two pulleys' motion by the belt acts to increase the effective inertia I_{eff} in resisting the applied moment.

Lecture 25. GOVERNING FORCE AND MOMENT EQUATIONS FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a *particle*.

Find: force and moment differential equations of motion for planar motion of a rigid body.

Force Equation

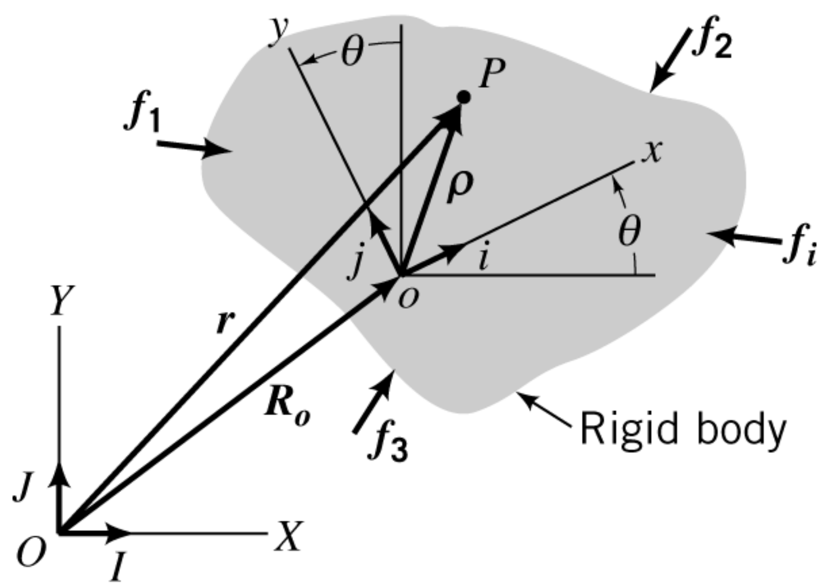


Figure 5.5 Rigid body acted on by external forces. The x,y,z coordinate system is fixed in the rigid body; the X, Y, Z system is an inertial coordinate system.

X, Y, Z inertial coordinate system

x, y, z coordinate system fixed in the rigid body.

θ defines the orientation of the rigid body (and the x, y, z coordinate system) with respect to the X, Y, Z system.

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ is the angular velocity of the rigid body and the x, y, z coordinate system, with respect to X, Y, Z coordinate system.

$\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$ locates the origin of the x, y, z system in the X, Y, Z system.

Point P in the rigid body is located in the X, Y, Z system by $\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y + \boldsymbol{K}r_Z$.

Point is located in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

Hence,

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \quad .$$

Force Equation. Applying Newton's second law to the particle at P gives

$$\boldsymbol{f_P} = dm\,\ddot{\boldsymbol{r}} = dm\,\frac{d^2\boldsymbol{r}}{dt^2}\Big|_{\boldsymbol{X,Y,Z}}\ , \tag{5.8}$$

where:

$\boldsymbol{f_P}$ is the *resultant* force

$\ddot{\boldsymbol{r}}$ is the acceleration of the particle with respect to the inertial $X,\ Y,\ Z$ system.

$dm = \gamma\, dx\, dy\, dz$ where γ is the mass density of the rigid body.

The resultant force at P is

$$\boldsymbol{f_P} = \Sigma \boldsymbol{f}_{\boldsymbol{external}} + \Sigma \boldsymbol{f}_{\boldsymbol{internal}}$$

On the left hand side of Eq.(5.8), integrating over the mass of the body gives

$$\int_m \boldsymbol{f_p}\, dm = \Sigma \int_m \boldsymbol{f}_{\boldsymbol{external}}\, dm + \Sigma \int_m \boldsymbol{f}_{\boldsymbol{internal}}\, dm = \Sigma\ \boldsymbol{f_i} + 0\ \ ;$$

i.e., when integrated over the whole body, the internal forces cancel.

The integral expression of Eq.(5.8) is then

$$\sum \boldsymbol{f_i} = \int_V \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \ddot{\boldsymbol{r}}\, dm \ , \qquad (5.9)$$

For the two points o and P in the rigid body

$$\boldsymbol{a_P} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oP}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oP}}) \ .$$

Since $\boldsymbol{r}$ and $\boldsymbol{R_o}$ locate points P and o, respectively, in the X, Y, Z system, and $\boldsymbol{\rho}$ is the vector from point o to P,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \ . \qquad (5.10)$$

Since $dm = \gamma\, dx\, dy\, dz$, integration extends over the volume of the rigid body.

Since $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are constant with respect to the x, y, z integration variables they can be brought outside the integral sign yielding

$$\sum \boldsymbol{f_i} = m \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \int_m \boldsymbol{\rho}\, dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\, dm) \ . \qquad (5.11)$$

The mass center is located in the x, y, z system by $\boldsymbol{b_{og}}$, defined by

$$m \boldsymbol{b_{og}} = \int_m \boldsymbol{\rho}\, dm = m(\boldsymbol{i} b_{ogx} + \boldsymbol{j} b_{ogy} + \boldsymbol{k} b_{ogz}) \ . \qquad (5.12)$$

Substituting from Eq.(5.12) into Eq.(5.11) gives

$$\sum \boldsymbol{f_i} = m[\ddot{\boldsymbol{R}}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}})] \ . \tag{5.13}$$

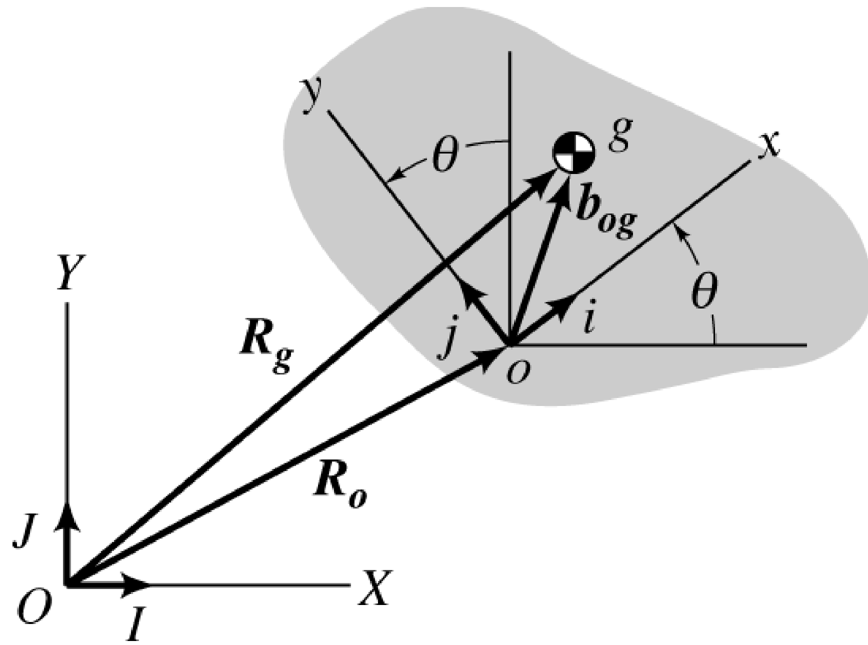


Figure 5.6 A rigid body with a mass center located in the body-fixed *x, y, z* coordinate system by the vector $\boldsymbol{b_{og}}$ and located in the inertial *X, Y, Z* system by $\boldsymbol{R_g}$.

Since *g* and *o* are fixed in the rigid body, their accelerations are related by

$$\boldsymbol{a_g} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{og}})$$

But $\boldsymbol{r_{og}} = \boldsymbol{b_{og}}$, and $\boldsymbol{a_o} = \ddot{\boldsymbol{R}}_o$; hence,

$$\ddot{\boldsymbol{R}}_g = \ddot{\boldsymbol{R}}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}}) \ ,$$

and the force equation can be written (finally) as

$$\sum \boldsymbol{f_i} = m\,\ddot{\boldsymbol{R}}_{\boldsymbol{g}} \ . \tag{5.14}$$

In words, Eq.(5.14) states that a rigid body can be treated like a particle, in that the summation of external forces acting on the rigid body equals the mass of the body times the acceleration of the mass center with respect to an inertial coordinate system.

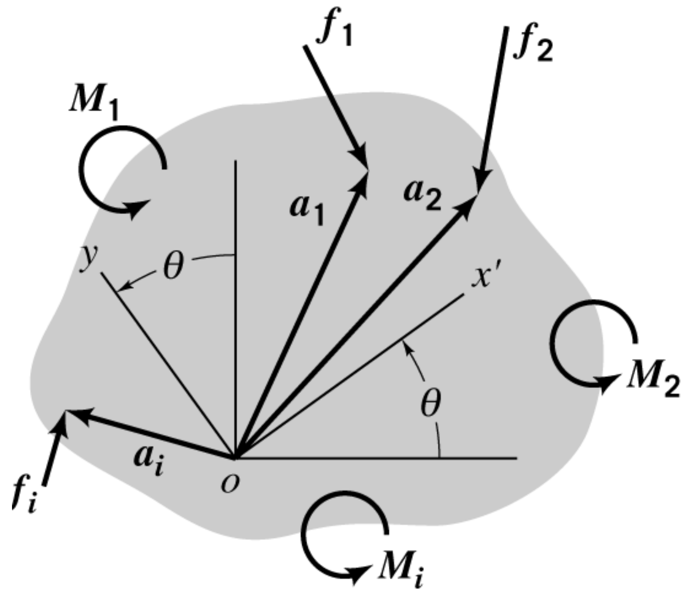
Cartesian component of Force equations:

$$\boxed{\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \qquad (5.15a)}$$

Polar version

$$\boxed{\sum f_{ir} = m(\ddot{r}_g - r_g\dot{\theta}^2) \ , \quad \sum f_{i\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) \ . \qquad (5.15b)}$$

Moment Equation



A rigid body acted on by several external forces $\boldsymbol{f_i}$ acting on the body at points located by the position vectors $\boldsymbol{a_i}$ and moments $\boldsymbol{M}_i$

In figure 5.5, the position vector $\boldsymbol{\rho}$ extends from o to a particle at point P. For moments about o, $\boldsymbol{\rho}$ is the moment arm, and the particle moment equation is

$$\boldsymbol{\rho} \times \boldsymbol{f_P} = \boldsymbol{\rho} \times dm\,\ddot{\boldsymbol{r}} \quad . \tag{5.16}$$

Integrating Eq.(5.16) over the mass of the rigid body yields

$$\sum (\boldsymbol{a_i} \times \boldsymbol{f_i}) + \sum \boldsymbol{M_i} = \boldsymbol{M_o} = \int_V \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, dm \tag{5.17}$$

The vector $\boldsymbol{M_o}$ on the left is the *resultant* external moment acting on the rigid body about point *o,* the origin of the *x, y, z* coordinate system.

Kinematics: Substituting from Eq.(5.10) gives

$$\boldsymbol{\rho}\times\ddot{\boldsymbol{r}}=(\boldsymbol{\rho}\times\ddot{\boldsymbol{R}}_{\boldsymbol{o}})+\boldsymbol{\rho}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho})+\boldsymbol{\rho}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})]\ .$$

The vector identity,

$$\boldsymbol{A}\times[\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{A})]=\boldsymbol{B}\times[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{A})]\ ,$$

gives

$$\boldsymbol{\rho}\times\ddot{\boldsymbol{r}}=(\boldsymbol{\rho}\times\ddot{\boldsymbol{R}}_{\boldsymbol{o}})+\boldsymbol{\rho}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho})+\boldsymbol{\omega}\times[\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})]\ . \qquad (5.18)$$

Since $\ddot{\boldsymbol{R}}_{\boldsymbol{o}},\dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are not functions of the variables of integration, substitution from Eq.(5.18) into Eq.(5.17) gives

$$\begin{aligned}\boldsymbol{M}_{\boldsymbol{o}}&=m(\boldsymbol{b}_{\boldsymbol{og}}\times\ddot{\boldsymbol{R}}_{\boldsymbol{o}})+\int\boldsymbol{\rho}\times(\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho})\,dm\\&+\boldsymbol{\omega}\times\int[\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})]\,dm\ .\end{aligned} \qquad (5.19)$$

with $\boldsymbol{b}_{\boldsymbol{og}}$ defined by Eq.(5.12).

To find component equations from Eq.(5.19)

$$\boldsymbol{b_{og}} \times m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = m\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_{ogx} & b_{ogy} & b_{ogz} \\ \ddot{R}_{ox} & \ddot{R}_{oy} & 0 \end{vmatrix}$$

$$\begin{aligned} &-\boldsymbol{i}\, m b_{ogz}\ddot{R}_{oy} \\ =\ &+\boldsymbol{j}\, m b_{ogz}\ddot{R}_{ox} \\ &+\boldsymbol{k} m(b_{ogx}\ \ddot{R}_{oy} - b_{ogy}\ \ddot{R}_{ox}) \end{aligned} \tag{5.20}$$

In carrying out the cross product, note that $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}$ is stated in terms of its components in the *x, y, z* coordinate system, versus the customary *X, Y, Z* system.

Defining the vectors in Eq.(5.19) in terms of their components gives

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \ , \ \boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} \ , \ \dot{\boldsymbol{\omega}} = \boldsymbol{k}\ddot{\theta} \ .$$

Hence,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\dot{\theta} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z) = \dot{\theta}(\boldsymbol{j}x - \boldsymbol{i}y)$$

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} = \ddot{\theta}\ (\boldsymbol{j}x - \boldsymbol{i}\ y) \ ,$$

and

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ -y & x & 0 \end{vmatrix} = \boldsymbol{k}\, \dot{\theta}\, (x^2 + y^2) \ . \quad (5.21)$$

Similarly,

$$\boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) = \boldsymbol{k}\, \ddot{\theta}\, (x^2 + y^2) \quad (5.22)$$

Substituting from Eqs.(5.20)-(5.22)into Eq.(5.19) gives the z component equation

$$\begin{aligned} \boldsymbol{k} M_{oz} &= \boldsymbol{k} m (\boldsymbol{b} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z \\ &+ \boldsymbol{k}\ddot{\theta} \int_m (x^2 + y^2)\, dm + \boldsymbol{k}\dot{\theta} \times \boldsymbol{k}\dot{\theta} \int_m (x^2 + y^2)\, dm \ . \end{aligned} \quad (5.23)$$

The last expression in this equation is zero because $\boldsymbol{k} \times \boldsymbol{k} = 0$. Since

$$I_o = \int_m (x^2 + y^2)\, dm \ ,$$

the moment Eq.(5.23) can be stated (finally) as

$$\sum M_{oz} = I_o \ddot{\theta} + m (\boldsymbol{b}_{\boldsymbol{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}} \ . \quad (5.24)$$

Summary of governing equations of motion for planar motion of a rigid body

Force-Equation Cartesian Components

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15}$$

Moment Equation

$$\sum M_{oz} = I_o\ddot{\theta} + m(\boldsymbol{b_{og}} \times \boldsymbol{\ddot{R}_o})_z \ . \tag{5.24}$$

Reduced Forms for the Moment Equation

Moments taken about the mass center. If the point *o* about which moments are taken coincides with the mass center *g*, $\boldsymbol{b_{og}} = 0$, and Eq.(5.24) reduces to

$$M_{gz} = I_g\ddot{\theta} \ . \tag{5.25}$$

This equation is *only* correct for moments taken about the mass center of the rigid body.

Moments taken about a fixed point in inertial space. When point o is fixed in the (inertial) X, Y, Z coordinate system, $\ddot{\boldsymbol{R}}_o = 0$, and the moment equation is

$$M_{oz} = I_o \ddot{\theta} \ . \tag{5.26}$$

Fixed-Axis-Rotation Applications of the Force and Moment equations for Planar Motion of a Rigid Body

Rotor in Bearings

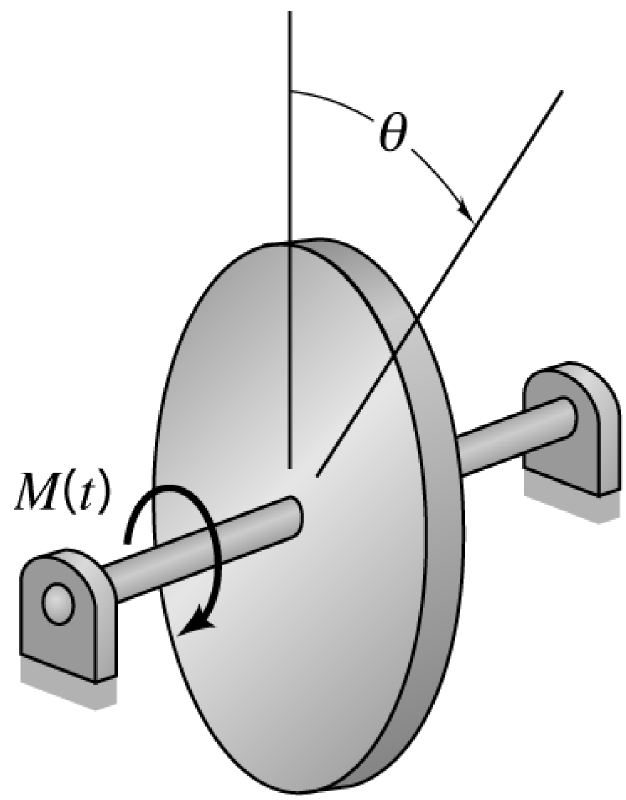


Figure 5.8 A disk mounted on a massless shaft, supported by two frictionless bearings, and acted on by the applied torque $M(t)$.

Derive the differential equation of motion for the rotor. The governing equation of motion for the present system is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \left(\frac{mr^2}{2}\right)\ddot{\theta} = M(t) \ .$$

The moment $M(t)$ is positive because it is acting in the same direction as $+\theta$. This is basically the same second-order differential equation obtained for a particle of mass m acted on by the force $f(t)$, namely, $m\ddot{x} = f(t)$, where x locates the particle in an inertial coordinate system.

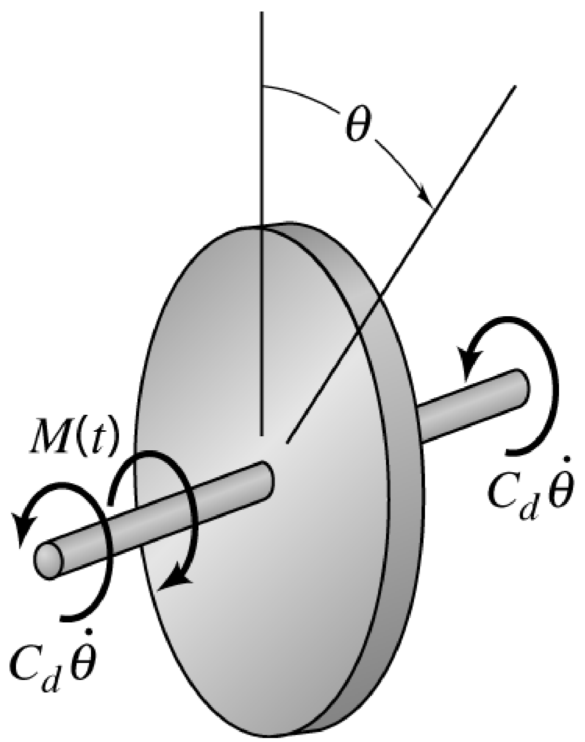


Figure 5.9 Free-body diagram for the rotor of figure 5.8 with a drag torque $C_d\dot{\theta}$ acting at each bearing.

The shaft is rotating in the $+\dot{\theta}$ direction; hence, the drag moment terms have negative signs because they are acting in -θ direction. The differential equation of motion to be obtained from the moment equation is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \quad \frac{mr^2}{2}\ddot{\theta} = M_{oz}(t) - 2C_d\dot{\theta} \;, or$$

$$\frac{mr^2}{2}\ddot{\theta} + 2C_d\dot{\theta} = M_{oz}(t)$$

This equation has the same form as a particle of mass *m* acted on by the force *f(t)* and a linear dashpot with a damping coefficient *c* ; namely, $m\ddot{x} + c\dot{x} = f(t)$.

A-One Degree of Freedom Torsional Vibration Example

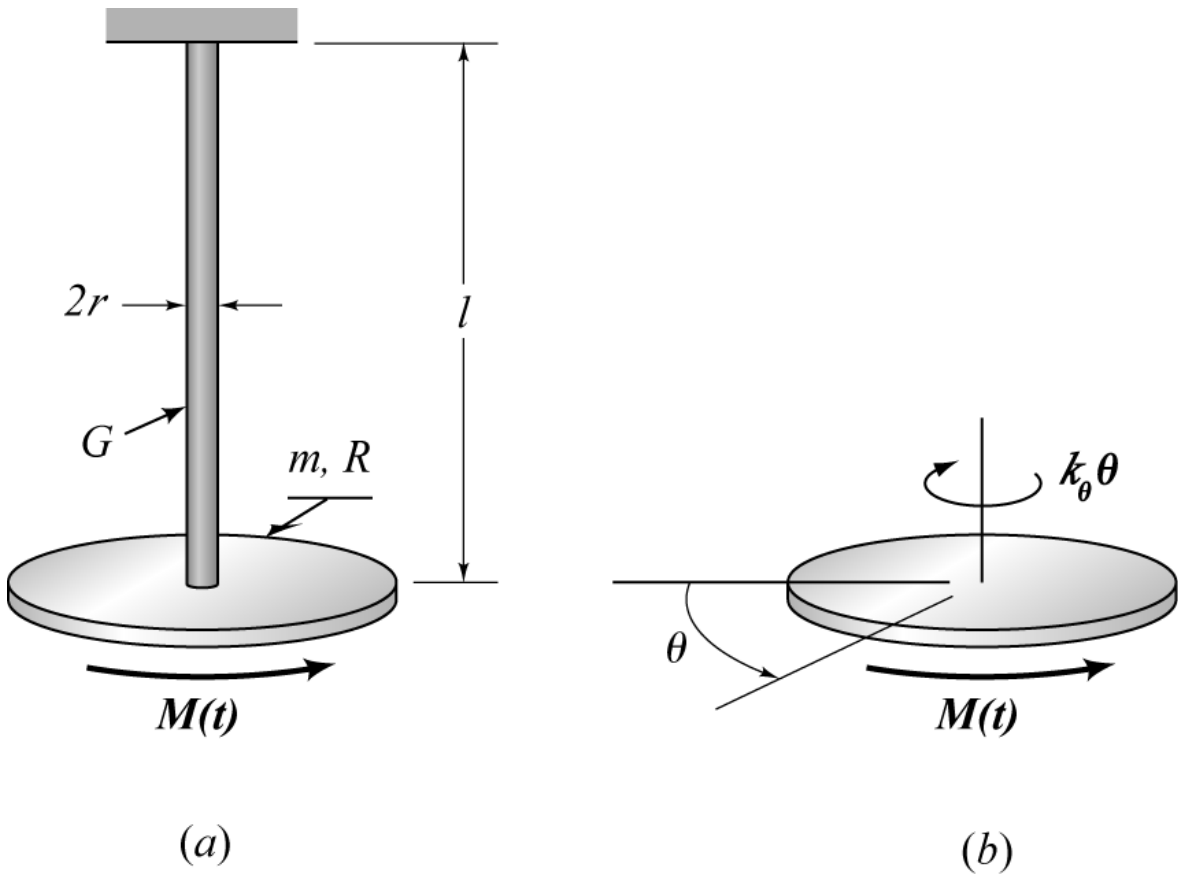


Figure 5.11 (a) Circular disk of mass *m* and radius *R*, supported by a slender rod of length *l*, radius *r*, and shear modulus *G*, (b) Free-body diagram for $\theta > 0$

Twisting the rod about its axis through an angle θ will create a reaction moment, related to θ by

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta.$$

$k_\theta = GJ/l$, where *G* is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. Recall that

the SI units for G is N/m^2; hence, k_θ has the units: *N-m/ radian*, i.e., moment per unit torsional rotation of the rod.

Derive the differential equation of motion for the disk. Applying Eq.(5.26) yields the moment equation

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \quad \frac{mR^2}{2}\ddot{\theta} = M(t) + M_\theta = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta \,,$$

The signs of the moments on the right hand side of this moment equation are positive or negative, depending on whether they are, respectively, in the $+\theta$ or $-\theta$ direction.

The differential equation of motion to be obtained from the moment equation is

$$\frac{mR^2}{2}\ddot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t) \,.$$

This result is analogous to the differential equation of motion for a particle of mass m, acted on by an external force $f(t)$, and supported by a linear spring with stiffness coefficient k ; viz., $m\ddot{x} + kx = f(t)$. For comparison, look at Eq.(3.13). This equation can be rewritten as

$$\ddot{\theta} + \omega_n^2 \theta = \frac{2M(t)}{mR^2} \,,$$

where the undamped natural frequency ω_n is defined by

$$\omega_n = \sqrt{\frac{\pi G r^4}{l m R^2}} \ .$$

Torsional Vibration Example with Viscous Damping

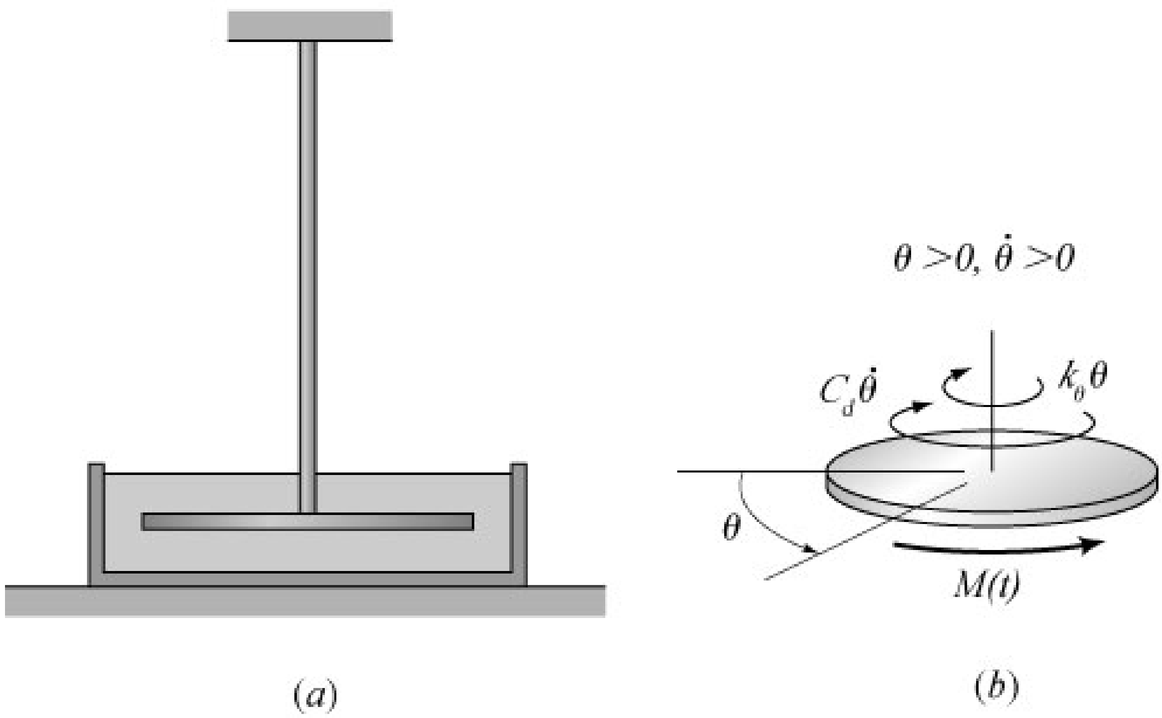


Figure 5.11 (a) The disk of figure 5.10 is now immersed in a viscous fluid, (b) Free-body diagram

Rotation of the disk at a finite rotational velocity $\dot{\theta}$ within the fluid causes the drag moment, $-C_d\dot{\theta}$, on the disk. The negative

sign for the drag term is chosen because it acts in the -θ direction. The complete moment equation is

$$\frac{mR^2}{2}\ddot{\theta} = \Sigma M_z = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta - C_d\dot{\theta},$$

with the governing differential equation

$$\frac{mR^2}{2}\ddot{\theta} + C_d\dot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t). \tag{5.27}$$

This differential equation has the same form as a particle of mass *m* supported by a parallel arrangement of a spring with stiffness coefficient *k* and a linear damper with damping coefficient *c*; namely, $m\ddot{x} + c\dot{x} + kx = f(t)$.

Eq.(5.27) can be restated as

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2},$$

where ζ is the damping factor, defined by

$$2\zeta\omega_n = \frac{2C_d}{mR^2}, \quad \omega_n = \sqrt{\frac{\pi G r^4}{l m R^2}}.$$

The models developed from figures 5.10 and 5.11 show the same

damped and undamped vibration possibilities for rotational motion of a disk that we reviewed earlier for linear motion of a particle. The same possibilities exist to define damped and undamped natural frequencies, damping factors, etc.

An example involving kinematics between a disk and a particle

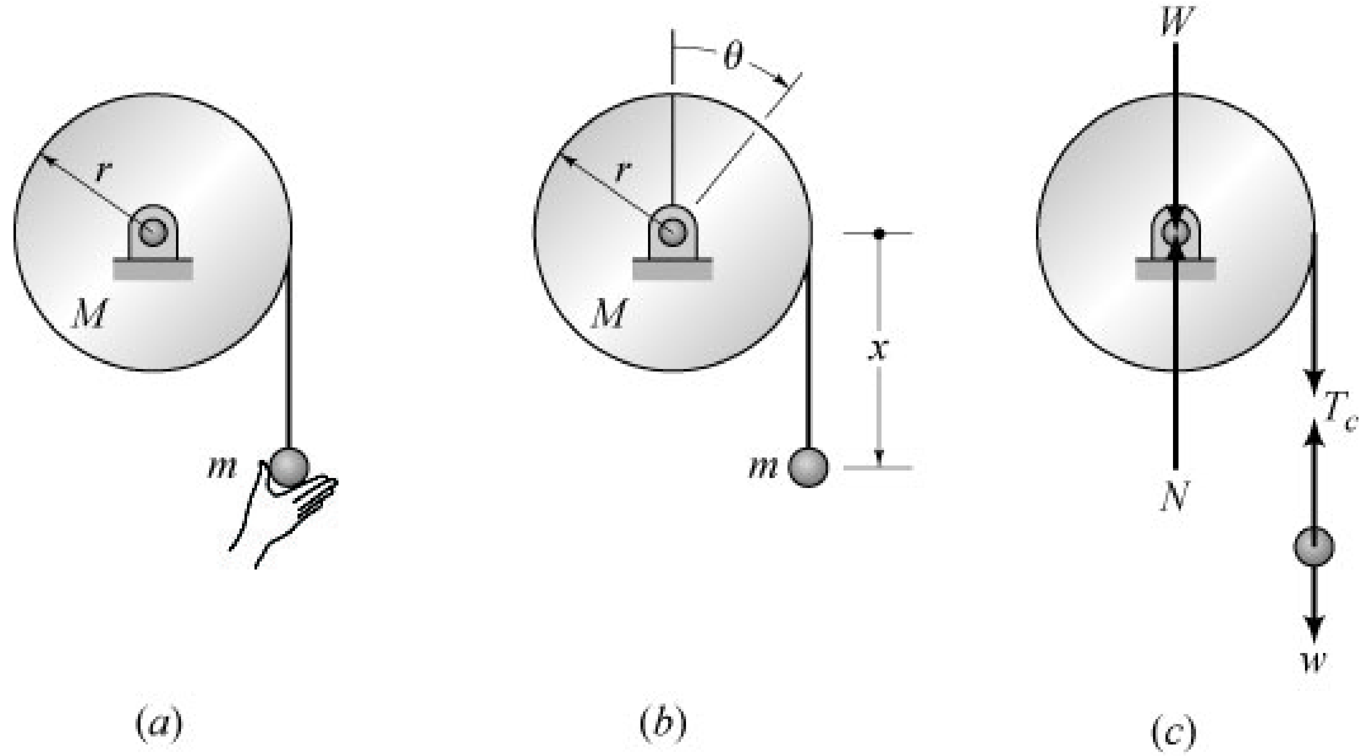


Figure 5.12 (a) Disk of mass *M* and radius *r* supported in frictionless bearings and connected to a particle of mass *m* by a light and inextensible cord, (b) Coordinates, (c) Free-body diagram.

Derive the differential equation of motion for the system.

Kinematics:

$$\delta x = r\delta\theta \;\Rightarrow\; \dot{x} = r\dot{\theta} \;,\; \ddot{x} = r\ddot{\theta}. \tag{5.28}$$

From the free-body diagram, the equation of motion for the disk is obtained by writing a moment equation about its axis of rotation. The equation of motion for mass m follows from $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a particle. The governing equations are:

$$\frac{Mr^2}{2}\ddot{\theta} = \Sigma M_{oz} = T_c r \;,\quad m\ddot{x} = \Sigma f = w - T_c, \tag{5.29}$$

where T_c is the tension in the cord. (The mass of the cord has been neglected in stating these equations.) In the first of Eq.(5.29), the moment term $T_c\, r$ is positive because it acts in the $+\theta$ direction. The sign of w is positive in the force equation because it acts in the $+x$ *direction* ; T_c has a negative sign because it is directed in the $-x$ direction.

Eqs.(5.29) provides two equations for the three unknowns: $\ddot{x}, \ddot{\theta}$, and T_c. Eliminating the tension T_c from Eqs.(5.29) gives

$$\frac{Mr^2}{2}\ddot{\theta} + rm\ddot{x} = wr. \tag{5.30}$$

Substituting from the last of Eq.(5.28) for $\ddot{x} = r\ddot{\theta}$ gives the final differential equation

$$(\frac{M}{2} + m)r^2\ddot{\theta} = wr.$$

Two driven pulleys connected by a belt

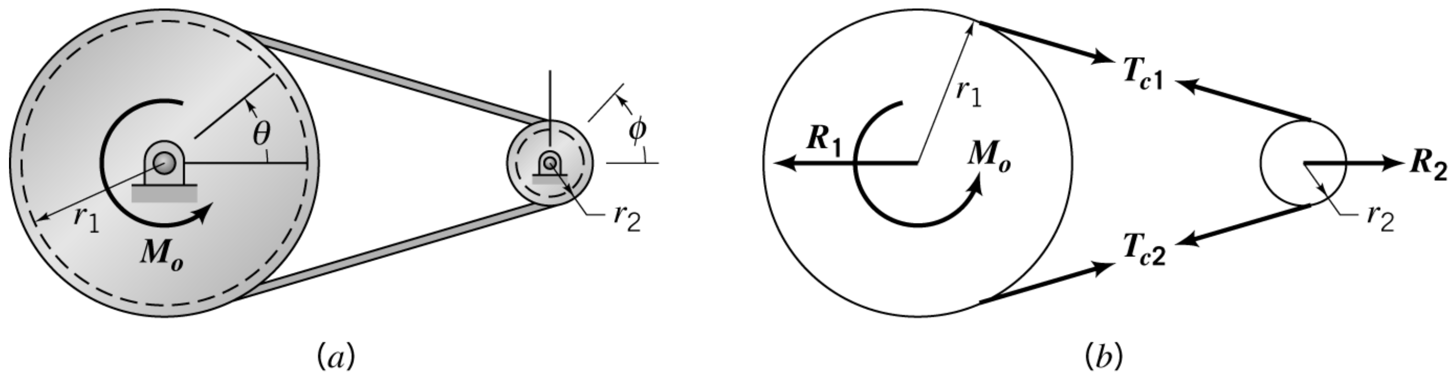


Figure 5.13 (a). Two disks connected by a belt. (b). Free-body diagram.

Figure 5.13A illustrates two pulleys that are connected to each other by a light and inextensible belt. The pulley at the left has mass m_1, radius of gyration k_{1g} about the pulley's axis of rotation, and is acted on by the counterclockwise moment M_o. The pulley at the right has mass m_2 and a radius of gyration k_{2g} about its axis of rotation. (The radius of gyration k_g defines the moment of inertia about the axis of rotation by $I = mk_g^2$.) The belt runs in a groove in pulley 1 with inner radius r_1. The inner radius of the belt groove for pulley 2 is r_2. The angle of rotation for pulleys 1 and 2 are, respectively, θ and φ.

Derive the governing differential equation of motion in terms of θ *and its derivatives.*

From the free-body diagram the fixed-axis rotation moment Eq.(5.26) gives:

$$\begin{aligned} I_1\ddot{\theta} &= \Sigma M_{1oz} = M_o(t) + r_1(T_{c2} - T_{c1}) \\ I_2\ddot{\varphi} &= \Sigma M_{2oz} = r_2(T_{c1} - T_{c2}) \ , \end{aligned} \tag{5.31}$$

where T_{c1} and T_{c2} are the tension components in the upper and lower belt segments. $M_o(t)$ has a positive sign because it is acting in the +θ direction; $r_1(T_{c1} - T_{c2})$ has a negative sign because it acts in the -θ direction. Similarly, $r_2(T_{c1} - T_{c2})$ has a positive sign in the second of Eq.(5.31) because it is acting in the +φ direction.

The moments of inertia in Eq.(5.31) are defined in terms of their masses and radii of gyrations by

$$I_1 = m_1 k_1^2 \quad ; \ I_2 = m_2 k_2^2 \ .$$

Returning to Eq.(5.31), we can eliminate the tension terms in the two equations, obtaining

$$I_1\ddot{\theta} = M_o - \frac{r_1}{r_2} I_2\ddot{\varphi} \quad \Rightarrow \quad I_1\ddot{\theta} + \frac{r_1}{r_2} I_2\ddot{\varphi} = M_o(t) \ . \tag{5.32}$$

We now have one equation for the two unknowns $\ddot{\theta}$ and $\ddot{\varphi}$, and need an additional kinematic equation relating these two angular acceleration terms. Given that the belt connecting the pulleys is

inextensible (can not stretch) the velocity v of the belt leaving both pulleys must be equal; hence,

$$v = r_1 \dot{\theta} = r_2 \dot{\varphi} \ \Rightarrow \ r_1 \ddot{\theta} = r_2 \ddot{\varphi} \ .$$

Substituting this result back into Eq.(5.32) gives the desired final result

$$[I_1 + (\frac{r_1}{r_2})^2 I_2]\ddot{\theta} = I_{eff}\ddot{\theta} = M_o(t) \ .$$

Note that coupling the two pulleys' motion by the belt acts to increase the effective inertia I_{eff} in resisting the applied moment.

Lecture 25. GOVERNING FORCE AND MOMENT EQUATIONS FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a *particle*.

Find: force and moment differential equations of motion for planar motion of a rigid body.

Force Equation

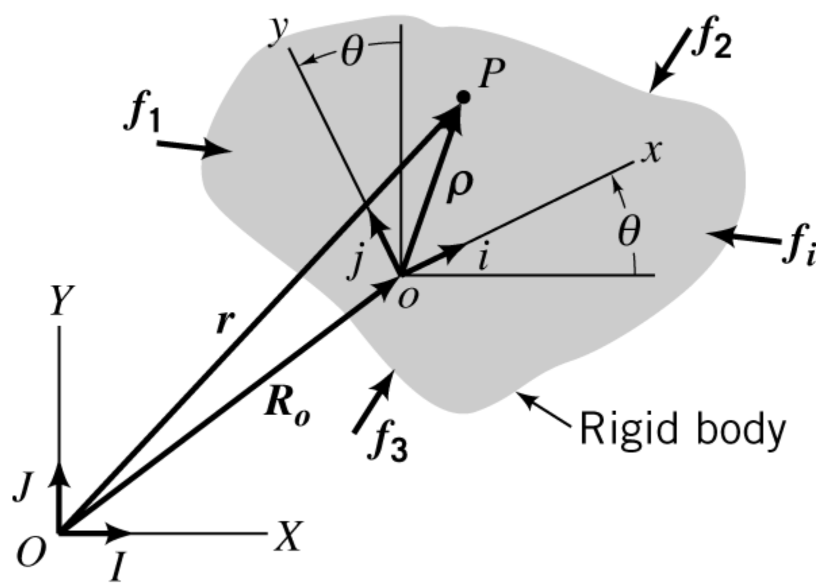


Figure 5.5 Rigid body acted on by external forces. The *x,y,z* coordinate system is fixed in the rigid body; the *X, Y, Z* system is an inertial coordinate system.

X, Y, Z inertial coordinate system

x, y, z coordinate system fixed in the rigid body.

θ defines the orientation of the rigid body (and the x, y, z coordinate system) with respect to the X, Y, Z system.

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ is the angular velocity of the rigid body and the x, y, z coordinate system, with respect to X, Y, Z coordinate system.

$\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$ locates the origin of the x, y, z system in the X, Y, Z system.

Point P in the rigid body is located in the X, Y, Z system by $\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y + \boldsymbol{K}r_Z$.

Point is located in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

Hence,

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \ .$$

Force Equation. Applying Newton's second law to the particle at P gives

$$\boldsymbol{f_P} = dm\,\ddot{\boldsymbol{r}} = dm\,\frac{d^2\boldsymbol{r}}{dt^2}\Big|_{\boldsymbol{X,Y,Z}}\ , \tag{5.8}$$

where:

$\boldsymbol{f_P}$ is the *resultant* force

$\ddot{\boldsymbol{r}}$ is the acceleration of the particle with respect to the inertial *X, Y, Z* system.

$dm = \gamma\, dx\, dy\, dz$ where γ is the mass density of the rigid body.

The resultant force at P is

$$\boldsymbol{f_P} = \Sigma \boldsymbol{f}_{\boldsymbol{external}} + \Sigma \boldsymbol{f}_{\boldsymbol{internal}}$$

On the left hand side of Eq.(5.8), integrating over the mass of the body gives

$$\int_m \boldsymbol{f_p}\, dm = \Sigma \int_m \boldsymbol{f}_{\boldsymbol{external}}\, dm + \Sigma \int_m \boldsymbol{f}_{\boldsymbol{internal}}\, dm = \Sigma\ \boldsymbol{f_i} + 0\ \ ;$$

i.e., when integrated over the whole body, the internal forces cancel.

The integral expression of Eq.(5.8) is then

$$\sum \boldsymbol{f_i} = \int_V \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \ddot{\boldsymbol{r}}\, dm \ , \tag{5.9}$$

For the two points o and P in the rigid body

$$\boldsymbol{a_P} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oP}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oP}}) \ .$$

Since $\boldsymbol{r}$ and $\boldsymbol{R_o}$ locate points P and o, respectively, in the X, Y, Z system, and $\boldsymbol{\rho}$ is the vector from point o to P,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \ . \tag{5.10}$$

Since $dm = \gamma\, dx\, dy\, dz$, integration extends over the volume of the rigid body.

Since $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are constant with respect to the x, y, z integration variables they can be brought outside the integral sign yielding

$$\sum \boldsymbol{f_i} = m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \int_m \boldsymbol{\rho}\, dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\, dm) \ . \tag{5.11}$$

The mass center is located in the x, y, z system by $\boldsymbol{b_{og}}$, defined by

$$m\boldsymbol{b_{og}} = \int_m \boldsymbol{\rho}\, dm = m(\boldsymbol{i} b_{ogx} + \boldsymbol{j} b_{ogy} + \boldsymbol{k} b_{ogz}) \ . \tag{5.12}$$

Substituting from Eq.(5.12) into Eq.(5.11) gives

$$\sum \boldsymbol{f_i} = m[\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}})] \ . \tag{5.13}$$

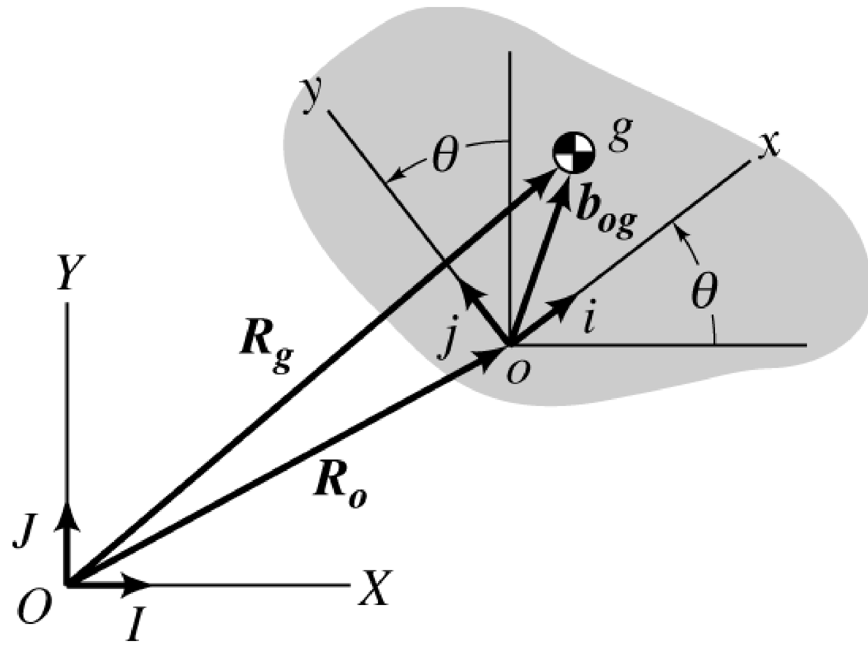


Figure 5.6 A rigid body with a mass center located in the body-fixed *x, y, z* coordinate system by the vector $\boldsymbol{b_{og}}$ and located in the inertial *X, Y, Z* system by $\boldsymbol{R_g}$.

Since *g* and *o* are fixed in the rigid body, their accelerations are related by

$$\boldsymbol{a_g} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{og}})$$

But $\boldsymbol{r_{og}} = \boldsymbol{b_{og}}$, and $\boldsymbol{a_o} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}}$; hence,

$$\ddot{\boldsymbol{R}}_{\boldsymbol{g}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}}) \ ,$$

and the force equation can be written (finally) as

$$\sum \boldsymbol{f_i} = m\,\ddot{\boldsymbol{R}}_{\boldsymbol{g}} \quad . \tag{5.14}$$

In words, Eq.(5.14) states that a rigid body can be treated like a particle, in that the summation of external forces acting on the rigid body equals the mass of the body times the acceleration of the mass center with respect to an inertial coordinate system.

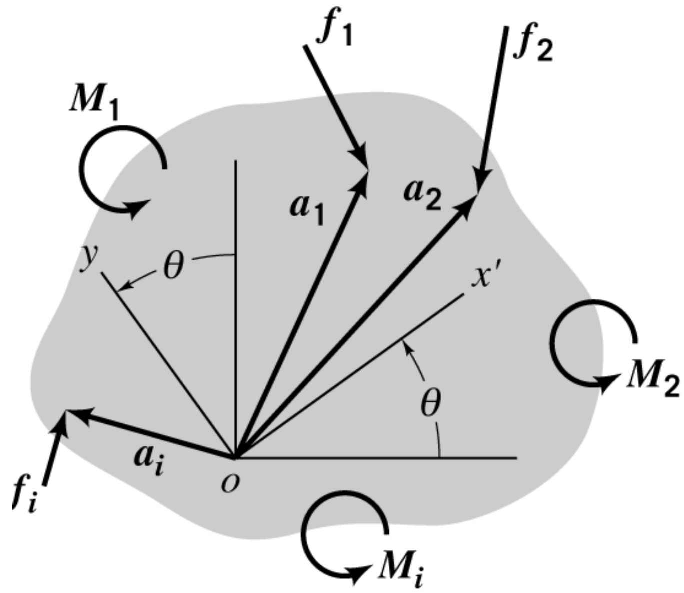
Cartesian component of Force equations:

$$\boxed{\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \qquad (5.15a)}$$

Polar version

$$\boxed{\sum f_{ir} = m(\ddot{r}_g - r_g\dot{\theta}^2) \ , \quad \sum f_{i\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) \ . \qquad (5.15b)}$$

Moment Equation



A rigid body acted on by several external forces $\boldsymbol{f_i}$ acting on the body at points located by the position vectors $\boldsymbol{a_i}$ and moments $\boldsymbol{M}_i$

In figure 5.5, the position vector $\boldsymbol{\rho}$ extends from o to a particle at point P. For moments about o, $\boldsymbol{\rho}$ is the moment arm, and the particle moment equation is

$$\boldsymbol{\rho} \times \boldsymbol{f_P} = \boldsymbol{\rho} \times dm\,\ddot{\boldsymbol{r}} \quad . \tag{5.16}$$

Integrating Eq.(5.16) over the mass of the rigid body yields

$$\sum (\boldsymbol{a_i} \times \boldsymbol{f_i}) + \sum \boldsymbol{M_i} = \boldsymbol{M_o} = \int_V \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, dm \tag{5.17}$$

The vector $\boldsymbol{M_o}$ on the left is the *resultant* external moment acting on the rigid body about point o, the origin of the x, y, z coordinate system.

Kinematics: Substituting from Eq.(5.10) gives

$$\boldsymbol{\rho} \times \ddot{\boldsymbol{r}} = (\boldsymbol{\rho} \times \ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\rho} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] \ .$$

The vector identity,

$$\boldsymbol{A} \times [\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{A})] = \boldsymbol{B} \times [\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{A})] \ ,$$

gives

$$\boldsymbol{\rho} \times \ddot{\boldsymbol{r}} = (\boldsymbol{\rho} \times \ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] \ . \tag{5.18}$$

Since $\ddot{\boldsymbol{R}}_o, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are not functions of the variables of integration, substitution from Eq.(5.18) into Eq.(5.17) gives

$$\begin{aligned} \boldsymbol{M}_o = m(\boldsymbol{b}_{og} \times \ddot{\boldsymbol{R}}_o) &+ \int \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho})\, dm \\ &+ \boldsymbol{\omega} \times \int [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]\, dm \ . \end{aligned} \tag{5.19}$$

with $\boldsymbol{b}_{og}$ defined by Eq.(5.12).

To find component equations from Eq.(5.19)

$$\boldsymbol{b_{og}} \times m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = m\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_{ogx} & b_{ogy} & b_{ogz} \\ \ddot{R}_{ox} & \ddot{R}_{oy} & 0 \end{vmatrix}$$

$$\begin{aligned} & -\boldsymbol{i}\, m b_{ogz}\ddot{R}_{oy} \\ = & +\boldsymbol{j}\, m b_{ogz}\ddot{R}_{ox} \\ & +\boldsymbol{k} m(b_{ogx}\ \ddot{R}_{oy} - b_{ogy}\ \ddot{R}_{ox}) \end{aligned} \tag{5.20}$$

In carrying out the cross product, note that $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}$ is stated in terms of its components in the *x, y, z* coordinate system, versus the customary *X, Y, Z* system.

Defining the vectors in Eq.(5.19) in terms of their components gives

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \ , \ \boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} \ , \ \dot{\boldsymbol{\omega}} = \boldsymbol{k}\ddot{\theta} \ .$$

Hence,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\dot{\theta} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z) = \dot{\theta}(\boldsymbol{j}x - \boldsymbol{i}y)$$

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} = \ddot{\theta}\ (\boldsymbol{j}x - \boldsymbol{i}\ y) \ ,$$

and

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ -y & x & 0 \end{vmatrix} = \boldsymbol{k}\, \dot{\theta}\, (x^2 + y^2) \; . \tag{5.21}$$

Similarly,

$$\boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) = \boldsymbol{k}\, \ddot{\theta}\, (x^2 + y^2) \tag{5.22}$$

Substituting from Eqs.(5.20)-(5.22)into Eq.(5.19) gives the z component equation

$$\begin{aligned} \boldsymbol{k} M_{oz} &= \boldsymbol{k} m (\boldsymbol{b} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z \\ &\quad + \boldsymbol{k} \ddot{\theta} \int_m (x^2 + y^2)\, dm + \boldsymbol{k} \dot{\theta} \times \boldsymbol{k} \dot{\theta} \int_m (x^2 + y^2)\, dm \; . \end{aligned} \tag{5.23}$$

The last expression in this equation is zero because $\boldsymbol{k} \times \boldsymbol{k} = 0$. Since

$$I_o = \int_m (x^2 + y^2)\, dm \; ,$$

the moment Eq.(5.23) can be stated (finally) as

$$\sum M_{oz} = I_o \ddot{\theta} + m (\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}} \; . \tag{5.24}$$

Summary of governing equations of motion for planar motion of a rigid body

Force-Equation Cartesian Components

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15}$$

Moment Equation

$$\sum M_{oz} = I_o\ddot{\theta} + m(\boldsymbol{b_{og}} \times \boldsymbol{\ddot{R}_o})_{\boldsymbol{z}} \ . \tag{5.24}$$

Reduced Forms for the Moment Equation

Moments taken about the mass center. If the point *o* about which moments are taken coincides with the mass center *g*, $\boldsymbol{b_{og}} = 0$, and Eq.(5.24) reduces to

$$M_{gz} = I_g\ddot{\theta} \ . \tag{5.25}$$

This equation is *only* correct for moments taken about the mass center of the rigid body.

Moments taken about a fixed point in inertial space. When point o is fixed in the (inertial) X, Y, Z coordinate system, $\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$, and the moment equation is

$$M_{oz} = I_o \ddot{\theta} \quad . \tag{5.26}$$

Fixed-Axis-Rotation Applications of the Force and Moment equations for Planar Motion of a Rigid Body

Rotor in Bearings

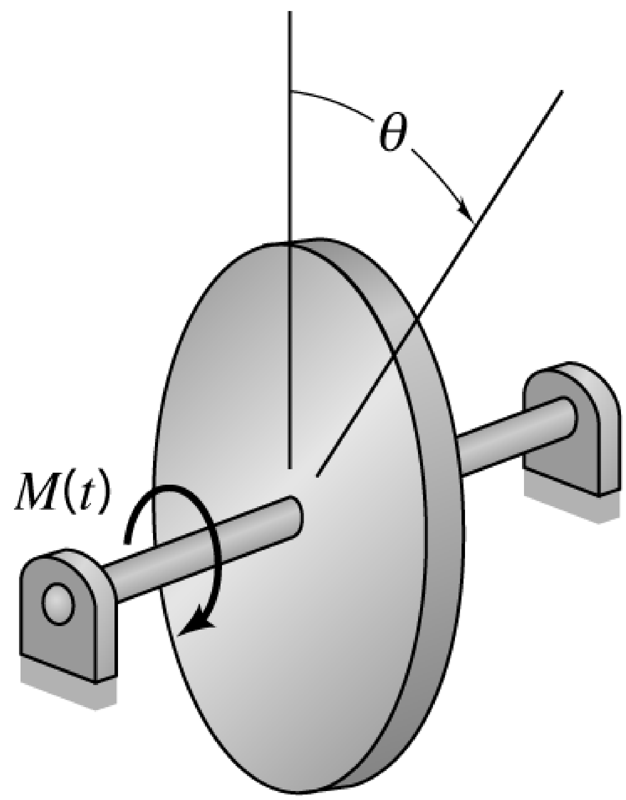


Figure 5.8 A disk mounted on a massless shaft, supported by two frictionless bearings, and acted on by the applied torque $M(t)$.

Derive the differential equation of motion for the rotor. The governing equation of motion for the present system is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \left(\frac{mr^2}{2}\right)\ddot{\theta} = M(t) \ .$$

The moment $M(t)$ is positive because it is acting in the same direction as $+\theta$. This is basically the same second-order differential equation obtained for a particle of mass m acted on by the force $f(t)$, namely, $m\ddot{x} = f(t)$, where x locates the particle in an inertial coordinate system.

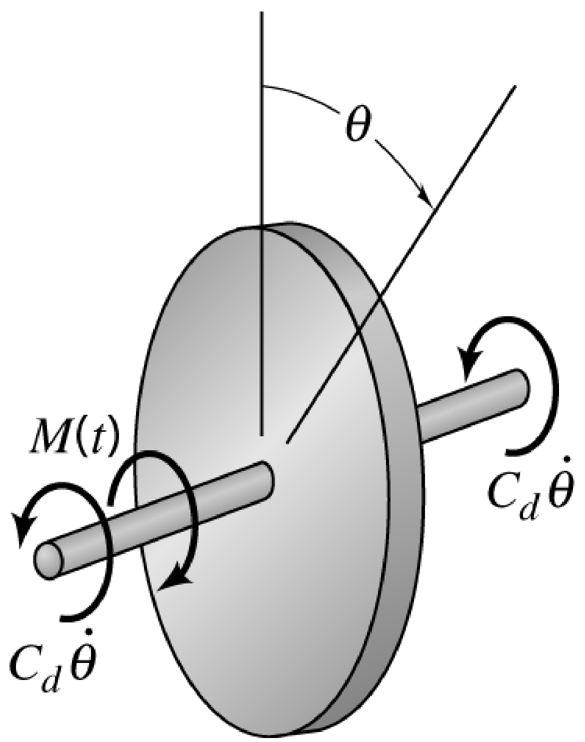


Figure 5.9 Free-body diagram for the rotor of figure 5.8 with a drag torque $C_d\dot{\theta}$ acting at each bearing.

The shaft is rotating in the $+\dot{\theta}$ direction; hence, the drag moment terms have negative signs because they are acting in -θ direction. The differential equation of motion to be obtained from the moment equation is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \quad \frac{mr^2}{2}\ddot{\theta} = M_{oz}(t) - 2C_d\dot{\theta} \ \ , or$$

$$\frac{mr^2}{2}\ddot{\theta} + 2C_d\dot{\theta} = M_{oz}(t)$$

This equation has the same form as a particle of mass *m* acted on by the force *f(t)* and a linear dashpot with a damping coefficient *c* ; namely, $m\ddot{x} + c\dot{x} = f(t)$.

A-One Degree of Freedom Torsional Vibration Example

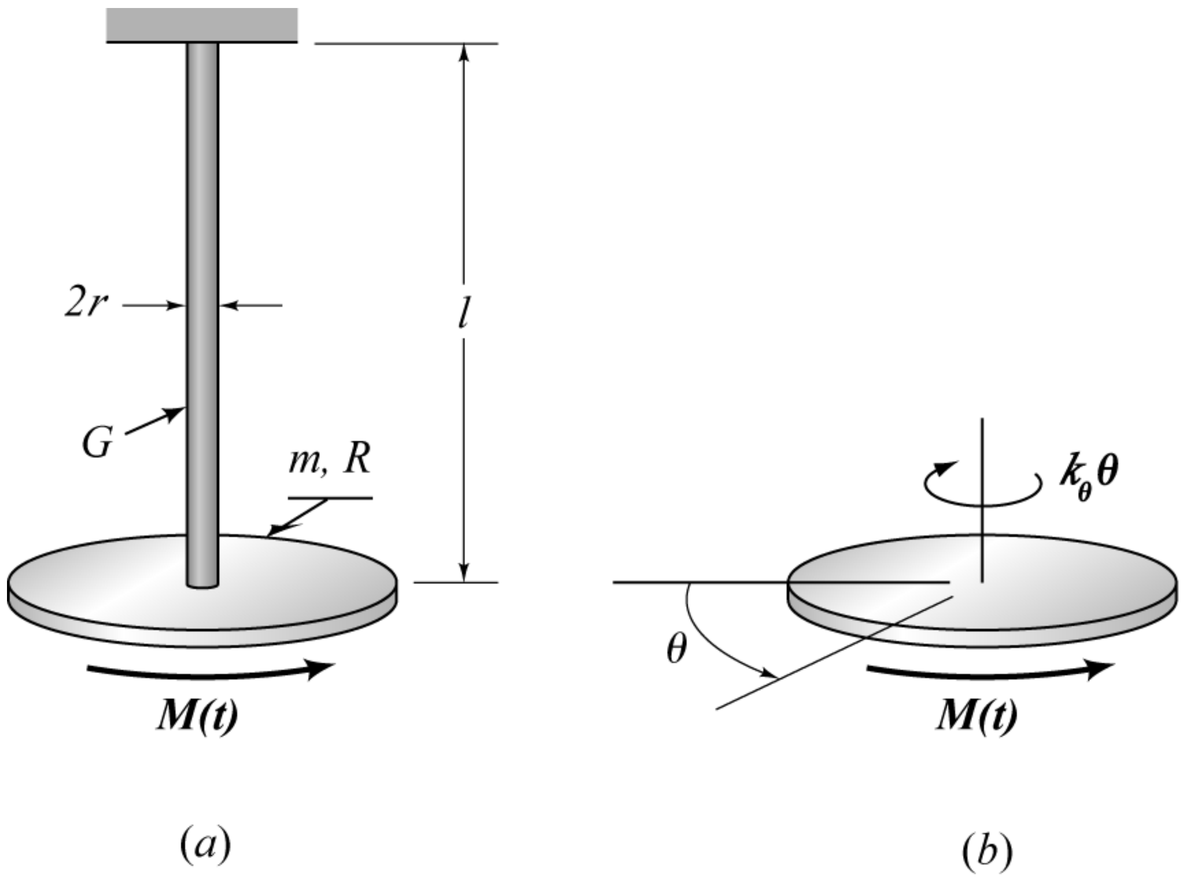


Figure 5.11 (a) Circular disk of mass *m* and radius *R*, supported by a slender rod of length *l*, radius *r*, and shear modulus *G*, (b) Free-body diagram for $\theta > 0$

Twisting the rod about its axis through an angle θ will create a reaction moment, related to θ by

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta.$$

$k_\theta = GJ/l$, where G is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. Recall that

the SI units for G is N/m^2; hence, k_θ has the units: *N-m/ radian*, i.e., moment per unit torsional rotation of the rod.

Derive the differential equation of motion for the disk. Applying Eq.(5.26) yields the moment equation

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \quad \frac{mR^2}{2} \ddot{\theta} = M(t) + M_\theta = M(t) - \frac{G}{l} \frac{\pi r^4}{2} \theta \, ,$$

The signs of the moments on the right hand side of this moment equation are positive or negative, depending on whether they are, respectively, in the $+\theta$ or $-\theta$ direction.

The differential equation of motion to be obtained from the moment equation is

$$\frac{mR^2}{2} \ddot{\theta} + \frac{\pi G r^4}{2l} \theta = M(t) \, .$$

This result is analogous to the differential equation of motion for a particle of mass m, acted on by an external force $f(t)$, and supported by a linear spring with stiffness coefficient k ; viz., $m\ddot{x} + kx = f(t)$. For comparison, look at Eq.(3.13). This equation can be rewritten as

$$\ddot{\theta} + \omega_n^2 \theta = \frac{2M(t)}{mR^2} \, ,$$

where the undamped natural frequency ω_n is defined by

$$\omega_n = \sqrt{\frac{\pi G r^4}{l m R^2}} \ .$$

Torsional Vibration Example with Viscous Damping

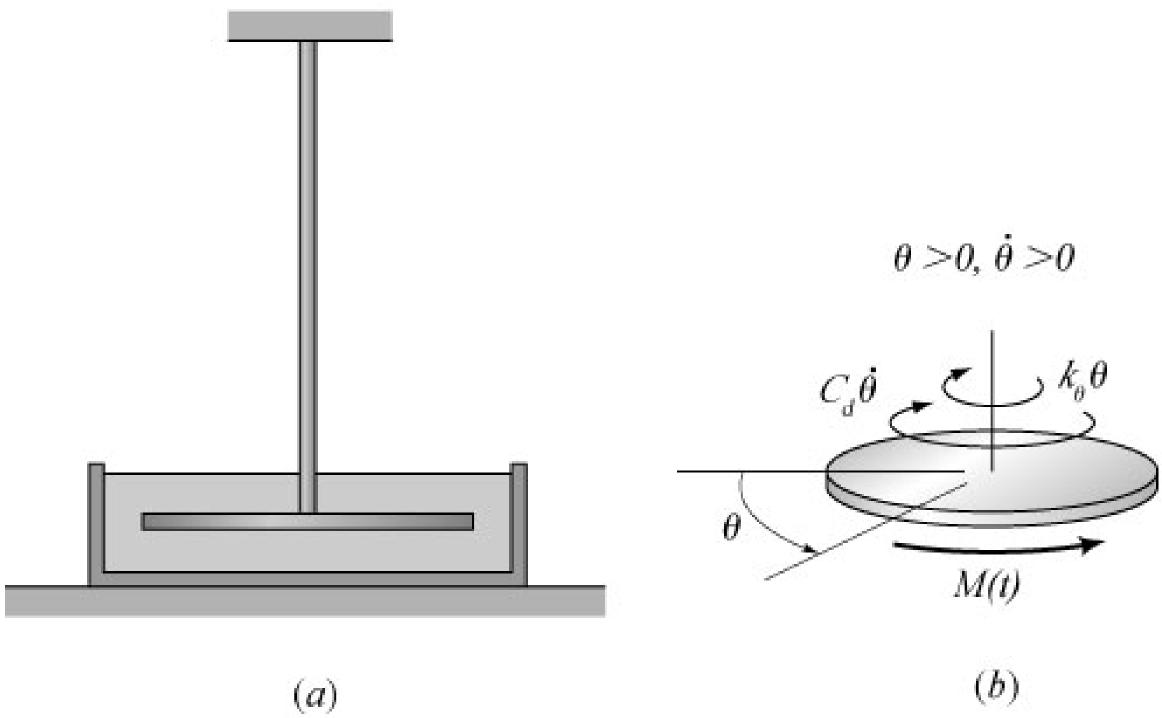


Figure 5.11 (a) The disk of figure 5.10 is now immersed in a viscous fluid, (b) Free-body diagram

Rotation of the disk at a finite rotational velocity $\dot{\theta}$ within the fluid causes the drag moment, $-C_d\dot{\theta}$, on the disk. The negative

sign for the drag term is chosen because it acts in the $-\theta$ direction. The complete moment equation is

$$\frac{mR^2}{2}\ddot{\theta} = \Sigma M_z = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta - C_d\dot{\theta},$$

with the governing differential equation

$$\frac{mR^2}{2}\ddot{\theta} + C_d\dot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t). \qquad (5.27)$$

This differential equation has the same form as a particle of mass *m* supported by a parallel arrangement of a spring with stiffness coefficient *k* and a linear damper with damping coefficient *c*; namely, $m\ddot{x} + c\dot{x} + kx = f(t)$.

Eq.(5.27) can be restated as

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2},$$

where ζ is the damping factor, defined by

$$2\zeta\omega_n = \frac{2C_d}{mR^2}, \quad \omega_n = \sqrt{\frac{\pi G r^4}{lmR^2}}.$$

The models developed from figures 5.10 and 5.11 show the same

damped and undamped vibration possibilities for rotational motion of a disk that we reviewed earlier for linear motion of a particle. The same possibilities exist to define damped and undamped natural frequencies, damping factors, etc.

An example involving kinematics between a disk and a particle

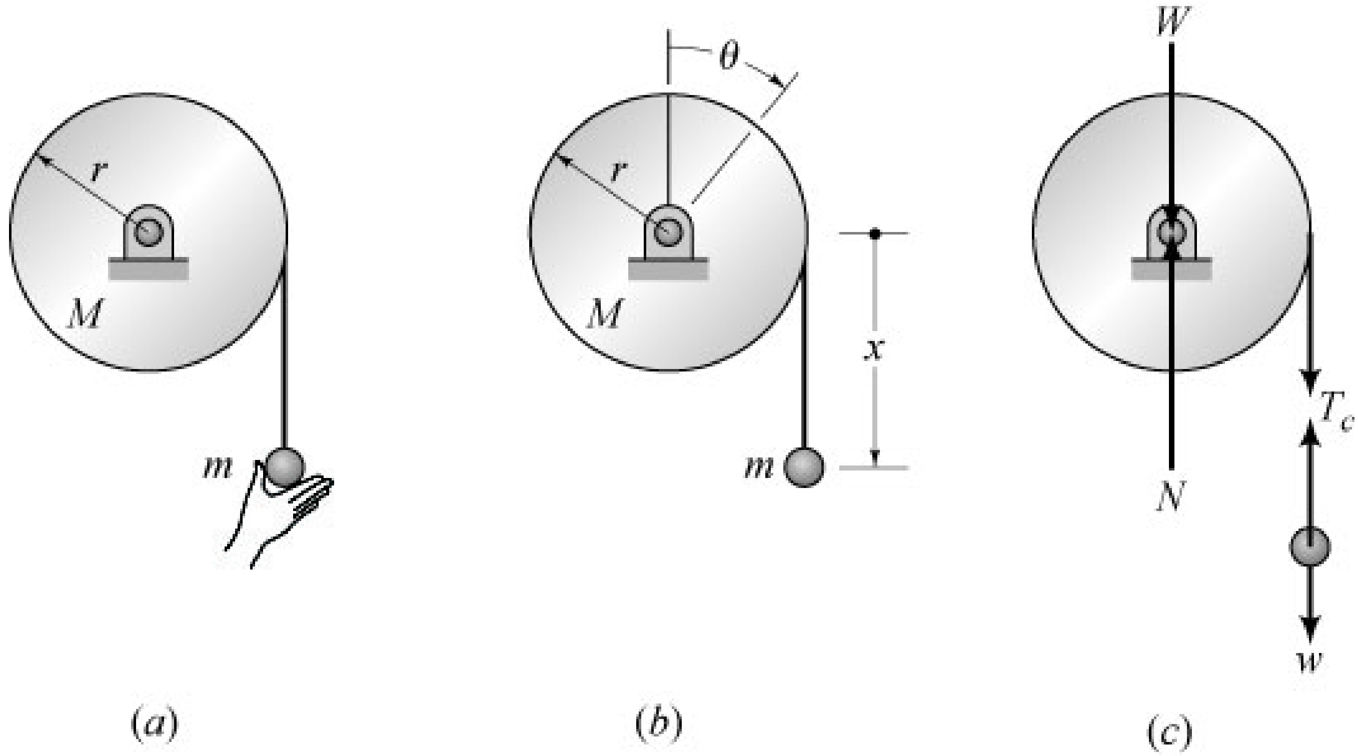


Figure 5.12 (a) Disk of mass *M* and radius *r* supported in frictionless bearings and connected to a particle of mass *m* by a light and inextensible cord, (b) Coordinates, (c) Free-body diagram.

Derive the differential equation of motion for the system.

Kinematics:

$$\delta x = r\delta\theta \;\Rightarrow\; \dot{x} = r\dot{\theta} \;,\; \ddot{x} = r\ddot{\theta}. \tag{5.28}$$

From the free-body diagram, the equation of motion for the disk is obtained by writing a moment equation about its axis of rotation. The equation of motion for mass m follows from $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a particle. The governing equations are:

$$\frac{Mr^2}{2}\ddot{\theta} = \Sigma M_{oz} = T_c r \;, \quad m\ddot{x} = \Sigma f = w - T_c, \tag{5.29}$$

where T_c is the tension in the cord. (The mass of the cord has been neglected in stating these equations.) In the first of Eq.(5.29), the moment term $T_c\, r$ is positive because it acts in the $+\theta$ direction. The sign of w is positive in the force equation because it acts in the $+x$ *direction* ; T_c has a negative sign because it is directed in the $-x$ direction.

Eqs.(5.29) provides two equations for the three unknowns: $\ddot{x}, \ddot{\theta}$, and T_c. Eliminating the tension T_c from Eqs.(5.29) gives

$$\frac{Mr^2}{2}\ddot{\theta} + rm\ddot{x} = wr. \tag{5.30}$$

Substituting from the last of Eq.(5.28) for $\ddot{x} = r\ddot{\theta}$ gives the final differential equation

$$(\frac{M}{2} + m)r^2\ddot{\theta} = wr.$$

Two driven pulleys connected by a belt

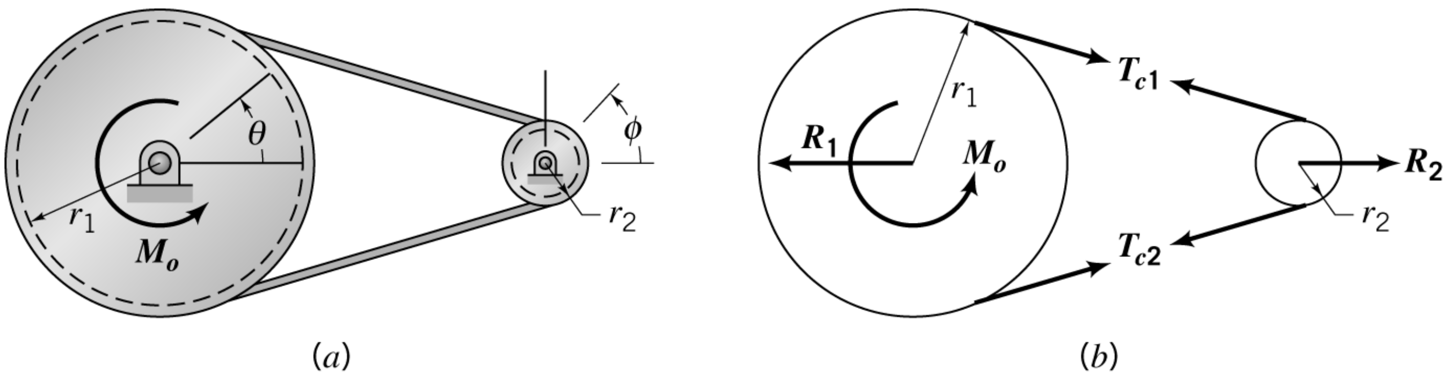


Figure 5.13 (a). Two disks connected by a belt. (b). Free-body diagram.

Figure 5.13A illustrates two pulleys that are connected to each other by a light and inextensible belt. The pulley at the left has mass m_1, radius of gyration k_{1g} about the pulley's axis of rotation, and is acted on by the counterclockwise moment M_o. The pulley at the right has mass m_2 and a radius of gyration k_{2g} about its axis of rotation. (The radius of gyration k_g defines the moment of inertia about the axis of rotation by $I = mk_g^2$.) The belt runs in a groove in pulley 1 with inner radius r_1. The inner radius of the belt groove for pulley 2 is r_2. The angle of rotation for pulleys 1 and 2 are, respectively, θ and φ.

Derive the governing differential equation of motion in terms of θ *and its derivatives.*

From the free-body diagram the fixed-axis rotation moment Eq.(5.26) gives:

$$
\begin{aligned}
I_1 \ddot{\theta} &= \Sigma M_{1oz} = M_o(t) + r_1 (T_{c2} - T_{c1}) \\
I_2 \ddot{\varphi} &= \Sigma M_{2oz} = r_2 (T_{c1} - T_{c2}) \ ,
\end{aligned}
\tag{5.31}
$$

where T_{c1} and T_{c2} are the tension components in the upper and lower belt segments. $M_o(t)$ has a positive sign because it is acting in the +θ direction; $r_1 (T_{c1} - T_{c2})$ has a negative sign because it acts in the -θ direction. Similarly, $r_2 (T_{c1} - T_{c2})$ has a positive sign in the second of Eq.(5.31) because it is acting in the +φ direction.

The moments of inertia in Eq.(5.31) are defined in terms of their masses and radii of gyrations by

$$
I_1 = m_1 k_1^2 \quad ; \ I_2 = m_2 k_2^2 \ .
$$

Returning to Eq.(5.31), we can eliminate the tension terms in the two equations, obtaining

$$
I_1 \ddot{\theta} = M_o - \frac{r_1}{r_2} I_2 \ddot{\varphi} \quad \Rightarrow \quad I_1 \ddot{\theta} + \frac{r_1}{r_2} I_2 \ddot{\varphi} = M_o(t) \ . \tag{5.32}
$$

We now have one equation for the two unknowns $\ddot{\theta}$ and $\ddot{\varphi}$, and need an additional kinematic equation relating these two angular acceleration terms. Given that the belt connecting the pulleys is

inextensible (can not stretch) the velocity v of the belt leaving both pulleys must be equal; hence,

$$v = r_1 \dot{\theta} = r_2 \dot{\varphi} \;\Rightarrow\; r_1 \ddot{\theta} = r_2 \ddot{\varphi} \;\; .$$

Substituting this result back into Eq.(5.32) gives the desired final result

$$[I_1 + (\frac{r_1}{r_2})^2 I_2]\ddot{\theta} = I_{eff}\ddot{\theta} = M_o(t) \;\; .$$

Note that coupling the two pulleys' motion by the belt acts to increase the effective inertia I_{eff} in resisting the applied moment.

Lecture 26. KINETIC-ENERGY FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $T = m\dfrac{v^2}{2}$ for a particle

Find: Kinetic Energy for a rigid body

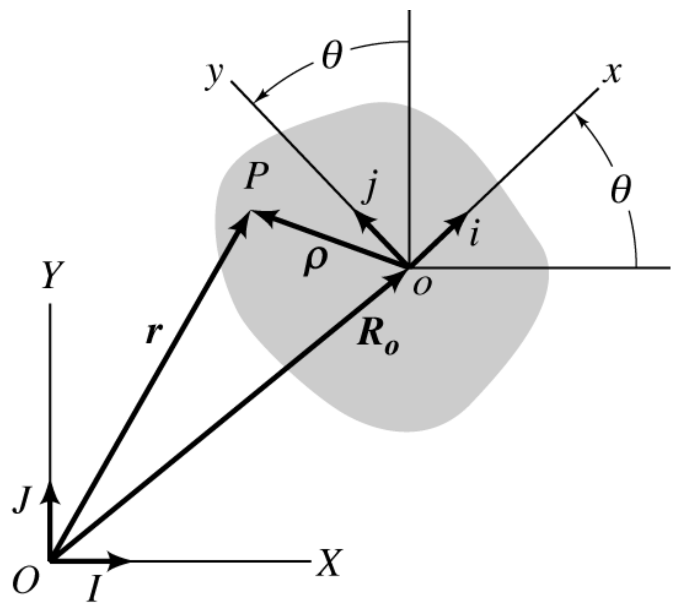


Figure 5.63 Rigid body with an imbedded *x, y, z* coordinate system. Point o , the origin of the x,y,z system, is located in the inertial *X, Y* system by the vector $\boldsymbol{R_o}$.

The mass center of the body is located in the *x, y, z* system by the position vector $\boldsymbol{b_{og}}$ defined earlier in section 5.2 as

$$m\boldsymbol{b_{og}} = m(\boldsymbol{i}b_{ogx} + \boldsymbol{j}b_{ogy} + \boldsymbol{k}b_{ogz}) = \int_V \boldsymbol{\rho}\gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho}\, dm \ , \quad (5.2)$$

where γ is the mass density of the body at point P. A point P in the body is located in the X, Y coordinate system by the position vector $\boldsymbol{r}$ and in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

The kinetic energy of the mass can be stated

$$T = \frac{1}{2}\int_m v^2 dm = \frac{1}{2}\int_m (\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})dm \ , \tag{5.178}$$

where $\dot{\boldsymbol{r}}$ is the velocity of a particle of mass dm at point P with respect to the X, Y coordinate system .

Since, points o and P are both fixed in the rigid body,

$$\boldsymbol{v}_{\boldsymbol{P}} = \boldsymbol{v}_{\boldsymbol{o}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\boldsymbol{oP}} \ \Rightarrow \ \dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

Hence,

$$\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = |\dot{\boldsymbol{R}}_{\boldsymbol{o}}|^2 + 2\ \dot{\boldsymbol{R}}_{\boldsymbol{o}} \cdot \boldsymbol{\omega} \times \boldsymbol{\rho} + |\boldsymbol{\omega} \times \boldsymbol{\rho}|^2 \ . \tag{5.179}$$

Substituting from Eq.(5.179) into the integral of Eq.(5.178) gives

$$\begin{aligned} T = {} & \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{o}}|^2}{2} + \dot{\boldsymbol{R}}_{\boldsymbol{o}} \cdot \int_m \boldsymbol{\omega} \times \boldsymbol{\rho}\ d\ m \\ & + \frac{1}{2}\int_m |\boldsymbol{\omega} \times \boldsymbol{\rho}|^2\ dm \end{aligned} \tag{5.180}$$

where $\ddot{\boldsymbol{R}}_o$ is not a function of the integration variables x, y, z and has been taken outside the integrals.

Continuing,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\,\dot{\theta} \times (\boldsymbol{i}\,x + \boldsymbol{j}\,y + \boldsymbol{k}z) = \dot{\theta}\,(\boldsymbol{j}\,x - \boldsymbol{i}\,y)\ ,$$

and

$$|\boldsymbol{\omega} \times \boldsymbol{\rho}|^2 = (\boldsymbol{\omega} \times \boldsymbol{\rho}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta}^2\,(x^2 + y^2)$$
$$\int (\boldsymbol{\omega} \times \boldsymbol{\rho})\,d\,m = \boldsymbol{\omega} \times \int \boldsymbol{\rho}\,d\,m = \boldsymbol{\omega} \times m\,\boldsymbol{b}_{og} \qquad (5.181)$$

Substituting from Eqs.(5.181) into (5.180) gives

$$T = \frac{m\,|\dot{\boldsymbol{R}}_o|^2}{2} + \dot{\boldsymbol{R}}_o \cdot (\boldsymbol{\omega} \times m\,\boldsymbol{b}_{og}) + \frac{I_o\,\dot{\theta}^2}{2}\ , \qquad (5.182)$$

where

$$I_{zzo} = I_o = \int (x^2 + y^2)\,dm$$

is the moment of inertia about a z axis through point o, the origin of the x, y, z system.

If o the origin of the x, y, z system coincides with g the body's mass center, $\boldsymbol{b}_{og} = 0$, and Eq.(5.182) reduces to

$$T = \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{g}}|^2}{2} + \frac{I_g\,\dot{\theta}^2}{2}\,. \tag{5.183}$$

This equation states that the kinetic energy of a rigid body is the sum of the following terms:

a. The translational energy of the body assuming that all of its mass is concentrated at the mass center, and

b. The rotational energy of the rigid body from rotation about the mass center.

Rotation about a Fixed Axis

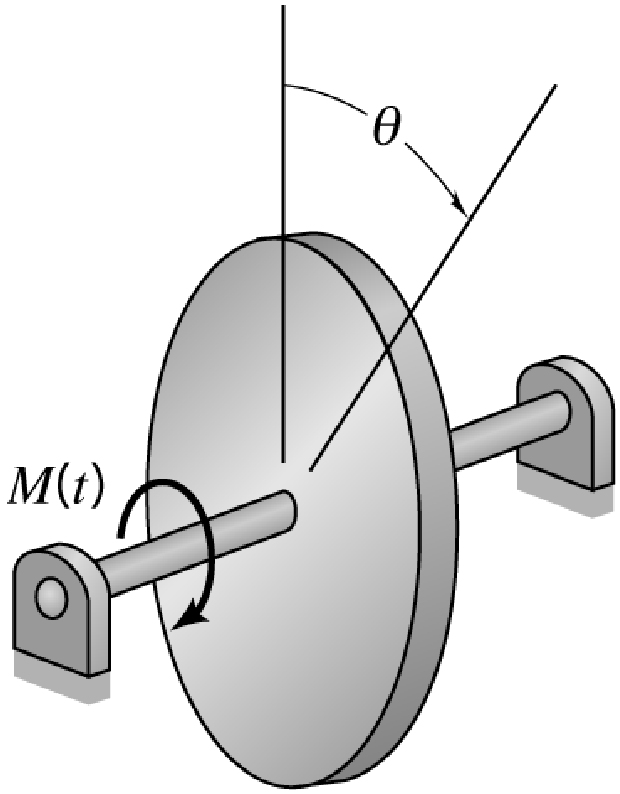
For pure rotation about *o*, $\dot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$ in Eq.(5.182) and the following simplified definition applies

$$T = \frac{I_o\dot{\theta}^2}{2}. \tag{5.184}$$

Eq.(5.184) defines the kinetic energy of the body for pure rotation about an axis through a point *o* fixed in space.

Applications of the Energy Equation

Rotor in Bearings



Assume that the rotor has an initial angular velocity of $\dot{\theta}(0)=\omega_0$, *and is acted on by a constant drag moment* $\bar{M}$, *how many revolutions will it take to come to rest?*

There is no change in potential energy, and the final kinetic energy is zero; hence, the energy equation $Work_{n.c.}=\Delta(T+V)$ gives

$$Work_{n.c.}=(0+0)-(\frac{I_o\omega_o^2}{2}+0) \quad . \qquad (5.207)$$

We need to calculate the work done by the resistance torque.

The differential work due to a applied force $\boldsymbol{f}$ acting through the differential distance $\boldsymbol{ds}$ is $dWk = \boldsymbol{f} \bullet \boldsymbol{ds}$. We can replace a moment M by a force acting at a fixed radius $\bar{r}$, such that $M = f\bar{r}$. When the moment M rotates through the differential angle $d\theta$, the force will act through the arc distance $ds = \bar{r}\,d\theta$, and the differential work will be

$$dWk = f\,ds = f\,\bar{r}\,d\theta = M d\theta \ . \tag{5.208}$$

Using Eq.(5.208), Eq.(5.207) becomes

$$\begin{aligned} \int_o^{\Delta\theta} -\bar{M} d\theta &= -\bar{M}\Delta\theta = -\left(\frac{I_o\omega_o^2}{2}\right) \\ \therefore\ \Delta revolutions &= \frac{\Delta\theta}{2\pi} = \frac{I_o\omega_o^2}{4\pi\bar{M}} \ . \end{aligned} \tag{5.209}$$

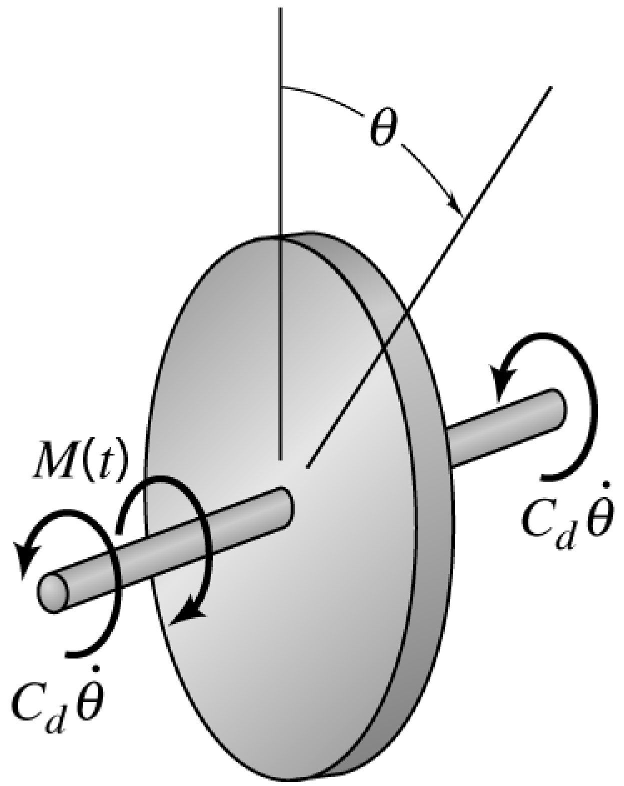
The work integral is negative because it decreases the energy of the system.

Assume that the rotor is acted on by the positive (in the direction of $+\theta$*) applied moment* $M(t)$, and derive the equation of motion. For this task, $Work_{n.c.} = \Delta(T+V)$ gives

$$\int_o^{\theta} M(t)\,dx = \frac{I_{oz}\dot{\theta}^2}{2} - T_o . \tag{5.210}$$

Differentiating Eq.(5.210) with respect to θ gives the differential equation of motion

$$I_{oz}\frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) = I_{oz}\ddot{\theta} = M(t) \quad .$$



Derive the governing equation of motion for the rotor including the applied moment M(t) and viscous drag moment $-C_d\dot{\theta}$.

For this task, $Work_{n.c.} = \Delta(T+V)$ becomes

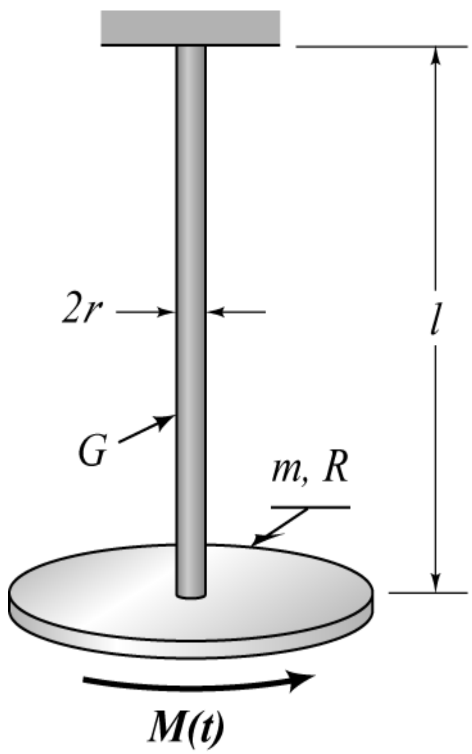
$$\int_o^{\theta}[M(t) - 2C_d\dot{\theta}]dx = \frac{I_{oz}\dot{\theta}^2}{2} - T_o,$$

and differentiation with respect to θ gives

$$I_{oz}\frac{d}{dt}(\frac{\dot{\theta}^2}{2}) = I_{oz}\ddot{\theta} = M(t) - 2C_d\dot{\theta} \Rightarrow I_{oz}\ddot{\theta} + 2C_d\dot{\theta} = M(t) \quad .$$

Using the work-energy equation has no particular advantage in developing these last two equations of motion. As with the Newtonian approach, a free-body diagram is required to define the applied moment, and the nonconservative moments can not be integrated with respect to θ.

A Torsional-Vibration Example



(*a*)

Derive the governing equation of motion. The external moment $M(t)$ is adding energy to the system; hence, the work-energy equation is

$$\int_0^\theta M(t)dx = \left(\frac{I_o\dot{\theta}^2}{2} + V\right) - (T_0 + V_0) \ . \tag{5.211}$$

In this example, the potential energy of the system is stored in the shaft due to the torsional rotation θ. Recall that the reaction moment is defined from

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta,$$

where G is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. The requirement that a potential force (or moment) be derivable as the negative gradient of a potential function gives

$$M_\theta = -k_\theta \theta = -\frac{dV_\theta}{d\theta} \quad \Rightarrow \quad V_\theta = k_\theta(\frac{\theta^2}{2}) \ . \qquad (5.191)$$

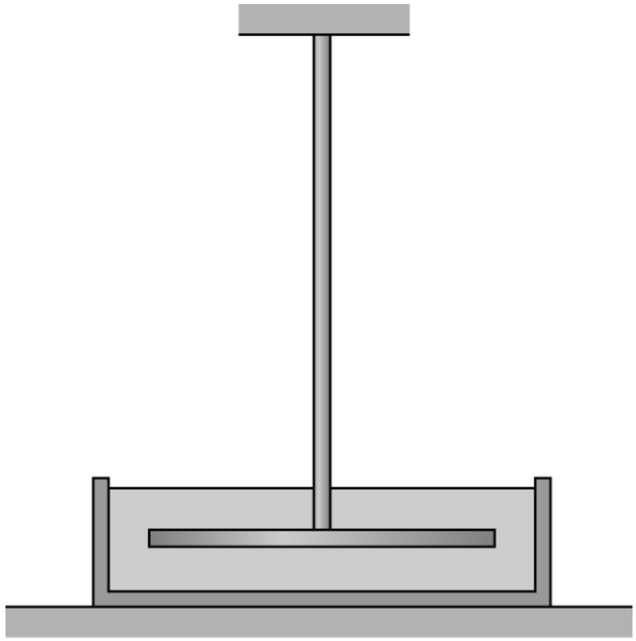
Substituting for V_θ into Eq.(5.211) gives

$$\int_0^\theta M(t)\,d\theta = (\frac{I_o\dot{\theta}^2}{2} + \frac{k_\theta \theta^2}{2}) - (T_0 + V_0) \ .$$

Differentiating with respect to θ gives the differential equation of motion

$$I_o \frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) + k_\theta \theta = M(t) \quad \Rightarrow \quad I_o \ddot{\theta} + k_\theta \theta = M(t) \ .$$

Torsional Vibration Example with viscous drag



For the viscous drag moment $-C_d\dot{\theta}$, the integrand in the work-energy Eq.(5.211) gives

$$\int_0^{\theta}[M(t)-C_d\dot{\theta}]\,dx=\frac{I_o\dot{\theta}^2}{2}+\frac{k_\theta\theta^2}{2}-(T_0+V_0)\ . \qquad (5.213)$$

Differentiating with respect to θ gives the final equation of motion

$$I_o\ddot{\theta}+C_d\dot{\theta}+k_\theta\theta=M(t)\,. \qquad (5.213a)$$

For this example, deriving the equation of motion using the moment or the work-energy equation requires about the same effort. The equation of motion can be derived by including the external driving moment $M(t)$ or viscous damping moment

$-C_d\dot{\theta}$ in the work integral, but the integral is now a function of t or $\dot{\theta}$, not θ, and can not be integrated.

Recall that the moment equation gave

$$I_o\ddot{\theta} = \Sigma M_o = M(t) - C_d\dot{\theta} - k_\theta\theta \ ,$$

where ΣM_o is the resultant moment about the vertical axis. Substituting the energy-integral substitution $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$ yields

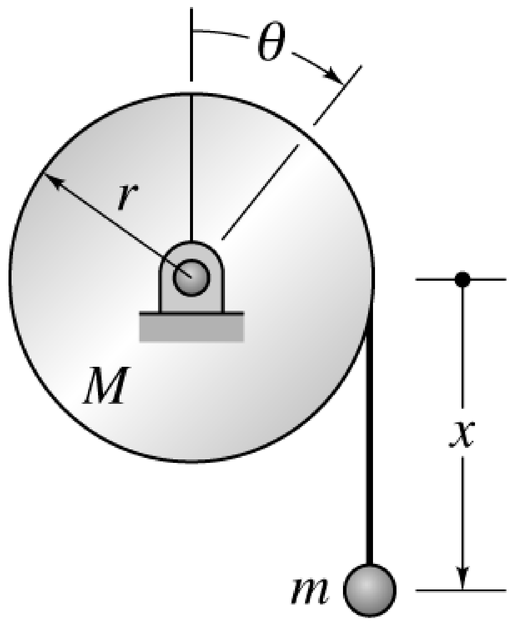
$$I_o\frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) = M(t) - C_d\dot{\theta} - k_\theta\theta \ . \tag{5.214}$$

Multiplying through by $d\theta$ and integrating gives

$$I_o\frac{\dot{\theta}^2}{2} + k_\theta\frac{\theta^2}{2} = \int_{\theta_0}^{\theta}[M(t) - C_d\dot{\theta}]\,dx - (I_o\frac{\dot{\theta}_0^2}{2} + k_\theta\frac{\theta_0^2}{2}) \ . \tag{5.215}$$

This result coincides with Eq.(5.213a), obtained from the work-energy-equation.

An Example Involving Connected Motion of a Disk and a Particle



Derive the governing equation of motion using conservation of energy.

There are no nonconservative forces ; hence, energy is conserved. Using a plane through the bearing as datum for potential energy due to gravity

$$T + V = T_0 + V_o \quad \Rightarrow \quad \frac{I_o\dot{\theta}^2}{2} + \frac{m\dot{x}^2}{2} - mgx = T_0 + V_0 \ .$$

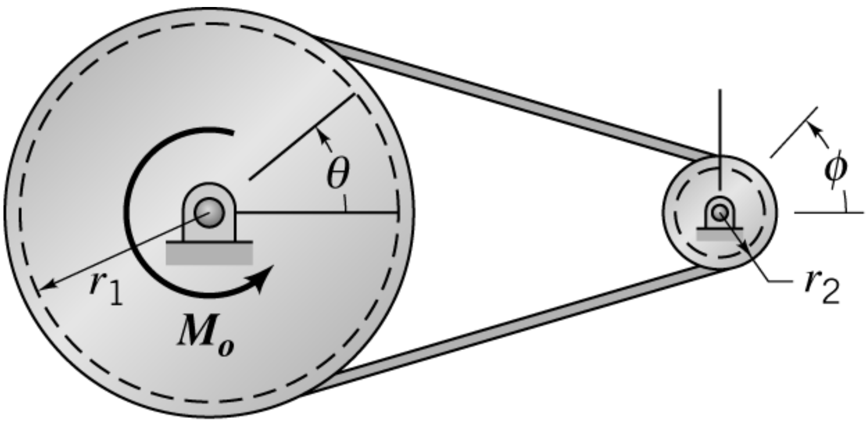
We need $I_o = Mr^2/2$, and the kinematics of Eq.(5.28), $x = r\theta, \dot{x} = r\dot{\theta}$, to obtain

$$\frac{\dot{\theta}^2 r^2}{2}\left(\frac{M}{2} + m\right) - mgr\theta = T_0 + V_0 \ .$$

Differentiating with respect to θ gives the differential equation of motion

$$(\frac{M}{2}+m)r^2\ddot{\theta}=wr \quad ,$$

Two Driven Pulleys Connected by a Belt



(*a*)

The left and right pulleys have radii and radii of gyrations (r_1,k_{g1}) and (r_2,k_{g2}), respectively. There is no energy dissipation (frictionless bearings), and the belt connecting the two pulleys does not slip.

a. If the system starts from rest, and the applied moment $M_o=\bar{M}$ is constant, find the angular velocity of both pulleys after 10 complete rotations of the left (driven)pulley.

b. Derive the differential equations of motion.

There is no change in the potential energy of this system; hence, $Work_{n.c.} = \Delta(T+V)$ yields

$$Work_{n.c.} = T - T_1 = \left(\frac{I_1\dot{\theta}^2}{2} + \frac{I_2\dot{\varphi}^2}{2}\right) - 0 = \frac{m_1 k_{g1}^2\dot{\theta}^2}{2} + \frac{m_2 k_{g2}^2\dot{\varphi}^2}{2} \quad (5.195)$$

Since the belt does not slip, the tangential velocities at the rims of the pulleys must equal, providing the kinematic condition, $r_1\dot{\theta} = r_2\dot{\varphi}$, which reduces Eq.(5.195) to

$$Work_{n.c.} = \frac{\dot{\theta}^2}{2}\left[m_1 k_{g1}^2 + \left(\frac{r_1}{r_2}\right)^2 m_2 k_{g2}^2\right] = \frac{I_{eff}\dot{\theta}^2}{2} \ . \quad (5.196)$$

Using $d\,Work = M\,d\theta$ to define the work integral on the left-hand side gives

$$\int_0^{\Delta\theta} \bar{M} d\theta = \bar{M}\Delta\theta = \frac{I_{eff}\dot{\theta}^2}{2} \ , \quad (5.197)$$

and the angular velocity after the moment has been applied for ten rotations $(\Delta\theta = 20\pi)$ is

$$\dot{\theta}(\Delta\theta = 20\pi) = \sqrt{\frac{2\bar{M}\cdot 20\pi}{I_{eff}}} \ ,$$

which concludes *Task a.* The work integral is positive in

Eq.(5.197) because it is increasing the mechanical energy of the system.

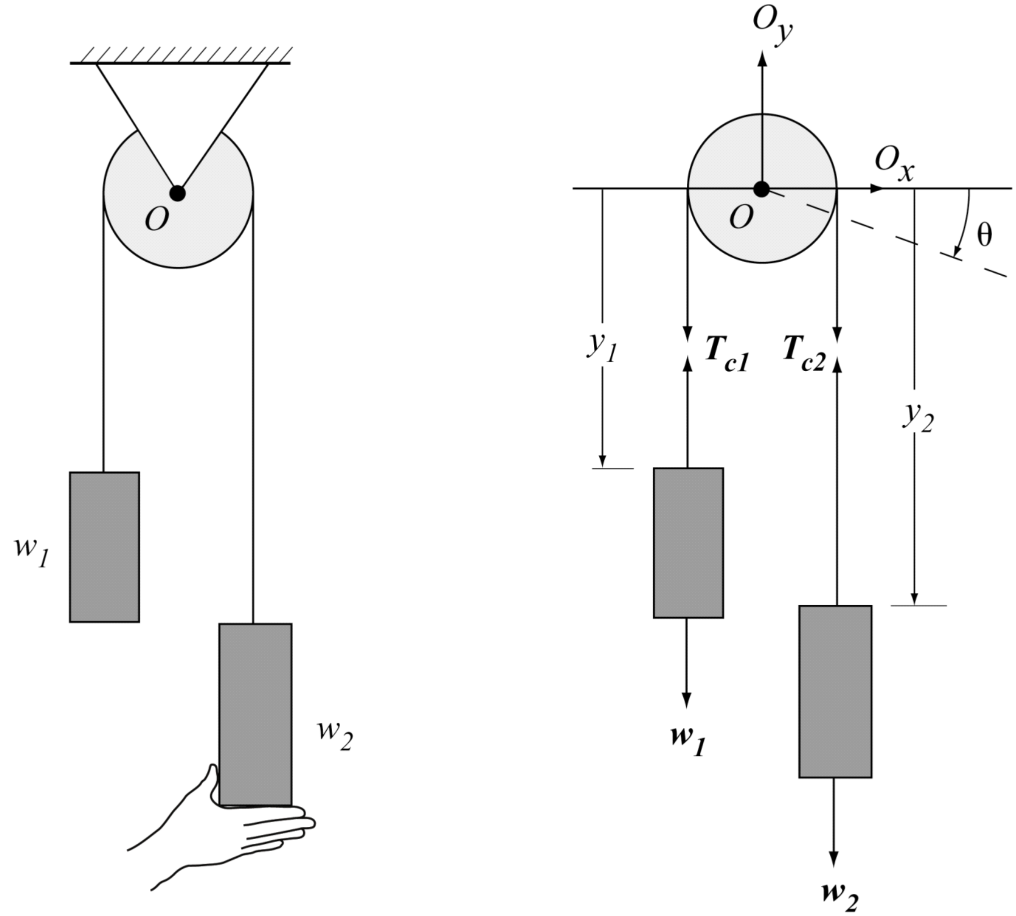
Differentiating Eq.(5.197) with respect to θ gives the governing differential equation of motion

$$I_{eff}\ddot{\theta} = M_o \ ,$$

which coincides with our earlier result and concludes *Task b*.

Particle Dynamics versus Rigid Body Dynamics

Derive the equation of motion for the system illustrated



Moment Equation

$$\sum M_0 = I_0 \ddot{\theta} = T_{2c} r - T_{1c} r \tag{72}$$

Force Equations

$$\begin{aligned} &Body\ \ 1:\ \sum f_{y1} = w_1 - T_{1c} = m_1 \ddot{y}_1 \\ &Body\ \ 2:\ \sum f_{y2} = w_2 - T_{2c} = m_2 \ddot{y}_2 \end{aligned} \tag{73}$$

For negligible pulley inertia, $I_0 = 0 \Rightarrow T_{1c} = T_{2c} = T_c$, and Eq.(2) becomes

$$w_1 - T_c = m_1 \ddot{y}_1 \ , \ w_2 - T_c = m_2 \ddot{y}_2 \qquad \textbf{(3)}$$

Subtracting the 1st from the 2nd gives

$$m_2 \ddot{y}_2 - m_1 \ddot{y}_1 = w_2 - w_1 \qquad \textbf{(4)}$$

Kinematic Constraint

$$y_1 + y_2 + constant = l_c \Rightarrow \ddot{y}_1 = -\ddot{y}_2 \qquad \textbf{(5)}$$

Substitute from Eq.(6) into (5) gives

$$[m_2 \ddot{y}_2 - m_1(-\ddot{y}_2)] = (m_1 + m_2)\ddot{y}_2 = w_2 - w_1 \qquad \textbf{(6)}$$

This is a particle dynamics result.

For finite moment of inertia, $I_0 > 0$, solve for T_{1c} and T_{2c} from Eq.(2) and substitute into Eq.(1) to obtain

$$I_0 \ddot{\theta} = r(w_2 - m_2 \ddot{y}_2) - r(w_1 - m_1 \ddot{y}_1) \ ,$$

Substitute the kinematics relationships,

$$\ddot{y}_1 = -r\ddot{\theta} \ , \ \ddot{y}_2 = r\ddot{\theta} \ ,$$

to obtain

$$I_0 \ddot{\theta} = r w_2 - r m_2 (r\ddot{\theta}) - r w_1 + r m_1 (-r\ddot{\theta}_1) \ ,$$

or

$$(I_0 + m_1 r^2 + m_2 r^2)\ddot{\theta} = I_{eff}\ddot{\theta} = r(w_2 - w_1) \ . \qquad \textbf{(7)}$$

All of the contributions to I_{eff} should be positive.

Compare (7) and (6). Setting I_0 equal to zero in (7) and substituting $\ddot{\theta} = \ddot{y}_2 / r$ gives

$$(m_1 + m_2) r^2 \left(\frac{\ddot{y}_2}{r}\right) = r(w_2 - w_1) \ \Rightarrow \ (m_1 + m_2)\ddot{y}_2 = w_2 - w_1$$

Equation of Motion from conservation of energy

$$T = I_0 \frac{\dot{\theta}^2}{2} + m_1 \frac{\dot{y}_1^2}{2} + m_2 \frac{\dot{y}_2^2}{2} \ , \ V = -w_1 y_1 - w_2 y_2$$

The datum for V goes through the center of the pulley.

Kinematics:

$$y_1 = y_{10} - r\theta \ \Rightarrow \ \dot{y}_1 = -r\dot{\theta} \ , \ y_2 = y_{20} + r\theta \ \Rightarrow \ \dot{y}_2 = r\dot{\theta}$$

Substitution gives

$$T = I_0 \frac{\dot{\theta}^2}{2} + m_1 \frac{(-r\dot{\theta})^2}{2} + m \frac{(r\dot{\theta})^2}{2} = (I_0 + m_1 r^2 + m_2 r^2) \frac{\dot{\theta}^2}{2}$$

$$V = -w_1 (y_{10} - r\theta) - w_2 (y_{20} + r\theta) \ .$$

Hence, $T + V = T_0 + V_0$ gives

$$(I_0 + m_1 r^2 + m_2 r^2) \frac{\dot{\theta}^2}{2} - w_1 (y_{10} - r\theta) - w_2 (y_{20} + r\theta) = T_0 + V_0$$

Differentiate w.r.t. θ gives

$$(I_0 + m_1 r^2 + m_2 r^2) \frac{d}{d\theta} \frac{\dot{\theta}^2}{2} + w_1 r - w_2 r = 0 \ ,$$

or

$$(I_0 + m_1 r^2 + m_2 r^2)\,\ddot{\theta} = w_2 r - w_1 r \ ,$$

which coincides with Eq.(7).

Lecture 27. THE COMPOUND PENDULUM

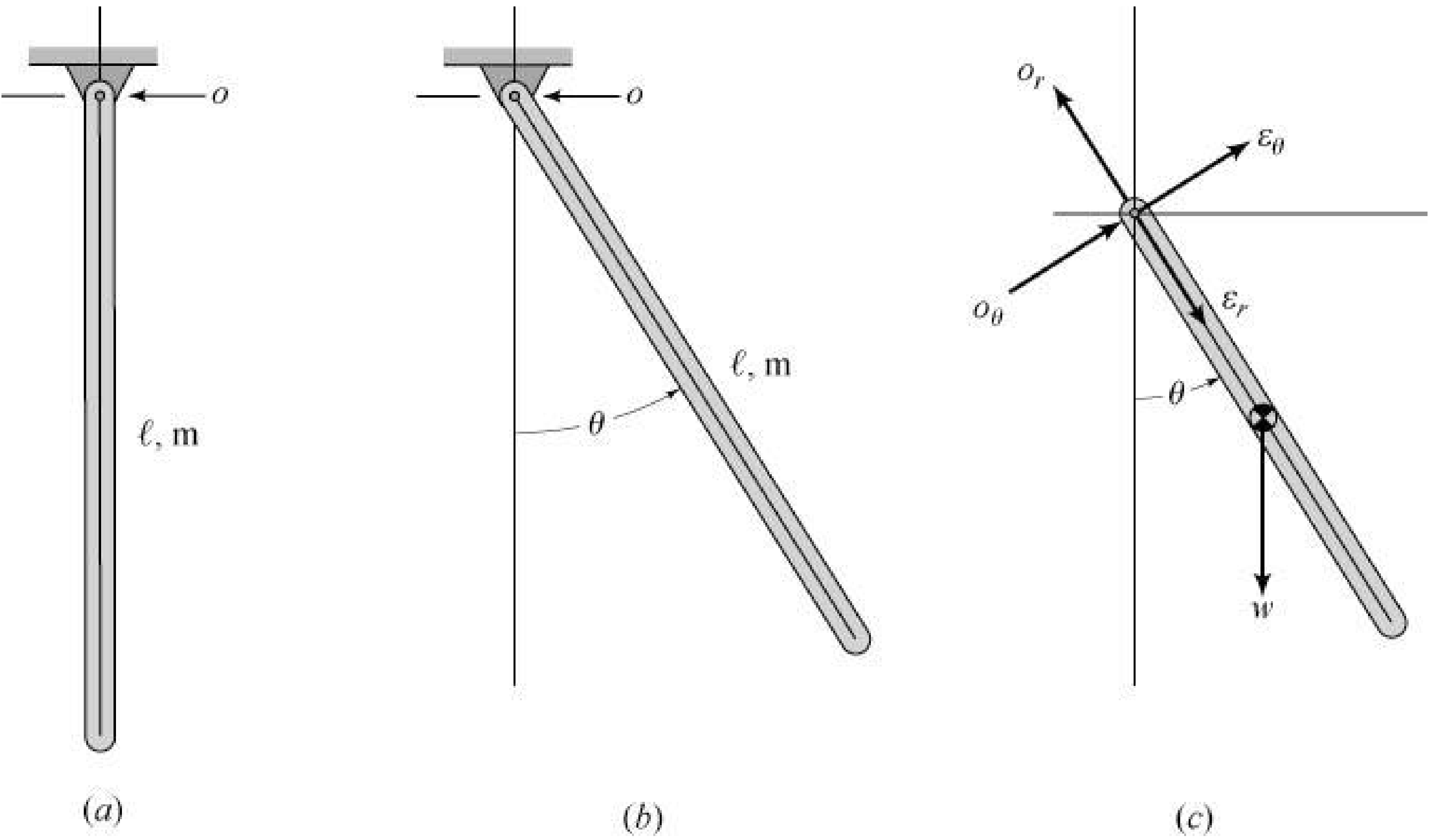


Figure 5.16 Compound pendulum: (a) At rest in equilibrium, (b) General position with coordinate θ, (c) Free-body diagram

The term "compound" is used to distinguish the present rigid-body pendulum from the "simple" pendulum of Section 3.4b, which consisted of a particle at the end of a massless string.

Derive the general differential equation of motion for the pendulum of figure 5.16a and determine its undamped natural frequency for small motion about the static equilibrium position.

From the free-body diagram the moment equation is

$$I_o\ddot{\theta} = \Sigma M_o = -w\frac{l}{2}\sin\theta \ . \tag{5.33}$$

The minus sign on the right-hand side term applies because the moment is acting in the -θ direction. For a uniform bar, $I_o = ml^2/3$; hence, the governing differential equation of motion is

$$\frac{ml^2}{3}\ddot{\theta} + \frac{mgl}{2}\sin\theta = 0 \ ,$$

or

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0 \ . \tag{5.34}$$

For small motion, sin θ ≅ θ, and the nonlinear differential equation reduces to

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \omega_n^2\theta = 0 \ . \tag{5.35}$$

This differential equation is the *rotation* analog of the single-degree-of-freedom, *displacement*, vibration problem of $\ddot{x} + \omega_n^2 x = 0$. The compound pendulum's natural frequency is

$$\omega_n = \sqrt{3g/2l} \ .$$

Assuming that the pendulum is released from rest at $\theta = \pi/2$ *radians* $= 90^o$*, define the reaction force components* (o_θ, o_r) *as a function of* θ (*only*).

From figure 5.16c applying $\Sigma \boldsymbol{f} = m \ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ in polar coordinates for the mass center of the rod gives:

$$\begin{aligned} \Sigma f_r &= -o_r + w\cos\theta = m\ddot{R}_{gr} = m(\ddot{r}_g - r_g\dot{\theta}^2) = -m\frac{l}{2}\dot{\theta}^2 \\ \Sigma f_\theta &= o_\theta - w\sin\theta = m\ddot{R}_{g\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) = m\frac{l}{2}\ddot{\theta} \ . \end{aligned} \qquad (5.36)$$

In the acceleration terms, $\dot{r}_g = \ddot{r}_g = 0$, because $r_g = l/2$ is a constant. Eqs.(5.36) define the reaction force components o_r, o_θ, but not as a function of θ alone. Direct substitution from Eq.(5.34) into the second of Eqs.(5.36) defines o_θ as

$$o_\theta = w\sin\theta + \frac{ml}{2}\ddot{\theta} = w\sin\theta - \frac{ml}{2}\left(\frac{3g}{2l}\right)\sin\theta = \frac{w}{4}\sin\theta \ . \qquad (5.37)$$

Finding a comparable relationship for o_r is more complicated, because the first of Eqs.(5.36) involves $\dot{\theta}^2$. We will need to integrate the differential equation of motion via the energy-integral substitution to obtain $\dot{\theta}^2$ as a function of θ, proceeding from

$$\ddot{\theta} = \frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) = -\frac{3g}{2l}\sin\theta \ .$$

Multiplying through by $d\theta$ and integrating both sides of this equation gives

$$\frac{\dot{\theta}^2}{2} - \frac{\dot{\theta}_o^2}{2} = \int_{\frac{\pi}{2}}^{\theta} -\frac{3g}{2l}\sin x \, dx$$
$$\frac{\dot{\theta}^2}{2} = \frac{3g}{2l}\,\Big|_{\frac{\pi}{2}}^{\theta}\cos x = \frac{3g}{2l}\cos\theta \ . \tag{5.38}$$

Substituting this result into the first of Eq.(5.36) gives

$$o_r = w\cos\theta + ml\frac{\dot{\theta}^2}{2} = w\cos\theta + ml(\frac{3g}{2l})\cos\theta = \frac{5w}{2}\cos\theta \ . \tag{5.39}$$

This result shows that the dynamic reaction force will be 2.5 times greater than the static weight w when the rod reached its lowest position ($\theta = 0$).

Alternative Moment Equation with Moments about g

Suppose that we had chosen to take moments about g, the mass center of the rod in figure 5.16c, obtaining

$$\Sigma M_g = I_g\ddot{\theta} = -o_\theta\frac{l}{2} \ .$$

The moment has a negative sign because it is acting in the -θ direction. In this equation, the required moment of inertia about g is $I_g = ml^2/12$. Substituting for o_θ from Eq.(5.37) gives

$$-\frac{l}{2}(m\frac{l}{2}\ddot{\theta} + w\sin\theta) = \frac{ml^2}{12}\ddot{\theta} \ ,$$

or

$$-\frac{wl}{2}\sin\theta = ml^2(\frac{1}{12} + \frac{1}{4})\ddot{\theta} = \frac{ml^2}{3}\ddot{\theta} \ . \qquad (5.40)$$

Writing the moment equation about g involves more work but gets tha same equation. Note in the intermediate step of Eq.(5.40) that we are accomplishing the parallel-axis formula in moving from $I_g = ml^2/12$ to $I_o = ml^2/3$, via $I_o = I_g + m|\boldsymbol{b}_{go}|^2 = I_g + m(l/2)^2$,where $\boldsymbol{b}_{go}$ is the vector from the mass center g to the pivot point o.

Deriving The Equation of Motion From The Energy Equation

There are no external time varying forces or moments and no energy dissipation; hence, mechanical energy is conserved; i.e., $T + V = T_0 + V_0$. Using a horizontal plane through the pendulum's pivot point as a datum for gravity potential energy gives

$$V = V_g = -w\frac{l}{2}\cos\theta \ .$$

For rotation about the fixed point *o*, the kinetic energy of the pendulum is defined by

$$T = I_o \frac{\dot{\theta}^2}{2} = \frac{ml^2}{3}\frac{\dot{\theta}^2}{2} \quad .$$

Hence,

$$\frac{ml^2}{3}\frac{\dot{\theta}^2}{2} - \frac{wl}{2}\cos\theta = T_0 + V_0 = 0 - \frac{wl}{2}\cos(\theta_0 = \frac{\pi}{2}) = 0$$

$$\therefore \quad \dot{\theta}^2 = \frac{3g}{l}\cos\theta \quad .$$

Differentiating w.r.t. θ gives

$$\frac{ml^2}{3}\frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) + \frac{mgl}{2}\sin\theta = 0 \;\Rightarrow\; \ddot{\theta} + \frac{3g}{2l}\sin\theta = 0 \quad .$$

Static stability about equilibrium points.

From

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0 \;\;,\quad \ddot{\theta} = 0 \;\Rightarrow\; \sin\bar{\theta} = 0 \;\;,\; \bar{\theta} = 0\,,\pi$$

Equilibrium for small Motion about the equilibrium position $\theta = 0$**.**

Small motion about the equilibrium position $\theta = 0$ gives the linearized differential equation of motion

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0 \;\; .$$

For the initial conditions, $\dot{\theta}(0) = \dot{\theta}_0 ; \theta(0) = \theta_0$, the solution to the linearized Eq.(5.35) can be stated

$$\theta(t) = \theta_0 \cos\omega_n t + \frac{\dot{\theta}_0}{\omega_n}\sin\omega_n t \;\;,\;\; \omega_n = \sqrt{\frac{3g}{2l}} \;\;, \qquad (5.41)$$

consisting of a *stable* oscillation at the natural frequency. Hence, $\theta = 0$ is said to be a *stable equilibrium point* for the body.

Equilibrium for small Motion about the equilibrium position $\theta = \pi$.

Motion is governed by the nonlinear equation of motion,

$$\ddot{\theta} = -\frac{3g}{2l}\sin\theta \quad . \tag{5.34}$$

Expanding $\sin\theta$ in a Taylor's series about $\theta = \pi$ gives

$$\begin{aligned} \sin\theta &= \sin(\pi + \delta\theta) = \sin\pi\cos\delta\theta + \cos\pi\sin\delta\theta \\ &= -\sin\delta\theta = -\delta\theta + \frac{(\delta\theta)^3}{6} - \frac{(\delta\theta)^5}{120} + \dots \quad . \end{aligned} \tag{5.42}$$

Retaining only the linear term in Eq.(5.42) and substituting back into Eq.(5.34) gives

$$\delta\ddot{\theta} - \frac{3g}{2l}\delta\theta = 0 \ , \ \theta = \pi + \delta\theta \ \Rightarrow \ \ddot{\theta} = \delta\ddot{\theta} \tag{5.43}$$

Observe the negative sign in the coefficient for $\delta\theta$. If this were a harmonic oscillator consisting of a spring supporting a mass, a comparable negative sign would imply a negative stiffness, yielding a differential equation of the form

$$m\ddot{x} - kx = 0 \quad .$$

Substituting the assumed solution $\delta\theta = Ae^{st}$ into Eq.(5.43) gives

$$(s^2 - \omega_n^2)Ae^{st} = 0 \Rightarrow s = \pm\omega_n \ .$$

Hence the solution to Eq.(5.43) for small motion about $\theta = \pi$ is

$$\delta\theta(t) = A_1 e^{\omega_n t} - A_2 e^{-\omega_n t}$$

The first term in this solution grows exponentially with time. Hence, any small disturbance of the pendulum from the equilibrium position $\theta = \pi$ will grow exponentially with time, and $\theta = \pi$ is a *statically unstable equilibrium point* for the pendulum.

A Swinging-Plate Problem

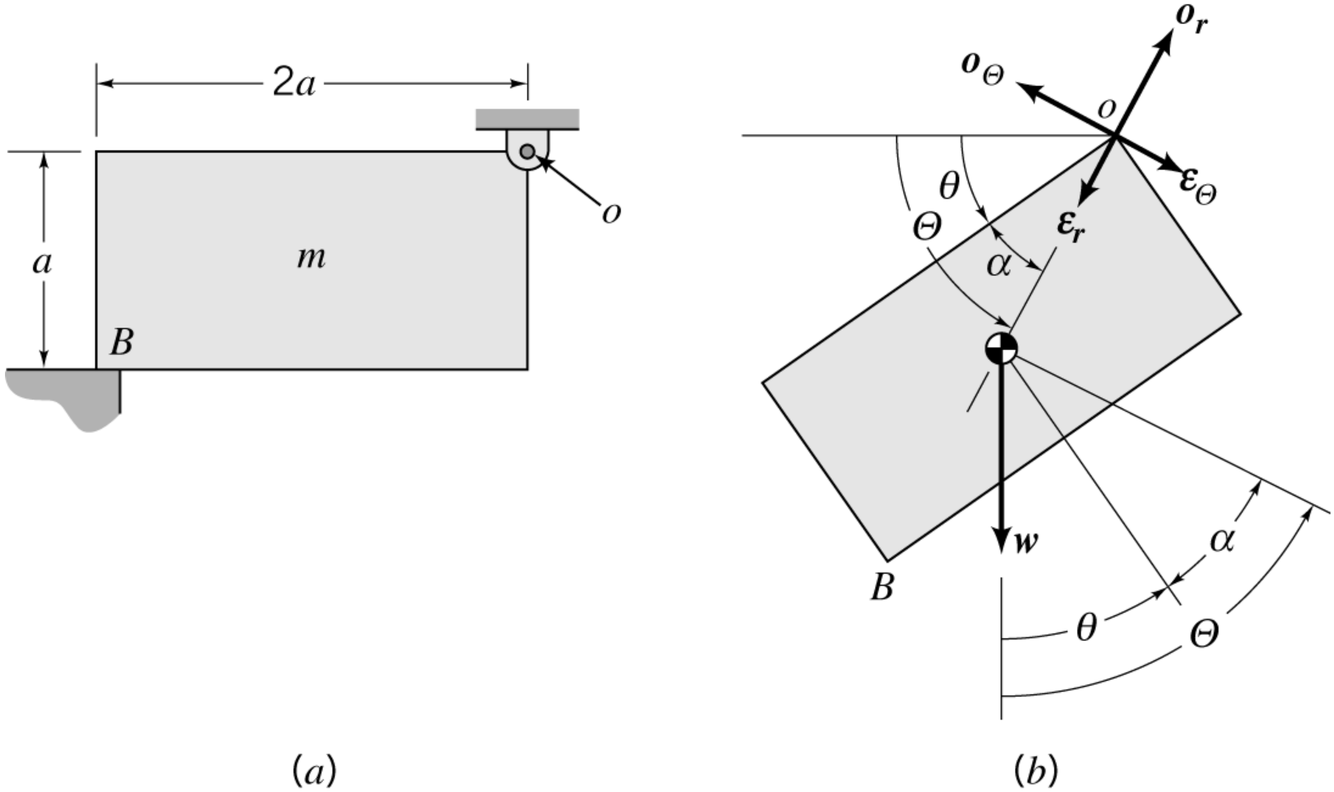


Figure XP5.2 (a) Rectangular plate supported at *o* by a frictionless pivot and at *B* by a ledge, (b) Free-body diagram that applies after the support at *B* has been removed.

The plate has mass *m* , length *2a*, and width *a* , and is supported by a frictionless pivot at *o* and a ledge at *B*. The engineering tasks associated with this problem follow. *Assuming that the support at B is suddenly removed, carry out the following steps.:*

a. Derive the governing differential equation of motion.

b. Develop relationships that define the components of the reaction force as a function of the rotation angle only.

c. Derive the governing differential equation of motion for small motion about the plate's equilibrium position . Determine the natural frequency of the plate for small motion about this position.

Kinematics: θ in the free-body diagram defines the plate's orientation with respect to the horizontal. The angle α lies between the top surface of the plate and a line running from the pivot point *o* through the plate's mass center at *g*, and is defined by

$$\alpha = \tan^{-1}\left(\frac{a/2}{a}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26.57^{o}$$

$\Theta = (\theta + \alpha)$ is the rotation angle (from the horizontal) of a line running from *o* through *g* .

Moment equation with moments taken about *o:*

$$I_o \ddot{\theta} = \Sigma M_{oz} = w a \frac{\sqrt{5}}{2} \cos(\theta + \alpha) \ . \qquad (5.56)$$

The distance from *o* to *g* is $a\sqrt{5}/2$, and the weight develops the external moment acting through the moment arm $a\sqrt{5}/2\cos(\theta+\alpha)$. The moment is positive because it is acting in the +θ direction.

Applying the parallel-axis formula gives

$$I_o = I_g + m\,|\,\boldsymbol{b_{go}}\,|^2 = \frac{m}{12}(a^2 + 4a^2) + 5m\frac{a^2}{4} = \frac{5ma^2}{3} \ . \tag{5.57}$$

Substitution gives the governing differential equation of motion

$$\ddot{\theta} = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta + \alpha) \ , \tag{5.58}$$

and we have completed *Task a*.

To define the reaction forces, we need to state $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$. The polar-coordinate version of this equation works best for the current problem, and the free-body diagram has been drawn using polar coordinates with o_r aligned with the radial acceleration component of point g, and o_Θ is aligned with the circumferential component.

Force equation components:

$$\begin{aligned}\Sigma f_r &= w\sin(\theta + \alpha) - o_r = m\ddot{R}_{gr} = m(\ddot{r} - r\dot{\Theta}^2) = -ma\frac{\sqrt{5}}{2}\dot{\theta}^2 \\ \Sigma f_\Theta &= w\cos(\theta + \alpha) - o_\Theta = m\ddot{R}_{g\Theta} = m(r\ddot{\Theta} + 2\dot{r}\dot{\Theta}) = ma\frac{\sqrt{5}}{2}\ddot{\theta} \ .\end{aligned} \tag{5.59}$$

$\dot{r} = \ddot{r} = 0$, because $r = a\sqrt{5}/2$ is a constant, and $\dot{\theta} = \dot{\Theta}, \ddot{\theta} = \ddot{\Theta}$

Eliminate $\ddot{\theta}$ by substitution

$$o_\Theta = w\cos(\theta+\alpha) - ma\frac{\sqrt{5}}{2}\cdot\frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta+\alpha) = \frac{w}{4}\cos(\theta+\alpha) \ .$$

This result states that o_Θ starts at $(w/4)\cos\alpha$ for $\theta = 0$, and is zero when the mass center is directly beneath the pivot point *o,* at $(\theta+\alpha) = \pi/2$.

To obtain $\dot{\theta}^2$ as a function of θ, use the energy-integral substitution to integrate Eq.(5.58). Starting with

$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{\dot{\theta}^2}{2}\right) = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta+\alpha) \ ,$$

multiplying through by $d\theta$, and integrating both sides of the equation gives

$$\frac{\dot{\theta}^2}{2} - \frac{\dot{\theta}_o^2}{2} = \int_0^\theta \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(x+\alpha)\,dx$$

$$\frac{\dot{\theta}^2}{2} = \frac{3\sqrt{5}}{10}\frac{g}{a}\Big|_0^\theta \sin(x+\alpha) = \frac{3\sqrt{5}}{10}\frac{g}{a}[\sin(\theta+\alpha) - \sin\alpha] \ .$$

Substituting this result back into the first of Eq.(5.59) gives

$$o_r = w\sin(\theta+\alpha) + ma\frac{\sqrt{5}}{2}\times\frac{3\sqrt{5}}{5}\frac{g}{a}[\sin(\theta+\alpha)-\sin\alpha]$$

$$= \frac{5w}{2}\sin(\theta+\alpha) - \frac{3w}{2}\sin\alpha \quad ,$$

and *Task b* is now completed .

The equilibrium condition for the pendulum is obtained by setting the right-hand side of Eq.(5.47) equal to zero, obtaining

$$\ddot{\theta} = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta+\alpha) = 0$$

$$\therefore \ \cos(\alpha+\theta) = 0 \ \Rightarrow \ \Theta = \theta+\alpha = \frac{\pi}{2},\frac{3\pi}{2} \quad .$$

To get the governing equation of motion for small motion about the equilibrium condition, start by substituting $\Theta = \pi/2 + \delta\Theta$ into Eq.(5.58), obtaining

$$\delta\ddot{\Theta} = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\frac{\pi}{2}+\delta\Theta) \quad .$$

Expanding the last term on the right in a Taylor's series gives

$$\cos(\frac{\pi}{2}+\delta\Theta)=\cos(\frac{\pi}{2})\cos\delta\Theta-\sin(\frac{\pi}{2})\sin\delta\Theta$$
$$=-\sin\delta\Theta=-\delta\Theta+\frac{(\delta\Theta)^3}{6}-\frac{(\delta\Theta)^5}{120}+\ldots$$

Retaining only the linear term in this expansion gives the linearized differential equation of motion

$$\delta\ddot{\Theta}+\frac{3\sqrt{5}}{10}\frac{g}{a}\delta\Theta=0\ ;$$

hence, the natural frequency is defined by

$$\omega_n^2=\frac{3\sqrt{5}}{10}\frac{g}{a}\quad\Rightarrow\quad\omega_n=.819\sqrt{\frac{g}{a}}\ . \tag{5.60}$$

Alternative Development for motion about equilibrium

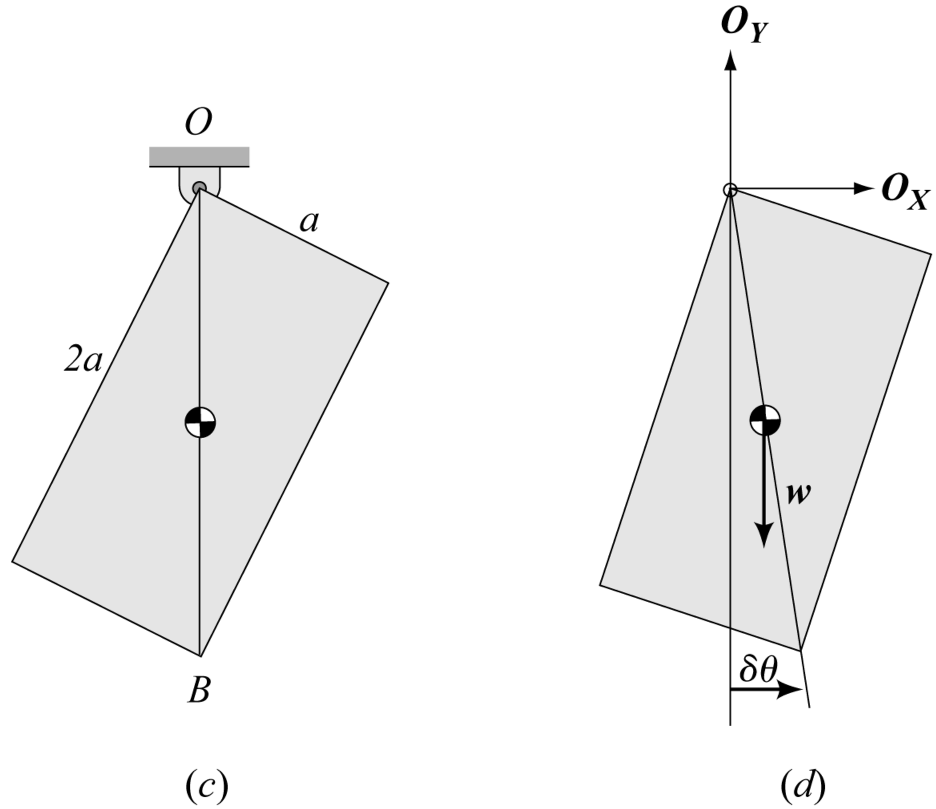


Figure XP5.2 (c) Plate in its equilibrium position, (d) Free-body diagram for motion away from equilibrium

Taking moments about O in figure XP5.2d gives

$$I_o \delta\ddot{\theta} = \Sigma M_{oz} = -\frac{wa\sqrt{5}}{2}\sin\delta\theta \Rightarrow \delta\ddot{\theta} + \frac{3\sqrt{5}}{10}\frac{g}{a}\delta\theta = 0 \ .$$

Task c is now completed.

Deriving the Equation of Motion from the Energy Equation

Energy is conserved; hence, $T+V=T_o+V_o$. Using a horizontal plane through the pivot point as the datum for gravity potential energy gives

$$I_o\frac{\dot{\theta}^2}{2}-wa\frac{\sqrt{5}}{2}\sin(\alpha+\theta)=0-w\frac{a}{2}$$

Substituting $I_o=5ma^2/3$ and differentiating w.r.t. θ gives

$$\frac{5ma^2}{2}\ddot{\theta}-wa\frac{\sqrt{5}}{2}\sin(\alpha+\theta)=0\ ;\ \ddot{\theta}=\frac{d(\dot{\theta}^2/2)}{d\theta}\ .$$

Lecture 28. MORE COMPOUND-PENDULUM EXAMPLES

Spring-Connection Vibration Examples

Nonlinear-Linear Spring relationships

We considered linearization of the pendulum equation earlier in this section. Linearization of connecting spring and damper forces for small motion of a pendulum is the subject of this lecture.

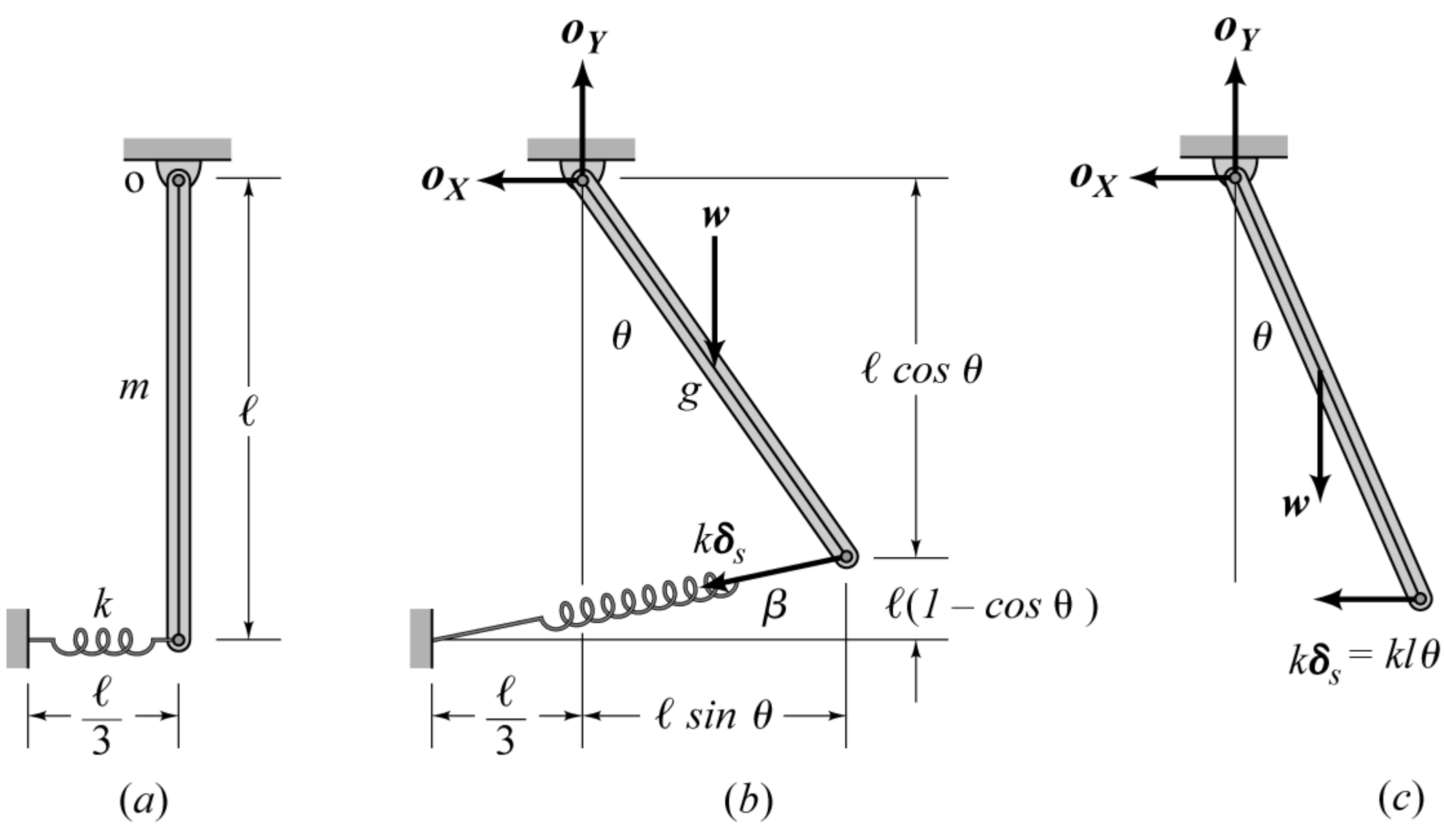


Figure 5.18 Compound pendulum with spring attachment to ground. (a) At rest in equilibrium, (b) General position, (c) Small-angle free-body diagram

The spring has length $l/3$ and is undeflected at $\theta = 0$. The following engineering-analysis tasks apply:

a. Draw free-body diagrams and derive the EOM

b. For small θ develop the linearized EOM.

Figure 5.18B provides the free-body diagram illustrating the stretched spring. The deflected spring length is

$$l_s^2 = (\frac{l}{3} + l\sin\theta)^2 + [l(1-\cos\theta)]^2$$

$$= l^2(\frac{1}{9} + \frac{2\sin\theta}{3} + \sin^2\theta + 1 - 2\cos\theta + \cos^2\theta) \tag{5.56}$$

$$= \frac{l^2}{9}(19 + 6\sin\theta - 18\cos\theta)\ .$$

Hence, the spring force is

$$f_s = k\delta_s = k(l_s - \frac{l}{3})\ ,$$

and it acts at the angle β from the horizontal defined by

$$\sin\beta = l(1-\cos\theta)/l_s\ , \quad \cos\beta = (\frac{l}{3} + l\sin\theta)/l_s\ . \tag{5.57}$$

The pendulum equation of motion is obtained by a moment equation about the pivot point, yielding

$$\Sigma M_o = I_o \ddot{\theta} = -w\frac{l}{2}\sin\theta - k\delta_s \cos\beta \times l\cos\theta - k\delta_s \sin\beta \times l \sin\theta$$

$$= -w\frac{l}{2}\sin\theta - k\delta_s l \cos(\theta - \beta)$$

Substituting for $\delta_s, \cos\beta, \sin\beta$ (plus a considerable amount of algebra) yields

$$\frac{ml^2}{3}\ddot{\theta} + \frac{wl}{2}\sin\theta + \frac{kl^2}{3}(\cos\theta + 3\sin\theta)[1 - \frac{1}{(19 + 6\sin\theta - 18\cos\theta)^{1/2}}] = 0 \ . \tag{5.58}$$

This is a "geometric" nonlinearity. The spring is linear, but the finite θ rotation causes a nonlinearity.

For small θ, expanding $\delta_s = l_s - l/3$ with l_s defined by Eq.(5.56) in a Taylor's series expansion gives $\delta_s \cong l\theta$. Also, for small θ, a Taylor series expansion gives $\beta \cong \sin\beta \cong -3\theta^2/2 \cong 0$; hence, for small θ, $\cos(\theta - \beta) \cong \cos\theta \cong 1$ the spring force acts perpendicular to the pendulum axis. For small θ, the moment equation reduces to

$$\frac{ml^2}{3}\ddot{\theta} = -w\frac{l}{2}\theta - k(l\theta)\times 1\times l$$

$$\therefore \quad \frac{ml^2}{3}\ddot{\theta} + (\frac{wl}{2} + kl^2)\theta = 0 \ . \tag{5.59}$$

For small θ, the spring deflection is $\delta_s = l\theta$, the spring force, $f_s \cong -k\delta_s = -kl\theta$, acts perpendicular to the pendulum, and the moment of the spring force about *o* is $kl^2\theta$. Also, note that the spring force is independent of its initial spring length. Figure 5.18c provides the small-angle free body diagram From Eq.(5.59), the natural frequency is

$$\omega_n = \sqrt{\frac{3g}{2l} + \frac{3k}{m}} \ ,$$

showing (as expected) an increase in the pendulum natural frequency due to the spring's stiffness.

Nonlinear-Linear Damper forces

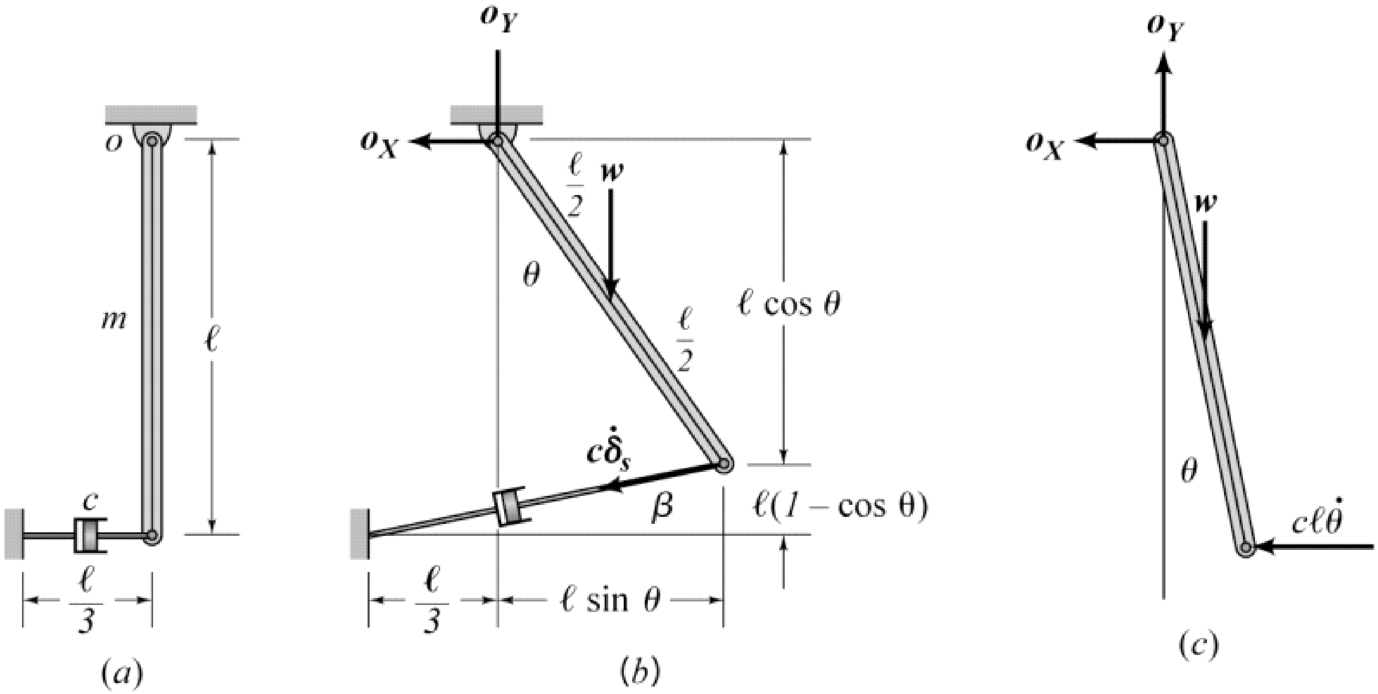


Figure 5.19 Compound pendulum. (a) At rest in equilibrium, (b) General-position free-body diagram, (c) Small rotation free-body diagram

For large θ the damper reaction force, $f_d = -c\dot{\delta}_s$, acts at the angle β from the horizontal. From Eq.(5.56),

$$\dot{\delta}_s = \dot{l}_s = l(\cos\theta/3 + \sin\theta)\dot{\theta}/\sqrt{\frac{19}{9} + \frac{2\sin\theta}{3} - 2\cos\theta)}$$

For small θ, $\beta \cong 0$, and the damping force acts perpendicular to the pendulum axis and reduces to $f_d = -cl\dot{\theta}$, where $v_\theta = l\dot{\theta}$ is the pendulum's circumferential velocity at the attachment point. Figure 5.19c provides a "small θ" free-body diagram, yielding the following equation of motion,

$$\Sigma M_o = I_o\ddot{\theta} = -w\frac{l}{2}\theta - l\times c l\dot{\theta} \;\Rightarrow\; \frac{ml^2}{3}\ddot{\theta} + cl^2\dot{\theta} + \frac{wl}{2}\theta = 0 \ ,$$

with $I_o = ml^2/3$ defined in Appendix C. The natural frequency and damping factor are:

$$\omega_n = \sqrt{\frac{3g}{2l}} \ , \quad 2\zeta\omega_n = cl^2\times\frac{3}{ml^2} = \frac{3c}{m} \ .$$

As with the spring, for small θ the damping force $f_d = -cl\dot{\theta}$ is independent of the initial damper length.

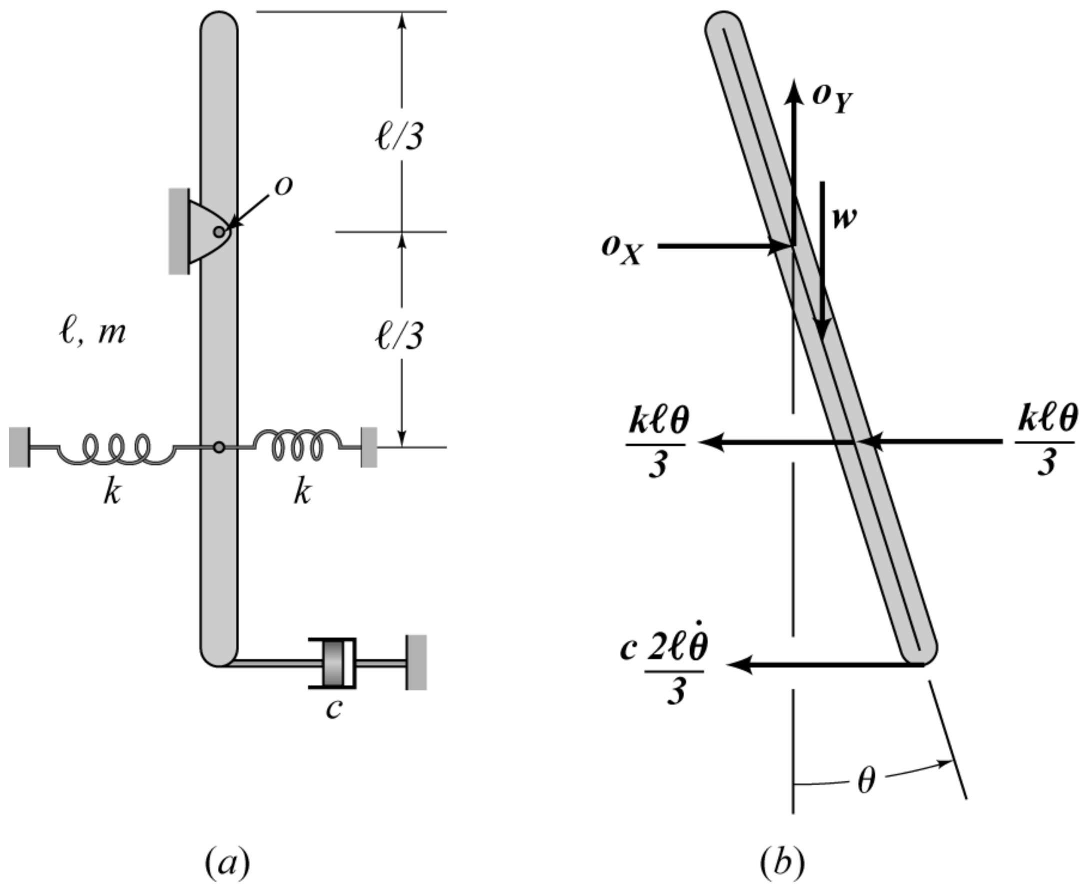


Figure XP5.3 (a) Pendulum attached to ground by two linear springs and a viscous damper, (b) Coordinate and free-body diagram

Figure XP5.3a shows a pendulum with mass $m = .5\,kg$ and length $l = 1\,m$ supported by a pivot point located $l/3$ from the pendulum's end. Two linear spring with stiffness coefficient $k = 15\,N/m$ are attached to the pendulum a distance $l/3$ down from the pivot point, and a linear damper with damping coefficient $c = .5\,N\sec/m$ is attached to the pendulum's end. The spring is undeflected when the pendulum is vertical. The following engineering analysis tasks apply to this system:

a. Draw a free-body diagram and derive the differential equation of motion.

b. Determine the natural frequency and damping factor.

A "small θ" free-body diagram is given in figure 5.22B.

Taking moments about the pivot point gives

$$I_o\ddot{\theta} = \Sigma M_o = -w\frac{l}{6}\sin\theta - 2\times k\frac{l\theta}{3}\times\frac{l}{3} - c\frac{2l\dot{\theta}}{3}\times\frac{2l}{3}$$

$$\therefore\ \frac{ml^2}{9}\ddot{\theta} = -\frac{wl}{6}\sin\theta - \frac{2kl^2}{9}\theta - \frac{4cl^2}{9}\dot{\theta}\ ,$$

From Appendix C and the parallel-axis formula, $I_o = ml^2/12 + m(l/6)^2 = ml^2/9$. For $\sin\theta \cong \theta$, the linearized EOM is

$$\ddot{\theta} + \frac{4c}{m}\dot{\theta} + (\frac{3g}{2l} + \frac{2k}{m})\theta = 0\ .$$

The natural frequency and damping factor are:

$$\omega_n = \left(\frac{3\times 9.81}{2\times 1} + \frac{30}{.5}\right)^{1/2} = 8.64\,\frac{rad}{\sec} \;\Rightarrow\; f_n = 1.38\,Hz$$

$$2\zeta\omega_n = \frac{4c}{m} \;\Rightarrow\; \zeta = \frac{4c}{2\omega_n m} = \frac{4\times .5}{2\times 8.64\times .5} = .231$$

EOM from Energy, neglecting damping

Select datum through pivot point; hence, $V = -wl/6\cos\theta$.

$$T + V = T_0 + V_0 \;\Rightarrow\; I_o\frac{\dot\theta^2}{2} + 2\times\frac{k}{2}\left(\frac{l\theta}{3}\right)^2 - w\frac{l}{6}\cos\theta = 0 + 0$$

Differentiating w.r.t. θ gives

$$I_o\ddot\theta + \frac{2kl^2}{9}\theta + \frac{wl}{6}\sin\theta = 0$$

$I_o = ml^2/9$, and for small θ, $\sin\theta \cong \theta$

$$\frac{ml^2}{9}\ddot\theta + \frac{kl^2}{9}\theta + \frac{wl}{6}\theta = 0 \;\Rightarrow\; \ddot\theta + \left(\frac{3g}{2l} + \frac{k}{m}\right)\theta = 0$$

Spring Supported Bar — Preload and Equilibrium

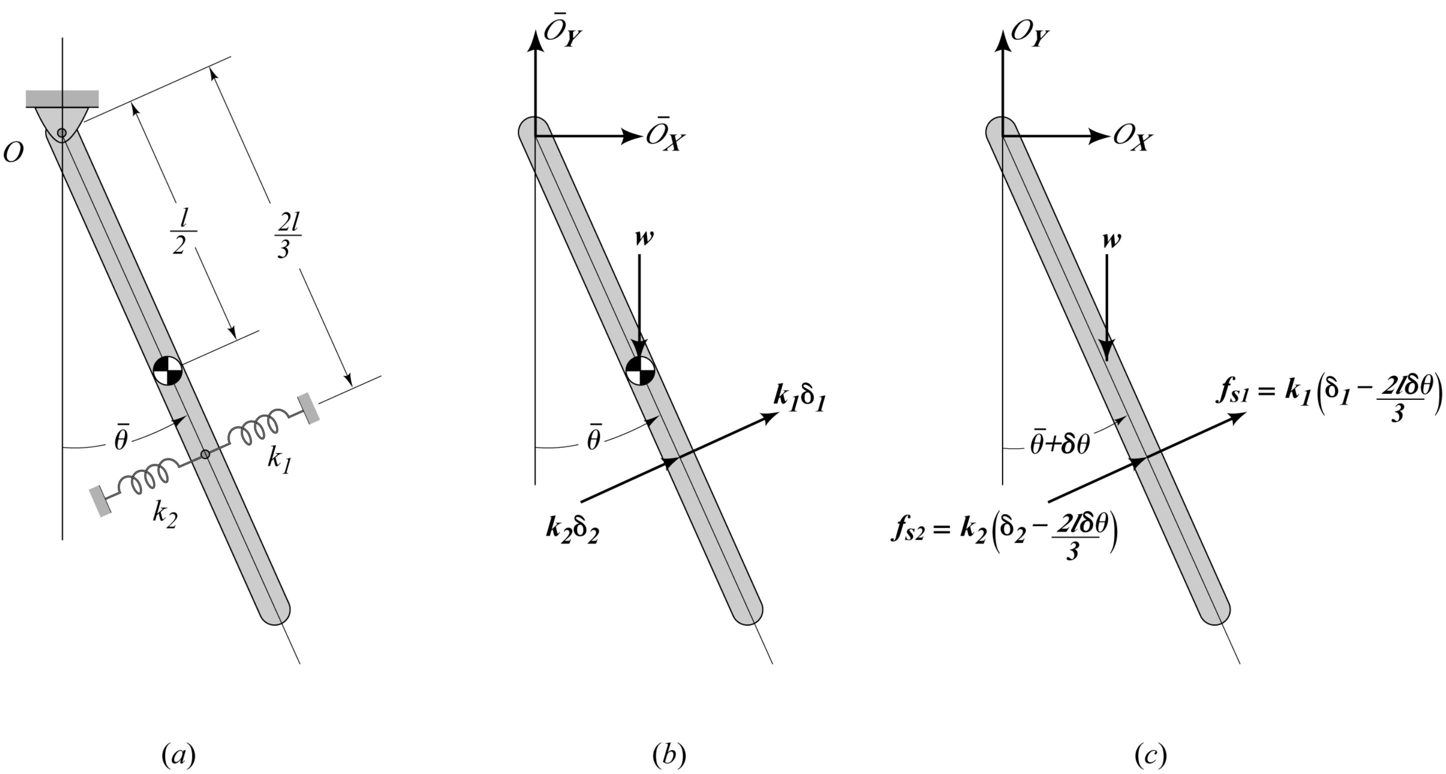


Figure 5.20 Uniform bar, (a) In equilibrium at the angle $\bar{\theta}$, (b) Equilibrium free-body diagram, (c) Displaced position free-body diagram

Bar of mass *m* and length *l* in equilibrium at $\theta = \bar{\theta}$ with linear springs having stiffness coefficients k_1, k_2 counteracting the weight *w*. The springs act at a distance *2l* / 3 from the pivot support point and have been preloaded (stretched or compressed) to maintain the bar in its equilibrium position.

Draw a free-body diagram, derive the EOM, and determine the natural frequency.

Equilibrium Conditions. Taking moments about O in Figure 5.20b gives

$$\Sigma M_O = 0 = w\frac{l}{2}\sin\bar{\theta} - k_1\delta_1\frac{2l}{3} - k_2\delta_2\frac{2l}{3} \ . \tag{5.84}$$

Non-equilibrium reaction forces.

Figure 5.20c provides a free-body diagram for a general displaced position defined by the rotation angle $\theta + \delta\theta$. For small $\delta\theta$ the spring-support point moves the perpendicular distance $\delta_s = (2l/3)\theta$. Hence, the stretch of the upper spring decreases from δ_1 to $\delta_1 - (2l/3)\theta$, and the compression of the lower spring decreases from δ_2 to $\delta_2 - (2l/3)\theta$. The spring reaction forces are:

$$f_{s1} = k_1(\delta_1 - \frac{2l}{3}\delta\theta) \ , \quad f_{s2} = k_2(\delta_2 - \frac{2l}{3}\delta\theta) \ .$$

Moment Equation about O

$$\Sigma M_O = I_O \delta\ddot{\theta} = -w\frac{l}{2}\sin(\overline{\theta}+\delta\theta) + \frac{2l}{3}k_1(\delta_1 - \frac{2l}{3}\delta\theta)$$
$$+\frac{2l}{3}k_2(\delta_2 - \frac{2l}{3}\delta\theta)$$

$$\cong -w\frac{l}{2}(\sin\overline{\theta} + \cos\overline{\theta}\,\delta\theta) + \frac{2l}{3}k_1\delta_1 + \frac{2l}{3}k_2\delta_2$$
$$-(\frac{2l}{3})^2(k_1+k_2)\delta\theta$$

$$= (-w\frac{l}{2}\sin\overline{\theta} + \frac{2l}{3}k_1\delta_1 + \frac{2l}{3}k_2\delta_2) - w\frac{l}{2}\cos\overline{\theta}\,\delta\theta$$
$$-(\frac{2l}{3})^2(k_1+k_2)\delta\theta \quad ,$$

after dropping second-order terms in $\delta\theta$. Rearranging provides the EOM ,

$$\frac{ml^2}{3}\delta\ddot{\theta} + [(\frac{2l}{3})^2(k_1+k_2) + w\frac{l}{2}\cos\overline{\theta}]\delta\theta = 0$$
$$= -w\frac{l}{2}\sin\overline{\theta} + \frac{2l}{3}k_1\delta_1 + \frac{2l}{3}k_2\delta_2 \quad . \tag{5.85}$$

The right-hand side of Eq.(5.85) is zero from the equilibrium result of Eq.(5.84). If the bar is in equilibrium in a vertical position $(\bar\theta = 0,\ \cos\bar\theta = 1)$, the weight contribution to the EOM reverts to the compound pendulum results of Eq.(3.60). For a horizontal equilibrium position, $(\bar\theta = \pi/2,\ \cos\bar\theta = 0)$, and the weight term is eliminated. The natural frequency is

$$\omega_n^2 = \frac{3}{ml^2}\left[\left(\frac{2l}{3}\right)^2(k_1 + k_2) + w\frac{l}{2}\cos\bar\theta\right]$$

$$\therefore\ \omega_n = \sqrt{\frac{4(k_1 + k_2)}{3m} + \frac{3g}{2l}\cos\bar\theta}\ . \tag{5.86}$$

Alternative Equilibrium Condition In figure 5.21a, the lower spring is also assumed to be in tension with a static stretch δ_2, developing the tension force $k_2\delta_2$ at equilibrium. Taking moments about O gives the static equilibrium requirement

$$\Sigma M_O = 0 = -w\frac{l}{2}\sin\bar\theta + k_1\delta_1\frac{2l}{3} - k_2\delta_2\frac{2l}{3}\ . \tag{5.87}$$

The $\delta\theta$ rotation increases the stretch in the lower spring from δ_2 to $\delta_2 + (2l/3)\delta\theta$, decreases the stretch in the upper spring from δ_1 to $\delta_1 - (2l/3)\delta\theta$, and the reaction forces are:

$$f_{s1} = k_1\left(\delta_1 - \frac{2l}{3}\delta\theta\right)\ ,\ f_{s2} = k_2\left(\delta_2 + \frac{2l}{3}\delta\theta\right)\ .$$

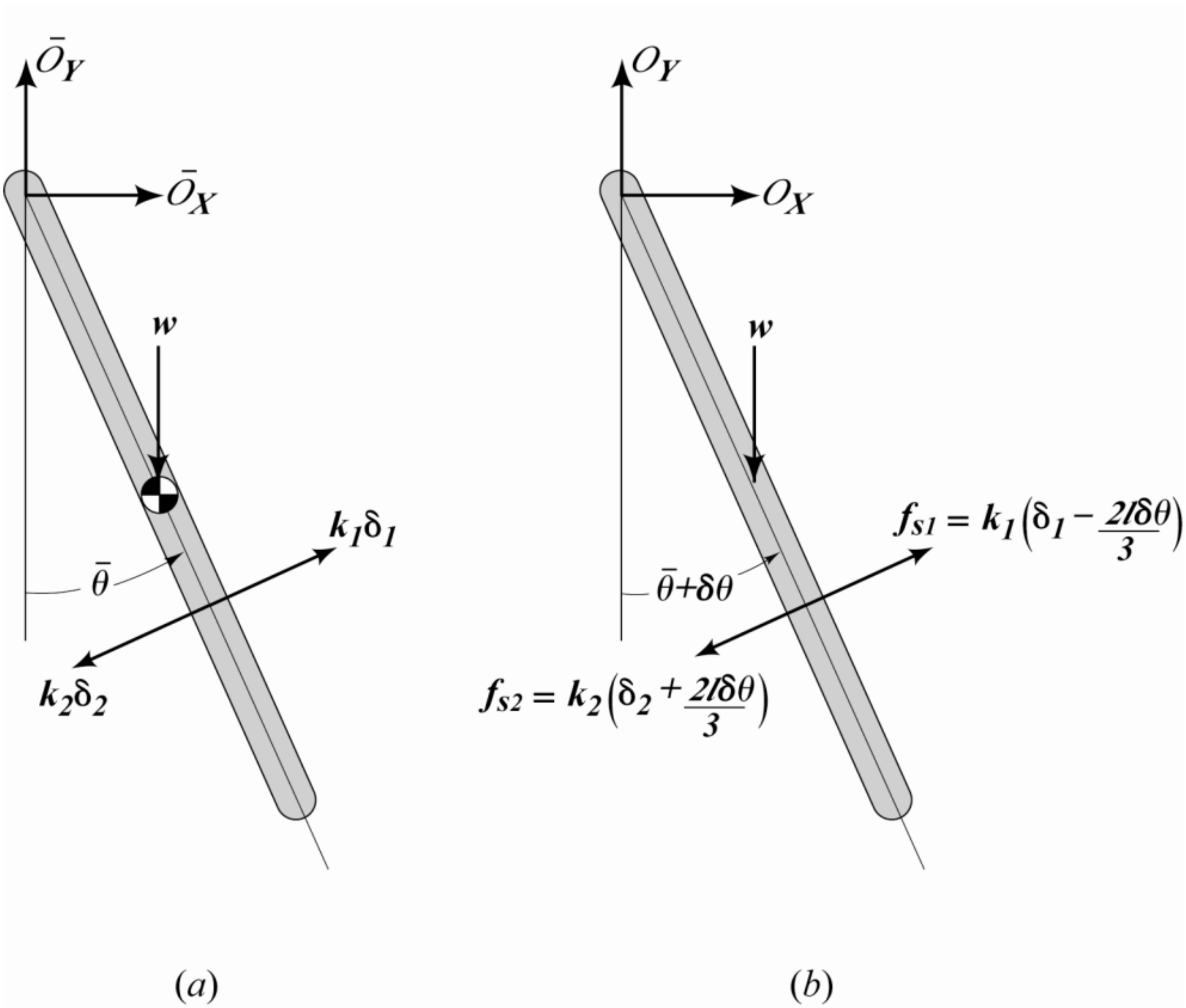


Figure 5.21 Uniform bar: (a) Alternative static equilibrium free-body diagram, (b) Displaced-position free-body diagram

From figure 5.21b,

$$\Sigma M_O = I_O \delta\ddot{\theta} = -w\frac{l}{2}\sin(\bar{\theta} + \delta\theta) + \frac{2l}{3}k_1(\delta_1 - \frac{2l}{3}\delta\theta) - \frac{2l}{3}k_2(\delta_2 + \frac{2l}{3}\delta\theta)$$

$$\cong -w\frac{l}{2}(\sin\bar{\theta} + \cos\bar{\theta}\,\delta\theta) + \frac{2l}{3}k_1\delta_1 - \frac{2l}{3}k_2\delta_2 - (\frac{2l}{3})^2(k_1 + k_2)\delta\theta$$

$$= (-w\frac{l}{2}\sin\bar{\theta} + \frac{2l}{3}k_1\delta_1 - \frac{2l}{3}k_2\delta_2) + [w\frac{l}{2}\cos\bar{\theta} - (\frac{2l}{3})^2(k_1 + k_2)]\delta\theta \quad ,$$

and the EOM is (again)

$$\frac{ml^2}{3}\delta\ddot{\theta} + [(\frac{2l}{3})^2(k_1 + k_2) + w\frac{l}{2}\cos\bar{\theta}]\delta\theta = 0 = -w\frac{l}{2}\sin\bar{\theta} + \frac{2l}{3}k_1\delta_1 - \frac{2l}{3}k_2\delta_2 \quad . \tag{5.88}$$

The right-hand side is zero from the equilibrium requirement of Eq.(5.87), and Eq.(5.88) repeats the EOM of Eq.(5.85).

The lesson from this second development is: For small motion about equilibrium, the same EOM is obtained *irrespective* of the initial equilibrium forces in the (linear) springs. The spring-force contributions to the differential equation arise from the *change* in the equilibrium forces due to a *change in position*. This is the same basic outcome that we obtained for a mass m supported by linear springs in figure 3.7. The change in equilibrium angle θ changes w's contribution to the EOM , because $-w(l/2)\sin\theta$, the moment due to w, is a *nonlinear* function of θ.

Prescribed acceleration of a Pivot Support Point

Moment equations for the fixed-axis rotation problems of the preceding section were taken about a *fixed* pivot point, employing the moment equation

$$M_{oz} = I_o \ddot{\theta} \ , \tag{5.26}$$

where o identifies the axis of rotation. The problems involved in this short section concerns situations where the pivot point is accelerating, and the general moment equation,

$$M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z \ , \tag{5.24}$$

is required. In applying Eq.(5.24), recall the following points:

a. Moments are being taken about the body-fixed axis o, and I_o is the moment of inertia through axis o.

b. The vector $\boldsymbol{b_{og}}$ goes from a z axis through o to a z axis through the mass center at g.

c. The positive rotation and moment sense in Eq.(5.24) correspond to a counter clockwise rotation for θ.

The last term in the moment equation is positive because the positive right-hand-rule convention for the cross-product in this term coincides with the $+\theta$ sense. For a rigid body with a positive clockwise rotation angle this last term requires a negative sign.

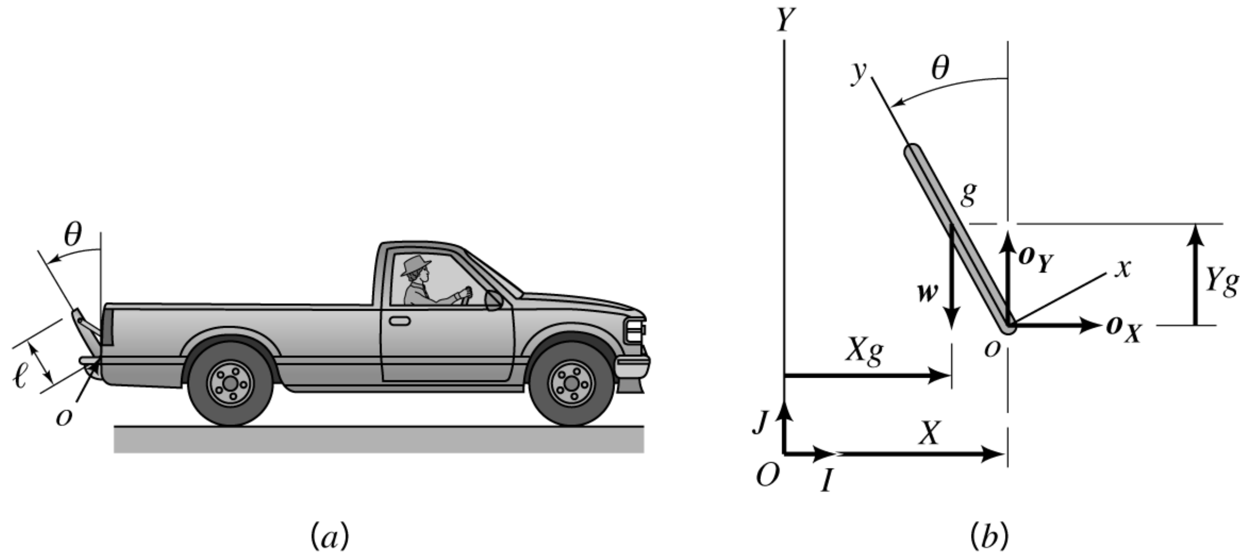


Figure 5.24 (a). An accelerating pickup truck with a loose tail gate. (b). Free-body diagram for the tail gate.

The pickup has a constant acceleration of $g/3$. Neglecting friction at the pivot and assuming that the tailgate can be modeled as a uniform plate of mass m, carry out the following engineering tasks:

a. Derive the governing equation of motion.

b. Assuming that the tailgate starts from rest at $\theta = 0$, *what will* $\dot{\theta}$ *be at* $\theta = \pi/2$?

c. Determine the reactions at pivot point o as a function of θ *(only).*

In applying Eq.(5.24) for moments about axis o, we can observe

$$\boldsymbol{b_{og}} = \frac{l}{2}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)$$

$$\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = \boldsymbol{I}\ddot{X} = \boldsymbol{I}\frac{g}{3}$$

$$\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}} = -\boldsymbol{K}\frac{gl}{6}\cos\theta \ .$$

Hence, Eq.(5.24) gives

$$\begin{aligned} \Sigma M_o = w\frac{l}{2}\sin\theta &= \frac{ml^2}{3}\ddot{\theta} - m\frac{gl}{6}\cos\theta \\ \therefore \quad \frac{ml^2}{3}\ddot{\theta} &= \frac{wl}{2}\sin\theta + \frac{wl}{6}\cos\theta \ . \end{aligned} \tag{5.61}$$

We have now completed *Task a*. We can use the energy-integral substitution to integrate this nonlinear equation of motion as

$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{\dot{\theta}^2}{2}\right) = \frac{3g}{2l}\sin\theta + \frac{g}{2l}\cos\theta. \tag{5.62}$$

Multiplying by $d\theta$ reduces both sides of this equation to exact differentials. Integrating both sides with the initial condition $\dot{\theta}(\theta = 0) = 0$ gives

$$\frac{\dot{\theta}^2}{2} = \frac{g}{l} \Big|_0^{\theta} \left(-\frac{3}{2}\cos u + \frac{1}{2}\sin u \right)$$
$$= \frac{g}{l}\left[\frac{1}{2}\sin\theta + \frac{3}{2}(1-\cos\theta) \right] \ . \tag{5.63}$$

Hence, at $\theta = \pi / 2$, $\dot{\theta}(\pi/2) = 2\sqrt{g/l}$,and we have completed *Task b*. We used the energy-integral substitution, <u>but note that the tail gate's mechanical energy energy is not conserved</u>. The truck's acceleration is adding energy to the tail gate.

Moving on to *Task c*, stating $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ for the mass center gives:

$$\Sigma f_X = o_X = m\ddot{X}_g$$
$$\Sigma f_Y = o_Y - w = m\ddot{Y}_g \tag{5.64}$$

We need to determine $\ddot{X}_g, \ddot{Y}_g$ in these equations. From figure 5.19B,

$$X_g = X - \frac{l}{2}\sin\theta \ \ ; \quad Y_g = \frac{l}{2}\cos\theta \ .$$

Differentiating twice with respect to time gives:

$$\ddot{X}_g = \ddot{X} - \frac{l}{2}\cos\theta\,\ddot{\theta} + \frac{l}{2}\sin\theta\,\dot{\theta}^2 = \frac{g}{3} - \frac{l}{2}\cos\theta\,\ddot{\theta} + \frac{l}{2}\sin\theta\,\dot{\theta}^2$$
$$\ddot{Y}_g = -\frac{l}{2}\sin\theta\,\ddot{\theta} - \frac{l}{2}\cos\theta\,\dot{\theta}^2 \ .$$

Substituting into Eqs.(5.64) gives:

$$\begin{aligned} o_X &= m\left(\frac{g}{3} - \frac{l}{2}\cos\theta\ \ddot{\theta} + \frac{l}{2}\sin\theta\ \dot{\theta}^2\right) \\ o_Y - w &= -m\left(\frac{l}{2}\sin\theta\ \ddot{\theta} + \frac{l}{2}\cos\theta\ \dot{\theta}^2\right)\ , \end{aligned} \tag{5.65}$$

where $\ddot{X}$ has been replaced with $g/3$, the pick-up truck's acceleration. Substituting from Eqs.(5.62) and (5.63) for $\ddot{\theta}$ and $\dot{\theta}^2$, respectively, (and some algebra) gives:

$$\begin{aligned} o_X &= w\left(\frac{1}{3} + 2\sin\theta - \frac{3}{2}\sin 2\theta\right) \\ o_Y &= w\left(\frac{11}{8} - \frac{3}{2}\cos\theta + \frac{9}{8}\cos 2\theta - \frac{3}{8}\sin 2\theta\right)\ , \end{aligned}$$

and completes *Task c.*

The decision to use the general moment Eq.(5.24) and sum moments about the pivot point *o* instead of the mass center *g* saves a great deal of effort in arriving at the differential equation of motion. To confirm this statement, consider the following moment equation about *g*

$$\Sigma M_g = o_X \frac{l}{2}\cos\theta + o_Y \frac{l}{2}\sin\theta = I_g \ddot{\theta}$$

Substituting from Eq.(5.65) for o_X, o_Y gives

$$I_g \ddot{\theta} = \frac{l}{2}\cos\theta\, m\left(\frac{g}{3} - \frac{l}{2}\cos\theta\,\ddot{\theta} + \frac{l}{2}\sin\theta\,\dot{\theta}^2\right)$$
$$+ \frac{l}{2}\sin\theta\left[w - m\left(\frac{l}{2}\sin\theta\,\ddot{\theta} + \frac{l}{2}\cos\theta\,\dot{\theta}^2\right)\right] .$$

Gathering like terms gives

$$\left[I_g + \frac{ml^2}{4}(\sin^2\theta + \cos^2\theta)\right]\ddot{\theta} = \frac{wl}{6}\cos\theta + \frac{wl}{2}\sin\theta$$
$$+ \frac{ml}{4}\dot{\theta}^2(\sin\theta\cos\theta - \sin\theta\cos\theta) .$$

Simplifying these equations gives Eq.(5.62) ,the original differential equation of motion.

The lesson from this short section is: In problems where a pivot support point has a prescribed acceleration, stating the moment equation (correctly) about the pivot point will lead to the governing equation of motion much more quickly and easily than taking moments about the mass center.

Note: Energy is not conserved with base acceleration!

Lecture 29. GENERAL MOTION/ROLLING-WITHOUT-SLIPPING EXAMPLES
General equations of Motion

Force Equation

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15}$$

General Moment Equation

$$\sum M_{oz} = I_o\ddot{\theta} + m(\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}} \ . \tag{5.24}$$

Moments About the Mass Center

$$M_{gz} = I_g\ddot{\theta} \ . \tag{5.25}$$

Moments About a Fixed axis

$$M_{oz} = I_o\ddot{\theta} \ . \tag{5.26}$$

The examples of this lecture will be analyzed using $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ and $\Sigma M_{gz} = I_g\ddot{\theta}$.

Kinetic Energy of a Rigid Body

$$T = \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{g}}|^2}{2} + \frac{I_g\ \dot{\theta}^2}{2} \ , \tag{5.205}$$

A Cylinder Rolling Down An Inclined Plane

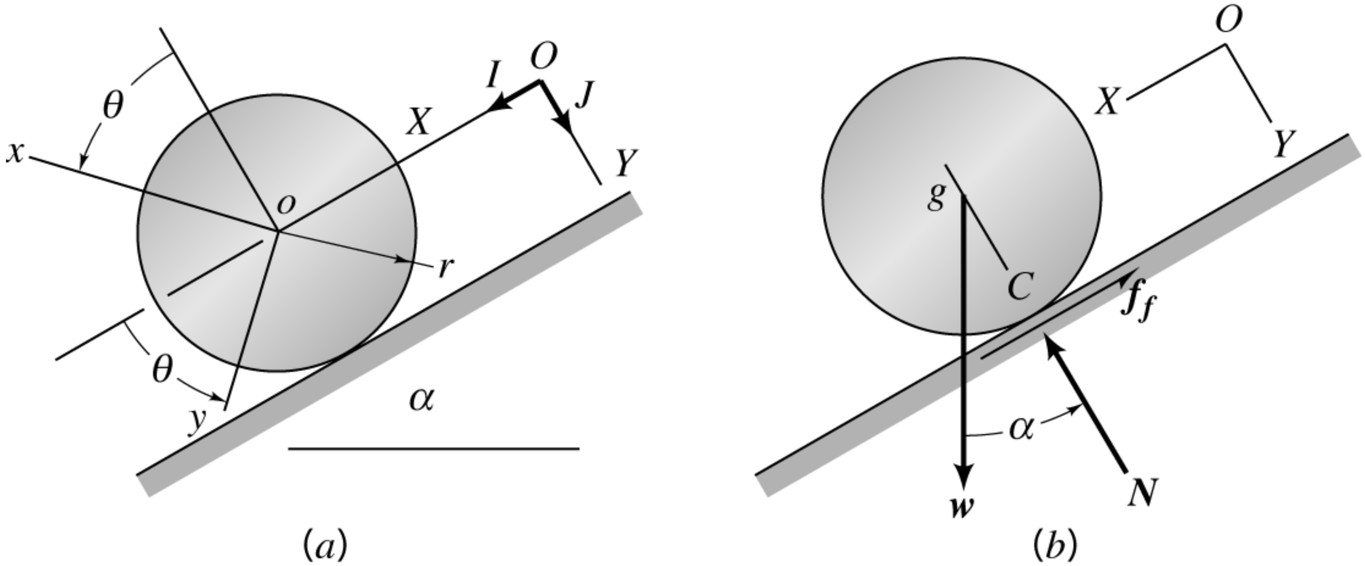


Figure 5.25 (*a*) Uniform disk of radius *r* and mass *m* rolling (without slipping) down an inclined plane, (*b*) Free-body diagram. The static Coulomb coefficient of friction for the plane is μ_s.

Derive the governing differential equation of motion for rolling without slipping.

Force Equation Components

$$\Sigma f_X = w \sin\alpha - f_f = m\ddot{X}$$
$$\Sigma f_Y = w \cos\alpha - N = m\ddot{Y} = 0 \;\Rightarrow\; N = W\cos\alpha. \tag{5.75}$$

Moment Equation

$$\Sigma M_{\theta g} = f_f r = I_g \ddot{\theta} = \frac{m r^2}{2} \ddot{\theta}. \tag{5.76}$$

Eq.(5.76) and the first of Eq.(5.75) constitute two equations in the three unknowns: $\ddot{\theta}, \ddot{X},$ and f_f.

Rolling-without-slipping kinematic constraint

$$\ddot{X} = r \ddot{\theta} \;. \tag{5.77}$$

Solving for f_f from Eq.(5.76), and substituting into the first of Eq.(5.75) gives

$$m\ddot{X} = w \sin\alpha - m \frac{r}{2} \ddot{\theta}.$$

Now, substituting for $\ddot{\theta} = \ddot{X}/r$ from Eq.(5.77) gives

$$\boxed{m\ddot{X} + m \frac{r}{2} \cdot \frac{\ddot{X}}{r} = \frac{3m}{2} \ddot{X} = w \sin\alpha \;\Rightarrow\; \ddot{X} = \frac{2g}{3} \sin\alpha. \qquad (5.78)}$$

Using θ as the dependent variable, solve the first of Eq.(5.75) for f_f and then substitute into Eq.(5.76) to obtain

$$I_g \ddot{\theta} = r(w \sin\alpha - m\ddot{X}) \;,$$

Substituting $\ddot{X} = r\ddot{\theta}$ from Eq.(5.77) gives

$$(I_g + mr^2)\ddot{\theta} = \frac{3mr^2}{2}\ddot{\theta} = wr\sin\alpha. \tag{5.79}$$

Comparisons of Eqs.(5.78) and (5.79) show basically the same equation.

Friction Force

$$\textit{From,} \quad rf_f = m\frac{r^2}{2}\ddot{\theta} \ , \quad f_f = 0 \ \Rightarrow \quad \ddot{\theta} = 0$$

$$\textit{From,} \quad w\sin\alpha - f_f = m\ddot{X} \ , \quad f_f = 0 \ \Rightarrow \quad \ddot{X} = g\sin\alpha \ .$$

The friction force causes the cylinder to rotate and reduces the acceleration to $\ddot{X} = (2/3)g\sin\alpha$.

How much Coulomb friction is required to prevent slipping ?

$$f_f = w\sin\alpha - m\ddot{X} = w\sin\alpha - m(\frac{2g}{3}\sin\alpha) = \frac{w}{3}\sin\alpha \ . \tag{5.80}$$

Since, $N = w\cos\alpha$, the "required" static Coulomb friction force coefficient to prevent slipping is

$$\mu_s(\textit{required}) = \frac{f_f}{N} = \frac{w\sin\alpha}{3} \times \frac{1}{w\cos\alpha} = \frac{\tan\alpha}{3}. \tag{5.81}$$

If this calculated value is less than or equal to μ_s, the wheel will roll without slipping. If it greater than μ_s, the wheel will slip, and $f_f = \mu_d N$.

Slipping Motion Equations of Motion. For slipping $f_f = \mu_d N = \mu_d w \cos\alpha$, the cylinder has two degrees of freedom, and the equations of motion are

$$w \sin\alpha - \mu_d w \cos\alpha = m\ddot{X}$$

$$\frac{mr^2}{2}\ddot{\theta} = r\mu_d w \cos\alpha \ .$$

Deriving the equation of Motion for Rolling-Without-Slipping from Conservation of Energy

Without slipping, energy is conserved and $T + V = T_0 + V_0$. Taking the origin of the *X, Y* system as the datum for potential energy,

$$V_g = -wX\sin\alpha \ . \tag{i}$$

The kinetic energy is defined to be

$$T = \frac{m}{2}\dot{X}^2 + \frac{mr^2}{4}\dot{\theta}^2 = \frac{m}{2}\dot{X}^2 + \frac{mr^2}{4}\left(\frac{\dot{X}}{r}\right)^2 = \frac{3m}{4}\dot{X}^2 \ , \tag{ii}$$

where the rolling without slipping relationship $\dot{X} = r\dot{\theta}$ has been used. Substituting from (i) and (ii) gives

$$\frac{3m}{2}\left(\frac{\dot{X}^2}{2}\right) - wX\sin\alpha = T_0 + V_0 \ .$$

Differentiating w.r.t. X

$$\frac{3m}{2}\frac{d}{dX}\left(\frac{\dot{X}^2}{2}\right) - w\sin\alpha = 0 \ \Rightarrow \ \frac{3m}{2}\ddot{X} = w\sin\alpha \ .$$

An Imbalanced Cylinder Rolling Down an Inclined Plane

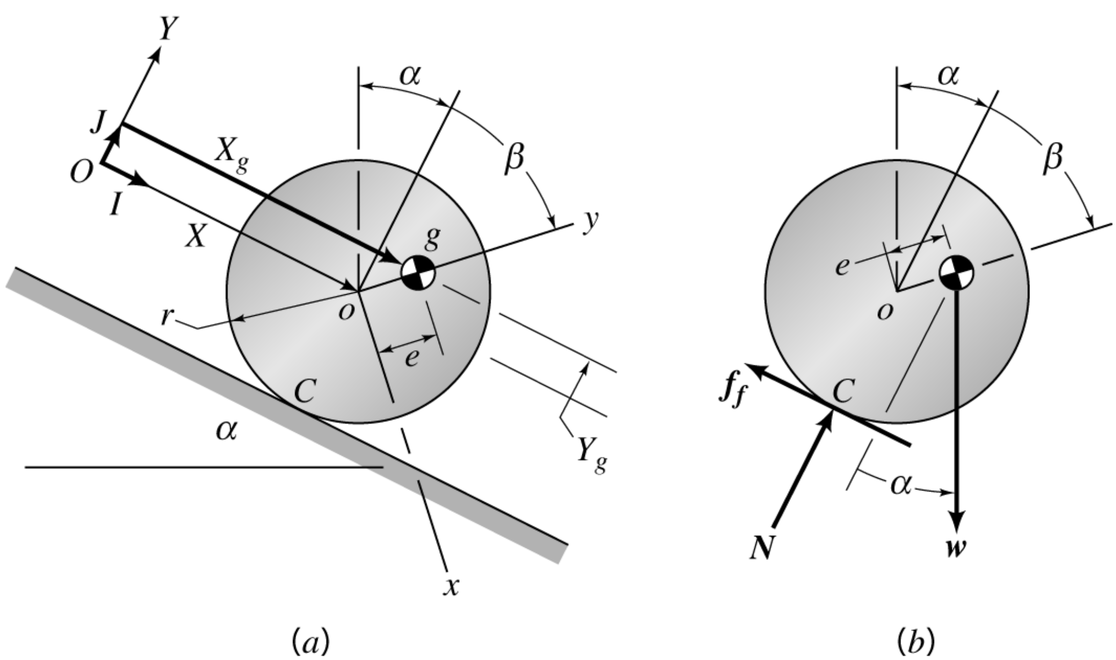


Figure 5.26 (a) Disk with its mass center displaced a distance e from its geometrical center, rolling down an inclined plane, (b) Free-body diagram.

The disk's mass center is located in the *X, Y* system by:

$$X_g = X + e\sin\beta \ , \quad Y_g = e\cos\beta \ . \tag{5.83}$$

The rolling-without-slipping kinematic condition for figure 5.26A is

$$\ddot{X} = r\ddot{\beta} \ . \tag{5.84}$$

Without slipping, the disk has two variables, *X* and β, but only one degree of freedom. The radius of gyration of the disk about point *o* is k_{og} ; hence, $I_o = m k_{og}^2$.

Derive the governing differential equation of motion.

Solution A. Take moments about the mass center.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ for the disk's mass center gives:

$$\begin{aligned} \Sigma f_X &= w\sin\alpha - f_f = m\ddot{X}_g \\ \Sigma f_Y &= N - w\cos\alpha = m\ddot{Y}_g \ . \end{aligned} \tag{5.85}$$

In reviewing these equations, note that $\ddot{Y}_g \neq 0$.

Taking moments about the mass center, the moment equation is

$$\Sigma M_g = Ne\sin\beta + f_f(r + e\cos\beta) = I_g\ddot{\beta} \ . \tag{5.86}$$

Note that positive moments are in the $+\beta$, clockwise direction. Also, from the parallel-axis formula, $I_g = I_o - me^2 = m(k_g^2 - e^2)$.

Eqs.(5.84), (5.85) and (5.86) provide four equations in the six unknowns: $\ddot{X}, \ddot{\beta}, \ddot{X}_g, \ddot{Y}_g, N,$ and f_f.

Kinematics. Differentiating Eqs.(5.83) once with respect to time gives

$$\dot{X}_g = \dot{X} + e\cos\beta\dot{\beta} \ , \quad \dot{Y}_g = -e\sin\beta\dot{\beta} \ .$$

Differentiating a second time and substituting for $\ddot{X} = r\ddot{\beta}$ gives

$$\ddot{X}_g = r\ddot{\beta} + e\cos\beta\ddot{\beta} - e\sin\beta\dot{\beta}^2 \ , \quad \ddot{Y}_g = -e\sin\beta\ddot{\beta} - e\cos\beta\dot{\beta}^2, \tag{5.87}$$

which provides our final two equations.

The governing equation of motion is obtained by the following steps:

a. Substitute for $\ddot{X}_g, \ddot{Y}_g$ from Eq.(5.87) into Eqs (5.85), obtaining

$$f_f = w\sin\alpha - m(r\ddot{\beta} + e\cos\beta\,\ddot{\beta} - e\sin\beta\,\dot{\beta}^2)$$
$$N = w\cos\alpha - m(e\sin\beta\,\ddot{\beta} + e\cos\beta\,\dot{\beta}^2)\ . \tag{5.88}$$

b. Substitute for N and f_f into Eq.(5.86), obtaining

$$I_g\ddot{\beta} = [w\cos\alpha - m(e\sin\beta\,\ddot{\beta} + e\cos\beta\,\beta)^2]e\sin\beta +$$
$$[w\sin\alpha - m(r\ddot{\beta} + e\cos\beta\,\ddot{\beta} - e\sin\beta\,\dot{\beta}^2)](r + e\cos\beta)\ .$$

After a fair amount of algebra, the governing equation is

$$m(k_{og}^2 + r^2 + 2re\cos\beta)\ddot{\beta} - mer\sin\beta\,\dot{\beta}^2$$
$$= w[r\sin\alpha + e\sin(\beta+\alpha)]. \tag{5.89}$$

Note: $I_o = mk_{og}^2 \Rightarrow I_g = I_o - me^2 = m(k_{og}^2 - e^2)$.

This equation reduces to Eq.(5.79) if $e = 0$, and $I_o = mr^2/2$.

The energy-integral substitution, $\ddot{\beta} = d(\dot{\beta}^2/2)/d\beta$, converts Eq.(5.89) to

$$\frac{d}{d\beta}[m(k_{og}^2 + r^2 + 2re\cos\beta)(\frac{\dot{\beta}^2}{2})] = w[r\sin\alpha + e\sin(\beta+\alpha)].$$

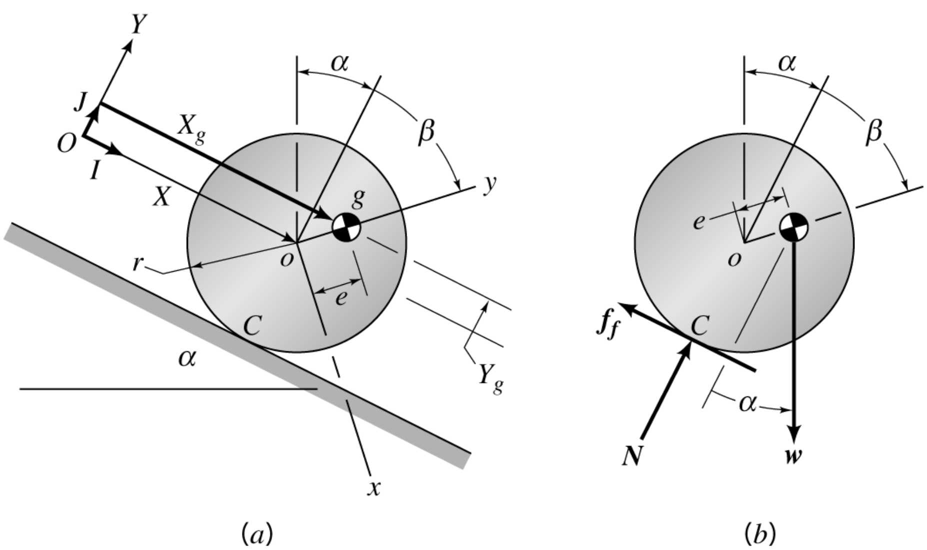
For the boundary conditions ($\beta(0)=0$; $\dot{\beta}(0)=0$); integration gives

$$m(k_{og}^2 + r^2 + 2re\cos\beta)(\frac{\dot{\beta}^2}{2}) \tag{5.90}$$

$$= w[r\sin\alpha\beta - e\cos(\beta+\alpha) + e\cos\alpha] \ .$$

Without slipping, there is no energy dissipation.

Solution B: Take moments about C.



For moments about C, the EOM is

$$\sum \boldsymbol{M_C} = I_C\ddot{\beta} - m(\boldsymbol{b_{Cg}} \times \ddot{\boldsymbol{R}}_C) \ ,$$

where clockwise moments are positive (positive β rotation).

Kinematics:

$$\boldsymbol{b}_{\boldsymbol{Cg}} = \boldsymbol{I} e \sin\beta + \boldsymbol{J}(r + e\cos\beta)\ ,\ \ddot{\boldsymbol{R}}_{\boldsymbol{c}} = \boldsymbol{J} r \dot{\beta}^2$$

$$\therefore\ \boldsymbol{b}_{\boldsymbol{Cg}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{C}} = \boldsymbol{K} e r \dot{\beta}^2 \sin\beta$$

Inertia properties:

$$I_g = I_o - me^2 = mk_{og}^2 - me^2$$

$$I_C = I_g + m\,|\,b_{gC}\,|^2 = (mk_{og}^2 - me^2) + m[(r + e\cos\beta)^2 + (e\sin\beta)^2]$$
$$= mk_{og}^2 + m(r^2 + 2re\cos\beta)$$

External moment about C due to weight

$$\begin{aligned}\boldsymbol{M}_{\boldsymbol{C}} &= -(\boldsymbol{r}_{\boldsymbol{Cg}} \times \boldsymbol{w}) \\ &= -[\boldsymbol{I} e\sin\beta + \boldsymbol{J}(r + e\cos\beta)] \times w(\boldsymbol{I}\sin\alpha + \boldsymbol{J}\cos\alpha)) \\ &= \boldsymbol{K}(we\sin\beta\cos\alpha + rw\sin\alpha + we\sin\alpha\, cos\,\beta \\ &= \boldsymbol{K}[wr\sin\alpha + we\sin(\alpha + \beta)]\end{aligned}$$

Plugging in the results gives (again)

$$m(k_{og}^2 + r^2 + 2re\cos\beta)\ddot{\beta} - mer\sin\beta\,\dot{\beta}^2$$
$$= w[r\sin\alpha + e\sin(\beta + \alpha)]. \tag{5.89}$$

This approach is obviously quicker.

Deriving the equation of Motion from Conservation of Energy

Applying $T+V=T_0+V_0$, and using the origin of the X, Y system for the gravity potential-energy function gives

$$\frac{I_g}{2}\dot{\beta}^2+\frac{m}{2}(\dot{X}_g^2+\dot{Y}_g^2)-wX\sin\alpha+we\cos(\alpha+\beta)$$
$$=(T_0+V_0)\ . \tag{5.223}$$

This example has three coordinates X_g, Y_g, β. To eliminate unwanted coordinates, we need the rolling-without-slipping condition $X=r\beta$, plus the kinematic conditions,

$$X_g=X+e\sin\beta=r\beta+e\sin\beta\ \Rightarrow\ \dot{X}_g=r\dot{\beta}+e\cos\beta\,\dot{\beta}$$

$$Y_g=e\cos\beta\ \Rightarrow\ \dot{Y}_g=-e\sin\beta\,\dot{\beta}\ .$$

Substituting into Eq.(5.223) gives

$$m(k_{og}^2+r^2+2er\cos\beta)\frac{\dot{\beta}^2}{2}$$
$$-w[r\beta\sin\alpha-e\cos(\alpha+\beta)]=(T_0+V_0)\ , \tag{5.224}$$

Differentiating with respect to β gives the DEQ. of motion

$$m(k_{og}^2+r^2+2er\cos\beta)\ddot{\beta}-mer\sin\beta\,\dot{\beta}^2$$
$$=w[r\sin\alpha+e\sin(\alpha+\beta)] \quad . \tag{5.204}$$

Lecture 30. MORE GENERAL-MOTION/ROLLING-WITHOUT-SLIPPING EXAMPLES

A Cylinder, Restrained by a Spring and Rolling on a Plane

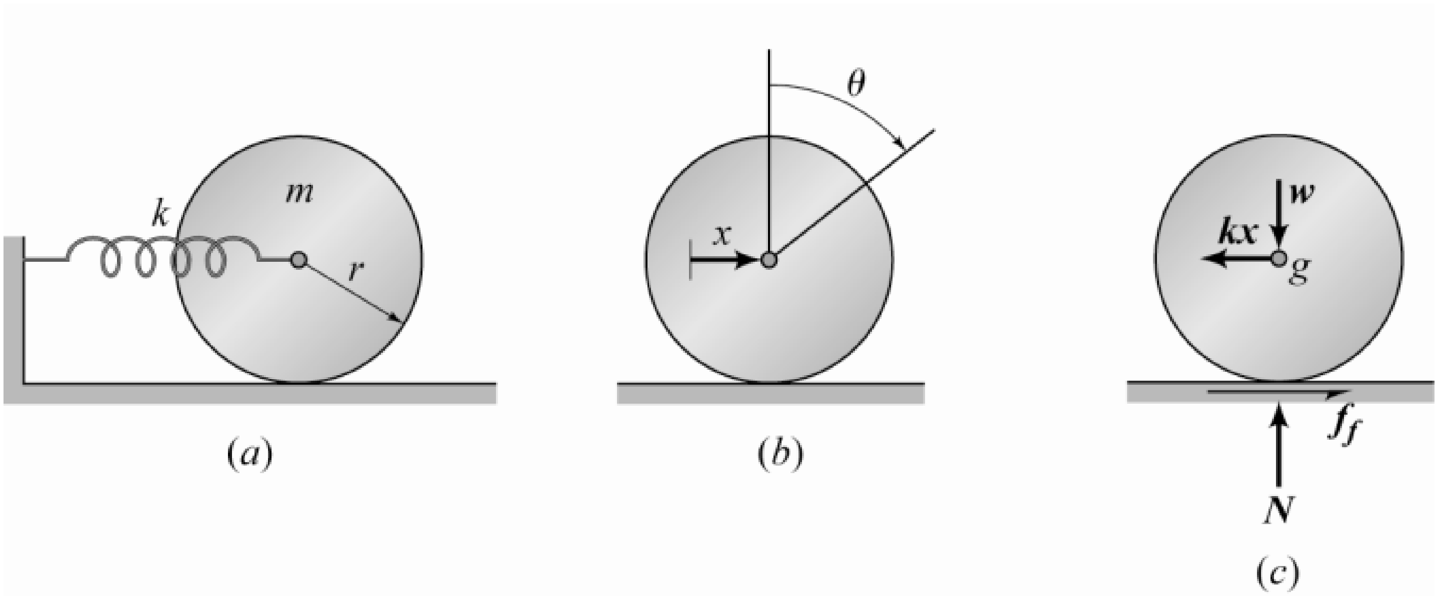


Figure 5.28 (a) Spring-restrained cylinder, (b) Kinematic variables, (c) Free-body diagram

The cylinder rolls without slipping. The spring is undeflected when $x = 0$.

The following engineering-analysis tasks apply:

a. Draw a free body diagram and derive the equation of motion, and

b. Determine the natural frequency for small amplitude vibrations.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ for the mass center of the cylinder nets

$$\Sigma f_X = f_f - kx = m\ddot{x} \ . \tag{5.117a}$$

Stating the moment equation about the cylinder's mass center gives

$$\Sigma M_g = -f_f r = I_g \ddot{\theta} = \frac{mr^2}{2}\ddot{\theta} \ . \tag{5.117b}$$

We now have two equations in the three unknowns $\ddot{x}, \ddot{\theta}, f_f$. The rolling-without-slipping kinematic condition,

$$x = r\theta \ \Rightarrow \ \ddot{x} = r\ddot{\theta} \ , \tag{5.117c}$$

provides the missing equation. Substituting for f_f from Eq.(5.117a) into Eq.(5.117b) gives

$$-r(kx + m\ddot{x}) = I_g \ddot{\theta}$$

We can use Eqs.(5.117c) to eliminate x and $\ddot{x}$, obtaining

$$-r(kr\theta + mr\ddot{\theta}) = \frac{mr^2}{2}\ddot{\theta} \ \Rightarrow \ \frac{3mr^2}{2}\ddot{\theta} + kr^2\theta = 0 \tag{5.118}$$

This result concludes *Task a*. The natural frequency is

$$\omega_n^2 = kr^2 / \frac{3mr^2}{2} = \frac{2k}{3m} \ \Rightarrow \ \omega_n = \sqrt{\frac{2k}{3m}} \ .$$

This result concludes *Task b*. The cylinder inertia has caused a substantial reduction in the natural frequency as compared to a simple spring-mass system that would yield $\omega_n = \sqrt{k/m}$.

Deriving The Equation of Motion From Conservation of Energy

Conservation of energy implies,

$$\frac{m\dot{x}^2}{2} + \frac{I_g\dot{\theta}^2}{2} + \frac{kx^2}{2} = T_0 + V_0 \ .$$

Substituting the rolling-without-slipping kinematic conditions, $x = r\theta, \dot{x} = r\dot{\theta}$ gives

$$m\frac{(r\dot{\theta})^2}{2} + \frac{mr^2}{2}\frac{\dot{\theta}^2}{2} + \frac{k}{2}(r\theta)^2 = T_0 + V_0 \ ,$$

where from Appendix C, $I_g = mr^2/2$. Differentiating with respect to θ gives the equation of motion

$$\frac{3mr^2}{2}\ddot{\theta} + kr^2\theta = 0 \ ,$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$.

A Cylinder Rolling Inside a Cylindrical Surface

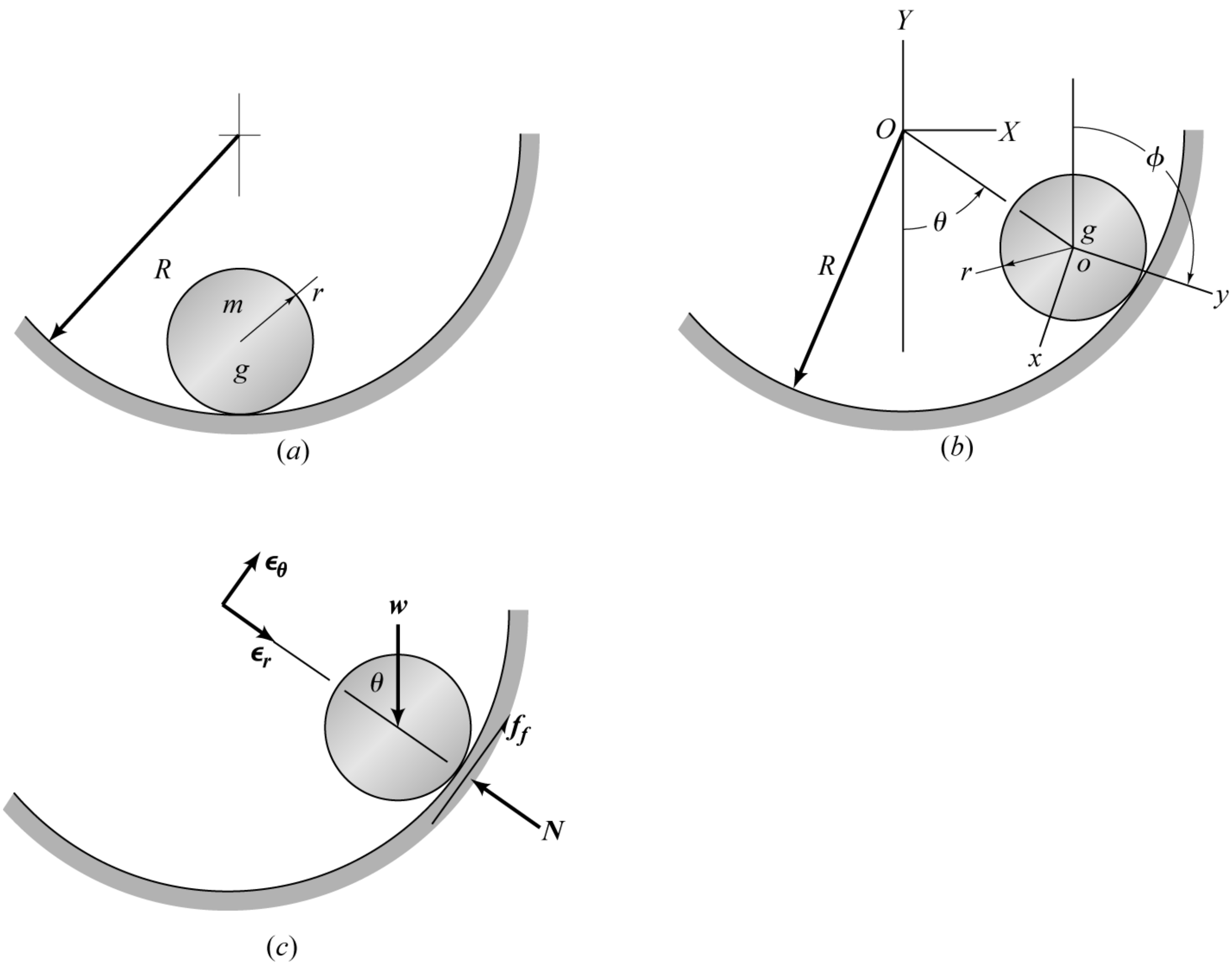


Figure 5.27 Cylinder rolling inside a cylinder. (a) Eqilibrium, (b) Coordinates, (c) Free-body diagram

O denotes the origin of the stationary *X,Y* coordinate system. The *x, y* coordinate system is fixed to the cylinder and its origin *o* coincides with the cylinder's mass center *g*. The angle θ defines the rotation of the line *O*-*g*, while φ defines the cylinder rotation with respect to ground.

The following engineering-analysis tasks apply:

a. Draw a free-body diagram and derive the equation of motion, and

b. For small motion about the bottom equilibrium position, determine the natural frequency.

c. Assuming that the cylinder is released from rest at $\theta = \pi/2$*, find* $\dot{\theta}$ *as a function of* θ*. Also define the normal reaction force as a function of* θ

In applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_g$, we will use the polar coordinate unit vectors. Starting in the $\boldsymbol{\varepsilon_\theta}$ direction,

$$\Sigma f_\theta = f_f - w \sin\theta = m a_\theta = m(R-r)\ddot{\theta} \ . \tag{5.119a}$$

$\Sigma f_r = m a_r$ will be needed to define the normal reaction force N, and gives

$$\Sigma f_r = w \cos\theta - N = m a_r = -m(R-r)\dot{\theta}^2 \ . \tag{5.119b}$$

Stating the moment equation about the mass center g gives

$$\Sigma M_{g\varphi} = -r f_f = I_g \ddot{\varphi} \ . \tag{5.120}$$

The moment due to the friction force is negative because it is acting in the - φ direction. We now have two equations in the three unknowns $\ddot{\theta}, \ddot{\varphi}, f_f$. From Eq.(4.14a), the required kinematic constraint equation between $\ddot{\theta}$ and $\ddot{\varphi}$ is

$$(R-r)\dot{\theta} = r\dot{\varphi} \;\Rightarrow\; (R-r)\ddot{\theta} = r\ddot{\varphi} \; . \tag{5.121}$$

Substituting for f_f from Eq.(5.119a) into Eq.(5.120) gives

$$-r[w\sin\theta + m(R-r)\ddot{\theta}] = \frac{mr^2}{2}\ddot{\varphi} \; .$$

Now substituting for $\ddot{\varphi}$ from Eq.(5.121) gives

$$\begin{aligned} -r[w\sin\theta + m(R-r)\ddot{\theta}] &= \frac{mr^2}{2}\frac{(R-r)}{r}\ddot{\theta} \\ \therefore \; \frac{3m}{2}(R-r)\ddot{\theta} + w\sin\theta &= 0 \; , \end{aligned} \tag{5.122}$$

For small θ, $\sin\theta \cong \theta$, the linearized equation of motion is

$$\ddot{\theta} + \frac{2g}{3(R-r)}\theta = 0 \; ,$$

and the natural frequency is $\omega_n = \sqrt{2g/3(R-r)}$.

The solution for $\dot{\theta}$ as a function of θ can be developed from Eq.(5.122) via the energy-integral substitution $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$ as

$$\frac{d}{d\theta}\left(\frac{\dot\theta^2}{2}\right) = -\frac{2g}{3(R-r)}\sin\theta \ .$$

Integration from the initial condition $\dot\theta(\theta=\pi/2)=0$ yields

$$\frac{\dot\theta^2}{2} - 0 = \frac{2g}{3(R-r)}\int_{\pi/2}^{\theta} -\sin x\, dx \ = \frac{2g}{3(R-r)}[\cos\theta - \cos(\frac{\pi}{2})] \ ,$$

and

$$\dot\theta^2 = \frac{4g}{3(R-r)}\cos\theta \qquad (5.123)$$

provides the requested solution. Substituting for $\dot\theta^2$ into Eq.(5.119b) defines N as

$$N = w\cos\theta + m(R-r)\frac{4g}{3(R-r)}\cos\theta = \frac{7w}{3}\cos\theta \ , \qquad (5.124)$$

and (*formally*) meets the requirements of Task c.

However, note from Eq.(5.124) that $N=0$ at $\theta=\pi/2$; hence, from our initial condition, the wheel will slip initially until N becomes large enough for the Coulomb friction force $f_f=\mu_d N$ to prevent slipping.

With slipping, the appropriate model as provided from Eqs.(5.119a-120) and $f_f = \mu_d N$ is:

$$\mu_d N - w\sin\theta = m(R-r)\ddot{\theta}$$

$$-r\mu_d N = I_g\ddot{\varphi} = \frac{mr^2}{2}\ddot{\varphi}$$

$$N = w\cos\theta + m(R-r)\dot{\theta}^2 \ .$$

Eliminating N gives the two coupled nonlinear equations of motion:

$$\begin{aligned} m(R-r)\ddot{\theta} &= \mu_d[w\cos\theta + m(R-r)\dot{\theta}^2] - w\sin\theta \\ I_g\ddot{\varphi} &= -r\mu_d[w\cos\theta + m(R-r)\dot{\theta}^2] \ . \end{aligned} \qquad (5.125)$$

These equations apply during slipping, provided that the direction of the friction force f_f in figure does not change[1]. With slipping, the cylinder has two degrees of freedom, θ and φ. Note from Eqs.(5.125) that initially, at $\theta = \pi/2$, $\ddot{\theta} < 0$ and $\ddot{\varphi} = 0$. As the cylinder rolls down the surface, $\dot{\theta}$ and $\dot{\varphi}$ increase in magnitude but are negative. The friction force f_f acts to slow down the magnitude increase in $\dot{\theta}$ and accelerate the magnitude increase in

---

[1] Note that the friction force would have a different sign if the cylinder were released from rest at $\theta = -\pi/2$.

$\dot{\varphi}$. When the kinematic condition $(R-r)\dot{\theta} = r\dot{\varphi}$ is met, slipping stops, Eqs.(5.125) become invalid, and Eq.(5.122) applies.

Deriving the Equation of Motion From Conservation of Energy

With *O*, the origin of the *X, Y* system as the gravity potential energy datum, $T + V = T_0 + V_0$ implies

$$m\frac{[(R-r)\dot{\theta}]^2}{2} + I_g\frac{\dot{\varphi}^2}{2} - w(R-r)\cos\theta = T_0 + V_0 \ .$$

Substituting $(R-r)\dot{\theta} = r\dot{\varphi}$ and $I_g = mr^2/2$ gives

$$m\frac{[(R-r)\dot{\theta}]^2}{2} + m\frac{r^2}{2}\times\frac{1}{2}[\frac{(R-r)\dot{\theta}}{r}]^2 - w(R-r)\cos\theta = T_0 + V_0 \ .$$

or

$$\frac{3m(R-r)^2}{4}\dot{\theta}^2 - w(R-r)\cos\theta = T_0 + V_0 \ .$$

Differentiating w.r.t. θ gives

$$\frac{3m(R-r)^2}{2}\ddot{\theta} + w(R-r)\sin\theta = 0 \ .$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$.

Pulley-Assembly Example

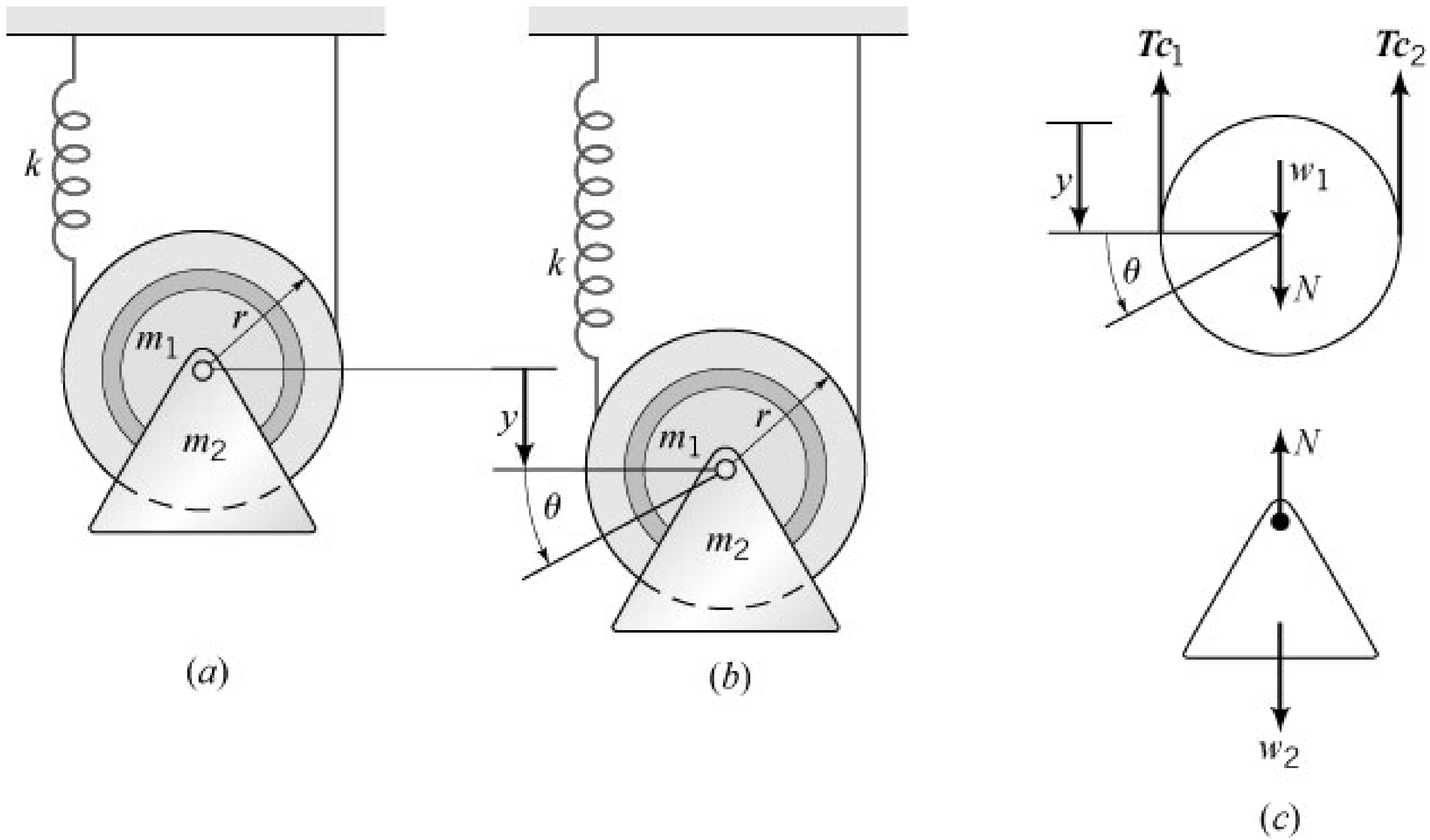


Figure 5.28 Pulley assembly consisting of a pulley of mass m_1 and an attached mass of mass m_2 . (a) Equilibrium, (b) Coordinates, (c) Free-body diagrams.

The assembly is supported by an inextensible cord in series with a linear spring with stiffness coefficient *k*. On the right, the cord's end is rigidly attached to a horizontal surface. On the left, the cord is attached to the spring which is attached to the same surface. The pulley has mass m_1 and a moment of inertia about its mass center of

$I_g = m_1 r^2/2$. The cord does not slip on the pulley.

The engineering analysis tasks are:

a. Draw free-body diagrams and derive the equation of motion.

b. Determine the natural frequency

The statement that the “cord does not slip on the pulley” introduces the rolling-without-slipping condition. The pulley can be visualized as rolling without slipping on the vertical surface defined by the right-hand-side cord line. The y coordinate locates the change in position of the pulley, and θ defines the pulley’s rotation angle. The spring is assumed to be undeflected when $y = 0$. The y and θ coordinates are related via the rolling-without-slipping kinematic condition,

$$y = r\theta \Rightarrow \ddot{y} = r\ddot{\theta} . \tag{5.126}$$

Figure 5.30B provides the appropriate free-body diagrams with the pulley and the lower assembly separated. The reaction force N acts between the two masses at the pivot connection point. The moment equation about the pulley’s mass center is

$$\Sigma M_{og} = rT_{c2} - rT_{c1} = I_g \ddot{\theta} \tag{5.127a}$$

Note that T_{c1} and T_{c2} are different. They must be different to induce the pulley's angular acceleration. The pulley's mass center and the lower assembly have the same acceleration; hence their equations of motion are:

$$(pulley)\ \Sigma f_y = w_1 + N - T_{c1} - T_{c2} = m_1 \ddot{y}$$
$$(lower\ mass)\ \Sigma f_y = w_2 - N = m_2 \ddot{y}\ . \tag{5.127b}$$

Adding these last equations eliminates N netting

$$w_1 + w_2 - T_{c1} - T_{c2} = (m_1 + m_2)\ddot{y}\ . \tag{5.128}$$

Eqs.(5.126), (5.127), and (5.128) provide three equations for the four unknowns $\ddot{\theta}, T_{c1}, T_{c2}, \ddot{y}$.

We need another kinematic constraint equation. Pulling the pulley down a distance y will pull the cord end attached to the spring down a distance $2y$. Hence, the cord tension T_{c1} is defined by $T_{c1} = k(2y) = 2ky$. Substituting this result and substituting $y = r\theta$ and $\ddot{y} = r\ddot{\theta}$ gives

$$rT_{c2} - r[2k(r\theta)] = I_g \ddot{\theta} = \frac{m_1 r^2}{2}\ddot{\theta}, and$$

$$w_1 + w_2 - 2k(r\theta) - T_{c2} = (m_1 + m_2) r\ddot{\theta}\ .$$

Eliminating T_{c2} by multiplying the second of these equations by

r and adding the result to the first gives

$$r(w_1 + w_2 - 2ky) - 2kry = (\frac{3m_1}{2} + m_2)r^2\ddot{\theta} + I_g\ddot{\theta} \ ,$$

and the equation of motion is

$$(\frac{3m_1}{2} + m_2)r^2\ddot{\theta} + 4kr^2\theta = r(w_1 + w_2) \ .$$

The natural frequency is defined by

$$\omega_n = \sqrt{\frac{8k}{(3m_1 + 2m_2)}} \ .$$

This example is "tricky" in that the rolling-without-slipping constraint and the second pulley constraint to define the spring deflection, $\delta_s = 2y$, are not immediately obvious.

Deriving the Equation of Motion From Conservation of Energy

Setting $y = 0$ as the zero potential energy for gravity means $T + V = T_0 + V_0$ implies

$$I_g \frac{\dot{\theta}^2}{2} + (m_1 + m_2)\frac{\dot{y}^2}{2} + k\frac{\delta_s^2}{2} - (w_1 + w_2)y = T_0 + V_0 \ ,$$

where δ_s is the spring deflection. Substituting: (i) $I_g = m_1 r^2/2$, (ii) the rolling without slipping condition $\dot{y} = r\dot{\theta}$,and (iii) the pulley condition $\delta_s = 2y = 2r\theta$ gives

$$\frac{m_1 r^2}{2}\frac{\dot{\theta}^2}{2} + (m_1 + m_2)\frac{r\dot{\theta}^2}{2} + \frac{k}{2}(2r\theta)^2 - (w_1 + w_2)(r\theta) = T_0 + V_0 \ .$$

Differentiating w.r.t. θ gives

$$(\frac{3m_1}{2} + m_2)r^2\ddot{\theta} + 4kr^2\theta = (w_1 + w_2) \ ,$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$.

Lecture 31. EXAMPLES: EQUATIONS OF MOTION USING NEWTON AND ENERGY APPROACHES

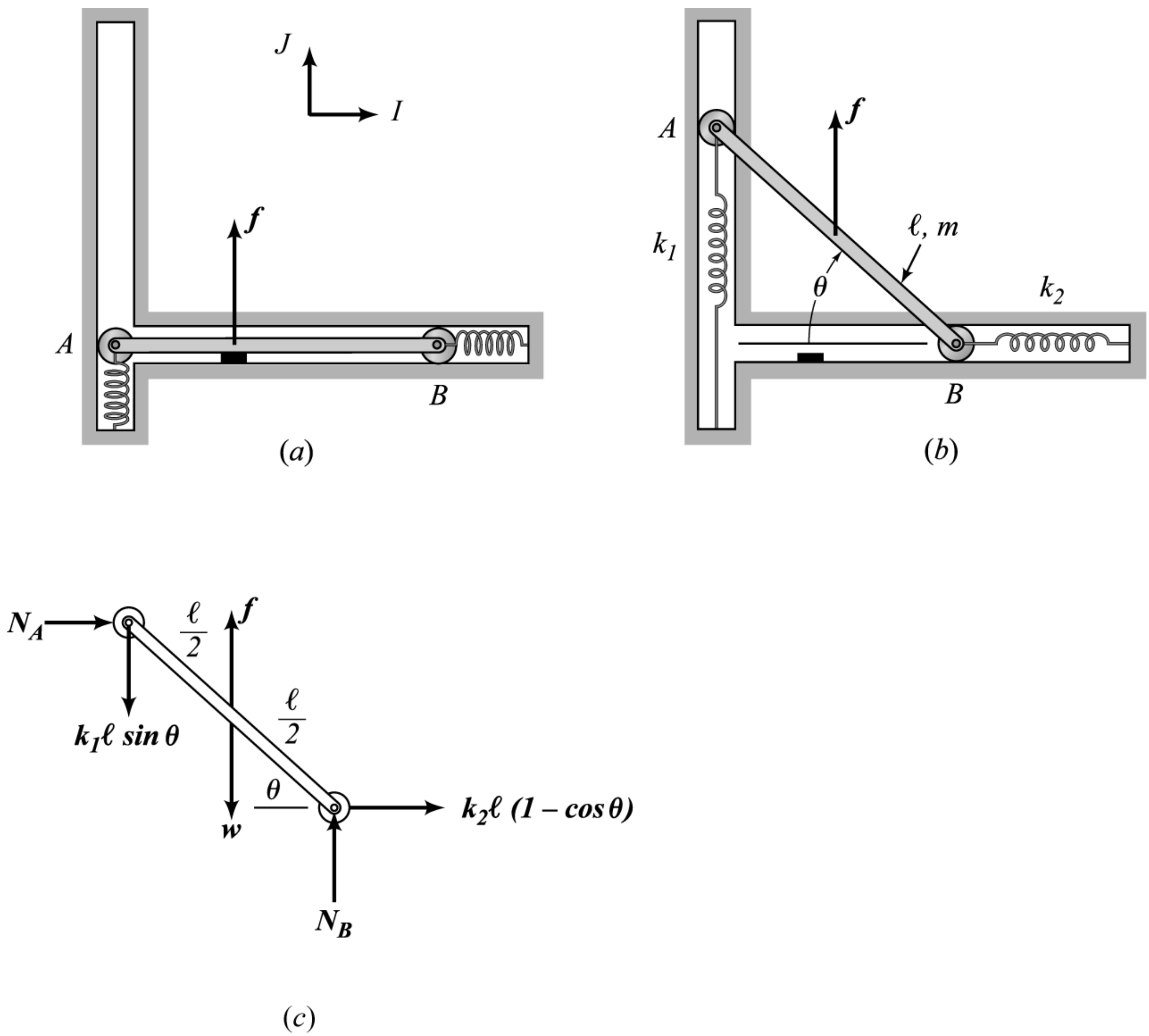


Figure 5.29 (a) Uniform beam moving in frictionless slots and attached to ground via springs at A and B. The vertical force $\boldsymbol{f}$ acts the bar's center, (b) Coordinate choice, (c) Free-body diagram. The springs are undeflected at $\theta = 0$.

Task: Derive the EOM

Solution from free-body diagrams

Stating the moment equation about the mass center gives

$$
\begin{aligned}
\Sigma M_g &= I_g \ddot{\theta} \\
&= N_A \frac{l}{2} \sin\theta - k_1 l \sin\theta \cdot \frac{l}{2} \cos\theta - N_B \frac{l}{2} \cos\theta \\
&\quad - k_2 l (1 - \cos\theta) \cdot \frac{l}{2} \sin\theta \ .
\end{aligned}
\tag{5.130}
$$

Stating $\Sigma \boldsymbol{f} = m \ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ for the mass center gives the component equations:

$$
\begin{aligned}
\Sigma f_X &= N_A + k_2 l (1 - \cos\theta) = m \ddot{X}_g \\
\Sigma f_Y &= f(t) + N_B - w - k_1 l \sin\theta = m \ddot{Y}_g .
\end{aligned}
\tag{5.131}
$$

We now have three equations in the five unknowns $N_A, N_B, \ddot{\theta}, \ddot{X}_g, \ddot{Y}_g$. The additional kinematic equations are obtained from the geometric relations

$$X_g = \frac{l}{2} \cos\theta \ , \ Y_g = \frac{l}{2} \sin\theta$$

Differentiating these equations twice with respect to time gives:

$$\ddot{X}_g = -\frac{l}{2}(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)\ , \quad \ddot{Y}_g = \frac{l}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) \quad (5.132)$$

Substitution into Eqs.(5.132) defines N_A, N_B as:

$$N_A = -k_2 l(1 - \cos\theta) - \frac{ml}{2}(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)$$

$$N_B = -f(t) + w + k_1 l \sin\theta + \frac{ml}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)\ . \quad (5.133)$$

Finally, substitution for N_A, N_B into the moment Eq.(5.130) gives

$$I_g\ddot{\theta} = \frac{l}{2}\sin\theta[-k_2 l(1 - \cos\theta) - \frac{ml}{2}(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)]$$

$$-\frac{k_1 l^2}{2}\sin\theta\cos\theta - \frac{l}{2}\cos\theta[-f(t) + w + k_1 l\sin\theta]$$

$$+\frac{ml}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)] - \frac{k_2 l^2}{2}(1 - \cos\theta)\sin\theta\ .$$

Gathering terms and simplifying gives the final equation of motion

$$\frac{ml^2}{3}\ddot{\theta} + k_2 l^2 \sin\theta(1 - \cos\theta)$$
$$+ k_1 l^2 \sin\theta\cos\theta + \frac{wl}{2}\cos\theta = \frac{fl}{2}\cos\theta \ , \tag{5.134}$$

Developments from Work Energy

The absence of energy-dissipation forces makes this example an ideal application for the work-energy equation $Work_{n.c} = (T_2 + V_2) - (T_1 + V_1)$.

The following engineering analysis tasks apply :

a. If the bar starts from rest at $\theta = 0$*, and* $f(t) = \bar{f}$*, determine its angular velocity when* $\theta = \pi/2$.

b. Derive the equation of motion.

First find $dWork_{n.c.}$ Figure 5.29c illustrates the force vector $\boldsymbol{f_1} = \boldsymbol{J} f_1$, located in the fixed *X, Y* system by the vector $\mathrm{r}_{\mathbf{1}} = l/2(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$**.** The differential work done by the force acting through a differential change in position is

$$dWork_{n.c} = \boldsymbol{f_1} \bullet \boldsymbol{dr} = \boldsymbol{J} f(t) \bullet \frac{l}{2}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)\,d\theta$$

$$= \frac{f(t)\,l}{2}\cos\theta\,d\theta = Q_{\theta 1}\,d\theta \ ,$$

where $Q_{\theta 1}$ is the generalized force associated with θ. The mass center of the body is also located by $\boldsymbol{r_1}$; hence, $\dot{\boldsymbol{R}}_{\boldsymbol{g}} = \dot{\boldsymbol{r}}_{\boldsymbol{1}} = l/2(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)\dot{\theta}$, and the kinetic energy is

$$T = \frac{m}{2}\,|\,\dot{\boldsymbol{R}}_{\boldsymbol{g}}|^2 + \frac{I_g}{2}\dot{\theta}^2 = (\frac{m l^4}{2} + \frac{m l^2}{12})\frac{\dot{\theta}^2}{2} = \frac{m l^2}{3}\frac{\dot{\theta}^2}{2} \ .$$

Establishing the gravity potential-energy datum in the center of the lower guide slot yields $V_g = w l/2\sin\theta$. The spring potential energy function is defined by

$$V_s = \frac{k_1}{2}\delta_1^2 + \frac{k_2}{2}\delta_2^2 = \frac{k_1}{2}(l\sin\theta)^2 + \frac{k_2}{2}(l - l\cos\theta)^2 \ .$$

Plugging all of these results into the work-energy equation gives

$$\int_0^{\theta} \frac{f(t)\, l}{2} \cos x\, dx = [\frac{m l^2}{6}\dot{\theta}^2 + \frac{w l}{2}\sin\theta$$

$$+ \frac{k_1 l_1^2}{2}\sin^2\theta + \frac{k_2 l^2}{2}(1-\cos\theta)^2] - (0+0), \tag{5.227}$$

where the initial potential and kinetic energy are zero. Evaluating Eq.(5.227) for $f(t)=\bar{f}$ and $\theta = \pi/2$ gives

$$\dot{\theta}(\theta = \frac{\pi}{2}) = \{\frac{6}{m l^2}[\frac{\bar{f} l}{2} - (\frac{w l}{2} + \frac{k_1 l_1^2}{2} + \frac{k_2 l_2^2}{2})]\}^{\frac{1}{2}} ,$$

and concludes *Task a*.

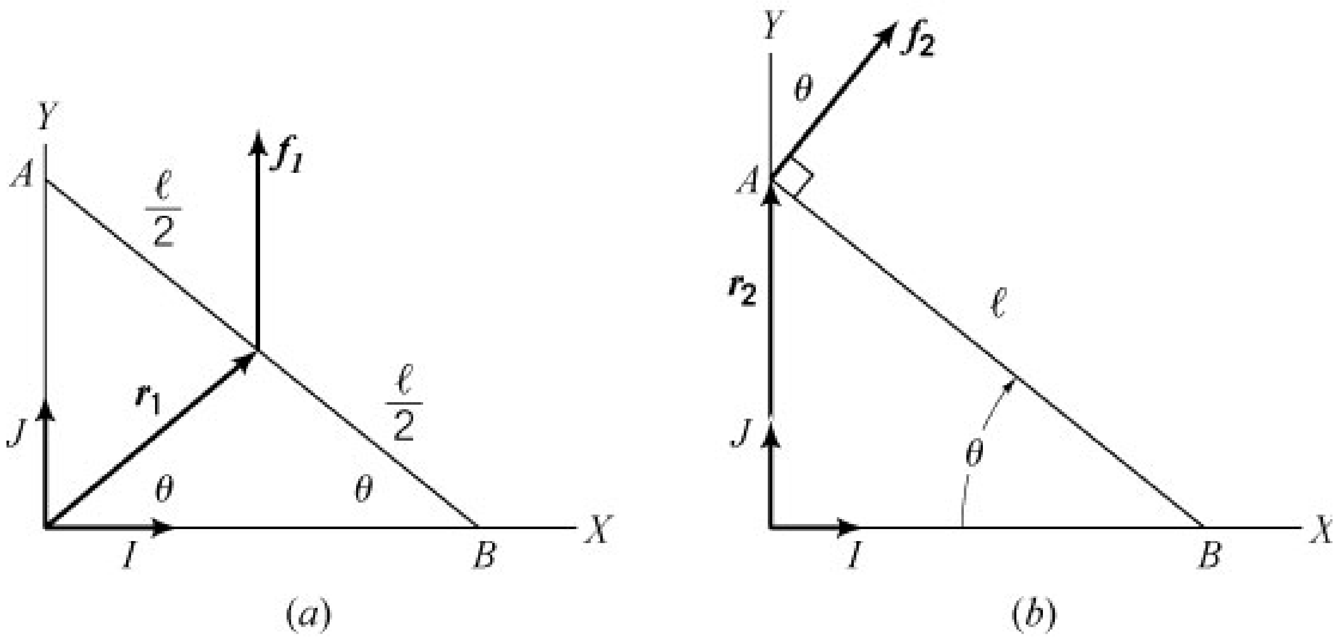
The equation of motion is obtained by differentiating Eq.(5.227) with respect to θ, obtaining,

$$\frac{m l^2}{3}\ddot{\theta} + \frac{w l}{2}\cos\theta + k_1 l_1^2 \sin\theta\cos\theta$$

$$+ k_2 l_2^2(1-\cos\theta)\sin\theta = \frac{f l}{2}\cos\theta = Q_{\theta 1} , \tag{5.228}$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$. This step concludes *Task b.*

Note the generalized force term $Q_{\theta 1}$ on the right-hand side of Eq.(5.228). Changing the external-force definition modifies only this term, leaving the left hand side of the equation unchanged.



Task: Derive the EOM for the force $\boldsymbol{f_2}$.

The external force $\boldsymbol{f_2} = f_2(\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)$ is located by the vector $\boldsymbol{r_2} = \boldsymbol{J}l\sin\theta$. Hence, the differential work due to $\boldsymbol{f_2}$ acting at $\boldsymbol{r_2}$ is

$$dWork_{n.c.} = \boldsymbol{f_2} \bullet d\boldsymbol{r_2} = f_2(\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) \bullet \boldsymbol{J}l\cos\theta\, d\theta$$

$$= f_2\, l\cos^2\theta\, d\theta = Q_{\theta 2}\, d\theta \ .$$

Substituting $Q_{\theta 2}$ for $Q_{\theta 1}$ in Eq.(5.228) provides the correct equation.

Energy Dissipatiom

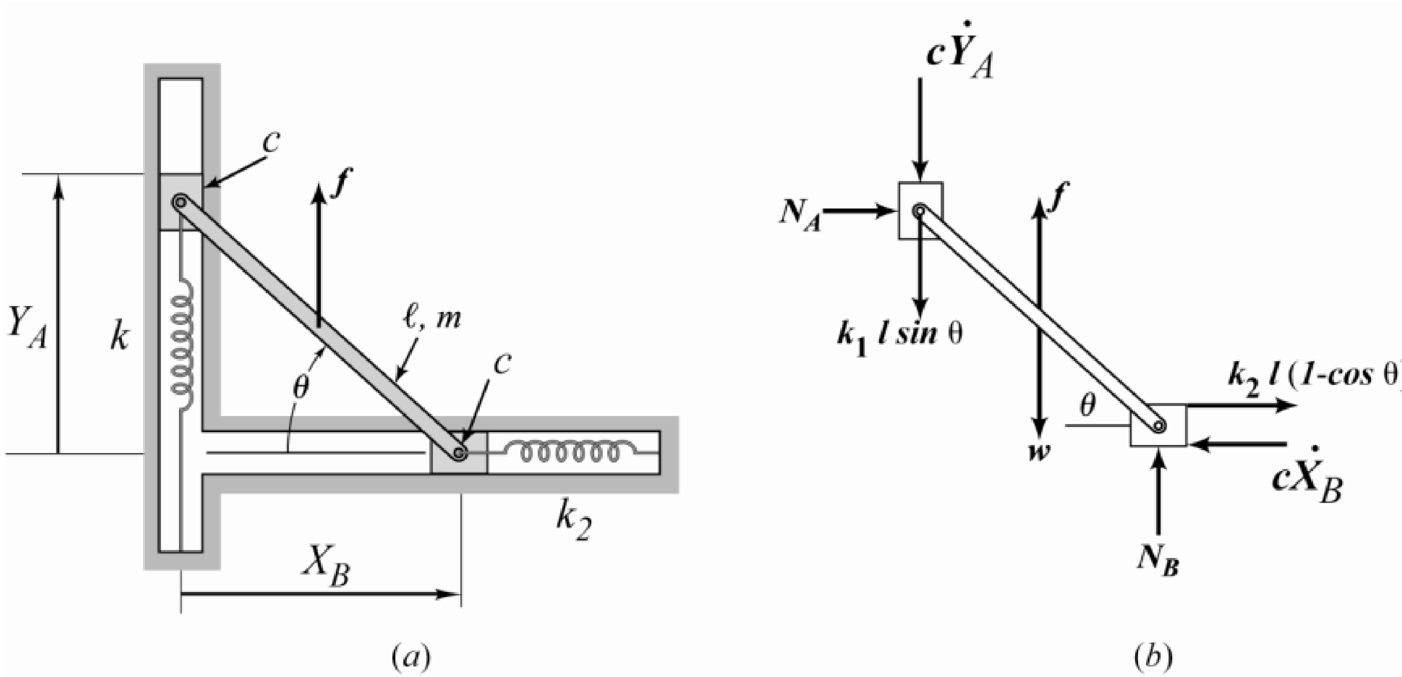


Figure 5.29 (a) The uniform bar of figure 5.31 with guides now replacing the end rollers, (b) Free-body diagram for viscous damping now replacing the end rollers within the slots.

Applying $\Sigma \boldsymbol{f} = m\ddot{R}_g$ to the free-body diagram of figure 5.29b provides:

$$\Sigma f_X = N_A - k_2 l(1 - \cos\theta) - c\dot{X}_B = m\ddot{X}_g$$

$$\Sigma f_Y = f + N_B - w - k_1 l \sin\theta - c\dot{Y}_A = m\ddot{Y}_g \ .$$

The acceleration components $\ddot{X}_g, \ddot{Y}_g$ continue to be defined by Eqs.(5.132). The velocity magnitudes $\dot{Y}_A, \dot{X}_B$ are obtained by differentiating the geometric relations ($Y_A = l\sin\theta$, $X_B = l\cos\theta$) to obtain ($\dot{Y}_A = l\cos\theta\dot{\theta}$, $\dot{X}_B = -l\sin\theta\dot{\theta}$). Substituting into the moment and force equations gives:

$$
\begin{aligned}
& I_g\ddot{\theta} + \frac{cl^2}{2}(\cos^2\theta + \sin^2\theta)\dot{\theta} = \frac{N_A l}{2}\sin\theta \\
& \quad - \frac{k_1 l^2}{2}\sin\theta\cos\theta - \frac{N_B l}{2}\cos\theta \\
& \quad - \frac{k_2 l^2}{2}(1-\cos\theta)\sin\theta \ , \\
& N_A = cl\sin\theta\dot{\theta} - k_2 l(1-\cos\theta) \\
& \quad - \frac{ml}{2}(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2) \ , \\
& N_B = -f + w + k_1 l\sin\theta + cl\cos\theta\dot{\theta} \\
& \quad + \frac{ml}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) \ .
\end{aligned}
\tag{5.135}
$$

Substitution for N_A, N_B into the moment equation and simplifying terms gives the final differential equation of motion

$$
\begin{aligned}
& \frac{ml^2}{3}\ddot{\theta} + cl^2\dot{\theta} + k_2 l^2\sin\theta(1-\cos\theta) \\
& + k_1 l^2\sin\theta\cos\theta + \frac{wl}{2}\cos\theta = \frac{fl}{2}\cos\theta \ .
\end{aligned}
\tag{5.136}
$$

The damping term on the left is the only addition to the prior moment Eq.(5.134).

Developing the equations of motion from work energy starts from

$$dWork_{n.c.} = \boldsymbol{J}f \cdot d\boldsymbol{r} + (-\boldsymbol{J}c\dot{Y}_A)\cdot d\boldsymbol{Y_A} + (-\boldsymbol{I}\dot{X}_B)\cdot d\boldsymbol{X_B}$$

where

$$\boldsymbol{r} = \frac{l}{2}(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \;\Rightarrow\; d\boldsymbol{r} = \frac{l}{2}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)\,d\theta$$

$$\boldsymbol{Y_A} = \boldsymbol{J}l\sin\theta \;\Rightarrow\; d\boldsymbol{Y_A} = \boldsymbol{J}l\cos\theta\,d\theta,\; \dot{Y}_A = l\cos\theta\,\dot{\theta}$$

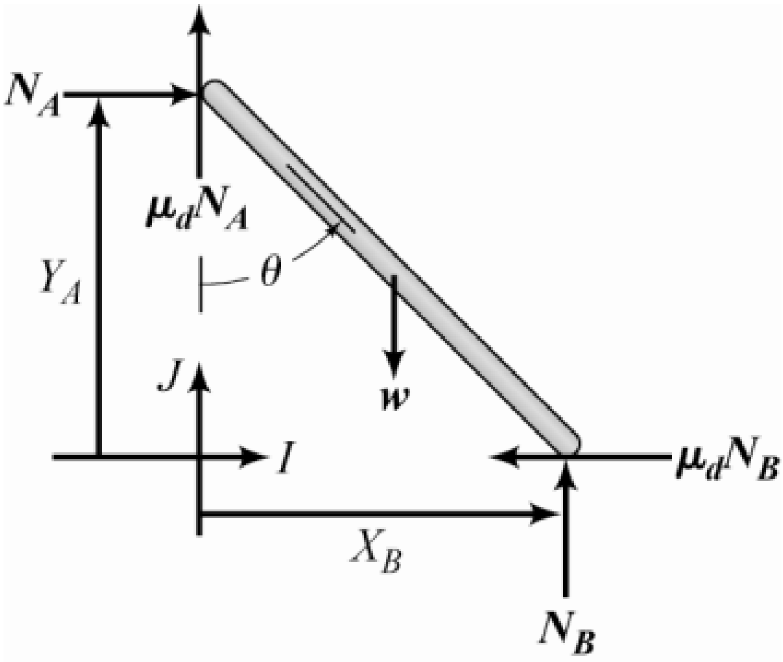
$$\boldsymbol{X_B} = \boldsymbol{I}l\cos\theta \;\Rightarrow\; d\boldsymbol{X_B} = -\boldsymbol{I}l\sin\theta\,d\theta,\; \dot{X}_B = -l\sin\theta\,\dot{\theta}\ .$$

Hence,

$$dWork_{n.c.} = [\frac{fl}{2}\cos\theta - cl^2(\cos^2\theta + \sin^2\theta)\dot{\theta}]\,d\theta = Q_\theta\,d\theta$$

Plugging this result into Eq.(5.128) gives the equation of motion. Accounting for energy dissipation due to viscous damping is easy since the damping force can be stated directly. Things are less pleasant with Coulomb damping.

Coulomb Damping



The free-body above reflects Coulomb damping at both supports with Coulomb damping coefficient μ. The force equations are:

$$\Sigma f_X = N_A + k_2 l(1 - \cos\theta) - \mu N_B sgn(\dot{X}_B) = m\ddot{X}_g$$

$$\Sigma f_Y = f - w + N_B - w - k_1 l \sin\theta - \mu N_A sgn(\dot{Y}_A) = m\ddot{Y}_g \quad . \tag{i}$$

.The moment equation is

$$\Sigma M_g = I_g \ddot{\theta}$$

$$\begin{aligned} &= N_A \cdot \frac{l}{2}\sin\theta - k_1 l \sin\theta \cdot \frac{l}{2}\cos\theta - N_B \cdot \frac{l}{2}\cos\theta \\ &- k_2 l(1-\cos\theta)\cdot\frac{l}{2}\sin\theta - \mu N_A sgn(\dot{Y}_A)\cdot\frac{l}{2}\cos\theta \\ &+ \mu N_B sgn(\dot{X}_B)\cdot\frac{l}{2}\sin\theta\ . \end{aligned} \qquad \textbf{(ii)}$$

Before preceding, note that $\dot{Y}_A = l\cos\theta\,\dot{\theta}$, $\dot{X}_A = -l\sin\theta\,\dot{\theta}$. Since, $0 \le \theta \le \pi/2$,

$$sgn(\dot{Y}_A) = sgn\,\dot{\theta}\ \ ,\ \ sgn(\dot{X}_B) = -sgn\,\dot{\theta}$$

We need to solve Eqs.(i) for N_A, N_B and substitute the results into Eq.(i). The equations for N_A, N_B can be stated

$$\begin{bmatrix} 1 & +\mu sgn\,\dot{\theta} \\ -\mu sgn\,\dot{\theta} & 1 \end{bmatrix} \begin{Bmatrix} N_A \\ N_B \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}.$$

where

$$g_1 = -k_2 l(1-\cos\theta) - \frac{ml}{2}(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)$$
$$g_2 = -f + w + k_1 l\sin\theta + \frac{ml}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) \ .$$

Solving for N_A, N_B gives

$$N_A = \frac{g_1 - \mu f_2\, sgn\dot{\theta}}{1+\mu^2} \ , \ N_B = \frac{g_2 + \mu f_1\, sgn\dot{\theta}}{1+\mu^2} \qquad \textbf{(iii)}$$

Substituting these result into Eq.(i) gives the equations of motion.

Trying to do this with work-energy starts reasonably. The differential work is

$$\begin{aligned}
dWork_{n.c.} &= \boldsymbol{J}f\cdot d\boldsymbol{r} + [-\boldsymbol{J}\mu N_A sgn(\dot{Y}_A)]\cdot d\boldsymbol{Y_A} \\
&\quad + -\boldsymbol{I}\mu N_B sgn(\dot{X}_B)\cdot d\boldsymbol{X_B} \\
&= \boldsymbol{J}f\cdot d\boldsymbol{r} - \boldsymbol{J}\mu N_A sgn\dot{\theta}\cdot \boldsymbol{J}l\cos\theta\, d\theta \\
&\quad - \boldsymbol{I}\mu N_B sgn\dot{\theta}\cdot -\boldsymbol{I}l\sin\theta\, d\theta \\
&= [\frac{fl}{2}\cos\theta - \mu l(N_A\cos\theta + N_B\sin\theta)sgn\dot{\theta}]\, d\theta \\
&= Q_\theta\, d\theta
\end{aligned}$$

However, applying this equation requires that you start with the force free-body diagrams and equations of motion to obtain the solutions for N_A, N_B provided by Eq.(iii).

Two Bars with an Applied Force and a Connecting Spring.

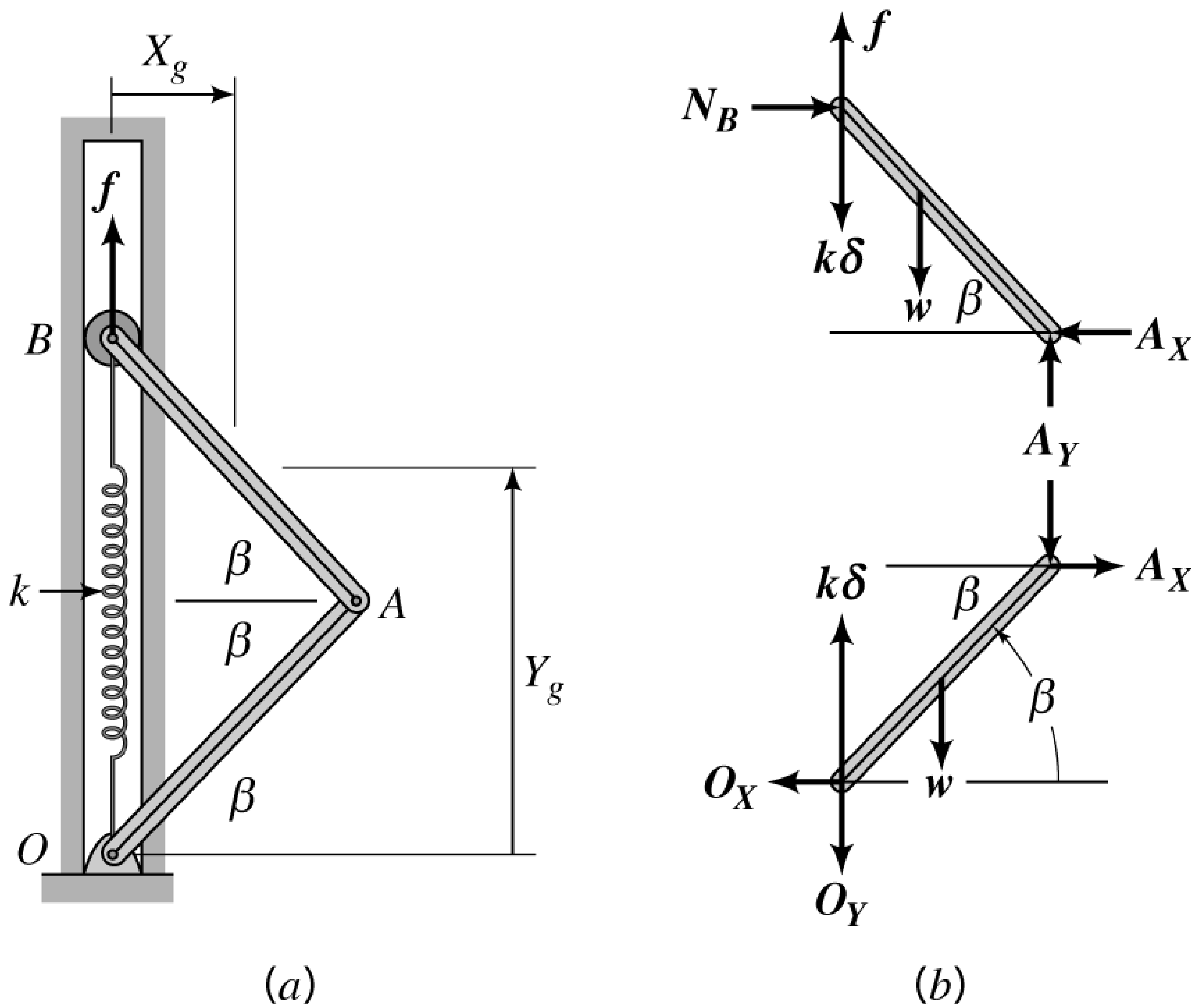


Figure 5 .35 (a) Two bars of mass m and length l, connected to each other by a linear spring and acted on by a vertical force, (b) Free-body diagram for the two bodies.

A linear spring, with spring coefficient *k*, connects points *O* and *B*. The spring is undeflected at $\beta = \beta_0$. The vertical external force $\boldsymbol{f} = \boldsymbol{J} f$ acts at point *B*. The following engineering-analysis tasks apply:

a. Develop a general work-energy equation.

b. If the bodies start from rest at $\beta = \beta_0$ and $f(t) = \bar{f} = constant$, determine the angular velocity of the bodies when $\beta = \pi/2$.

c. Derive the differential equation of motion.

Because of the external force, energy is not conserved, and the general work-energy equation $Work_{n.c.} = \Delta(T + V)$ applies. The kinetic energy of the two bars is

$$T = T_{bar1} + T_{bar2} = \frac{I_O \dot{\beta}^2}{2} + [\frac{I_g \dot{\beta}^2}{2} + \frac{m}{2}(\dot{X}_g^2 + \dot{Y}_g^2)] \ . \tag{5.231}$$

The velocity of the upper bar's mass center is defined from

$$X_g = \frac{l}{2}\cos\beta \ \Rightarrow \ \dot{X}_g = -\frac{l}{2}\sin\beta\,\dot{\beta}$$
$$Y_g = \frac{3l}{2}\sin\beta \ \Rightarrow \ \dot{Y}_g = \frac{3l}{2}\cos\beta\,\dot{\beta} \ . \tag{5.232}$$

Substituting these results plus ($I_O = ml^2/3, I_g = ml^2/12$) into Eq.(5.231) gives

$$T = \frac{ml^2}{6}\dot{\beta}^2 + \frac{ml^2}{24}\dot{\beta}^2 + \frac{m}{2}\left[\frac{l^2}{4}\sin^2\beta + \frac{9\ l^2}{4}\cos^2\beta\right]\dot{\beta}^2$$

$$= ml^2\left(\frac{1}{6} + \frac{1}{24} + \frac{1}{8}\right)\dot{\beta}^2 + ml^2\cos^2\beta\,\dot{\beta}^2$$

$$= \frac{ml^2}{3}\dot{\beta}^2 + ml2\ \cos^2\ \beta\dot{\beta}^2$$

For the gravity potential-energy datum through O, the mechanical potential energy is

$$V = \frac{wl}{2}\sin\beta + \frac{3wl}{2}\sin\beta + \frac{k\delta^2}{2}$$
$$= 2wl\sin\beta + \frac{k}{2}(2l\sin\beta - 2l\sin\beta_0)^2\ .$$

The differential work due to the external force is

$$dWork_{n.c.} = \boldsymbol{f}\bullet\boldsymbol{dr} = \boldsymbol{J}f(t)\bullet d(\boldsymbol{J}2l\sin\beta)$$
$$= 2f(t)l\cos\beta\, d\beta = Q_\beta d\beta\ .$$

Substituting into the work-energy equation gives

$$\int_0^{\beta} 2f(t)\, l \cos x\, dx = m\ l^2 \dot{\beta}^2 (\frac{1}{3}\ +\ \cos^2\beta\ )$$
$$+ [2wl\sin\beta + \frac{k}{2}(2l\sin\beta - 2l\sin\beta_0)^2] \qquad (5.233)$$
$$-(T_0 + V_0)\ ,$$

and concludes *Task a.*

Task b is accomplished by substituting $f(t) = \bar{f}$ and $\beta = \pi/2$ into Eq.(5.233) to obtain

$$\int_0^{\pi/2} 2\bar{f}\, l \cos\beta\, d\beta = 2\bar{f}l = \frac{m\ l^2}{3}\ \dot{\beta}^2(\beta = \pi/2)$$
$$+ 2wl(1 - \sin\beta_0) + 2kl^2(1 - \sin\beta_0)^2\ ,$$

and the requested answer is

$$\dot{\beta}(\beta = \pi/2)\ = \{\frac{6}{ml}[\bar{f} - w(1 - \sin\beta_0) - kl(1 - \sin\beta_0)^2]\}^{1/2}\ .$$

The differential equation of motion is obtained by differentiating Eq.(5.233) with respect to β, obtaining

$$2ml^2\ddot{\beta}(\frac{1}{3}\ +\ \cos^2\ \beta\ ) - m\ l^2\dot{\beta}^2\sin 2\beta$$
$$+ 2wl\cos\beta + 4kl^2(\sin\beta - \sin\beta_0)\cos\beta = Q_\beta = 2f(t)\, l\cos\beta\ .$$

This result concludes *Task c*.

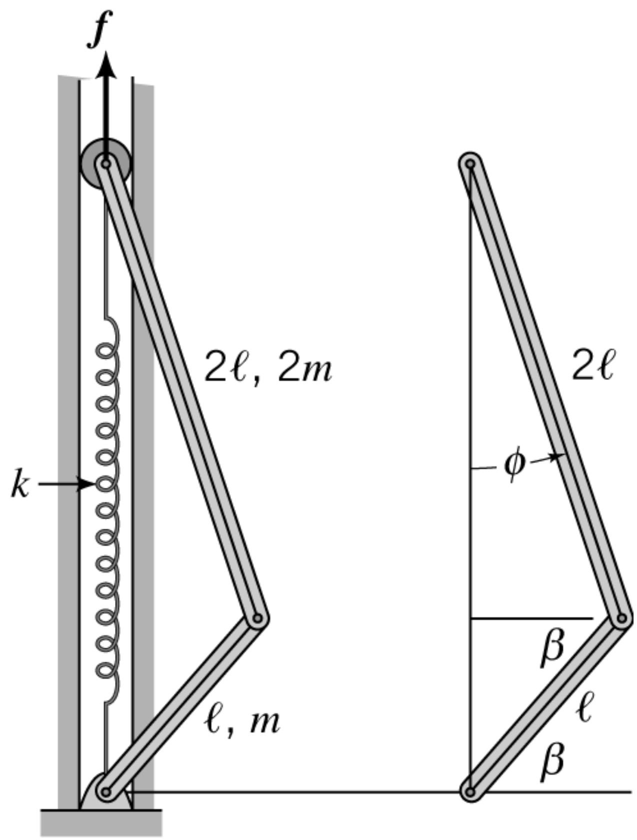


Figure 5.36 A modification to the example of figure 5.65, with the upper bar now having mass $2m$ and length $2l$.

The modification above complicates considerably our present approach. The complications are handled nicely by Lagrange's equations with Lagrange multipliers in Chapter 6.

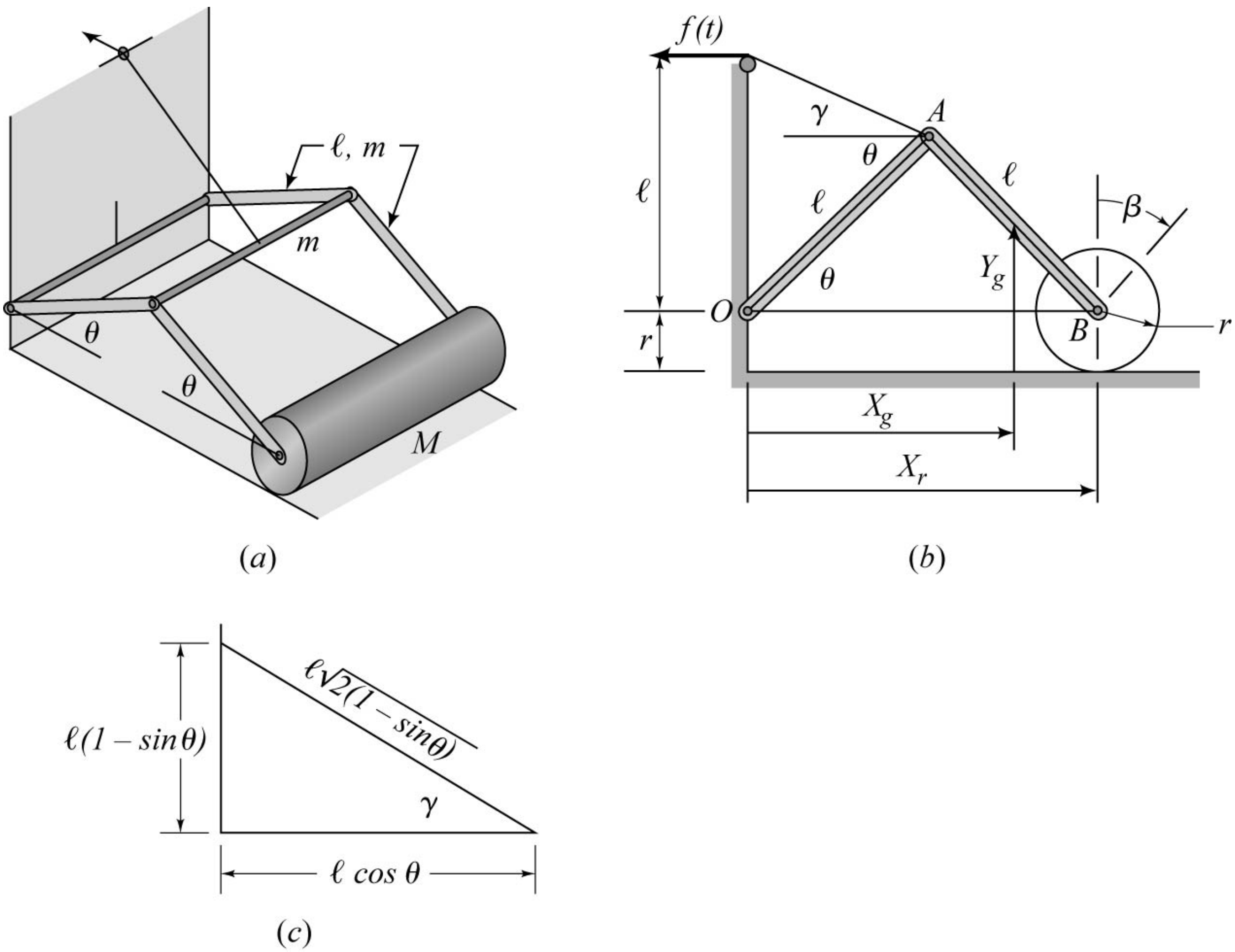


Figure 5.38 (a) Parallel, double-bar arrangement for retracting a cylinder, (b) Side view, (c) Load-angle geometry.

A Parallel, Double-Bar Arrangement for Retracting a Cylinder

Figure 5.68A illustrates a bar assembly for retracting a cylinder. Each of the four bars in the assembly has mass m and length l. The cylinder has radius r , length L and mass M. A time-varying force with magnitude $f(t)$ is acting through a cable that extends over a pulley and is attached at the center of the assembly's connecting bar. The cylinder rolls without slipping on a horizontal plane, and its rotation angle is β. The following engineering-analysis tasks apply:

a. Develop a general work-energy equation.

b. Develop the equation of motion.

Because of the external force $f(t)$, energy is not conserved, and the general work-energy equation $Work_{n.c.} = \Delta(T+V)$ applies. The kinetic energy is

$$T = 2\cdot\frac{I_O\dot{\theta}^2}{2} + 2\cdot\left[\frac{I_g\dot{\theta}^2}{2} + \frac{m}{2}(\dot{X}_g^2 + \dot{Y}_g^2)\right] + \frac{m}{2}(l\dot{\theta})^2$$
$$+\left(\frac{I_B\dot{\beta}^2}{2} + \frac{M\dot{X}_B^2}{2}\right) .$$

The first, second, and third terms define the kinetic energy of the left-hand bars, the right-hand bars, and the cylinder , respectively. From figure 5.38b,

$$X_g = \frac{3l}{2}\cos\theta \Rightarrow \dot{X}_g = -\frac{3l}{2}\sin\theta\,\dot{\theta}$$

$$Y_g = \frac{l}{2}\sin\theta \Rightarrow \dot{Y}_g = \frac{l}{2}\cos\theta\,\dot{\theta} \qquad (5.242)$$

$$X_B = 2l\cos\theta \Rightarrow \dot{X}_B = -2l\sin\theta\,\dot{\theta}$$

The rolling-without-slipping condition is

$$r\beta = X_B = 2l\cos\theta \;\Rightarrow\; r\dot{\beta} = \dot{X}_B = -2l\sin\theta\,\dot{\theta}$$

$$\therefore\; \dot{\beta} = -\frac{2l}{r}\sin\theta\,\dot{\theta}\;. \tag{5.243}$$

Substituting the kinematic results from Eqs.(5.242 and 5.243) plus the moment of inertia results ($I_O = ml^2/3$, $I_g = ml^2/12$, and $I_B = Mr^2/2$) from Appendix C into T gives

$$T = \dot{\theta}^2\Big[\frac{ml^2}{3} + \frac{ml^2}{12} + ml^2(1+2\sin^2\theta) + \frac{ml^2}{2}$$
$$+\frac{Mr^2}{4}\cdot\frac{4l^2\sin^2\theta}{r^2} + \frac{M}{2}\cdot 4l^2\sin^2\theta\Big]$$

$$= \dot{\theta}^2 l^2\Big[\frac{23m}{12} + (2m+3M)\sin^2\theta\Big]\;.$$

Defining the datum for gravity potential energy by a plane through O and B yields the potential-energy function

$$V = 4\cdot w\frac{l}{2}\sin\theta = 2wl\sin\theta\;.$$

Working out the differential-work function due to the force $f(t)$ acting at A is a little complicated for this example. The force acts at the angle γ from the horizontal. Figure 5.38c shows the right triangle defining γ and provides:

$$\sin\gamma = \frac{(1-\sin\theta)}{\sqrt{2(1-\sin\theta)}} \ , \ \cos\gamma = \frac{\cos\theta}{\sqrt{2(1-\sin\theta)}} \ .$$

Hence, the force vector is $\boldsymbol{f} = f(t)(-\boldsymbol{I}\cos\gamma + \boldsymbol{J}\sin\gamma)$. The force acts at the position $\boldsymbol{r} = l(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$, and the differential nonconservative work is

$$\begin{aligned} dWork_{n.c.} &= \boldsymbol{f}\bullet d\boldsymbol{r} = f(t)(-\boldsymbol{I}\cos\gamma + \boldsymbol{J}\sin\gamma)\bullet \\ & \quad l(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta)\, d\theta \\ &= \frac{f(t)\, l[\sin\theta\cos\theta + \cos\theta(1-\sin\theta)]}{\sqrt{2(1-\sin\theta)}}\, d\theta \\ &= \frac{f(t)\, l\cos\theta\, d\theta}{\sqrt{2(1-\sin\theta)}} = Q_\theta\, d\theta \ . \end{aligned}$$

Substituting for T, V, and $dWork_{n.c.}$ into the work-energy equation gives

$$\begin{aligned} \int_{\theta_0}^{\theta} \frac{f(t)\, l\cos x\, dx}{\sqrt{2(1-\sin x)}} &= \dot\theta^2 l^2 [\frac{23m}{12} + (2m+3M)\sin^2\theta] \\ & \quad + 2wl\sin\theta - (T_0 + v_0) \ , \end{aligned} \tag{5.244}$$

and concludes *Task a*.

Differentiating Eq.(5.244) with respect to θ gives the differential equation of motion

$$\ddot{\theta} l^2 [\frac{23m}{6} + 2(2m + 3M)\sin^2\theta] + \dot{\theta}^2 l^2 [2(2m + 3M)\sin\theta\cos\theta]$$
$$+ 2wl\cos\theta = Q_\theta = \frac{f(t)\,l\cos\theta}{\sqrt{2(1 - \sin\theta)}}\,,$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$, and concludes *Task c*.