Lecture 24. INERTIA PROPERTIES AND THE PARALLEL-AXIS FORMULA

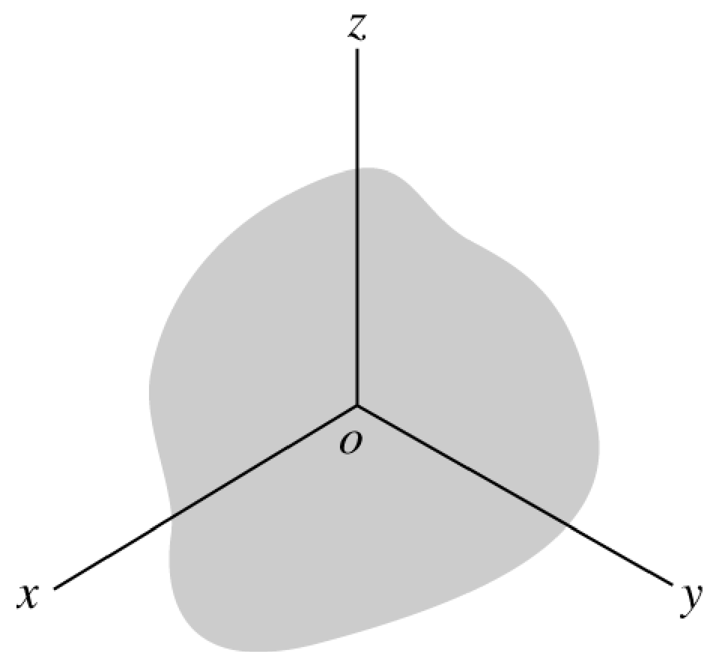


Figure 5.1 Rigid body with an imbedded x,y,z coordinate system.

The body's mass is defined by

$$m = \int_V \gamma(x,y,z)\, dx\, dy\, dz \quad , \tag{5.1}$$

where $\gamma(x,y,z)$ is the body's density at point x, y, z .

With the position vector of a point in the rigid body defined by

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \quad ,$$

the body's mass center is located in the x, y, z system by the vector $\boldsymbol{b}_{og}$, defined by

$$m\boldsymbol{b}_{og} = m(\boldsymbol{i}b_{ogx} + \boldsymbol{j}b_{ogy} + \boldsymbol{k}b_{ogz}) = \int_V \boldsymbol{\rho}\gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho}\, dm \ . \quad (5.2)$$

The mass moment of inertia about a z axis through o is defined by

$$I_{zzo} = \int_V (x^2 + y^2)\gamma\, dx\, dy\, dz = \int_m (x^2 + y^2)\, dm. \quad (5.3)$$

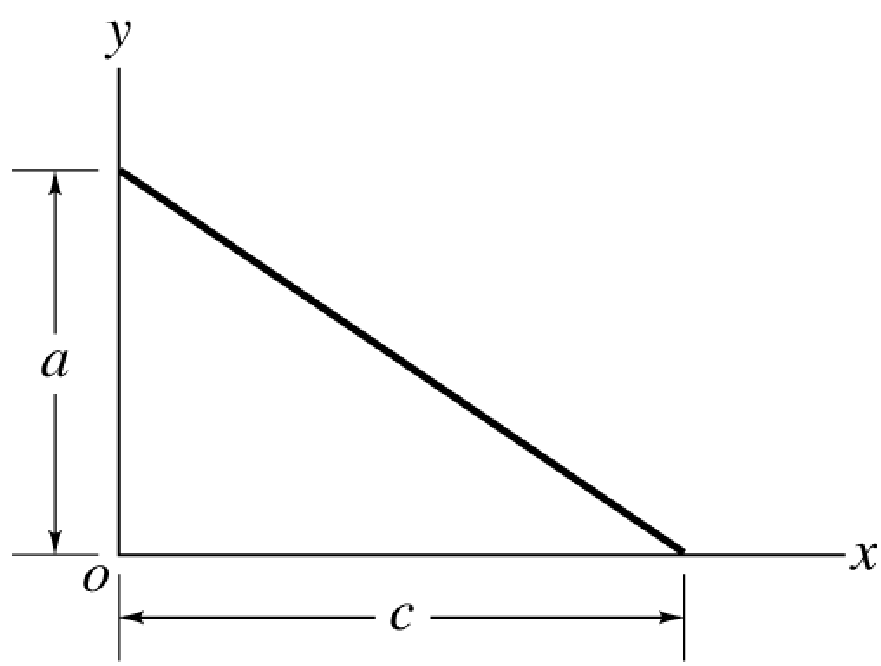


Figure 5.2 Triangular mass of unit depth and uniform mass per unit area $\bar{\gamma}$.

Applying Eq.(5.1) to the example of Figure 5.2 gives:

$$m = \bar{\gamma} \int_0^c y\,dx = \bar{\gamma} \int_0^c (a - \frac{a}{c}x)\,dx$$

$$= \bar{\gamma} \,|_0^c\, (ax - \frac{ax^2}{2c}) = \frac{ac\bar{\gamma}}{2} \quad .$$

Applying Eq.(5.2) to find the mass center gives

$$m\boldsymbol{b_{og}} = m(\boldsymbol{i}b_{ogx} + \boldsymbol{j}b_{ogy}) = \int_A \boldsymbol{\rho}\bar{\gamma}\,dx\,dy = \int_A (\boldsymbol{i}x + \boldsymbol{j}y)\bar{\gamma}\,dx\,dy \quad .$$

Hence,

$$m b_{ogx} = \bar{\gamma} \int_A x\,dx\,dy \quad , \quad m b_{ogy} = \bar{\gamma} \int_A y\,dx\,dy \quad ,$$

and:

$$m b_{ogx} = \bar{\gamma} \int_0^c xy\,dx = \bar{\gamma} \int_0^c x(a - \frac{ax}{c})\,dx = \bar{\gamma} \,|^c{}_0 (\frac{ax^2}{2} - \frac{ax^3}{3c})$$

$$= \bar{\gamma} \frac{ac^2}{6} = m\frac{c}{3}$$

$$(5.4)$$

$$m b_{ogy} = \bar{\gamma} \int_0^c yx\,dy = \bar{\gamma} \int_0^a y(c - \frac{cy}{a})\,dy = \bar{\gamma} \,|^a{}_0 (\frac{cy^2}{2} - \frac{cy^3}{3a})$$

$$= \gamma \frac{ca^2}{6} = m\frac{a}{3} \quad .$$

The mass center is located in the *x, y* system by

$\boldsymbol{b}_{og} = \boldsymbol{i}c/3 + \boldsymbol{j}a/3 .$

Proceeding from Eq.(5.3) for the moment-of-inertia definition about o is,

$$I_{zzo} = \int_A \bar{\gamma}(x^2 + y^2)\,dx\,dy = \bar{\gamma}\int_0^a \Big[\int_0^{c(1-\frac{y}{a})} (x^2 + y^2)dx\Big]dy \qquad (5.5)$$
$$= m(c^2 + a^2)/6 \ .$$

A considerable amount of work is hidden in getting across the last equality sign.

The radius of gyration is defined as the radius at which all of the mass could be concentrated to obtain the correct moment of inertia. For this example,

$$I_{zzo} = m k_g^2 \ \Rightarrow \ k_g = \sqrt{\frac{c^2 + a^2}{6}}$$

A particle has all of its mass concentrated at a point and has negligible dimensions of length, breadth, depth, etc. Rigid bodies have finite dimensions, yielding properties such as area, volume, and moment of inertia. Observe that continuing to reduce the dimensions of the triangular plate in figure 5.2 will cause the moment of inertia defined by Eq.(5.5) to rapidly approach zero, which is consistent with a particle.

The Parallel-Axis Formula

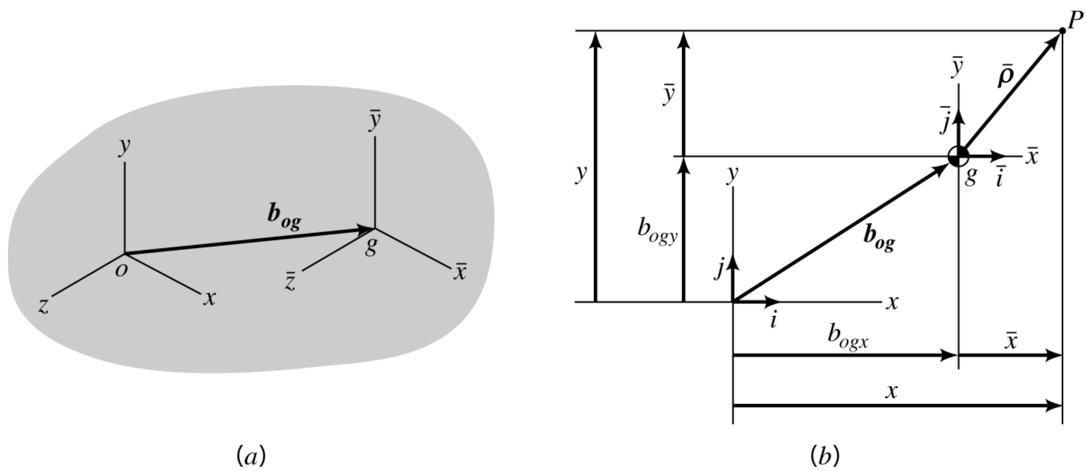


Figure 5.3 (a) Two coordinate systems fixed in a rigid body, (b) End view looking in along the z axis.

The *x, y, z* axes are parallel, respectively, to the $\bar{x}, \bar{y}, \bar{z}$ axes. The mass center of the body is located at the origin of the $\bar{x}, \bar{y}, \bar{z}$ coordinate system and is located in the *x, y, z* coordinate system by the vector $\boldsymbol{b}_{og} = \boldsymbol{i} b_{ogx} + \boldsymbol{j} b_{ogy} + \boldsymbol{k} b_{ogz}$. The question of interest is:

Suppose that we know the moment of inertia about the $\bar{z}$ *axis, what is it about the z axis ?*

Figure 5.3B provides an end view along the *z* axis of the $\bar{x}, \bar{y}$ and *x, y* axes. A point that is located in the $\bar{x}, \bar{y}, \bar{z}$ coordinate system by the vector $\bar{\boldsymbol{\rho}} = \bar{\boldsymbol{i}}\,\bar{x} + \bar{\boldsymbol{j}}\,\bar{y} + \bar{\boldsymbol{k}}\,\bar{z}$ is located in the *x, y, z* system by $\boldsymbol{\rho} = \bar{\boldsymbol{\rho}} + \boldsymbol{b}_{og}$, or $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z =$

$\boldsymbol{i}(b_{ogx}+\bar{x})+\boldsymbol{j}(b_{ogy}+\bar{y})+\boldsymbol{k}z$; hence,

$$x=b_{ogx}+\bar{x}, \quad y=b_{ogy}+\bar{y} \ .$$

The moment of inertia about the z axis is defined to be

$$I_{zzo}=\int_m (x^2+y^2)\, dm \ .$$

Substituting for x and y gives

$$\begin{aligned} I_{zzo} &= \int_m (b_{ogx}^2+2b_{ogx}\,\bar{x}+\bar{x}^2+b_{ogy}^2+2\,b_{ogy}\,\bar{y}+\bar{y}^2)\,dm \\ &= \int_m (\bar{x}^2+\bar{y}^2)\, dm + (b_{ogx}^2+b_{ogy}^2)\, m \\ &\quad + 2\, b_{ogx}\int_m \bar{x}\, dm + 2\, b_{ogy}\int_m \bar{y}\, dm \\ &= I_{\bar{z}\bar{z}}+m\,|\boldsymbol{b_{og}}|^2+2b_{ogx}\int_m \bar{x}\, dm+2b_{ogy}\int_m \bar{y}\, dm \ . \end{aligned} \tag{5.6}$$

Because the mass center is at the origin of the $\bar{x},\bar{y},\bar{z}$ coordinate system, the last two integrals in Eq.(5.6) are zero, and we obtain

$$I_{zz}=I_{\bar{z}\bar{z}}+m\,|\boldsymbol{b_{og}}|^2=I_g+m\,|\boldsymbol{b_{og}}|^2 \ . \tag{5.7}$$

Note that this expression is only valid when the mass center of the body is at the origin of the $\bar{x},\bar{y},\bar{z}$ *coordinate system.*

Example 1

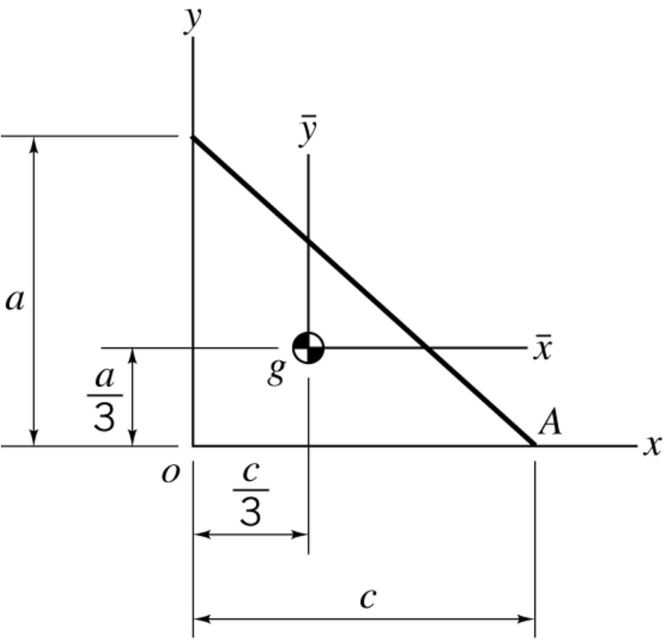


Figure 5.4 Two coordinate systems in the triangular plate of figure 5.2.

From Eq.(5.5)

$$I_{zzo} = I_o = m(c^2 + a^2)/6$$

Wanted: the moment of inertia about a *z* axis (perpendicular to the plate) through point *A* at the right-hand corner.

Procedure:

1. Use Eq.(5.7) to go from *o* to *g* and find I_g.

2. Use Eq.(5.7) to go from g to *A* and find I_A.

The vector from point *o* to *g* is $\boldsymbol{b_{og}} = \boldsymbol{i}c/3 + \boldsymbol{j}a/3$; hence, step 1 gives

$$I_g = I_o - m\,|\,\boldsymbol{b_{og}}\,|^2$$
$$= \frac{m}{6}(c^2 + a^2) - \frac{m}{9}(a^2 + c^2) = \frac{m}{18}(c^2 + a^2)\ .$$

The vector from point g to point A is $\boldsymbol{b_{gA}} = \boldsymbol{i}2c/3 - \boldsymbol{j}a/3$. Hence, step 2 gives

$$I_A = I_g + m\,|\,\boldsymbol{b_{gA}}\,|^2 = \frac{m}{18}(c^2 + a^2) + m\left(\frac{4c^2}{9} + \frac{a^2}{9}\right)$$
$$= \frac{m}{18}(9c^2 + 3a^2) = \frac{mc^2}{2} + \frac{ma^2}{6}\ .$$

Note

$$I_A \neq I_o + m\,|\,\boldsymbol{b_{oA}}\,|^2$$

$$I_o + m\,|\,\boldsymbol{b_{oA}}\,|^2 = \frac{m}{6}(c^2 + a^2) + mc^2 = \frac{7mc^2}{6} + \frac{ma^2}{6}\ .$$

Example 2

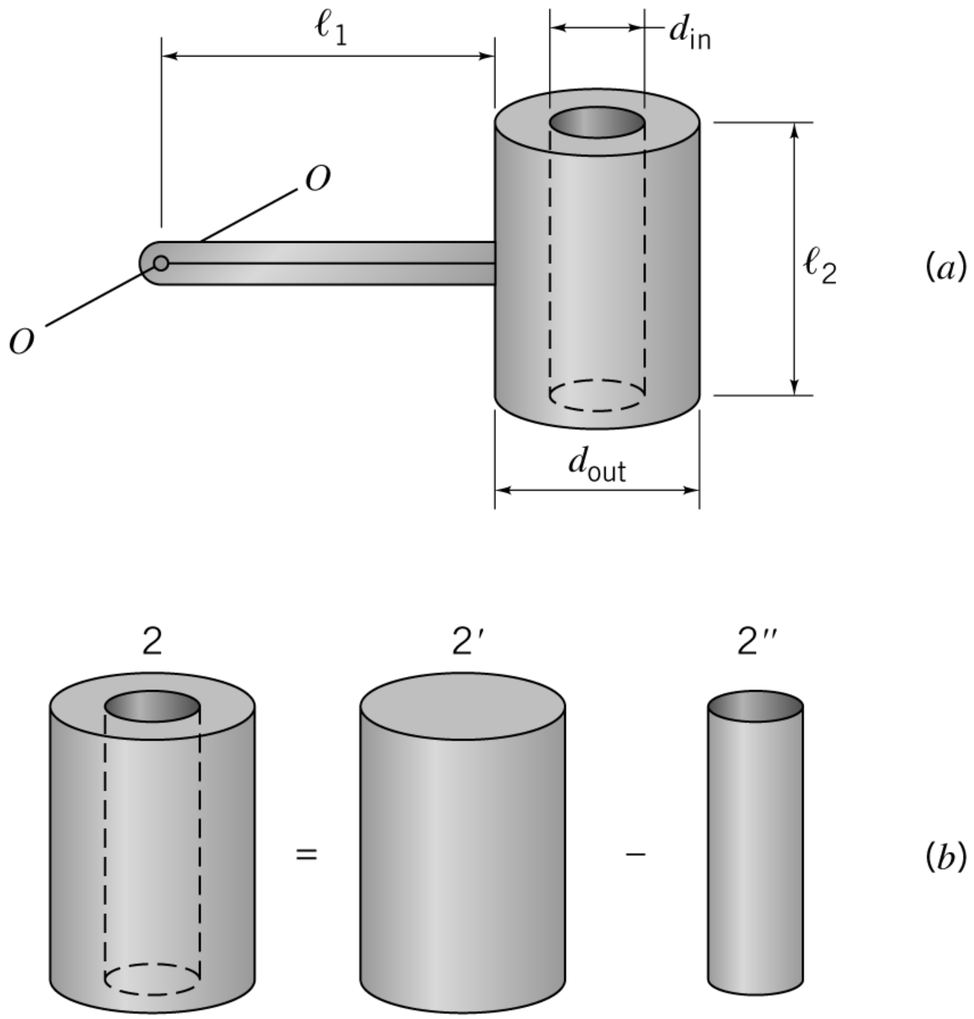


Figure XP5.1 (a) Assembly rotating about axis *o-o*, (b) Modeling approach for the hollow cylinder.

Figure XP5.1A illustrates a welded assembly consisting of a uniform bar with dimensions $l_1 = 25\,cm$, $d_1 = 2.5\,cm$ attached to a hollow cylinder with length $l_2 = 20\,cm$ and inner and outer radii $d_{out} = 150\,mm$; $d_{in} = 75\,mm$, respectively. The assembly is made from steel with density $\gamma = 7750\,kg/m^3$. The assembly rotates about the *o-o* axis.

The following engineering-analysis tasks apply:

a. Determine the moment of inertia of the assembly about the o-o axis.

b. Determine the assembly's radius of gyration for rotation about the o-o axis.

b. Determine the assembly's mass-center location.

Solution Break the assembly into two pieces and analyze the bar and hollow cylinder separately.

Slender Bar: The moment of inertia for a slender bar about a transverse axis at its end is $I_{end} = ml^2/3$, and the moment of inertia for a transverse axis through the mass center is $I_g = ml^2/12$. These results are related to each other via the parallel-axis formula as

$$I_{end} = I_g + m(\frac{l}{2})^2 = \frac{ml^2}{12} + \frac{ml^2}{4} = \frac{ml^2}{3} \quad . \qquad (\text{i})$$

You should work at committing the bar's inertia-property definition to memory. The bar's mass is

$$m_1 = (\frac{\pi d_1^2}{4}) l_1 \gamma = \frac{3.14159 \times .025^2}{4} .25 \times 7750 \ = .951\, kg \ . \quad \text{(ii)}$$

Hence, from Eq.(i),

$$I_{1o} = \frac{m_1 l_1^2}{3} = \frac{.951 \times .25^2}{3} = .0198\, kg m^2 \ . \quad \text{(iii)}$$

Hollow Cylinder: figure XP5.1B shows that the hollow cylinder can be "constructed" by subtracting a solid cylinder (denoted $2''$) with the inner diameter d_{in} from a solid cylinder (denoted $2'$) with the outer radius d_{out}. Starting with the inner cylinder

$$m_{2''} = (\frac{\pi d_{in}^2}{4}) \times l_2 \times \gamma = (\frac{3.14159 \times .075^2}{4}) \times .2 \times 7550 = 6.671\, kg \quad \text{(iv)}$$

From Appendix C, the moment of inertia for a transverse axis through the inner cylinder's mass center is

$$I_{2''g} = m_{2''} \times (\frac{r_{in}^2}{4} + \frac{l_2^2}{12}) = 6.671 \times (\frac{.0375^2}{4} + \frac{.2^2}{12}) = .0241\, kg m^2 . \quad \text{(v)}$$

For the solid outer cylinder,

$$m_{2'} = (\frac{\pi d_{out}^2}{4}) \times l_2 \times \gamma = (\frac{3.14159 \times .15^2}{4}) \times .2 \times 7550 = 27.39\, kg , \quad \text{(vi)}$$

and

$$I_{2'g} = m_{2'} \times \left(\frac{r_{out}^2}{4} + \frac{l_2^2}{12}\right) = 27.39 \times \left(.\frac{075^2}{4} + \frac{.2^2}{12}\right) = .1298\,kg\,m^2 \quad \text{(vii)}$$

From Eqs.(iv) through (vii),

$$m_2 = m_2' - m_2'' = 27.39 - 6.671 = 20.72\,kg$$

$$I_2 = I_2' - I_2'' = .1298 - .0246 = .1019\,kg\,m^2 \;.$$

These values conclude the individual results for the hollow cylinder.

Combining the results for the bar and the hollow cylinder via the parallel-axis formula, the assembly moment of inertia is

$$I_o = I_{1o} + I_2 + m_2 \times b_{og}^2 = .0198 + .1019 + 20.72 \times .325^2 = 2.310\,kg\,m^2,$$

and concludes *Task a*. The radius of gyration k_g is obtained from $I_o = m k_g^2$ where m is the assembly moment of inertia; hence,

$$2.310 = (.951 + 20.72)k_g^2 = 21.67\,k_g^2 \Rightarrow \quad k_g = .326\,m \;, \qquad \text{(viii)}$$

and concludes *Task b*.

The assembly mass location is found from

$$md = (m_1 + m_2)d = 21.67\,d$$

$$= (m_1 d_1 + m_2 d_2) = .951 \times \frac{.25}{2} + 20.72 \times .325 = 6.853 \;\; ;$$

hence,

$$d = .316\,m \;\; .$$

Note that the mass center location defined by *d* is not related to the radius of gyration k_g .

Lecture 25. GOVERNING FORCE AND MOMENT EQUATIONS FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a *particle*.

Find: force and moment differential equations of motion for planar motion of a rigid body.

Force Equation

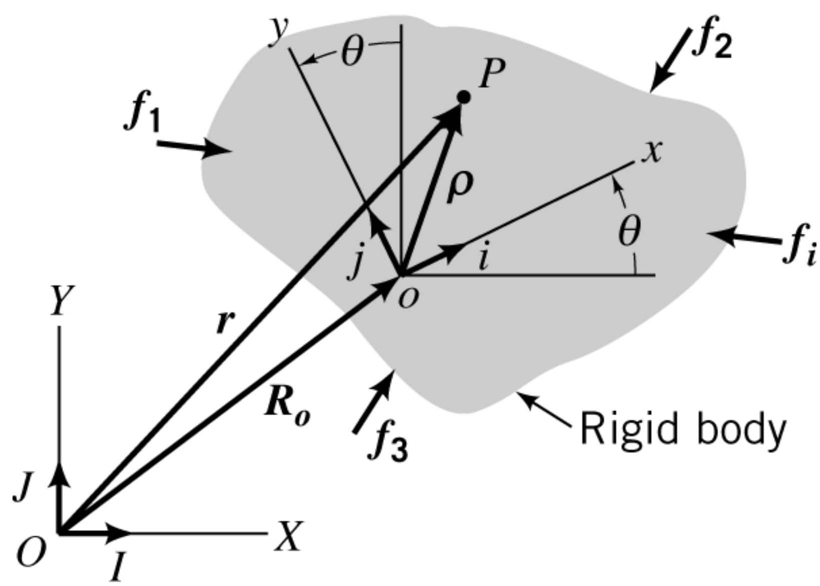


Figure 5.5 Rigid body acted on by external forces. The x,y,z coordinate system is fixed in the rigid body; the X, Y, Z system is an inertial coordinate system.

X, Y, Z inertial coordinate system

x, y, z coordinate system fixed in the rigid body.

θ defines the orientation of the rigid body (and the x, y, z coordinate system) with respect to the X, Y, Z system.

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ is the angular velocity of the rigid body and the x, y, z coordinate system, with respect to X, Y, Z coordinate system.

$\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$ locates the origin of the x, y, z system in the X, Y, Z system.

Point P in the rigid body is located in the X, Y, Z system by $\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y + \boldsymbol{K}r_Z$.

Point is located in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

Hence,

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \quad .$$

Force Equation. Applying Newton's second law to the particle at P gives

$$\boldsymbol{f_P} = dm\,\ddot{\boldsymbol{r}} = dm\,\frac{d^2\boldsymbol{r}}{dt^2}\Big|_{X,Y,Z}\ , \tag{5.8}$$

where:

$\boldsymbol{f_P}$ is the *resultant* force

$\ddot{\boldsymbol{r}}$ is the acceleration of the particle with respect to the inertial X, Y, Z system.

$dm = \gamma\, dx\, dy\, dz$ where γ is the mass density of the rigid body.

The resultant force at P is

$$\boldsymbol{f_P} = \Sigma \boldsymbol{f}_{external} + \Sigma \boldsymbol{f}_{internal}$$

On the left hand side of Eq.(5.8), integrating over the mass of the body gives

$$\int_m \boldsymbol{f_p}\, dm = \Sigma \int_m \boldsymbol{f}_{external}\, dm + \Sigma \int_m \boldsymbol{f}_{internal}\, dm = \Sigma\, \boldsymbol{f_i} + 0\ ;$$

i.e., when integrated over the whole body, the internal forces cancel.

The integral expression of Eq.(5.8) is then

$$\sum \boldsymbol{f_i} = \int_V \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \ddot{\boldsymbol{r}}\, dm \ , \tag{5.9}$$

For the two points o and P in the rigid body

$$\boldsymbol{a_P} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oP}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oP}}) \ .$$

Since $\boldsymbol{r}$ and $\boldsymbol{R_o}$ locate points P and o, respectively, in the X, Y, Z system, and $\boldsymbol{\rho}$ is the vector from point o to P,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \ . \tag{5.10}$$

Since $dm = \gamma\, dx\, dy\, dz$, integration extends over the volume of the rigid body.

Since $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are constant with respect to the x, y, z integration variables they can be brought outside the integral sign yielding

$$\sum \boldsymbol{f_i} = m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \int_m \boldsymbol{\rho}\, dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\, dm) \ . \tag{5.11}$$

The mass center is located in the x, y, z system by $\boldsymbol{b_{og}}$, defined by

$$m\boldsymbol{b_{og}} = \int_m \boldsymbol{\rho}\, dm = m(\boldsymbol{i} b_{ogx} + \boldsymbol{j} b_{ogy} + \boldsymbol{k} b_{ogz}) \ . \tag{5.12}$$

Substituting from Eq.(5.12) into Eq.(5.11) gives

$$\sum \boldsymbol{f_i} = m[\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}})] \ . \tag{5.13}$$

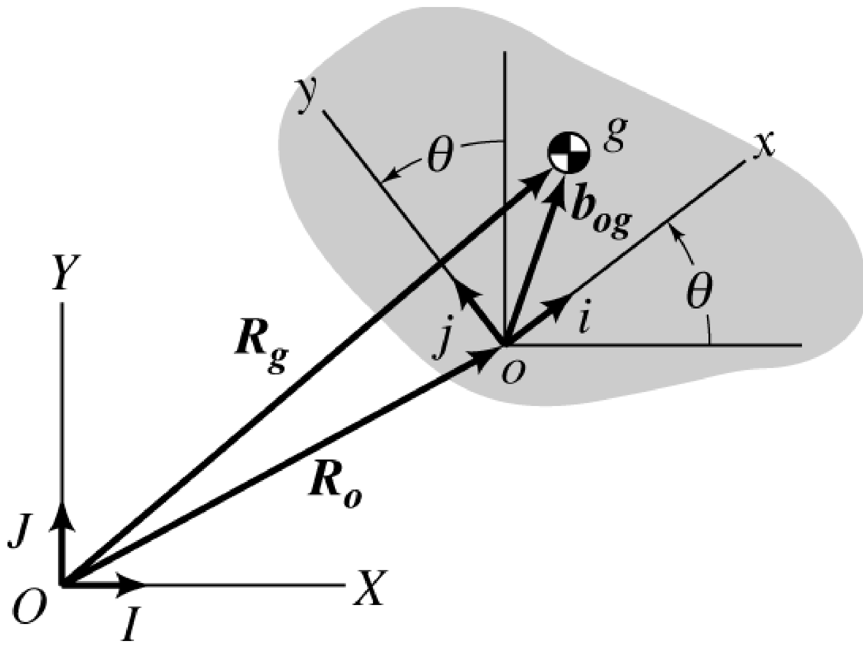


Figure 5.6 A rigid body with a mass center located in the body-fixed *x, y, z* coordinate system by the vector $\boldsymbol{b_{og}}$ and located in the inertial *X, Y, Z* system by $\boldsymbol{R_g}$.

Since *g* and *o* are fixed in the rigid body, their accelerations are related by

$$\boldsymbol{a_g} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{og}})$$

But $\boldsymbol{r_{og}} = \boldsymbol{b_{og}}$, and $\boldsymbol{a_o} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}}$; hence,

$$\ddot{\boldsymbol{R}}_{\boldsymbol{g}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}}) \ ,$$

and the force equation can be written (finally) as

$$\sum \boldsymbol{f_i} = m\,\ddot{\boldsymbol{R}}_{\boldsymbol{g}} \quad . \tag{5.14}$$

In words, Eq.(5.14) states that a rigid body can be treated like a particle, in that the summation of external forces acting on the rigid body equals the mass of the body times the acceleration of the mass center with respect to an inertial coordinate system.

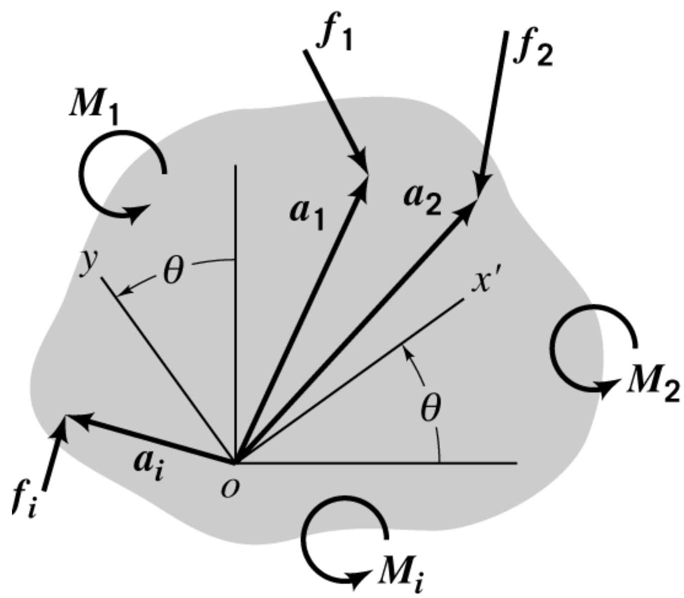
Cartesian component of Force equations:

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15a}$$

Polar version

$$\sum f_{ir} = m(\ddot{r}_g - r_g\dot{\theta}^2) \ , \quad \sum f_{i\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) \ . \tag{5.15b}$$

Moment Equation



A rigid body acted on by several external forces $\boldsymbol{f_i}$ acting on the body at points located by the position vectors $\boldsymbol{a_i}$ and moments $\boldsymbol{M}_i$

In figure 5.5, the position vector $\boldsymbol{\rho}$ extends from *o* to a particle at point *P*. For moments about *o*, $\boldsymbol{\rho}$ is the moment arm, and the particle moment equation is

$$\boldsymbol{\rho} \times \boldsymbol{f_P} = \boldsymbol{\rho} \times dm\,\ddot{\boldsymbol{r}} \quad . \tag{5.16}$$

Integrating Eq.(5.16) over the mass of the rigid body yields

$$\sum (\boldsymbol{a_i} \times \boldsymbol{f_i}) + \sum \boldsymbol{M_i} = \boldsymbol{M_o} = \int_V \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\,\gamma\,dx\,dy\,dz = \int_m \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\,dm \tag{5.17}$$

The vector $\boldsymbol{M_o}$ on the left is the *resultant* external moment acting on the rigid body about point *o,* the origin of the *x, y, z* coordinate system.

Kinematics: Substituting from Eq.(5.10) gives

$$\boldsymbol{\rho} \times \ddot{\boldsymbol{r}} = (\boldsymbol{\rho} \times \ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\rho} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] \ .$$

The vector identity,

$$\boldsymbol{A} \times [\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{A})] = \boldsymbol{B} \times [\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{A})] \ ,$$

gives

$$\boldsymbol{\rho} \times \ddot{\boldsymbol{r}} = (\boldsymbol{\rho} \times \ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] \ . \qquad (5.18)$$

Since $\ddot{\boldsymbol{R}}_o, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are not functions of the variables of integration, substitution from Eq.(5.18) into Eq.(5.17) gives

$$\begin{aligned} \boldsymbol{M}_o = m(\boldsymbol{b}_{og} \times \ddot{\boldsymbol{R}}_o) &+ \int \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho})\, dm \\ &+ \boldsymbol{\omega} \times \int [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]\, dm \ . \end{aligned} \qquad (5.19)$$

with $\boldsymbol{b}_{og}$ defined by Eq.(5.12).

To find component equations from Eq.(5.19)

$$\boldsymbol{b_{og}} \times m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} = m \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_{ogx} & b_{ogy} & b_{ogz} \\ \ddot{R}_{ox} & \ddot{R}_{oy} & 0 \end{vmatrix}$$

$$= \begin{aligned} &-\boldsymbol{i}\, m b_{ogz} \ddot{R}_{oy} \\ &+\boldsymbol{j}\, m b_{ogz} \ddot{R}_{ox} \\ &+\boldsymbol{k} m (b_{ogx}\ \ddot{R}_{oy} - b_{ogy}\ \ddot{R}_{ox}) \end{aligned} \tag{5.20}$$

In carrying out the cross product, note that $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}$ is stated in terms of its components in the *x, y, z* coordinate system, versus the customary *X, Y, Z* system.

Defining the vectors in Eq.(5.19) in terms of their components gives

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \ , \ \boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} \ , \ \dot{\boldsymbol{\omega}} = \boldsymbol{k}\ddot{\theta} \ .$$

Hence,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\dot{\theta} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z) = \dot{\theta}(\boldsymbol{j}x - \boldsymbol{i}y)$$

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} = \ddot{\theta}\ (\boldsymbol{j}x - \boldsymbol{i}\ y) \ ,$$

and

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ -y & x & 0 \end{vmatrix} = \boldsymbol{k}\,\dot{\theta}\,(x^2 + y^2)\ . \quad (5.21)$$

Similarly,

$$\boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) = \boldsymbol{k}\,\ddot{\theta}\,(x^2 + y^2) \quad (5.22)$$

Substituting from Eqs.(5.20)-(5.22)into Eq.(5.19) gives the z component equation

$$\begin{aligned} \boldsymbol{k} M_{oz} &= \boldsymbol{k} m (\boldsymbol{b} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z \\ &\quad + \boldsymbol{k}\ddot{\theta} \int_m (x^2 + y^2)\, dm + \boldsymbol{k}\dot{\theta} \times \boldsymbol{k}\dot{\theta} \int_m (x^2 + y^2)\, dm\ . \end{aligned} \quad (5.23)$$

The last expression in this equation is zero because $\boldsymbol{k} \times \boldsymbol{k} = 0$. Since

$$I_o = \int_m (x^2 + y^2)\, dm\ ,$$

the moment Eq.(5.23) can be stated (finally) as

$$\sum M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b}_{\boldsymbol{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}}\ . \quad (5.24)$$

Summary of governing equations of motion for planar motion of a rigid body

Force-Equation Cartesian Components

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \qquad (5.15)$$

Moment Equation

$$\sum M_{oz} = I_o\ddot{\theta} + m(\boldsymbol{b_{og}} \times \boldsymbol{\ddot{R}_o})_z \ . \qquad (5.24)$$

Reduced Forms for the Moment Equation

Moments taken about the mass center. If the point *o* about which moments are taken coincides with the mass center *g*, $\boldsymbol{b_{og}} = 0$, and Eq.(5.24) reduces to

$$M_{gz} = I_g\ddot{\theta} \ . \qquad (5.25)$$

This equation is *only* correct for moments taken about the mass center of the rigid body.

Moments taken about a fixed point in inertial space. When point o is fixed in the (inertial) X, Y, Z coordinate system, $\ddot{\boldsymbol{R}}_o = 0$, and the moment equation is

$$M_{oz} = I_o \ddot{\theta} \quad . \tag{5.26}$$

Fixed-Axis-Rotation Applications of the Force and Moment equations for Planar Motion of a Rigid Body

Rotor in Bearings

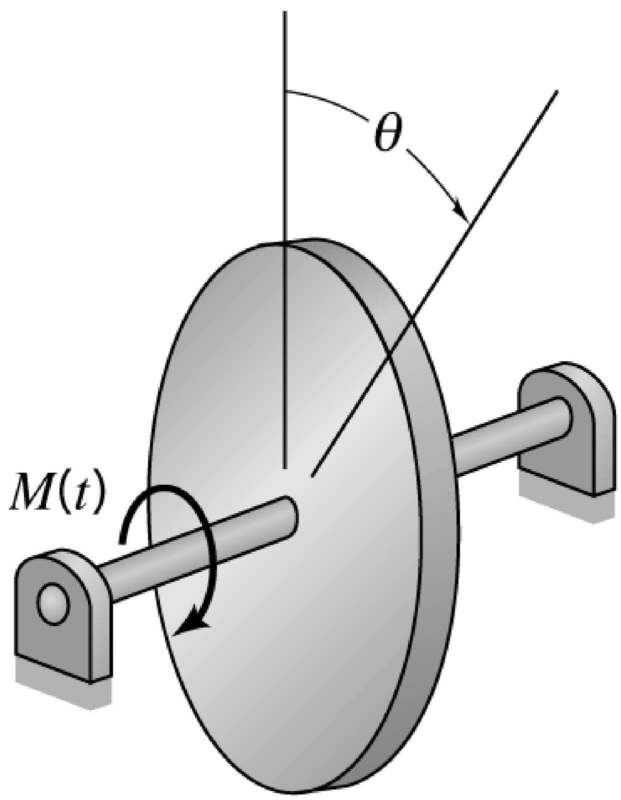


Figure 5.8 A disk mounted on a massless shaft, supported by two frictionless bearings, and acted on by the applied torque $M(t)$.

Derive the differential equation of motion for the rotor. The governing equation of motion for the present system is

$$M_{oz} = I_o\ddot{\theta} \Rightarrow \left(\frac{mr^2}{2}\right)\ddot{\theta} = M(t) \ .$$

The moment $M(t)$ is positive because it is acting in the same direction as $+\theta$. This is basically the same second-order differential equation obtained for a particle of mass m acted on by the force $f(t)$, namely, $m\ddot{x} = f(t)$, where x locates the particle in an inertial coordinate system.

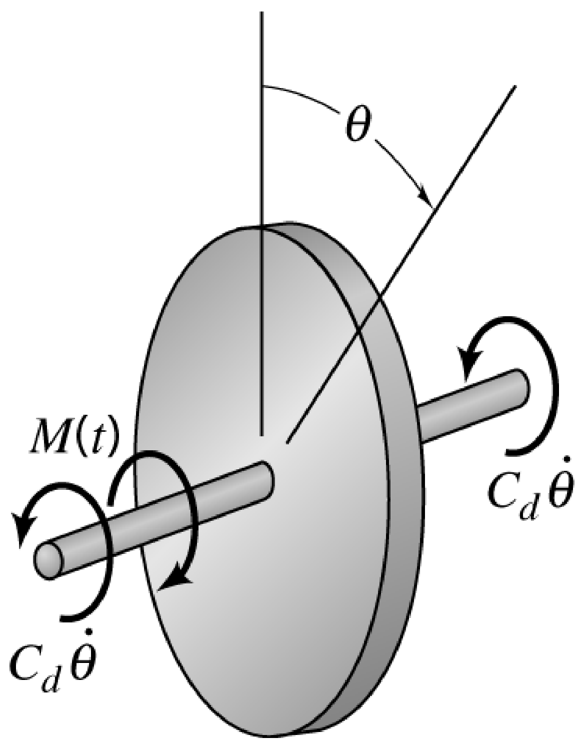


Figure 5.9 Free-body diagram for the rotor of figure 5.8 with a drag torque $C_d\dot{\theta}$ acting at each bearing.

The shaft is rotating in the $+\dot{\theta}$ direction; hence, the drag moment terms have negative signs because they are acting in -θ direction. The differential equation of motion to be obtained from the moment equation is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \frac{mr^2}{2}\ddot{\theta} = M_{oz}(t) - 2C_d\dot{\theta} \ , or$$

$$\frac{mr^2}{2}\ddot{\theta} + 2C_d\dot{\theta} = M_{oz}(t)$$

This equation has the same form as a particle of mass *m* acted on by the force *f(t)* and a linear dashpot with a damping coefficient *c* ; namely, $m\ddot{x} + c\dot{x} = f(t)$.

A-One Degree of Freedom Torsional Vibration Example

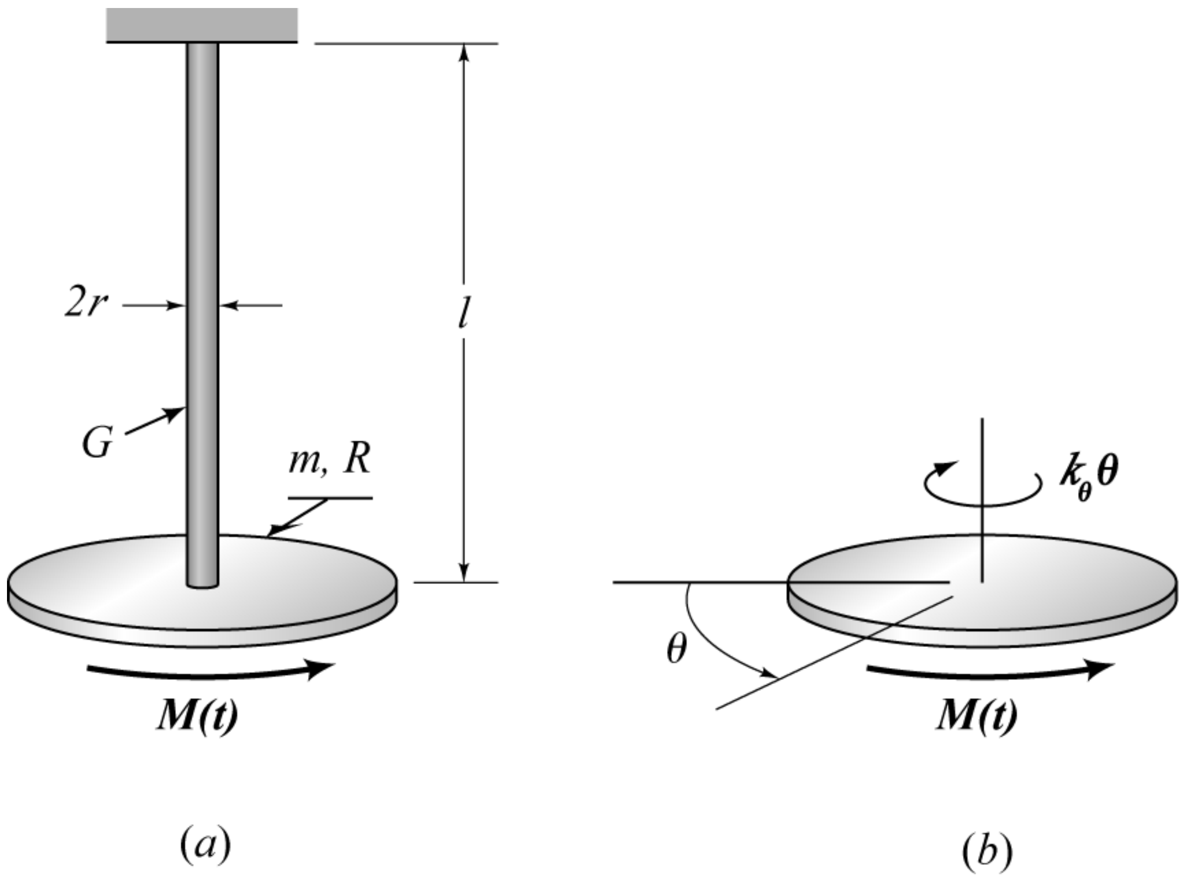


Figure 5.11 (a) Circular disk of mass *m* and radius *R*, supported by a slender rod of length *l*, radius *r*, and shear modulus *G*, (b) Free-body diagram for $\theta > 0$

Twisting the rod about its axis through an angle θ will create a reaction moment, related to θ by

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta.$$

$k_\theta = GJ/l$, where *G* is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. Recall that the SI units for G is N/m^2; hence, k_θ has the units: *N-m/ radian*, i.e., moment per unit torsional rotation of the rod.

Derive the differential equation of motion for the disk. Applying Eq.(5.26) yields the moment equation

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \frac{mR^2}{2}\ddot{\theta} = M(t) + M_\theta = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta ,$$

The signs of the moments on the right hand side of this moment equation are positive or negative, depending on whether they are, respectively, in the $+\theta$ or $-\theta$ direction.

The differential equation of motion to be obtained from the moment equation is

$$\frac{mR^2}{2}\ddot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t) .$$

This result is analogous to the differential equation of motion for a particle of mass m, acted on by an external force $f(t)$, and supported by a linear spring with stiffness coefficient k ; viz., $m\ddot{x} + kx = f(t)$. For comparison, look at Eq.(3.13). This equation can be rewritten as

$$\ddot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2} ,$$

where the undamped natural frequency ω_n is defined by

$$\omega_n = \sqrt{\frac{\pi G r^4}{l m R^2}} \; .$$

Torsional Vibration Example with Viscous Damping

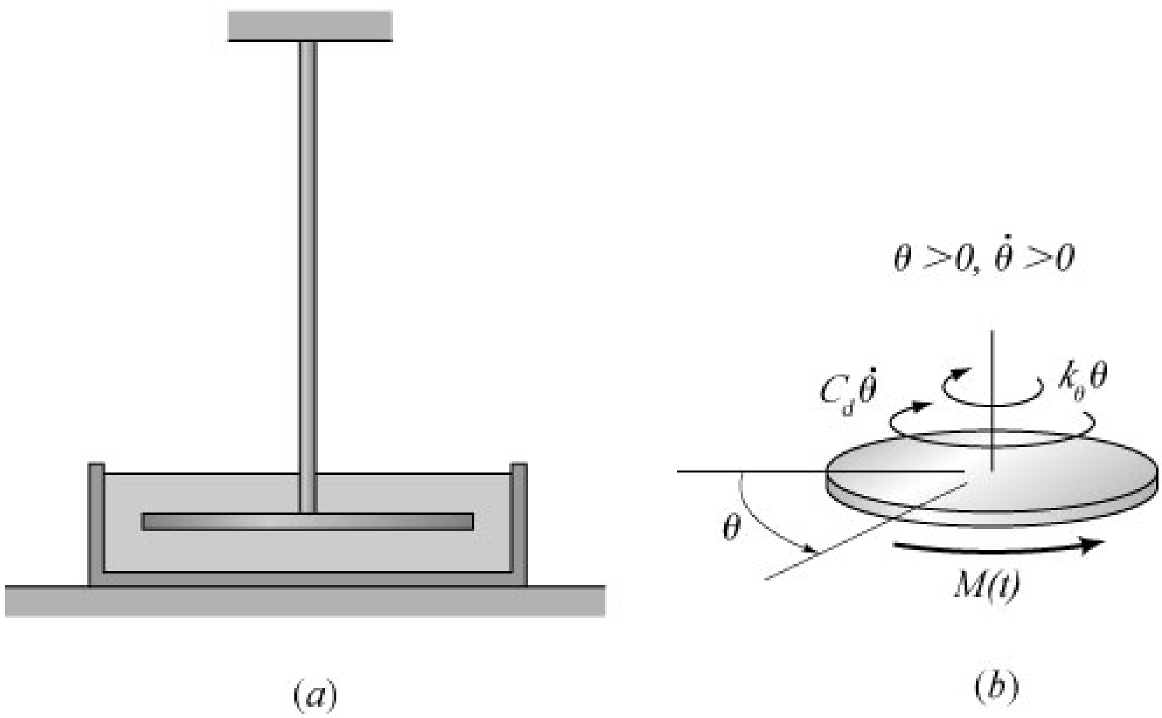


Figure 5.11 (a) The disk of figure 5.10 is now immersed in a viscous fluid, (b) Free-body diagram

Rotation of the disk at a finite rotational velocity $\dot{\theta}$ within the fluid causes the drag moment, $-C_d\dot{\theta}$, on the disk. The negative sign for the drag term is chosen because it acts in the -θ direction. The complete moment equation is

$$\frac{mR^2}{2}\ddot{\theta} = \Sigma M_z = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta - C_d\dot{\theta},$$

with the governing differential equation

$$\frac{mR^2}{2}\ddot{\theta} + C_d\dot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t). \qquad (5.27)$$

This differential equation has the same form as a particle of mass *m* supported by a parallel arrangement of a spring with stiffness coefficient *k* and a linear damper with damping coefficient *c*; namely, $m\ddot{x} + c\dot{x} + kx = f(t)$.

Eq.(5.27) can be restated as

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2},$$

where ζ is the damping factor, defined by

$$2\zeta\omega_n = \frac{2C_d}{mR^2}, \quad \omega_n = \sqrt{\frac{\pi G r^4}{lmR^2}}.$$

The models developed from figures 5.10 and 5.11 show the same damped and undamped vibration possibilities for rotational motion of a disk that we reviewed earlier for linear motion of a particle. The same possibilities exist to define damped and undamped natural frequencies, damping factors, etc.

An example involving kinematics between a disk and a particle

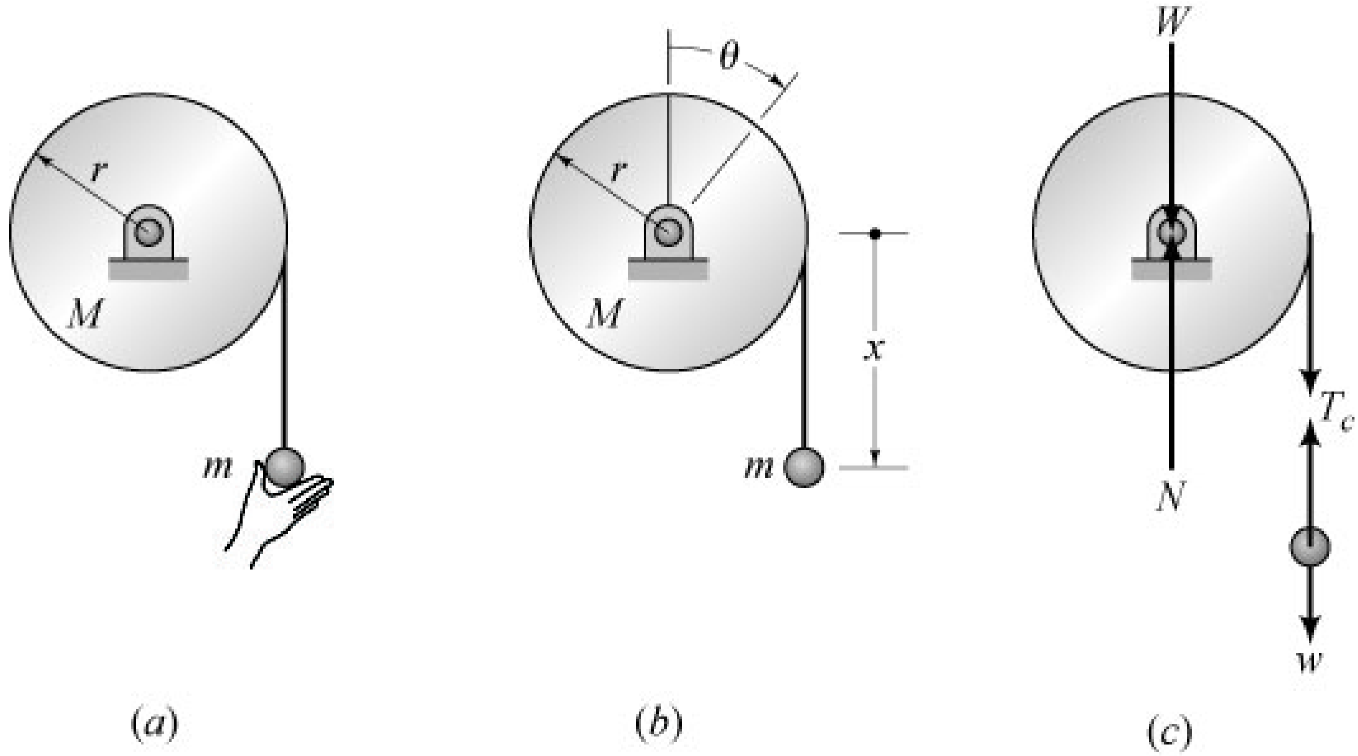


Figure 5.12 (a) Disk of mass *M* and radius *r* supported in frictionless bearings and connected to a particle of mass *m* by a light and inextensible cord, (b) Coordinates, (c) Free-body diagram.

Derive the differential equation of motion for the system.

Kinematics:

$$\delta x = r\delta\theta \;\Rightarrow\; \dot{x} = r\dot{\theta} \;,\; \ddot{x} = r\ddot{\theta}. \tag{5.28}$$

From the free-body diagram, the equation of motion for the disk is obtained by writing a moment equation about its axis of rotation. The equation of motion for mass m follows from $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a particle. The governing equations are:

$$\frac{Mr^2}{2}\ddot{\theta} = \Sigma M_{oz} = T_c r \;,\quad m\ddot{x} = \Sigma f = w - T_c, \tag{5.29}$$

where T_c is the tension in the cord. (The mass of the cord has been neglected in stating these equations.) In the first of Eq.(5.29), the moment term $T_c\, r$ is positive because it acts in the $+\theta$ direction. The sign of w is positive in the force equation because it acts in the $+x$ *direction* ; T_c has a negative sign because it is directed in the $-x$ direction.

Eqs.(5.29) provides two equations for the three unknowns: $\ddot{x}, \ddot{\theta}$, and T_c. Eliminating the tension T_c from Eqs.(5.29) gives

$$\frac{Mr^2}{2}\ddot{\theta} + rm\ddot{x} = wr. \tag{5.30}$$

Substituting from the last of Eq.(5.28) for $\ddot{x} = r\ddot{\theta}$ gives the final differential equation

$$(\frac{M}{2} + m)r^2\ddot{\theta} = wr.$$

Two driven pulleys connected by a belt

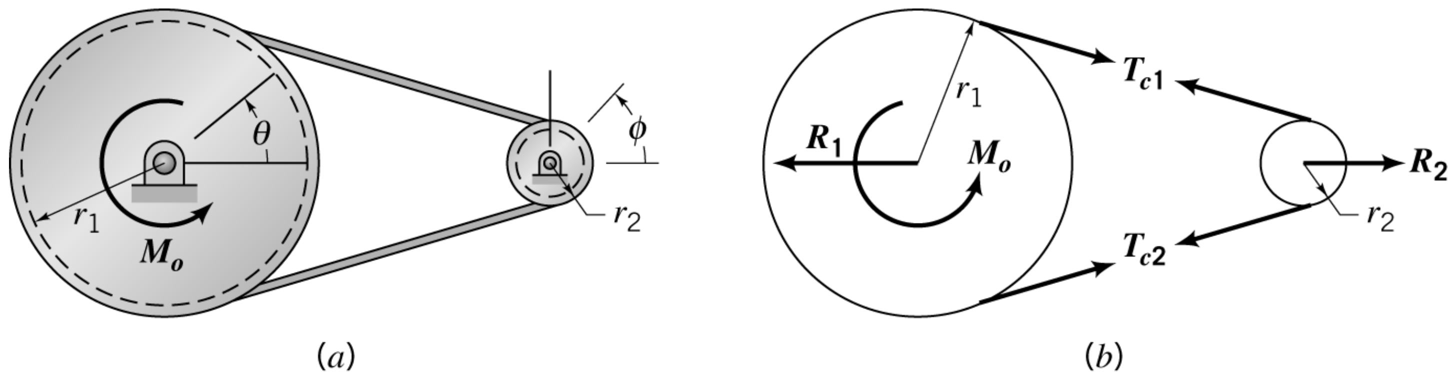


Figure 5.13 (a). Two disks connected by a belt. (b). Free-body diagram.

Figure 5.13A illustrates two pulleys that are connected to each other by a light and inextensible belt. The pulley at the left has mass m_1, radius of gyration k_{1g} about the pulley's axis of rotation, and is acted on by the counterclockwise moment M_o. The pulley at the right has mass m_2 and a radius of gyration k_{2g} about its axis of rotation. (The radius of gyration k_g defines the moment of inertia about the axis of rotation by $I = mk_g^2$.) The belt runs in a groove in pulley 1 with inner radius r_1. The inner radius of the belt groove for pulley 2 is r_2. The angle of rotation for pulleys 1 and 2 are, respectively, θ and φ.

Derive the governing differential equation of motion in terms of θ and its derivatives.

From the free-body diagram the fixed-axis rotation moment Eq.(5.26) gives:

$$I_1 \ddot{\theta} = \Sigma M_{1oz} = M_o(t) + r_1 (T_{c2} - T_{c1})$$
$$I_2 \ddot{\varphi} = \Sigma M_{2oz} = r_2 (T_{c1} - T_{c2}) \ , \tag{5.31}$$

where T_{c1} and T_{c2} are the tension components in the upper and lower belt segments. $M_o(t)$ has a positive sign because it is acting in the +θ direction; $r_1(T_{c1} - T_{c2})$ has a negative sign because it acts in the -θ direction. Similarly, $r_2(T_{c1} - T_{c2})$ has a positive sign in the second of Eq.(5.31) because it is acting in the +φ direction.

The moments of inertia in Eq.(5.31) are defined in terms of their masses and radii of gyrations by

$$I_1 = m_1 k_1^2 \quad ; \ I_2 = m_2 k_2^2 \ .$$

Returning to Eq.(5.31), we can eliminate the tension terms in the two equations, obtaining

$$I_1 \ddot{\theta} = M_o - \frac{r_1}{r_2} I_2 \ddot{\varphi} \quad \Rightarrow \quad I_1 \ddot{\theta} + \frac{r_1}{r_2} I_2 \ddot{\varphi} = M_o(t) \ . \tag{5.32}$$

We now have one equation for the two unknowns $\ddot{\theta}$ and $\ddot{\varphi}$, and need an additional kinematic equation relating these two angular acceleration terms. Given that the belt connecting the pulleys is inextensible (can not stretch) the velocity v of the belt leaving both pulleys must be equal; hence,

$$v = r_1 \dot{\theta} = r_2 \dot{\varphi} \;\Rightarrow\; r_1 \ddot{\theta} = r_2 \ddot{\varphi} \; .$$

Substituting this result back into Eq.(5.32) gives the desired final result

$$[I_1 + (\frac{r_1}{r_2})^2 I_2]\ddot{\theta} = I_{eff}\ddot{\theta} = M_o(t) \; .$$

Note that coupling the two pulleys' motion by the belt acts to increase the effective inertia I_{eff} in resisting the applied moment.

Lecture 25. GOVERNING FORCE AND MOMENT EQUATIONS FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a *particle*.

Find: force and moment differential equations of motion for planar motion of a rigid body.

Force Equation

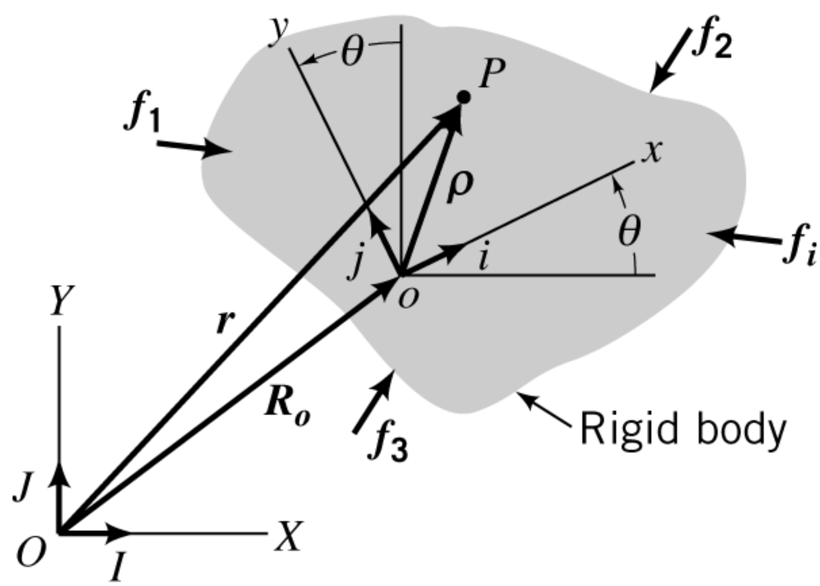


Figure 5.5 Rigid body acted on by external forces. The *x,y,z* coordinate system is fixed in the rigid body; the *X, Y, Z* system is an inertial coordinate system.

X, Y, Z inertial coordinate system

x, y, z coordinate system fixed in the rigid body.

θ defines the orientation of the rigid body (and the x, y, z coordinate system) with respect to the X, Y, Z system.

$\boldsymbol{\omega} = \boldsymbol{k}\dot{\theta}$ is the angular velocity of the rigid body and the x, y, z coordinate system, with respect to X, Y, Z coordinate system.

$\boldsymbol{R_o} = \boldsymbol{I}R_{oX} + \boldsymbol{J}R_{oY}$ locates the origin of the x, y, z system in the X, Y, Z system.

Point P in the rigid body is located in the X, Y, Z system by $\boldsymbol{r} = \boldsymbol{I}r_X + \boldsymbol{J}r_Y + \boldsymbol{K}r_Z$.

Point is located in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

Hence,

$$\boldsymbol{r} = \boldsymbol{R_o} + \boldsymbol{\rho} \quad .$$

Force Equation. Applying Newton's second law to the particle at P gives

$$\boldsymbol{f_P} = dm\,\ddot{\boldsymbol{r}} = dm\frac{d^2\boldsymbol{r}}{dt^2}\Big|_{X,Y,Z}\ , \tag{5.8}$$

where:

$\boldsymbol{f_P}$ is the *resultant* force

$\ddot{\boldsymbol{r}}$ is the acceleration of the particle with respect to the inertial X, Y, Z system.

$dm = \gamma\, dx\, dy\, dz$ where γ is the mass density of the rigid body.

The resultant force at P is

$$\boldsymbol{f_P} = \Sigma \boldsymbol{f}_{\boldsymbol{external}} + \Sigma \boldsymbol{f}_{\boldsymbol{internal}}$$

On the left hand side of Eq.(5.8), integrating over the mass of the body gives

$$\int_m \boldsymbol{f_p}\, dm = \Sigma \int_m \boldsymbol{f}_{\boldsymbol{external}}\, dm + \Sigma \int_m \boldsymbol{f}_{\boldsymbol{internal}}\, dm = \Sigma\ \boldsymbol{f_i} + 0\ \ ;$$

i.e., when integrated over the whole body, the internal forces cancel.

The integral expression of Eq.(5.8) is then

$$\sum \boldsymbol{f_i} = \int_V \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \ddot{\boldsymbol{r}}\, dm \ , \tag{5.9}$$

For the two points o and P in the rigid body

$$\boldsymbol{a_P} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{oP}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{oP}}) \ .$$

Since $\boldsymbol{r}$ and $\boldsymbol{R_o}$ locate points P and o, respectively, in the X, Y, Z system, and $\boldsymbol{\rho}$ is the vector from point o to P,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \ . \tag{5.10}$$

Since $dm = \gamma\, dx\, dy\, dz$, integration extends over the volume of the rigid body.

Since $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are constant with respect to the x, y, z integration variables they can be brought outside the integral sign yielding

$$\sum \boldsymbol{f_i} = m\ddot{\boldsymbol{R}}_{\boldsymbol{o}} + \dot{\boldsymbol{\omega}} \times \int_m \boldsymbol{\rho}\, dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\, dm) \ . \tag{5.11}$$

The mass center is located in the x, y, z system by $\boldsymbol{b_{og}}$, defined by

$$m\boldsymbol{b_{og}} = \int_m \boldsymbol{\rho}\, dm = m(\boldsymbol{i} b_{ogx} + \boldsymbol{j} b_{ogy} + \boldsymbol{k} b_{ogz}) \ . \tag{5.12}$$

Substituting from Eq.(5.12) into Eq.(5.11) gives

$$\sum \boldsymbol{f_i} = m[\ddot{\boldsymbol{R}}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}})] \ . \qquad (5.13)$$

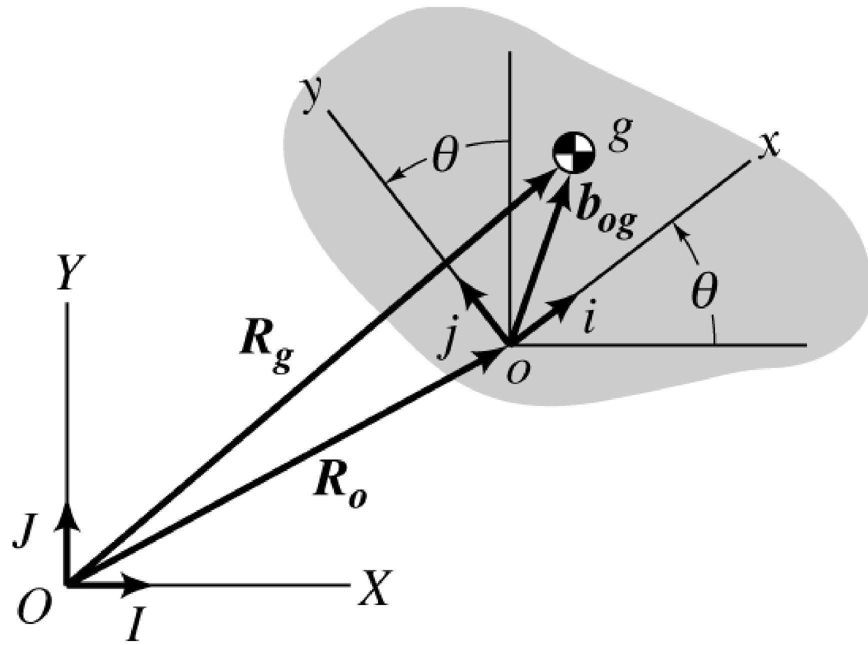


Figure 5.6 A rigid body with a mass center located in the body-fixed *x, y, z* coordinate system by the vector $\boldsymbol{b_{og}}$ and located in the inertial *X, Y, Z* system by $\boldsymbol{R_g}$.

Since *g* and *o* are fixed in the rigid body, their accelerations are related by

$$\boldsymbol{a_g} = \boldsymbol{a_o} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r_{og}})$$

But $\boldsymbol{r_{og}} = \boldsymbol{b_{og}}$, and $\boldsymbol{a_o} = \ddot{\boldsymbol{R}}_o$; hence,

$$\ddot{\boldsymbol{R}}_g = \ddot{\boldsymbol{R}}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{b_{og}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b_{og}}) \ ,$$

and the force equation can be written (finally) as

$$\sum \boldsymbol{f_i} = m\,\ddot{\boldsymbol{R}}_{\boldsymbol{g}} \quad . \tag{5.14}$$

In words, Eq.(5.14) states that a rigid body can be treated like a particle, in that the summation of external forces acting on the rigid body equals the mass of the body times the acceleration of the mass center with respect to an inertial coordinate system.

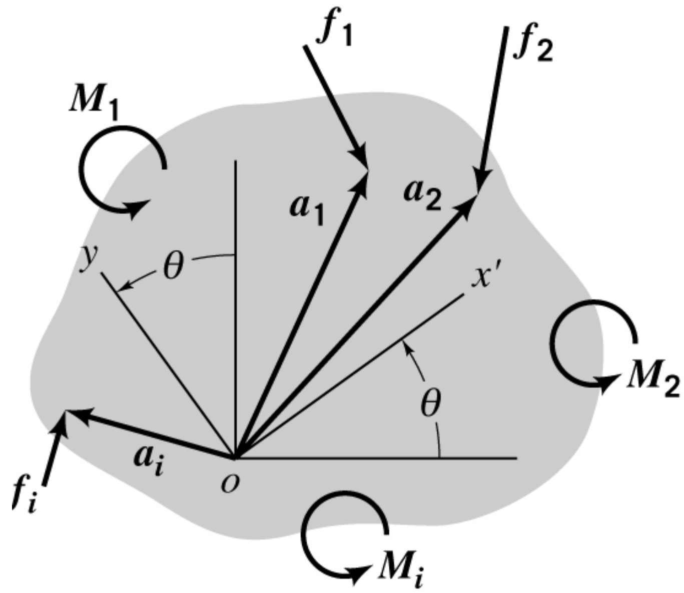
Cartesian component of Force equations:

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15a}$$

Polar version

$$\sum f_{ir} = m(\ddot{r}_g - r_g\dot{\theta}^2) \ , \quad \sum f_{i\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) \ . \tag{5.15b}$$

Moment Equation



A rigid body acted on by several external forces $\boldsymbol{f_i}$ acting on the body at points located by the position vectors $\boldsymbol{a_i}$ and moments $\boldsymbol{M}_i$

In figure 5.5, the position vector **ρ** extends from *o* to a particle at point *P*. For moments about *o*, **ρ** is the moment arm, and the particle moment equation is

$$\boldsymbol{\rho} \times \boldsymbol{f_P} = \boldsymbol{\rho} \times dm\,\ddot{\boldsymbol{r}} \quad . \tag{5.16}$$

Integrating Eq.(5.16) over the mass of the rigid body yields

$$\sum (\boldsymbol{a_i} \times \boldsymbol{f_i}) + \sum \boldsymbol{M_i} = \boldsymbol{M_o} = \int_V \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, \gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho} \times \ddot{\boldsymbol{r}}\, dm \tag{5.17}$$

The vector $\boldsymbol{M_o}$ on the left is the *resultant* external moment acting on the rigid body about point *o,* the origin of the *x, y, z* coordinate system.

Kinematics: Substituting from Eq.(5.10) gives

$$\boldsymbol{\rho} \times \ddot{\boldsymbol{r}} = (\boldsymbol{\rho} \times \ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\rho} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] \ .$$

The vector identity,

$$\boldsymbol{A} \times [\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{A})] = \boldsymbol{B} \times [\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{A})] \ ,$$

gives

$$\boldsymbol{\rho} \times \ddot{\boldsymbol{r}} = (\boldsymbol{\rho} \times \ddot{\boldsymbol{R}}_o) + \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] \ . \tag{5.18}$$

Since $\ddot{\boldsymbol{R}}_o, \dot{\boldsymbol{\omega}}$, and $\boldsymbol{\omega}$ are not functions of the variables of integration, substitution from Eq.(5.18) into Eq.(5.17) gives

$$\begin{aligned} \boldsymbol{M}_o = m(\boldsymbol{b}_{og} \times \ddot{\boldsymbol{R}}_o) + \int \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho})\, dm \\ + \boldsymbol{\omega} \times \int [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]\, dm \ . \end{aligned} \tag{5.19}$$

with $\boldsymbol{b}_{og}$ defined by Eq.(5.12).

To find component equations from Eq.(5.19)

$$\boldsymbol{b}_{og} \times m\ddot{\boldsymbol{R}}_o = m \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_{ogx} & b_{ogy} & b_{ogz} \\ \ddot{R}_{ox} & \ddot{R}_{oy} & 0 \end{vmatrix}$$

$$\begin{aligned} &-\boldsymbol{i}\, m b_{ogz} \ddot{R}_{oy} \\ = &+\boldsymbol{j}\, m b_{ogz} \ddot{R}_{ox} \\ &+\boldsymbol{k}\, m (b_{ogx}\, \ddot{R}_{oy} - b_{ogy}\, \ddot{R}_{ox}) \end{aligned} \tag{5.20}$$

In carrying out the cross product, note that $\ddot{\boldsymbol{R}}_o$ is stated in terms of its components in the *x, y, z* coordinate system, versus the customary *X, Y, Z* system.

Defining the vectors in Eq.(5.19) in terms of their components gives

$$\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \;,\; \boldsymbol{\omega} = \boldsymbol{k}\dot{\theta} \;,\; \dot{\boldsymbol{\omega}} = \boldsymbol{k}\ddot{\theta} \;.$$

Hence,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\dot{\theta} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z) = \dot{\theta}(\boldsymbol{j}x - \boldsymbol{i}y)$$

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} = \ddot{\theta}\;(\boldsymbol{j}x - \boldsymbol{i}\;y) \;,$$

and

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ -y & x & 0 \end{vmatrix} = \boldsymbol{k}\, \dot{\theta}\, (x^2 + y^2) \ . \quad (5.21)$$

Similarly,

$$\boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}) = \boldsymbol{k}\, \ddot{\theta}\, (x^2 + y^2) \quad (5.22)$$

Substituting from Eqs.(5.20)-(5.22)into Eq.(5.19) gives the z component equation

$$\begin{aligned} \boldsymbol{k} M_{oz} &= \boldsymbol{k} m (\boldsymbol{b} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z \\ &+ \boldsymbol{k} \ddot{\theta} \int_m (x^2 + y^2)\, dm + \boldsymbol{k} \dot{\theta} \times \boldsymbol{k} \dot{\theta} \int_m (x^2 + y^2)\, dm \ . \end{aligned} \quad (5.23)$$

The last expression in this equation is zero because $\boldsymbol{k} \times \boldsymbol{k} = 0$. Since

$$I_o = \int_m (x^2 + y^2)\, dm \ ,$$

the moment Eq.(5.23) can be stated (finally) as

$$\sum M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}} \ . \quad (5.24)$$

Summary of governing equations of motion for planar motion of a rigid body

Force-Equation Cartesian Components

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15}$$

Moment Equation

$$\sum M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b_{og}} \times \boldsymbol{\ddot{R}_o})_{\boldsymbol{z}} \ . \tag{5.24}$$

Reduced Forms for the Moment Equation

Moments taken about the mass center. If the point *o* about which moments are taken coincides with the mass center *g*, $\boldsymbol{b_{og}} = 0$, and Eq.(5.24) reduces to

$$M_{gz} = I_g \ddot{\theta} \ . \tag{5.25}$$

This equation is *only* correct for moments taken about the mass center of the rigid body.

Moments taken about a fixed point in inertial space. When point o is fixed in the (inertial) X, Y, Z coordinate system, $\ddot{\boldsymbol{R}}_o = 0$, and the moment equation is

$$M_{oz} = I_o \ddot{\theta} \ . \qquad (5.26)$$

Fixed-Axis-Rotation Applications of the Force and Moment equations for Planar Motion of a Rigid Body

Rotor in Bearings

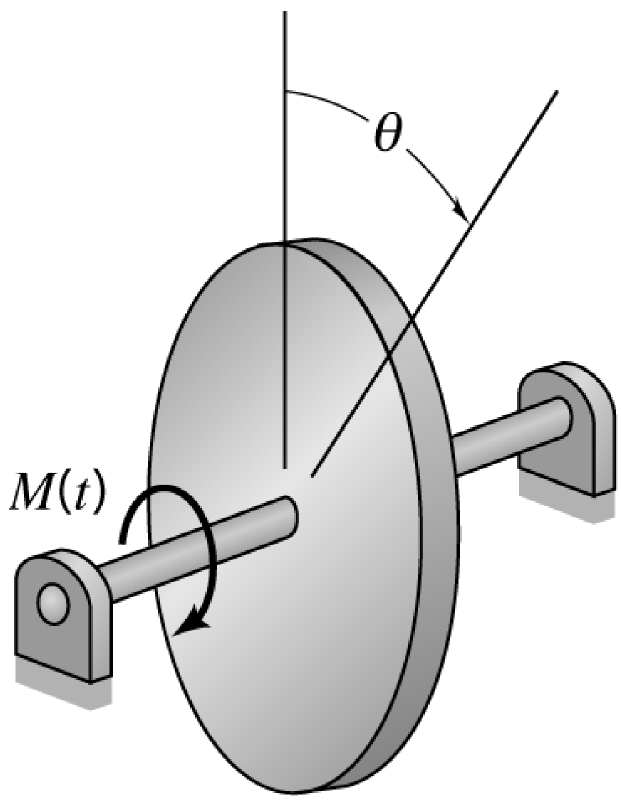


Figure 5.8 A disk mounted on a massless shaft, supported by two frictionless bearings, and acted on by the applied torque $M(t)$.

Derive the differential equation of motion for the rotor. The governing equation of motion for the present system is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \left(\frac{mr^2}{2}\right)\ddot{\theta} = M(t) \ .$$

The moment $M(t)$ is positive because it is acting in the same direction as $+\theta$. This is basically the same second-order differential equation obtained for a particle of mass m acted on by the force $f(t)$, namely, $m\ddot{x} = f(t)$, where x locates the particle in an inertial coordinate system.

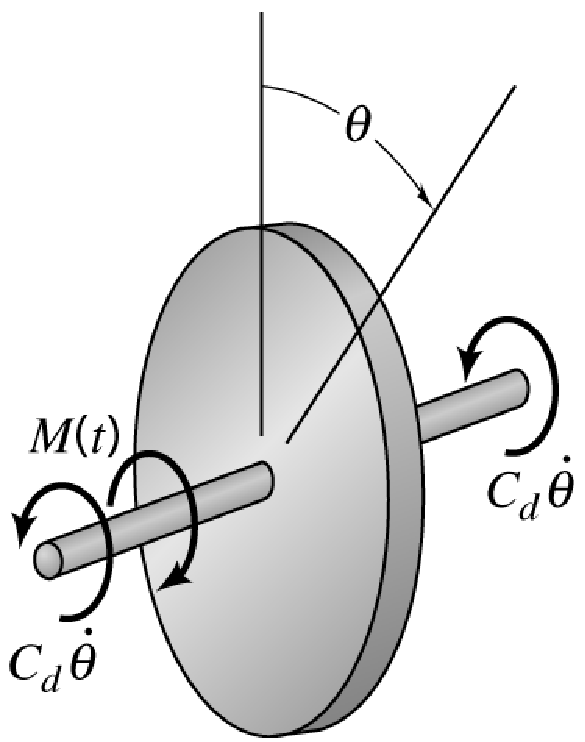


Figure 5.9 Free-body diagram for the rotor of figure 5.8 with a drag torque $C_d\dot{\theta}$ acting at each bearing.

The shaft is rotating in the $+\dot{\theta}$ direction; hence, the drag moment terms have negative signs because they are acting in -θ direction. The differential equation of motion to be obtained from the moment equation is

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \frac{mr^2}{2}\ddot{\theta} = M_{oz}(t) - 2C_d\dot{\theta} \;, or$$

$$\frac{mr^2}{2}\ddot{\theta} + 2C_d\dot{\theta} = M_{oz}(t)$$

This equation has the same form as a particle of mass *m* acted on by the force *f(t)* and a linear dashpot with a damping coefficient *c* ; namely, $m\ddot{x} + c\dot{x} = f(t)$.

A-One Degree of Freedom Torsional Vibration Example

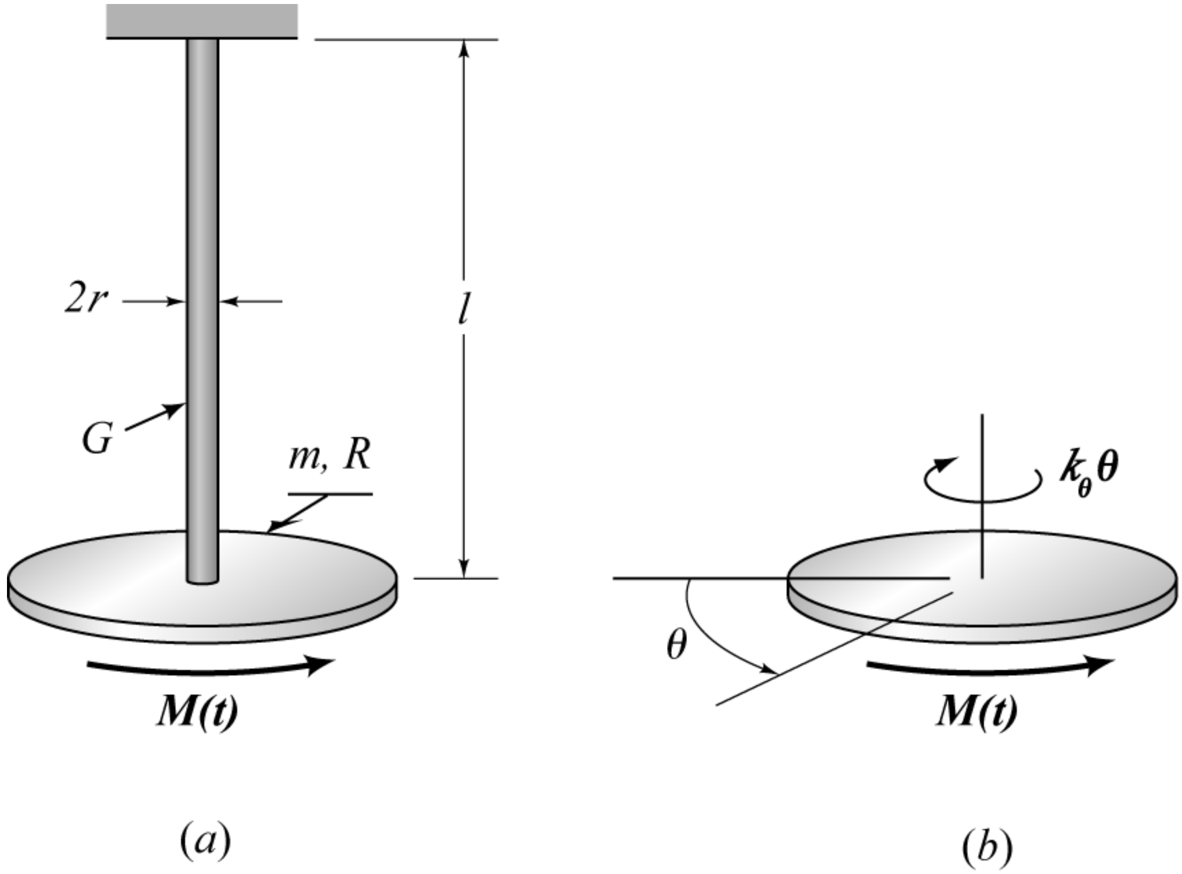


Figure 5.11 (a) Circular disk of mass *m* and radius *R*, supported by a slender rod of length *l*, radius *r*, and shear modulus *G*, (b) Free-body diagram for $\theta > 0$

Twisting the rod about its axis through an angle θ will create a reaction moment, related to θ by

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta.$$

$k_\theta = GJ/l$, where G is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. Recall that the SI units for G is N/m^2; hence, k_θ has the units: *N-m/ radian*, i.e., moment per unit torsional rotation of the rod.

Derive the differential equation of motion for the disk. Applying Eq.(5.26) yields the moment equation

$$M_{oz} = I_o \ddot{\theta} \Rightarrow \frac{mR^2}{2}\ddot{\theta} = M(t) + M_\theta = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta ,$$

The signs of the moments on the right hand side of this moment equation are positive or negative, depending on whether they are, respectively, in the +θ or -θ direction.

The differential equation of motion to be obtained from the moment equation is

$$\frac{mR^2}{2}\ddot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t) .$$

This result is analogous to the differential equation of motion for a particle of mass *m*, acted on by an external force $f(t)$, and supported by a linear spring with stiffness coefficient k ; viz., $m\ddot{x} + kx = f(t)$. For comparison, look at Eq.(3.13). This equation can be rewritten as

$$\ddot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2} ,$$

where the undamped natural frequency ω_n is defined by

$$\omega_n = \sqrt{\frac{\pi G r^4}{l m R^2}} \ .$$

Torsional Vibration Example with Viscous Damping

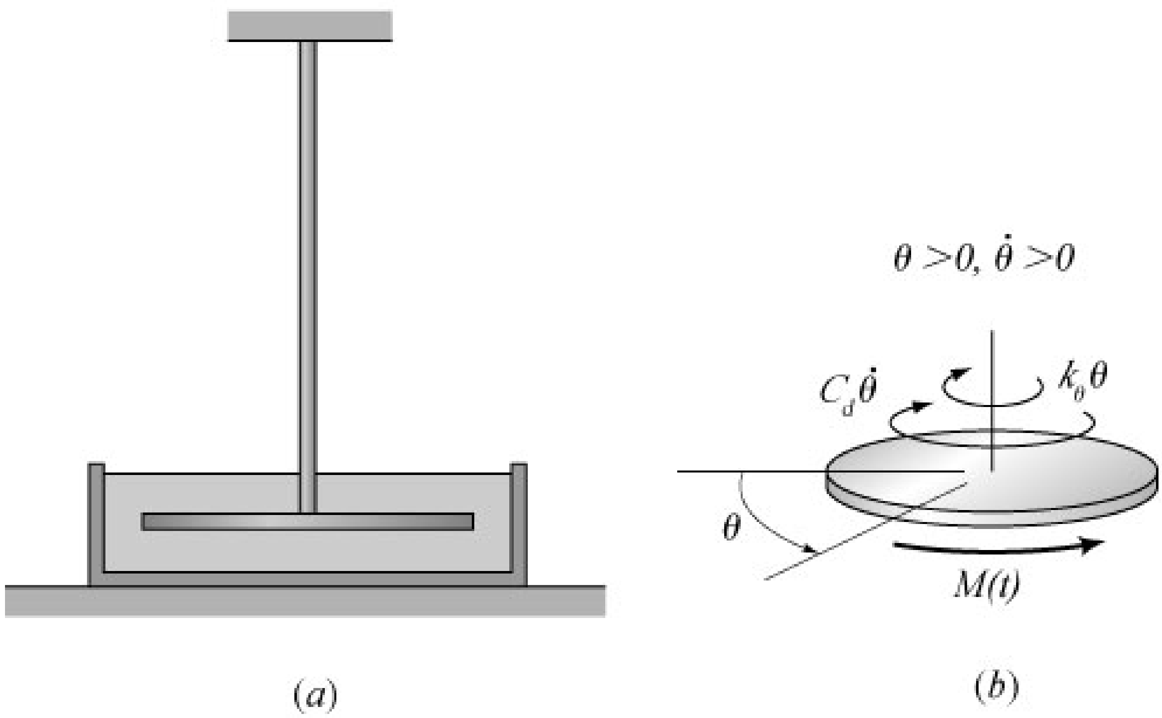


Figure 5.11 (a) The disk of figure 5.10 is now immersed in a viscous fluid, (b) Free-body diagram

Rotation of the disk at a finite rotational velocity $\dot{\theta}$ within the fluid causes the drag moment, $-C_d\dot{\theta}$, on the disk. The negative sign for the drag term is chosen because it acts in the -θ direction. The complete moment equation is

$$\frac{mR^2}{2}\ddot{\theta} = \Sigma M_z = M(t) - \frac{G}{l}\frac{\pi r^4}{2}\theta - C_d\dot{\theta},$$

with the governing differential equation

$$\frac{mR^2}{2}\ddot{\theta} + C_d\dot{\theta} + \frac{\pi G r^4}{2l}\theta = M(t). \qquad (5.27)$$

This differential equation has the same form as a particle of mass *m* supported by a parallel arrangement of a spring with stiffness coefficient *k* and a linear damper with damping coefficient *c*; namely, $m\ddot{x} + c\dot{x} + kx = f(t)$.

Eq.(5.27) can be restated as

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{2M(t)}{mR^2},$$

where ζ is the damping factor, defined by

$$2\zeta\omega_n = \frac{2C_d}{mR^2}, \quad \omega_n = \sqrt{\frac{\pi G r^4}{lmR^2}}.$$

The models developed from figures 5.10 and 5.11 show the same damped and undamped vibration possibilities for rotational motion of a disk that we reviewed earlier for linear motion of a particle. The same possibilities exist to define damped and undamped natural frequencies, damping factors, etc.

An example involving kinematics between a disk and a particle

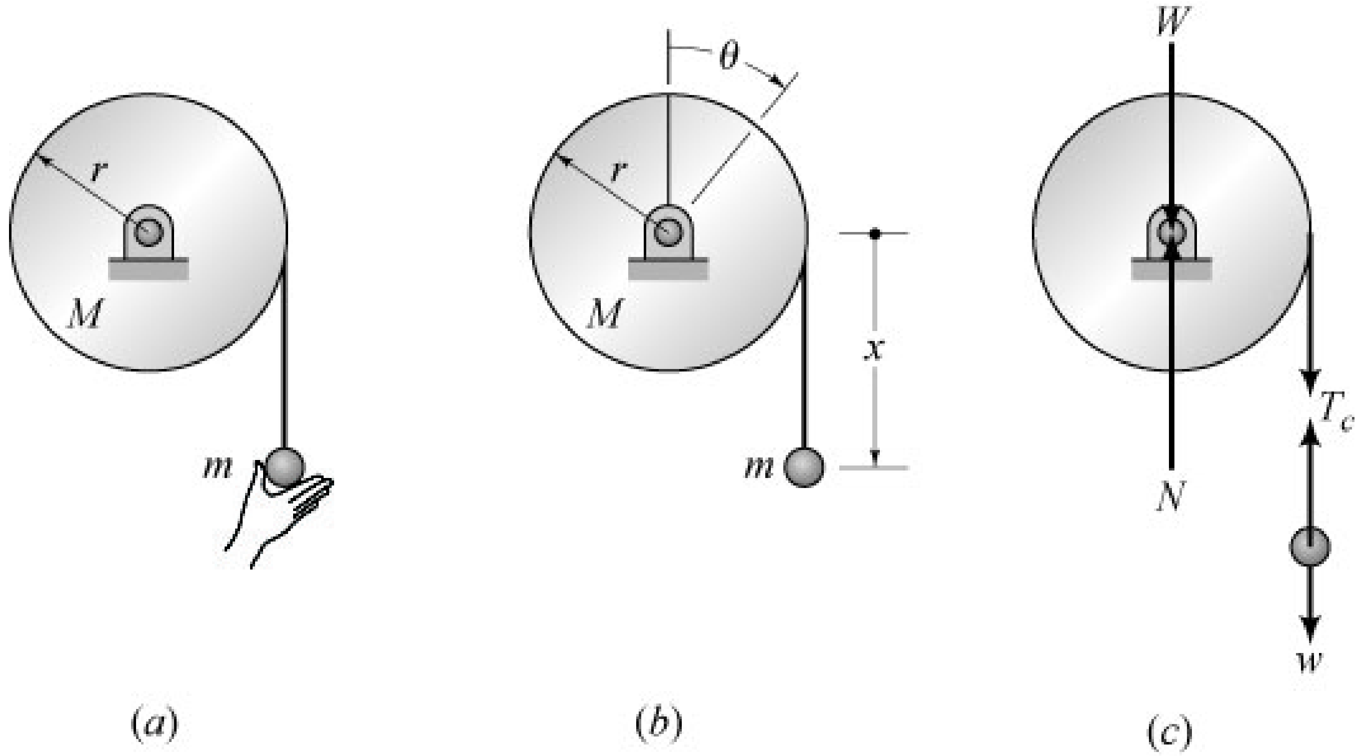


Figure 5.12 (a) Disk of mass *M* and radius *r* supported in frictionless bearings and connected to a particle of mass *m* by a light and inextensible cord, (b) Coordinates, (c) Free-body diagram.

Derive the differential equation of motion for the system.

Kinematics:

$$\delta x = r\delta\theta \;\Rightarrow\; \dot{x} = r\dot{\theta} \;,\; \ddot{x} = r\ddot{\theta}. \tag{5.28}$$

From the free-body diagram, the equation of motion for the disk is obtained by writing a moment equation about its axis of rotation. The equation of motion for mass m follows from $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ for a particle. The governing equations are:

$$\frac{Mr^2}{2}\ddot{\theta} = \Sigma M_{oz} = T_c r \;, \quad m\ddot{x} = \Sigma f = w - T_c, \tag{5.29}$$

where T_c is the tension in the cord. (The mass of the cord has been neglected in stating these equations.) In the first of Eq.(5.29), the moment term $T_c\, r$ is positive because it acts in the $+\theta$ direction. The sign of w is positive in the force equation because it acts in the $+x$ *direction* ; T_c has a negative sign because it is directed in the $-x$ direction.

Eqs.(5.29) provides two equations for the three unknowns: $\ddot{x}, \ddot{\theta}$, and T_c. Eliminating the tension T_c from Eqs.(5.29) gives

$$\frac{Mr^2}{2}\ddot{\theta} + rm\ddot{x} = wr. \tag{5.30}$$

Substituting from the last of Eq.(5.28) for $\ddot{x} = r\ddot{\theta}$ gives the final differential equation

$$(\frac{M}{2} + m)r^2\ddot{\theta} = wr.$$

Two driven pulleys connected by a belt

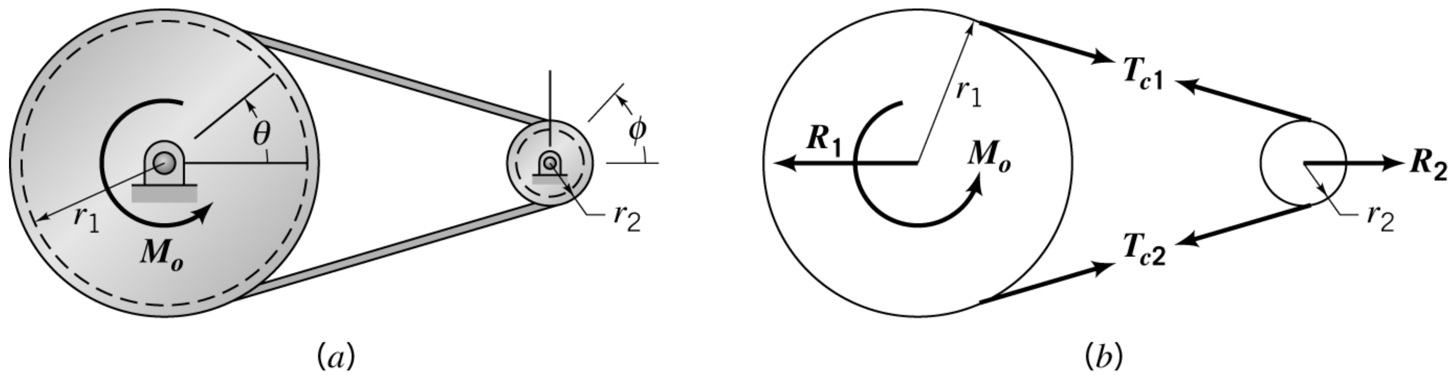


Figure 5.13 (a). Two disks connected by a belt. (b). Free-body diagram.

Figure 5.13A illustrates two pulleys that are connected to each other by a light and inextensible belt. The pulley at the left has mass m_1, radius of gyration k_{1g} about the pulley's axis of rotation, and is acted on by the counterclockwise moment M_o . The pulley at the right has mass m_2 and a radius of gyration k_{2g} about its axis of rotation. (The radius of gyration k_g defines the moment of inertia about the axis of rotation by $I = mk_g^2$.) The belt runs in a groove in pulley 1 with inner radius r_1. The inner radius of the belt groove for pulley 2 is r_2. The angle of rotation for pulleys 1 and 2 are, respectively, θ and φ .

Derive the governing differential equation of motion in terms of θ *and its derivatives.*

From the free-body diagram the fixed-axis rotation moment Eq.(5.26) gives:

$$I_1\ddot{\theta} = \Sigma M_{1oz} = M_o(t) + r_1(T_{c2} - T_{c1})$$
$$I_2\ddot{\varphi} = \Sigma M_{2oz} = r_2(T_{c1} - T_{c2}) \ , \qquad (5.31)$$

where T_{c1} and T_{c2} are the tension components in the upper and lower belt segments. $M_o(t)$ has a positive sign because it is acting in the +θ direction; $r_1(T_{c1} - T_{c2})$ has a negative sign because it acts in the -θ direction. Similarly, $r_2(T_{c1} - T_{c2})$ has a positive sign in the second of Eq.(5.31) because it is acting in the +φ direction.

The moments of inertia in Eq.(5.31) are defined in terms of their masses and radii of gyrations by

$$I_1 = m_1 k_1^2 \quad ; \ I_2 = m_2 k_2^2 \ .$$

Returning to Eq.(5.31), we can eliminate the tension terms in the two equations, obtaining

$$I_1\ddot{\theta} = M_o - \frac{r_1}{r_2} I_2 \ddot{\varphi} \quad \Rightarrow \quad I_1\ddot{\theta} + \frac{r_1}{r_2} I_2 \ddot{\varphi} = M_o(t) \ . \qquad (5.32)$$

We now have one equation for the two unknowns $\ddot{\theta}$ and $\ddot{\varphi}$, and need an additional kinematic equation relating these two angular acceleration terms. Given that the belt connecting the pulleys is inextensible (can not stretch) the velocity v of the belt leaving both pulleys must be equal; hence,

$$v = r_1 \dot{\theta} = r_2 \dot{\varphi} \;\Rightarrow\; r_1 \ddot{\theta} = r_2 \ddot{\varphi} \;\; .$$

Substituting this result back into Eq.(5.32) gives the desired final result

$$[I_1 + (\frac{r_1}{r_2})^2 I_2]\ddot{\theta} = I_{eff}\ddot{\theta} = M_o(t) \;\; .$$

Note that coupling the two pulleys' motion by the belt acts to increase the effective inertia I_{eff} in resisting the applied moment.