Lecture 27. THE COMPOUND PENDULUM

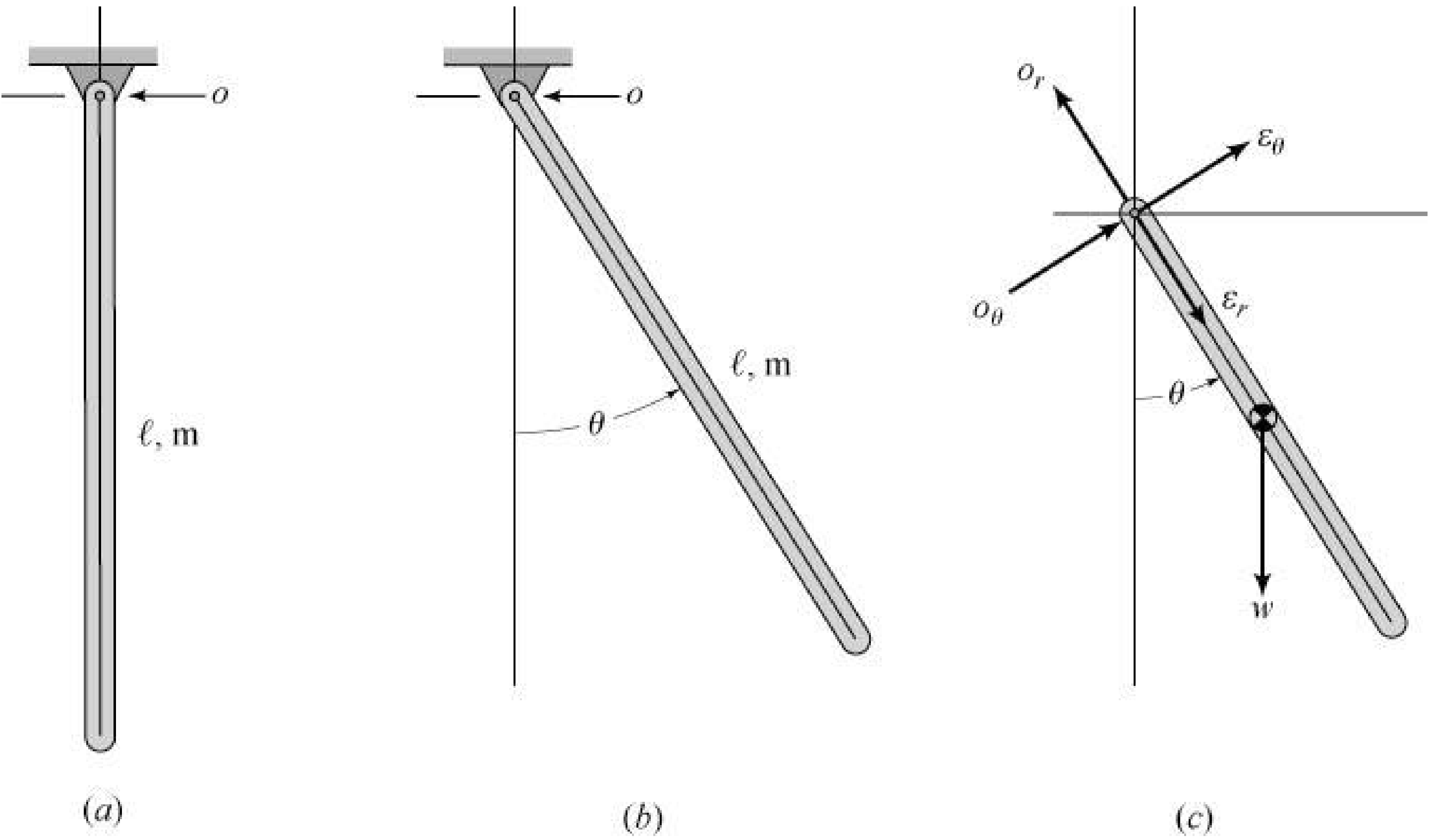

Figure 5.16 Compound pendulum: (a) At rest in equilibrium, (b) General position with coordinate θ, © Free-body diagram

The term "compound" is used to distinguish the present rigid-body pendulum from the "simple" pendulum of Section 3.4b, which consisted of a particle at the end of a massless string.

Derive the general differential equation of motion for the pendulum of figure 5.16a and determine its undamped natural frequency for small motion about the static equilibrium position.

From the free-body diagram the moment equation is

$$I_o \ddot{\theta} = \Sigma M_o = -w \frac{l}{2} \sin\theta \, . \tag{5.33}$$

The minus sign on the right-hand side term applies because the moment is acting in the -θ direction. For a uniform bar, $I_o = ml^2/3$; hence, the governing differential equation of motion is

$$\frac{ml^2}{3}\ddot{\theta} + \frac{mgl}{2}\sin\theta = 0 \ ,$$

or

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0 \ . \tag{5.34}$$

For small motion, sin θ ≅ θ, and the nonlinear differential equation reduces to

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \omega_n^2 \theta = 0 \ . \tag{5.35}$$

This differential equation is the *rotation* analog of the single-degree-of-freedom, *displacement*, vibration problem of $\ddot{x} + \omega_n^2 x = 0$. The compound pendulum's natural frequency is

$$\omega_n = \sqrt{3g/2l} \ .$$

Assuming that the pendulum is released from rest at $\theta = \pi/2$ *radians* $= 90^o$*, define the reaction force components* (o_θ, o_r) *as a function of* θ *(only).*

From figure 5.16c applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ in polar coordinates for the mass center of the rod gives:

$$\begin{aligned} \Sigma f_r &= -o_r + w\cos\theta = m\ddot{R}_{gr} = m(\ddot{r}_g - r_g\dot{\theta}^2) = -m\frac{l}{2}\dot{\theta}^2 \\ \Sigma f_\theta &= o_\theta - w\sin\theta = m\ddot{R}_{g\theta} = m(r_g\ddot{\theta} + 2\dot{r}_g\dot{\theta}) = m\frac{l}{2}\ddot{\theta} \ . \end{aligned} \qquad (5.36)$$

In the acceleration terms, $\dot{r}_g = \ddot{r}_g = 0$, because $r_g = l/2$ is a constant. Eqs.(5.36) define the reaction force components o_r, o_θ, but not as a function of θ alone. Direct substitution from Eq.(5.34) into the second of Eqs.(5.36) defines o_θ as

$$o_\theta = w\sin\theta + \frac{ml}{2}\ddot{\theta} = w\sin\theta - \frac{ml}{2}\left(\frac{3g}{2l}\right)\sin\theta = \frac{w}{4}\sin\theta \ . \qquad (5.37)$$

Finding a comparable relationship for o_r is more complicated, because the first of Eqs.(5.36) involves $\dot{\theta}^2$. We will need to integrate the differential equation of motion via the energy-integral substitution to obtain $\dot{\theta}^2$ as a function of θ, proceeding from

$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{\dot{\theta}^2}{2}\right) = -\frac{3g}{2l}\sin\theta \ .$$

Multiplying through by $d\theta$ and integrating both sides of this equation gives

$$\frac{\dot{\theta}^2}{2} - \frac{\dot{\theta}_o^2}{2} = \int_{\frac{\pi}{2}}^{\theta} -\frac{3g}{2l}\sin x\, dx$$

$$\frac{\dot{\theta}^2}{2} = \frac{3g}{2l}\Big|_{\frac{\pi}{2}}^{\theta}\cos x = \frac{3g}{2l}\cos\theta \ . \tag{5.38}$$

Substituting this result into the first of Eq.(5.36) gives

$$o_r = w\cos\theta + ml\frac{\dot{\theta}^2}{2} = w\cos\theta + ml\left(\frac{3g}{2l}\right)\cos\theta = \frac{5w}{2}\cos\theta \ . \tag{5.39}$$

This result shows that the dynamic reaction force will be 2.5 times greater than the static weight w when the rod reached its lowest position ($\theta = 0$).

Alternative Moment Equation with Moments about g

Suppose that we had chosen to take moments about g, the mass center of the rod in figure 5.16c, obtaining

$$\Sigma M_g = I_g\ddot{\theta} = -o_\theta\frac{l}{2} \ .$$

The moment has a negative sign because it is acting in the -θ direction. In this equation, the required moment of inertia about g is $I_g = ml^2/12$. Substituting for o_θ from Eq.(5.37) gives

$$-\frac{l}{2}\left(m\frac{l}{2}\ddot{\theta} + w\sin\theta\right) = \frac{ml^2}{12}\ddot{\theta} \ ,$$

or

$$-\frac{wl}{2}\sin\theta = ml^2\left(\frac{1}{12} + \frac{1}{4}\right)\ddot{\theta} = \frac{ml^2}{3}\ddot{\theta} \ . \tag{5.40}$$

Writing the moment equation about g involves more work but gets the same equation. Note in the intermediate step of Eq.(5.40) that we are accomplishing the parallel-axis formula in moving from $I_g = ml^2/12$ to $I_o = ml^2/3$, via $I_o = I_g + m\,|\boldsymbol{b_{go}}|^2 = I_g + m(l/2)^2$,where $\boldsymbol{b_{go}}$ is the vector from the mass center g to the pivot point o.

Deriving The Equation of Motion From The Energy Equation

There are no external time varying forces or moments and no energy dissipation; hence, mechanical energy is conserved; i.e., $T + V = T_0 + V_0$. Using a horizontal plane through the pendulum's pivot point as a datum for gravity potential energy gives

$$V = V_g = -w\frac{l}{2}\cos\theta \ .$$

For rotation about the fixed point *o,* the kinetic energy of the pendulum is defined by

$$T = I_o \frac{\dot{\theta}^2}{2} = \frac{ml^2}{3} \frac{\dot{\theta}^2}{2} \quad .$$

Hence,

$$\frac{ml^2}{3} \frac{\dot{\theta}^2}{2} - \frac{wl}{2} \cos\theta = T_0 + V_0 = 0 - \frac{wl}{2} \cos(\theta_0 = \frac{\pi}{2}) = 0$$

$$\therefore \quad \dot{\theta}^2 = \frac{3g}{l} \cos\theta \quad .$$

Differentiating w.r.t. θ gives

$$\frac{ml^2}{3} \frac{d}{d\theta} (\frac{\dot{\theta}^2}{2}) + \frac{mgl}{2} \sin\theta = 0 \;\Rightarrow\; \ddot{\theta} + \frac{3g}{2l} \sin\theta = 0 \quad .$$

Static stability about equilibrium points.

From

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0 \ , \quad \ddot{\theta} = 0 \ \Rightarrow \ \sin\bar{\theta} = 0 \ , \ \bar{\theta} = 0, \pi$$

Equilibrium for small Motion about the equilibrium position $\theta = 0$**.**

Small motion about the equilibrium position $\theta = 0$ gives the linearized differential equation of motion

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0 \ .$$

For the initial conditions, $\dot{\theta}(0) = \dot{\theta}_0 ; \theta(0) = \theta_0$, the solution to the linearized Eq.(5.35) can be stated

$$\theta(t) = \theta_0 \cos\omega_n t + \frac{\dot{\theta}_0}{\omega_n}\sin\omega_n t \ , \ \omega_n = \sqrt{\frac{3g}{2l}} \ , \qquad (5.41)$$

consisting of a *stable* oscillation at the natural frequency. Hence, $\theta = 0$ is said to be a *stable equilibrium point* for the body.

Equilibrium for small Motion about the equilibrium position $\theta = \pi$.

Motion is governed by the nonlinear equation of motion,

$$\ddot{\theta} = -\frac{3g}{2l}\sin\theta \quad . \tag{5.34}$$

Expanding $\sin\theta$ in a Taylor's series about $\theta = \pi$ gives

$$\begin{aligned} \sin\theta &= \sin(\pi + \delta\theta) = \sin\pi\cos\delta\theta + \cos\pi\sin\delta\theta \\ &= -\sin\delta\theta = -\delta\theta + \frac{(\delta\theta)^3}{6} - \frac{(\delta\theta)^5}{120} + \dots \quad . \end{aligned} \tag{5.42}$$

Retaining only the linear term in Eq.(5.42) and substituting back into Eq.(5.34) gives

$$\delta\ddot{\theta} - \frac{3g}{2l}\delta\theta = 0 \ , \ \theta = \pi + \delta\theta \ \Rightarrow \ \ddot{\theta} = \delta\ddot{\theta} \tag{5.43}$$

Observe the negative sign in the coefficient for $\delta\theta$. If this were a harmonic oscillator consisting of a spring supporting a mass, a comparable negative sign would imply a negative stiffness, yielding a differential equation of the form

$$m\ddot{x} - kx = 0 \quad .$$

Substituting the assumed solution $\delta\theta = Ae^{st}$ into Eq.(5.43) gives

$$(s^2 - \omega_n^2)Ae^{st} = 0 \Rightarrow s = \pm\omega_n \ .$$

Hence the solution to Eq.(5.43) for small motion about $\theta = \pi$ is

$$\delta\theta(t) = A_1 e^{\omega_n t} - A_2 e^{-\omega_n t}$$

The first term in this solution grows exponentially with time. Hence, any small disturbance of the pendulum from the equilibrium position $\theta = \pi$ will grow exponentially with time, and $\theta = \pi$ is a *statically unstable equilibrium point* for the pendulum.

A Swinging-Plate Problem

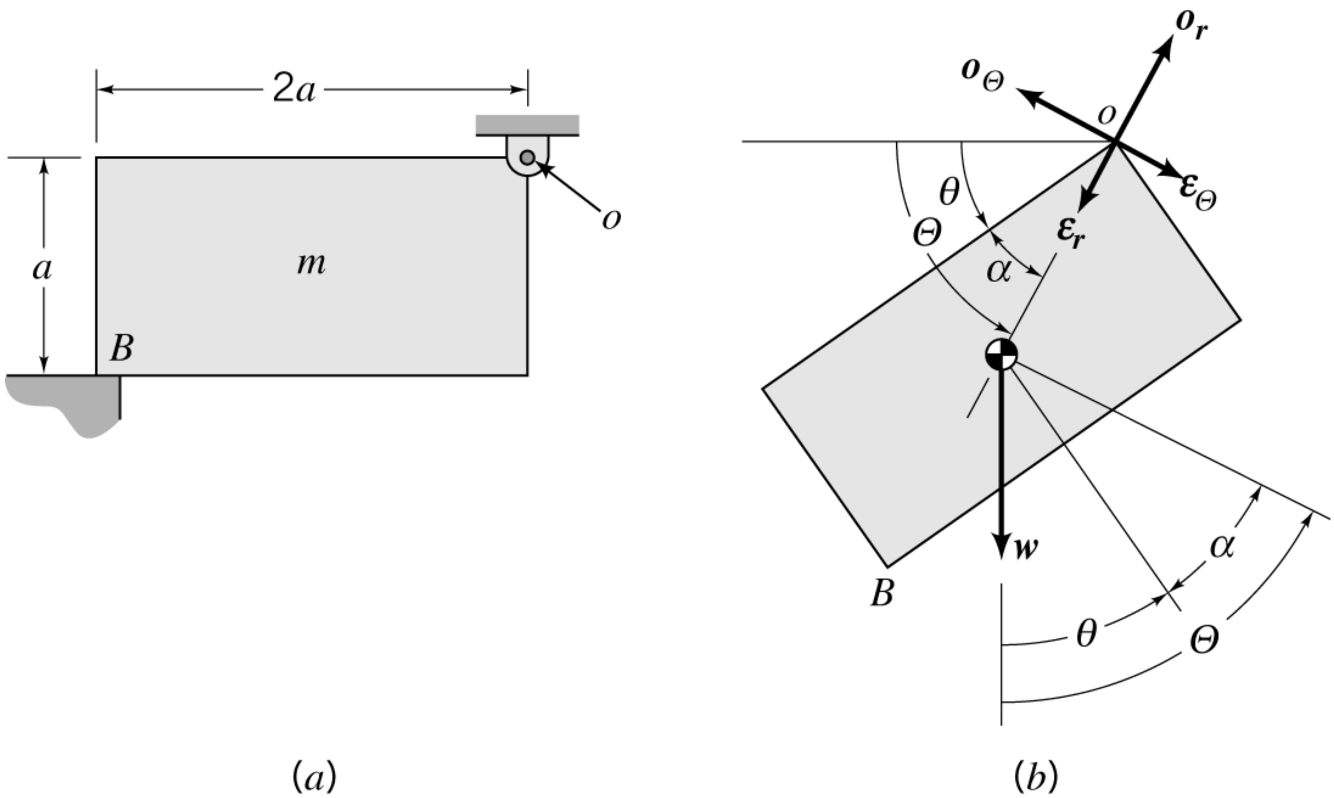

Figure XP5.2 (a) Rectangular plate supported at *o* by a frictionless pivot and at *B* by a ledge, (b) Free-body diagram that applies after the support at *B* has been removed.

The plate has mass *m* , length *2a*, and width *a* , and is supported by a frictionless pivot at *o* and a ledge at *B*. The engineering tasks associated with this problem follow. *Assuming that the support at B is suddenly removed, carry out the following steps.:*

a. Derive the governing differential equation of motion.

b. Develop relationships that define the components of the reaction force as a function of the rotation angle only.

c. Derive the governing differential equation of motion for small motion about the plate's equilibrium position . Determine the natural frequency of the plate for small motion about this position.

Kinematics: θ in the free-body diagram defines the plate's orientation with respect to the horizontal. The angle α lies between the top surface of the plate and a line running from the pivot point *o* through the plate's mass center at *g*, and is defined by

$$\alpha = \tan^{-1}\left(\frac{a/2}{a}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$$

$\Theta = (\theta + \alpha)$ is the rotation angle (from the horizontal) of a line running from *o* through *g* .

Moment equation with moments taken about *o:*

$$I_o \ddot{\theta} = \Sigma M_{oz} = wa\frac{\sqrt{5}}{2}\cos(\theta + \alpha) \ . \tag{5.56}$$

The distance from *o* to *g* is $a\sqrt{5}/2$, and the weight develops the external moment acting through the moment arm $a\sqrt{5}/2\cos(\theta + \alpha)$. The moment is positive because it is acting in the +θ direction.

Applying the parallel-axis formula gives

$$I_o = I_g + m\,|\,\boldsymbol{b_{go}}\,|^2 = \frac{m}{12}(a^2 + 4a^2) + 5m\frac{a^2}{4} = \frac{5ma^2}{3} \quad . \tag{5.57}$$

Substitution gives the governing differential equation of motion

$$\ddot{\theta} = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta + \alpha) \ , \tag{5.58}$$

and we have completed *Task a*.

To define the reaction forces, we need to state $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$. The polar-coordinate version of this equation works best for the current problem, and the free-body diagram has been drawn using polar coordinates with o_r aligned with the radial acceleration component of point g, and o_Θ is aligned with the circumferential component.

Force equation components:

$$\begin{aligned} \Sigma f_r &= w\sin(\theta + \alpha) - o_r = m\ddot{R}_{gr} = m(\ddot{r} - r\dot{\Theta}^2) = -ma\frac{\sqrt{5}}{2}\dot{\theta}^2 \\ \Sigma f_\Theta &= w\cos(\theta + \alpha) - o_\Theta = m\ddot{R}_{g\Theta} = m(r\ddot{\Theta} + 2\dot{r}\dot{\Theta}) = ma\frac{\sqrt{5}}{2}\ddot{\theta} \ . \end{aligned} \tag{5.59}$$

$\dot{r} = \ddot{r} = 0$, because $r = a\sqrt{5}/2$ is a constant, and $\dot{\theta} = \dot{\Theta}, \ddot{\theta} = \ddot{\Theta}$

Eliminate $\ddot{\theta}$ by substitution

$$o_\Theta = w\cos(\theta+\alpha) - ma\frac{\sqrt{5}}{2}\cdot\frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta+\alpha) = \frac{w}{4}\cos(\theta+\alpha) \ .$$

This result states that o_Θ starts at $(w/4)\cos\alpha$ for $\theta = 0$, and is zero when the mass center is directly beneath the pivot point *o,* at $(\theta+\alpha) = \pi/2$.

To obtain $\dot{\theta}^2$ as a function of θ, use the energy-integral substitution to integrate Eq.(5.58). Starting with

$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{\dot{\theta}^2}{2}\right) = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta+\alpha) \ ,$$

multiplying through by $d\theta$, and integrating both sides of the equation gives

$$\frac{\dot{\theta}^2}{2} - \frac{\dot{\theta}_o^2}{2} = \int_0^\theta \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(x+\alpha)\,dx$$

$$\frac{\dot{\theta}^2}{2} = \frac{3\sqrt{5}}{10}\frac{g}{a}\Big|_0^\theta \sin(x+\alpha) = \frac{3\sqrt{5}}{10}\frac{g}{a}[\sin(\theta+\alpha) - \sin\alpha] \ .$$

Substituting this result back into the first of Eq.(5.59) gives

$$o_r = w\sin(\theta+\alpha) + ma\frac{\sqrt{5}}{2}\times\frac{3\sqrt{5}}{5}\frac{g}{a}[\sin(\theta+\alpha)-\sin\alpha]$$

$$= \frac{5w}{2}\sin(\theta+\alpha) - \frac{3w}{2}\sin\alpha \ ,$$

and *Task b* is now completed .

The equilibrium condition for the pendulum is obtained by setting the right-hand side of Eq.(5.47) equal to zero, obtaining

$$\ddot{\theta} = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\theta+\alpha) = 0$$

$$\therefore \ \cos(\alpha+\theta) = 0 \ \Rightarrow \ \Theta = \theta+\alpha = \frac{\pi}{2}, \frac{3\pi}{2} \ .$$

To get the governing equation of motion for small motion about the equilibrium condition, start by substituting $\Theta = \pi/2 + \delta\Theta$ into Eq.(5.58), obtaining

$$\delta\ddot{\Theta} = \frac{3\sqrt{5}}{10}\frac{g}{a}\cos(\frac{\pi}{2}+\delta\Theta) \ .$$

Expanding the last term on the right in a Taylor's series gives

$$\cos\left(\frac{\pi}{2}+\delta\Theta\right)=\cos\left(\frac{\pi}{2}\right)\cos\delta\Theta-\sin\left(\frac{\pi}{2}\right)\sin\delta\Theta$$

$$=-\sin\delta\Theta=-\delta\Theta+\frac{(\delta\Theta)^3}{6}-\frac{(\delta\Theta)^5}{120}+\ldots$$

Retaining only the linear term in this expansion gives the linearized differential equation of motion

$$\delta\ddot{\Theta}+\frac{3\sqrt{5}}{10}\frac{g}{a}\delta\Theta=0\ ;$$

hence, the natural frequency is defined by

$$\omega_n^2=\frac{3\sqrt{5}}{10}\frac{g}{a}\quad\Rightarrow\quad\omega_n=.819\sqrt{\frac{g}{a}}\ . \tag{5.60}$$

Alternative Development for motion about equilibrium

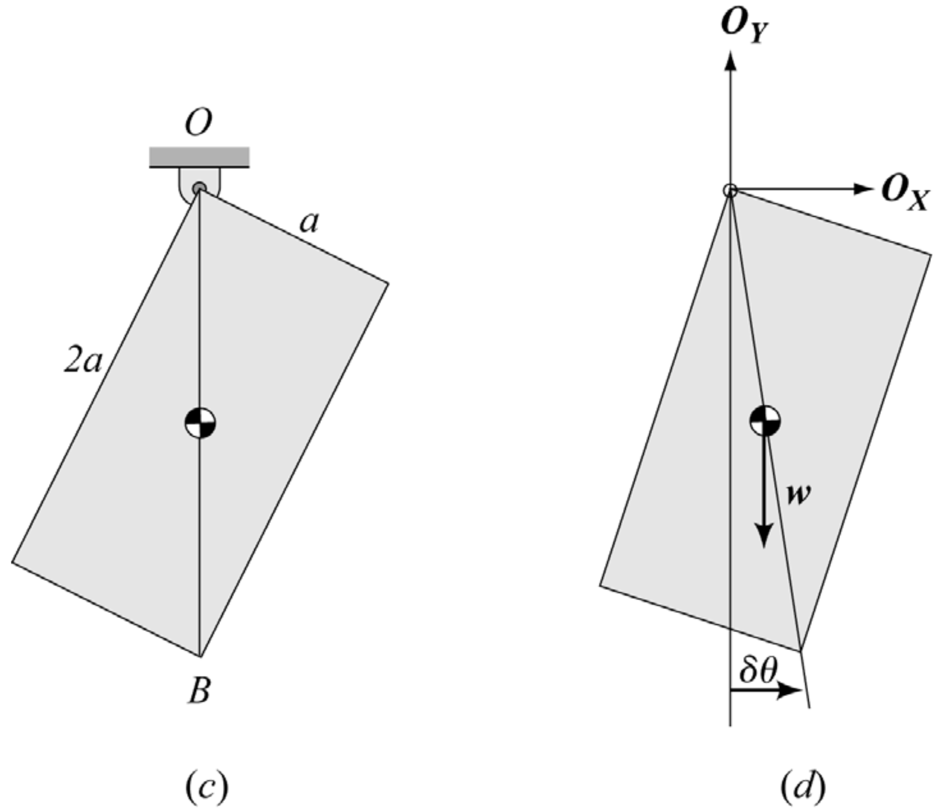

Figure XP5.2 (c) Plate in its equilibrium position, (d) Free-body diagram for motion away from equilibrium

Taking moments about O in figure XP5.2d gives

$$I_o\delta\ddot{\theta}=\Sigma M_{oz}=-\frac{wa\sqrt{5}}{2}\sin\delta\theta\;\Rightarrow\;\delta\ddot{\theta}+\frac{3\sqrt{5}}{10}\frac{g}{a}\delta\theta=0\;\;.$$

Task c is now completed.

Deriving the Equation of Motion from the Energy Equation

Energy is conserved; hence, $T+V=T_o+V_o$. Using a horizontal plane through the pivot point as the datum for gravity potential energy gives

$$I_o\frac{\dot{\theta}^2}{2}-wa\frac{\sqrt{5}}{2}\sin(\alpha+\theta)=0-w\frac{a}{2}$$

Substituting $I_o=5ma^2/3$ and differentiating w.r.t. θ gives

$$\frac{5ma^2}{2}\ddot{\theta}-wa\frac{\sqrt{5}}{2}\cos(\alpha+\theta)=0\ ;\ \ddot{\theta}=\frac{d(\dot{\theta}^2/2)}{d\theta}\ .$$