Lecture 30. MORE GENERAL-MOTION/ROLLING-WITHOUT-SLIPPING EXAMPLES

A Cylinder, Restrained by a Spring and Rolling on a Plane

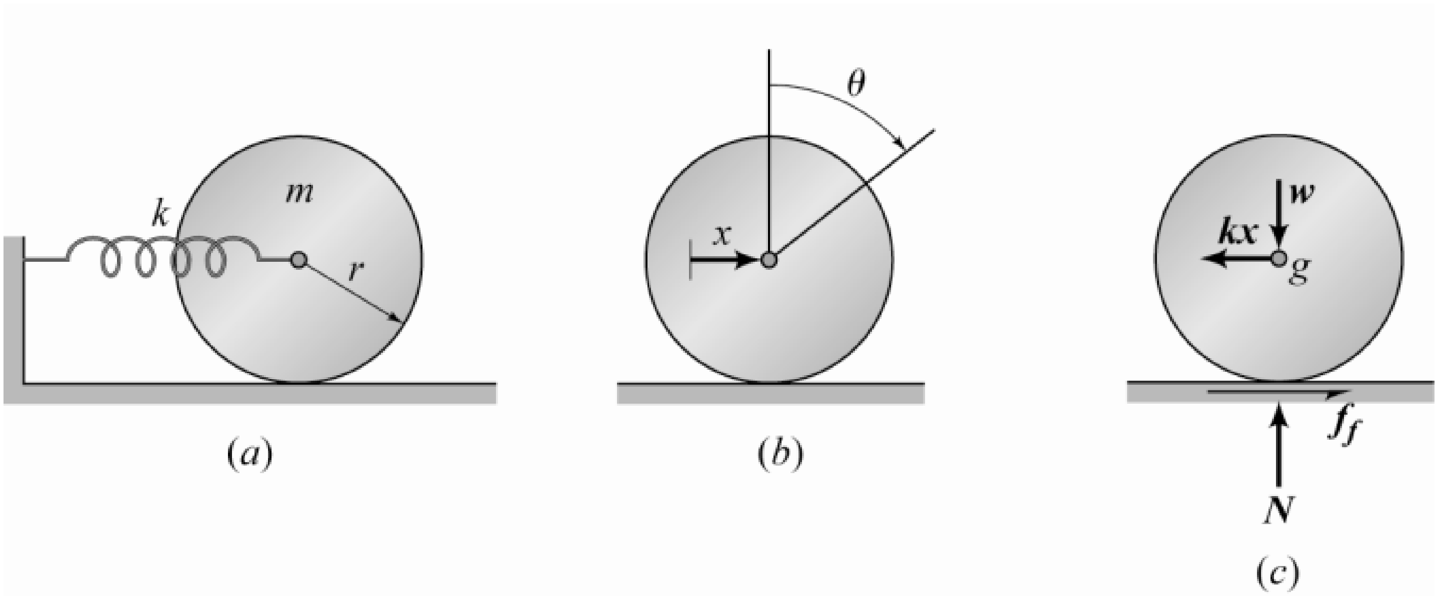

Figure 5.28 (a) Spring-restrained cylinder, (b) Kinematic variables, (c) Free-body diagram

The cylinder rolls without slipping. The spring is undeflected when $x = 0$.

The following engineering-analysis tasks apply:

a. Draw a free body diagram and derive the equation of motion, and

b. Determine the natural frequency for small amplitude vibrations.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_g$ for the mass center of the cylinder nets

$$\Sigma f_X = f_f - kx = m\ddot{x} \ . \qquad (5.117a)$$

Stating the moment equation about the cylinder's mass center gives

$$\Sigma M_g = -f_f r = I_g \ddot{\theta} = \frac{mr^2}{2}\ddot{\theta} \ . \qquad (5.117b)$$

We now have two equations in the three unknowns $\ddot{x}, \ddot{\theta}, f_f$. The rolling-without-slipping kinematic condition,

$$x = r\theta \ \Rightarrow \ \ddot{x} = r\ddot{\theta} \ , \qquad (5.117c)$$

provides the missing equation. Substituting for f_f from Eq.(5.117a) into Eq.(5.117b) gives

$$-r(kx + m\ddot{x}) = I_g \ddot{\theta}$$

We can use Eqs.(5.117c) to eliminate x and $\ddot{x}$, obtaining

$$-r(kr\theta + mr\ddot{\theta}) = \frac{mr^2}{2}\ddot{\theta} \ \Rightarrow \ \frac{3mr^2}{2}\ddot{\theta} + kr^2\theta = 0 \qquad (5.118)$$

This result concludes *Task a*. The natural frequency is

$$\omega_n^2 = kr^2 / \frac{3mr^2}{2} = \frac{2k}{3m} \ \Rightarrow \ \omega_n = \sqrt{\frac{2k}{3m}} \ .$$

This result concludes *Task b*. The cylinder inertia has caused a substantial reduction in the natural frequency as compared to a simple spring-mass system that would yield $\omega_n = \sqrt{k/m}$.

Deriving The Equation of Motion From Conservation of Energy

Conservation of energy implies,

$$\frac{m\dot{x}^2}{2} + \frac{I_g\dot{\theta}^2}{2} + \frac{kx^2}{2} = T_0 + V_0 \ .$$

Substituting the rolling-without-slipping kinematic conditions, $x = r\theta, \dot{x} = r\dot{\theta}$ gives

$$m\frac{(r\dot{\theta})^2}{2} + \frac{mr^2}{2}\frac{\dot{\theta}^2}{2} + \frac{k}{2}(r\theta)^2 = T_0 + V_0 \ ,$$

where from Appendix C, $I_g = mr^2/2$. Differentiating with respect to θ gives the equation of motion

$$\frac{3mr^2}{2}\ddot{\theta} + kr^2\theta = 0 \ ,$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$.

A Cylinder Rolling Inside a Cylindrical Surface

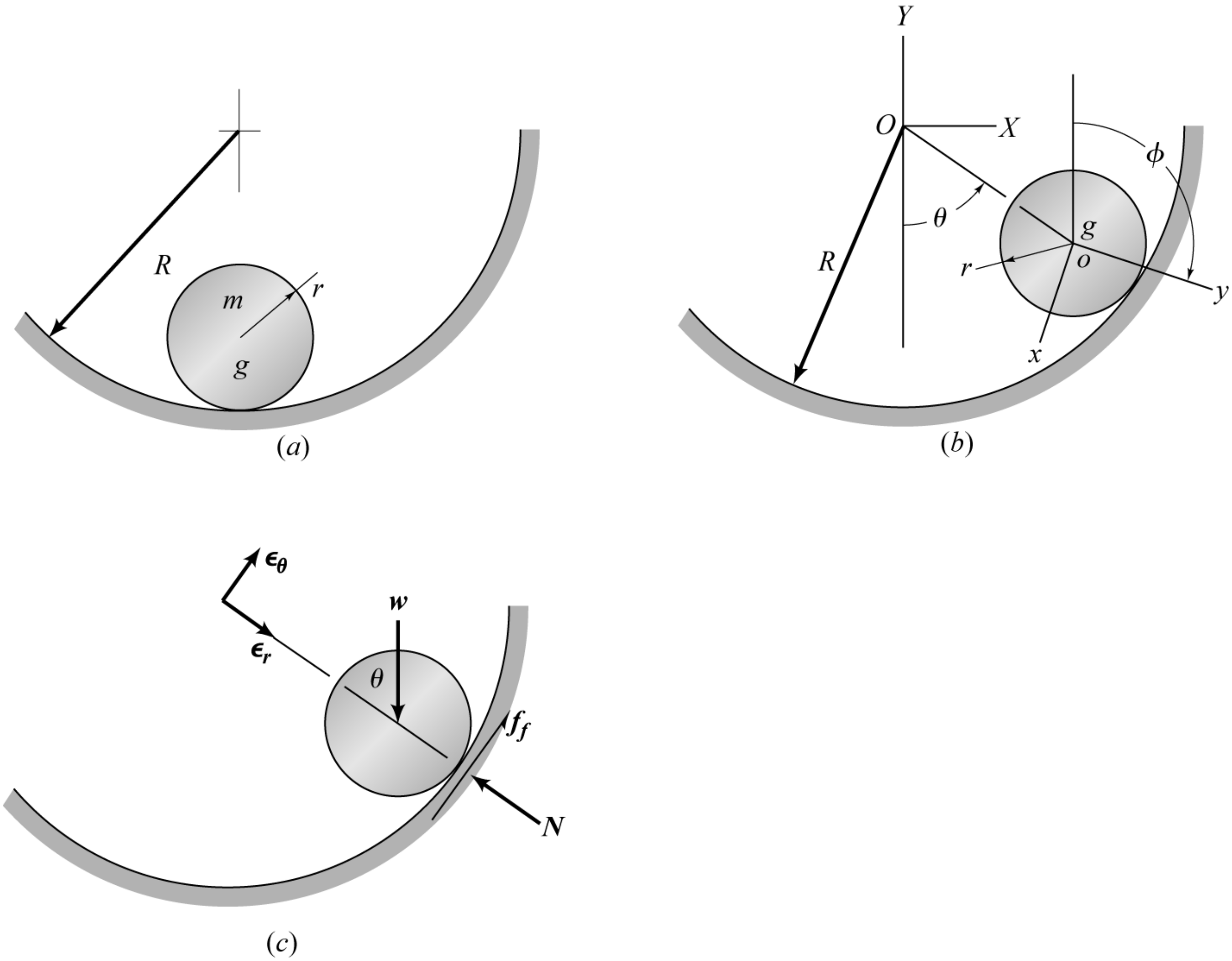

Figure 5.27 Cylinder rolling inside a cylinder. (a) Eqilibrium, (b) Coordinates, (c) Free-body diagram

O denotes the origin of the stationary *X,Y* coordinate system. The *x, y* coordinate system is fixed to the cylinder and its origin *o* coincides with the cylinder's mass center *g*. The angle θ defines the rotation of the line *O*-*g*, while φ defines the cylinder rotation with respect to ground.

The following engineering-analysis tasks apply:

a. Draw a free-body diagram and derive the equation of motion, and

b. For small motion about the bottom equilibrium position, determine the natural frequency.

c. Assuming that the cylinder is released from rest at $\theta = \pi/2$*, find* $\dot{\theta}$ *as a function of* θ*. Also define the normal reaction force as a function of* θ

In applying $\Sigma \boldsymbol{f} = m \ddot{\boldsymbol{R}}_g$, we will use the polar coordinate unit vectors. Starting in the $\boldsymbol{\varepsilon_\theta}$ direction,

$$\Sigma f_\theta = f_f - w \sin\theta = m a_\theta = m(R-r)\ddot{\theta} \ . \qquad (5.119a)$$

$\Sigma f_r = m a_r$ will be needed to define the normal reaction force N, and gives

$$\Sigma f_r = w\cos\theta - N = m a_r = -m(R-r)\dot{\theta}^2 \ . \qquad (5.119b)$$

Stating the moment equation about the mass center g gives

$$\Sigma M_{g\varphi} = -r f_f = I_g \ddot{\varphi} \ . \qquad (5.120)$$

The moment due to the friction force is negative because it is acting in the - φ direction. We now have two equations in the three unknowns $\ddot{\theta}, \ddot{\varphi}, f_f$. From Eq.(4.14a), the required kinematic constraint equation between $\ddot{\theta}$ and $\ddot{\varphi}$ is

$$(R-r)\dot{\theta} = r\dot{\varphi} \Rightarrow (R-r)\ddot{\theta} = r\ddot{\varphi} \ . \tag{5.121}$$

Substituting for f_f from Eq.(5.119a) into Eq.(5.120) gives

$$-r[w\sin\theta + m(R-r)\ddot{\theta}] = \frac{mr^2}{2}\ddot{\varphi} \ .$$

Now substituting for $\ddot{\varphi}$ from Eq.(5.121) gives

$$\begin{aligned} -r[w\sin\theta + m(R-r)\ddot{\theta}] &= \frac{mr^2}{2}\frac{(R-r)}{r}\ddot{\theta} \\ \therefore \ \frac{3m}{2}(R-r)\ddot{\theta} + w\sin\theta &= 0 \ , \end{aligned} \tag{5.122}$$

For small θ, $\sin\theta \cong \theta$, the linearized equation of motion is

$$\ddot{\theta} + \frac{2g}{3(R-r)}\theta = 0 \ ,$$

and the natural frequency is $\omega_n = \sqrt{2g/3(R-r)}$.

The solution for $\dot{\theta}$ as a function of θ can be developed from Eq.(5.122) via the energy-integral substitution $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$ as

$$\frac{d}{d\theta}\left(\frac{\dot\theta^2}{2}\right) = -\frac{2g}{3(R-r)}\sin\theta \ .$$

Integration from the initial condition $\dot\theta(\theta = \pi/2) = 0$ yields

$$\frac{\dot\theta^2}{2} - 0 = \frac{2g}{3(R-r)}\int_{\pi/2}^{\theta} -\sin x \, dx \ = \frac{2g}{3(R-r)}\left[\cos\theta - \cos\left(\frac{\pi}{2}\right)\right] \ ,$$

and

$$\dot\theta^2 = \frac{4g}{3(R-r)}\cos\theta \qquad (5.123)$$

provides the requested solution. Substituting for $\dot\theta^2$ into Eq.(5.119b) defines N as

$$N = w\cos\theta + m(R-r)\frac{4g}{3(R-r)}\cos\theta = \frac{7w}{3}\cos\theta \ , \qquad (5.124)$$

and (*formally*) meets the requirements of Task c.

However, note from Eq.(5.124) that $N = 0$ at $\theta = \pi/2$; hence, from our initial condition, the wheel will slip initially until N becomes large enough for the Coulomb friction force $f_f = \mu_d N$ to prevent slipping.

With slipping, the appropriate model as provided from Eqs.(5.119a-120) and $f_f = \mu_d N$ is:

$$\mu_d N - w\sin\theta = m(R-r)\ddot{\theta}$$

$$-r\mu_d N = I_g\ddot{\varphi} = \frac{mr^2}{2}\ddot{\varphi}$$

$$N = w\cos\theta + m(R-r)\dot{\theta}^2 \ .$$

Eliminating N gives the two coupled nonlinear equations of motion:

$$m(R-r)\ddot{\theta} = \mu_d[w\cos\theta + m(R-r)\dot{\theta}^2] - w\sin\theta$$

$$I_g\ddot{\varphi} = -r\mu_d[w\cos\theta + m(R-r)\dot{\theta}^2] \ . \qquad (5.125)$$

These equations apply during slipping, provided that the direction of the friction force f_f in figure does not change[1]. With slipping, the cylinder has two degrees of freedom, θ and φ. Note from Eqs.(5.125) that initially, at $\theta = \pi/2$, $\ddot{\theta} < 0$ and $\ddot{\varphi} = 0$. As the cylinder rolls down the surface, $\dot{\theta}$ and $\dot{\varphi}$ increase in magnitude but are negative. The friction force f_f acts to slow down the magnitude increase in $\dot{\theta}$ and accelerate the magnitude increase in

[1] Note that the friction force would have a different sign if the cylinder were released from rest at $\theta = -\pi/2$.

$\dot{\varphi}$. When the kinematic condition $(R-r)\dot{\theta} = r\dot{\varphi}$ is met, slipping stops, Eqs.(5.125) become invalid, and Eq.(5.122) applies.

Deriving the Equation of Motion From Conservation of Energy

With O, the origin of the X, Y system as the gravity potential energy datum, $T + V = T_0 + V_0$ implies

$$m\frac{[(R-r)\dot{\theta}]^2}{2} + I_g\frac{\dot{\varphi}^2}{2} - w(R-r)\cos\theta = T_0 + V_0 \ .$$

Substituting $(R-r)\dot{\theta} = r\dot{\varphi}$ and $I_g = mr^2/2$ gives

$$m\frac{[(R-r)\dot{\theta}]^2}{2} + m\frac{r^2}{2}\times\frac{1}{2}[\frac{(R-r)\dot{\theta}}{r}]^2 - w(R-r)\cos\theta = T_0 + V_0 \ .$$

or

$$\frac{3m(R-r)^2}{4}\dot{\theta}^2 - w(R-r)\cos\theta = T_0 + V_0 \ .$$

Differentiating w.r.t. θ gives

$$\frac{3m(R-r)^2}{2}\ddot{\theta} + w(R-r)\sin\theta = 0 \ .$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$.

Pulley-Assembly Example

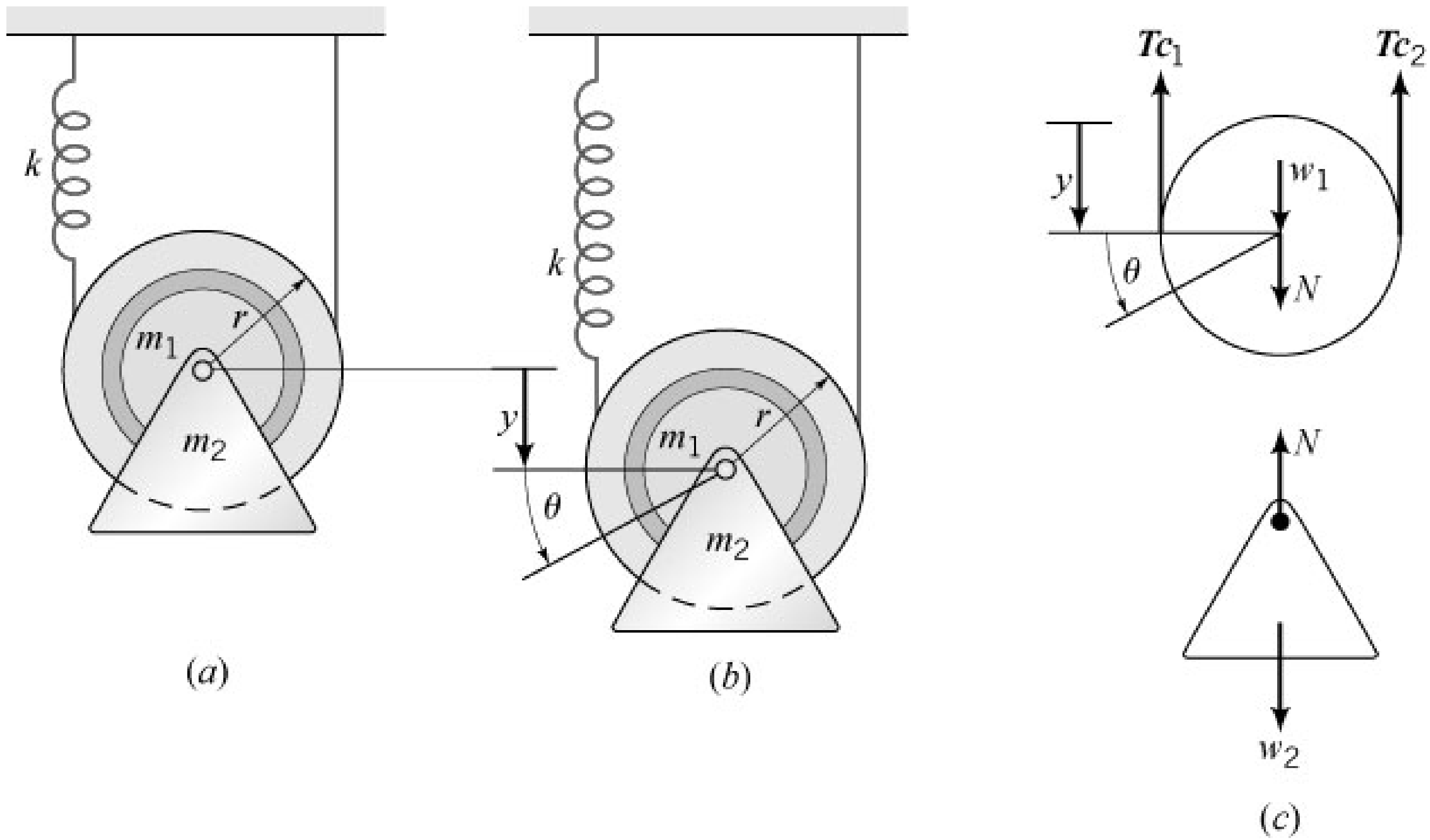

Figure 5.28 Pulley assembly consisting of a pulley of mass m_1 and an attached mass of mass m_2 . (a) Equilibrium, (b) Coordinates, (c) Free-body diagrams.

The assembly is supported by an inextensible cord in series with a linear spring with stiffness coefficient *k*. On the right, the cord's end is rigidly attached to
a horizontal surface. On the left, the cord is attached to the spring which is attached to the same surface. The pulley has mass m_1 and a moment of inertia about its mass center of

$I_g = m_1 r^2/2$. The cord does not slip on the pulley.

The engineering analysis tasks are:

a. Draw free-body diagrams and derive the equation of motion.

b. Determine the natural frequency

The statement that the "cord does not slip on the pulley" introduces the rolling-without-slipping condition. The pulley can be visualized as rolling without slipping on the vertical surface defined by the right-hand-side cord line. The y coordinate locates the change in position of the pulley, and θ defines the pulley's rotation angle. The spring is assumed to be undeflected when $y = 0$. The y and θ coordinates are related via the rolling-without-slipping kinematic condition,

$$y = r\theta \Rightarrow \ddot{y} = r\ddot{\theta} \ . \tag{5.126}$$

Figure 5.30B provides the appropriate free-body diagrams with the pulley and the lower assembly separated. The reaction force N acts between the two masses at the pivot connection point. The moment equation about the pulley's mass center is

$$\Sigma M_{og} = rT_{c2} - rT_{c1} = I_g \ddot{\theta} \tag{5.127a}$$

Note that T_{c1} and T_{c2} are different. They must be different to induce the pulley's angular acceleration. The pulley's mass center and the lower assembly have the same acceleration; hence their equations of motion are:

$$
\begin{aligned}
&(pulley)\ \Sigma f_y = w_1 + N - T_{c1} - T_{c2} = m_1 \ddot{y} \\
&(lower\ mass)\ \Sigma f_y = w_2 - N = m_2 \ddot{y}\ .
\end{aligned}
\qquad (5.127b)
$$

Adding these last equations eliminates N netting

$$
w_1 + w_2 - T_{c1} - T_{c2} = (m_1 + m_2)\ddot{y}\ . \qquad (5.128)
$$

Eqs.(5.126), (5.127), and (5.128) provide three equations for the four unknowns $\ddot{\theta}, T_{c1}, T_{c2}, \ddot{y}$.

We need another kinematic constraint equation. Pulling the pulley down a distance y will pull the cord end attached to the spring down a distance $2y$. Hence, the cord tension T_{c1} is defined by $T_{c1} = k(2y) = 2ky$. Substituting this result and substituting $y = r\theta$ and $\ddot{y} = r\ddot{\theta}$ gives

$$
rT_{c2} - r[2k(r\theta)] = I_g\ddot{\theta} = \frac{m_1 r^2}{2}\ddot{\theta}, and
$$

$$
w_1 + w_2 - 2k(r\theta) - T_{c2} = (m_1 + m_2)r\ddot{\theta}\ .
$$

Eliminating T_{c2} by multiplying the second of these equations by

r and adding the result to the first gives

$$r(w_1+w_2-2ky)-2kry=(\frac{3m_1}{2}+m_2)r^2\ddot{\theta}+I_g\ddot{\theta}\ ,$$

and the equation of motion is

$$(\frac{3m_1}{2}+m_2)r^2\ddot{\theta}+4kr^2\theta=r(w_1+w_2)\ .$$

The natural frequency is defined by

$$\omega_n=\sqrt{\frac{8k}{(3m_1+2m_2)}}\ .$$

This example is "tricky" in that the rolling-without-slipping constraint and the second pulley constraint to define the spring deflection, $\delta_s=2y$, are not immediately obvious.

Deriving the Equation of Motion From Conservation of Energy

Setting $y = 0$ as the zero potential energy for gravity means $T + V = T_0 + V_0$ implies

$$I_g \frac{\dot{\theta}^2}{2} + (m_1 + m_2)\frac{\dot{y}^2}{2} + k\frac{\delta_s^2}{2} - (w_1 + w_2)y = T_0 + V_0 \ ,$$

where δ_s is the spring deflection. Substituting: (i) $I_g = m_1 r^2/2$, (ii) the rolling without slipping condition $\dot{y} = r\dot{\theta}$, and (iii) the pulley condition $\delta_s = 2y = 2r\theta$ gives

$$\frac{m_1 r^2}{2}\frac{\dot{\theta}^2}{2} + (m_1 + m_2)\frac{r\dot{\theta}^2}{2} + \frac{k}{2}(2r\theta)^2 - (w_1 + w_2)(r\theta) = T_0 + V_0 \ .$$

Differentiating w.r.t. θ gives

$$(\frac{3m_1}{2} + m_2)r^2\ddot{\theta} + 4kr^2\theta = (w_1 + w_2) \ ,$$

where $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$.