Appendix A: Conservation of Mechanical Energy = Conservation of Linear Momentum

Consider the motion of a 2nd order mechanical system comprised of the fundamental mechanical elements: inertia or mass \( M \), stiffness \( K \), and viscous damping coefficient, \( D \). The Principle of Conservation of Linear Momentum (Newton’s 2nd Law of Motion) leads to the following 2nd order differential equation:

\[
M \ddot{X} + D \dot{X} + K X = F(t)
\]  

(1)

where the coordinate \( X(t) \) describes the system motion. \( X \) has its origin at the system static equilibrium position (SEP).

In the free body diagram above, \( F_{(t)} = F_{ext} \) is the external force acting on the system,

\[
F_k = -KY = -K(X - \delta_s)
\]

is the reaction force from the spring. \( \delta_s = W/K \) represents the static deflection. \( Y = (X - \delta_s) \) is the total deflection of the spring from its unstretched position.

\[
F_D = -D\dot{X}
\]

is the reaction force from the dashpot element.
(1) is recast as

\[ M \ddot{X} + D \dot{X} + K (X - \delta_s) = F(t) - W \]  

(1)

Now, integrate this Eq. (1) between two arbitrary displacements \( X_1 = X(t_1), X_2 = X(t_2) \) occurring at times \( t_1 \) and \( t_2 \), respectively. At these times the system velocities are \( \dot{X}_1 = \dot{X}(t_1), \dot{X}_2 = \dot{X}(t_2) \), respectively.

The process gives:

\[
\begin{align*}
\int_{X_1}^{X_2} M \ddot{X} \, dX + \int_{X_1}^{X_2} D \dot{X} \, dX + \int_{X_1}^{X_2} K (X - \delta_s) \, dX &= \int_{X_1}^{X_2} (F(t) - W) \, dX \\
\end{align*}
\]

(2a)

Since \( Y = (X - \delta_s) \) then \( dY = dX \), then write Eq. (2a) as

\[
\begin{align*}
\int_{X_1}^{X_2} M \ddot{X} \, dX + \int_{X_1}^{X_2} D \dot{X} \, dX + \int_{Y_1}^{Y_2} K Y \, dY &= \int_{X_1}^{X_2} (F(t) - W) \, dX \\
\end{align*}
\]

(2b)

The acceleration and velocity are \( \ddot{X} = \frac{d \dot{X}}{dt}, \dot{X} = \frac{d X}{d t} \), respectively. Using these definitions, write Eq. (2b) as:

\[
\begin{align*}
\int_{t_1}^{t_2} M \frac{d \dot{X}}{dt} \frac{dX}{dt} \, dt + \int_{t_1}^{t_2} D \frac{\dot{X}}{dt} \frac{dX}{dt} \, dt + \int_{Y_1}^{Y_2} K d \left( \frac{1}{2} Y^2 \right) = \int_{X_1}^{X_2} F(t) \, dX - \int_{X_1}^{X_2} W \, dX \\
\end{align*}
\]

or,

\[
\begin{align*}
\int_{t_1}^{t_2} M \frac{d \dot{X}}{dt} \dot{X} \, dt + \int_{t_1}^{t_2} D \dot{X} \dot{X} \, dt + \int_{Y_1}^{Y_2} K d \left( \frac{1}{2} Y^2 \right) + \int_{X_1}^{X_2} W \, dX = \int_{X_1}^{X_2} F(t) \, dX \\
\end{align*}
\]
\[
\int_{X_1}^{X_2} M \, d\left(\frac{1}{2} \dot{X}^2\right) + \int_{t_1}^{t_2} D \ddot{X} \, dt + K \left(\frac{1}{2} Y^2\right)_{\gamma_2}^{\gamma_1} + W \left(X_2 - X_1\right) = \int_{X_1}^{X_2} F(t) \, dX
\]  
(3)

and since \((M, K, D)\) are constant parameters, express Eq. (3) as:

\[
\frac{1}{2} M \left(\dot{X}_2^2 - \dot{X}_1^2\right) + \int_{t_1}^{t_2} D \ddot{X} \, dt + \frac{1}{2} K \left(Y_2^2 - Y_1^2\right) + W \left(X_2 - X_1\right) = \int_{X_1}^{X_2} F(t) \, dX
\]  
(4)

Let’s recognize several of the terms in the equation above. These are known as:

**Change in kinetic energy,**

\[
T_2 - T_1 = \frac{1}{2} M ^2 - \frac{1}{2} M \dot{X}_1^2
\]  
(5.a)

**Change in potential energy** (elastic strain and gravitational)

\[
V_2 - V_1 = \frac{1}{2} K Y_2^2 - \frac{1}{2} K Y_1^2 + W X_2 - W X_1
\]  
(5.b)

**Total work from external force input into the system,**

\[
W_{1-2} = \int_{X_1}^{X_2} F(t) \, dX
\]  
(5.c)

Set \(P_v = D \dot{X}^2\) as the viscous power dissipation, Then, **the dissipated viscous energy (removed from system)** is,

\[
E_{v_{1-2}} = \int_{t_1}^{t_2} D \ddot{X} \, dt = \int_{t_1}^{t_2} P_v \, dt
\]  
(5.d)

With these definitions, write Eq. (4) as

\[
(T_2 - T_1) + (V_2 - V_1) + E_{v_{1-2}} = W_{1-2}
\]  
(6)
That is, the \textbf{change in (kinetic energy + potential energy)} + \textbf{the viscous dissipated energy} = \text{External work}. This is also known as the \textbf{Principle of Conservation of Mechanical Energy (PCME)}.

Note that Eq. (1) and Eq. (6) are \textbf{NOT} independent. They actually represent the same physical law. Note also that Eq. (6) is not to be mistaken with the first-law of thermodynamics since it does not account for heat flows and/or changes in temperature.

One can particularize Eqn. (6) for the initial time $t_0$ with initial displacement and velocities given as $(X_0, \dot{X}_0)$, and at an arbitrary time $(t)$ with displacements and velocities equal to $(X_{(t)}, \dot{X}_{(t)})$, respectively, i.e., Thus, the PCME states

$$
(T_{(t)} + V_{(t)}) = W_{(0\rightarrow t)} - E_v(0\rightarrow t) + (T_0 + V_0)
$$

where $(T_0 + V_0)$ is the initial state of energy for the system at time $t=0$ s. Eqn. (7a) is also written as

$$
\frac{1}{2}M \dot{X}_{(t)}^2 + \frac{1}{2}KY_{(t)}^2 + W X = \int_{X_0}^{X_{(t)}} F_{(t)} dX - \int_{t_0}^{t} D \dot{X}^2 dt + \frac{1}{2}M \dot{X}_0^2 + \frac{1}{2}KY_0^2 + W X_0
$$

Taking the time derivative of Eq. (7) gives

$$
\frac{d}{dt} \left( T_{(t)} + V_{(t)} \right) = \frac{dW}{dt} - \frac{dE_v}{dt} = \phi_{ext} - \phi_v
$$
where $\mathcal{P}_{ext}$, $\mathcal{P}_{v}$ are the mechanical power from external forces acting on the system and the power dissipated by a viscous-type forces, respectively.

Work with Eq. (8) to obtain

$$\frac{d}{dt} \left[ \frac{1}{2} M \dddot{x}(t) + \frac{1}{2} K \dot{y}(t) + W X = \int_{x_0}^{x(t)} F(t) \, dX - \int_{t_0}^{t} D \dot{X}^2 \, dt + \frac{1}{2} M \dot{X}_0^2 + \frac{1}{2} K \dot{y}_0^2 + W X_0 \right]$$

$$\frac{1}{2} M \dddot{x}(t) \frac{d\dddot{x}(t)}{dt} + \frac{1}{2} K \dot{y}(t) \frac{d\dot{y}(t)}{dt} + W \frac{dX(t)}{dt} = F(t) \frac{d\dot{X}(t)}{dt} - D \dot{X}^2 \quad (10)$$

Recall that the derivative of an integral function is just the integrand.

To obtain

$$M \dddot{x}(t) + K \dot{y}(t) \dot{y}(t) + W \dddot{x}(t) = F(t) \dddot{x} - D \dot{X}^2 \quad (11a)$$

Since $Y = (X - \delta_s)$ and $\dot{Y} = \dot{X}$, Eq. (11) becomes

$$\dddot{x}(t) \left( M \dddot{x}(t) + K \left[ X(t) - \delta_s \right] + W \right) = F(t) \dddot{x} - D \dot{X}^2$$

Canceling the static load balance terms, $W = K \delta_s$, and factoring out the velocity, obtain

$$\left[ M \dddot{x}(t) + K X(t) + D \dot{X} \right] \dddot{x}(t) = F(t) \dddot{x} \quad (11)$$

Since for most times the system velocity is different from zero, i.e., $\dot{X}(t) \neq 0$; that is, the system is moving; then

$$M \dddot{x} + D \dot{X} + K X = F(t) \quad (1)$$

i.e., the original equation derived from Newton’s Law (conservation of linear momentum).
Suggestion/recommended work:
Rework the problem for a rotational (torsional) mechanical system and show the equivalence of conservation of mechanical energy to the principle of angular momentum, i.e. start with the following Eqn.

\[ I \ddot{\theta} + D_{\theta} \dot{\theta} + K_{\theta} \theta = T(t) \]

where \( (I, D_{\theta}, K_{\theta}) \) are the equivalent mass moment of inertia, rotational viscous damping and stiffness coefficients, \( T(t)=T_{\text{ext}} \) is an applied external moment or torque, and \( \theta(t) \) is the angular displacement of the rotational system.