Important design issues and engineering applications of SDOF system Frequency response Functions

The following descriptions show typical questions related to the design and dynamic performance of a second-order mechanical system operating under the action of an external force of periodic nature, i.e. $\boldsymbol{F_{(t)}=F_o \cos(\Omega t)}$ *or* $\boldsymbol{F_{(t)}=F_o \sin(\Omega t)}$

The system EOM is: $M\ddot{X} + D\dot{X} + KX = F_o \cos(\Omega t)$

Recall that the system response is governed by its parameters, i.e. stiffness ***(K)***, mass ***(M)*** and viscous damping ***(D)*** coefficients. These parameters determine the fundamental natural frequency, $\omega_n = \sqrt{K/M}$, and viscous damping ratio, $\zeta = D/D_c$, with $D_c = 2\sqrt{KM}$

In all design cases below, let $r=(\Omega/\omega_n)$ as the frequency ratio. This ratio (excitation frequency/system natural frequency) largely determines the system periodic forced performance.

PROBLEM TYPE 1

Consider a system excited by a periodic force of magnitude $\boldsymbol{F_o}$ with external frequency $\boldsymbol{\Omega}$.

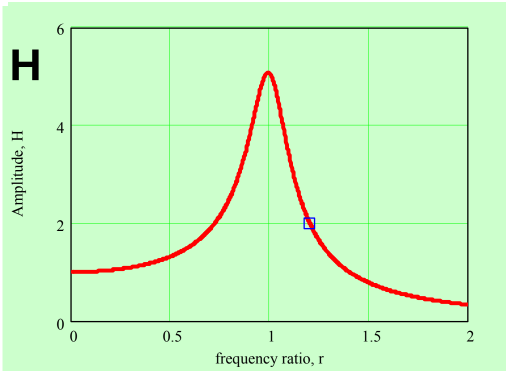
a) Determine the damping ratio ζ needed such that the amplitude of motion does not ever exceed (say) twice the displacement ($X_s=F_o/K$) for operation at a frequency (say) 20% above the natural frequency of the system ($\Omega=1.2\omega_n$).

b) With the result of (a), determine the amplitude of motion for operation with an excitation frequency coinciding with the system natural frequency. Is this response the maximum ever expected? Explain.

Recall that system periodic response is

$$X(t)=X_s H_{(r)}\cos(\Omega t+\psi)$$

Solution. From the amplitude of FRF



$$\left|\frac{X}{X_s}\right|=H_{(r)}=\frac{1}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

Set $r=r_a=1.2$ and $|X/X_s|=H_a=2$.

Find the damping ratio ζ from the algebraic equation:

$$H_a^2\left(\left(1-r_a^2\right)^2+(2\zeta r_a)^2\right)=1$$

$$(2\zeta r_a)^2=\frac{1}{H_a^2}-\left(1-r_a^2\right)^2$$

$$\Rightarrow \zeta=\frac{1}{2r_a}\left[\frac{1}{H_a^2}-\left(1-r_a^2\right)^2\right]^{1/2}=0.099$$

Finally, calculate the viscous damping coefficient $D=\zeta D_c$

For excitation at the natural frequency, i.e., at resonance, then $r=1$, $|X/X_s|=1/(2\zeta)=Q$. *Thus* $|X|=Q\,X_s$

The maximum amplitude of motion does not necessarily occur at r=1. In actuality, the magnitude of the frequency ratio (r_*) which maximizes the response, $\left(\partial \left| \frac{X}{X_s} \right| \Big/ \partial r \right) = 0$, is (after some algebraic manipulation):

$$r_* = \sqrt{\left(1 - 2\zeta^2\right)}; \text{ and } \left| \frac{X}{X_s} \right|_{\max} = \frac{1}{2\zeta} \frac{1}{\sqrt{\left(1 - \zeta^2\right)}}$$ Corrected 2/19/13

Note that for small values of damping $\left| \frac{X}{X_s} \right|_{\max} \approx \frac{1}{2\zeta}$

PROBLEM TYPE 2

Consider a system exited by an imbalance (u), giving an amplitude of force excitation equal to $F_o = M\, u\, \mathbf{\Omega}^2$. Recall that $u = m\, e/M$, where m is the imbalance mass and e is its radial location

$$M\,\ddot{X} + D\,\dot{X} + K\,X = M\,u\,\Omega^2 \cos(\Omega t)$$

Recall that system periodic response is

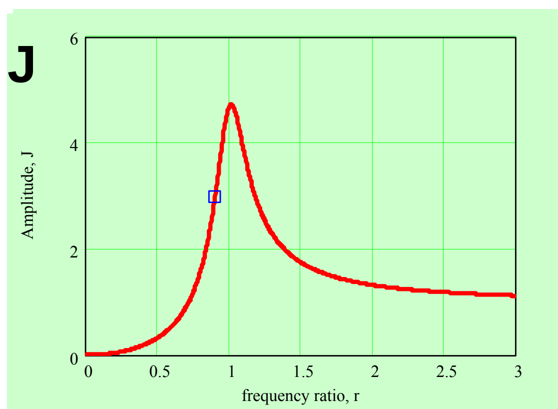
$$X(t) = u\,J_{(r)} \cos(\Omega t + \psi)$$

a) What is the value of damping ζ necessary so that the system response never exceeds (say) three times the imbalance ***u*** for operation at a frequency (say) 10% below the natural frequency of the system ($\Omega = 0.9\omega_n$).

b) With the result of (a), determine the amplitude of motion for operation with an excitation frequency coinciding with the system natural frequency.

Solution From the fundamental FRF amplitude ratio

$$\left|\frac{X}{u}\right| = J_{(r)} = \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta\, r)^2}}$$



Set $r = 0.9$ and $|X/u| = J_a = 3$. Calculate the damping ratio ζ from the algebraic equation.

$$\Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{r_a^4}{J_a^2} - \left(1 - r_a^2\right)^2\right]^{1/2} = 0.107$$

Finally, calculate the viscous damping coefficient, $D = \zeta D_c$.

Note that for forced operation with frequency = natural frequency, i.e., at resonance,

$r=1$, $|X/u| = 1/(2\zeta) = Q$. *Thus* $|X| = Q\,u$

The maximum amplitude of motion does not occur at $r=1$. The value of frequency ratio (r_*) which maximizes the response is obtained from

$$\left(\frac{\partial\left|\frac{X}{u}\right|}{\partial r}\right) = 0$$ then

$$r_* = \frac{1}{\sqrt{\left(1-2\zeta^2\right)}}; \text{ and } \left|\frac{X}{u}\right|_{\max} = \frac{1}{2\zeta}\frac{1}{\sqrt{\left(1-\zeta^2\right)}}$$ corrected 2/19/13

Note that for small values of damping $\left|\frac{X}{u}\right|_{\max} \approx \frac{1}{2\zeta}$

PROBLEM TYPE 3

Consider a system excited by a periodic force of magnitude F_o and frequency Ω. Assume that the spring and dashpot connect to ground.

a) Determine the damping ratio needed such that the **transmitted force** to ground does not ever exceed (say) two times the input force for operation at a frequency (say) = 75% of natural frequency ($\Omega=0.75\omega_n$).

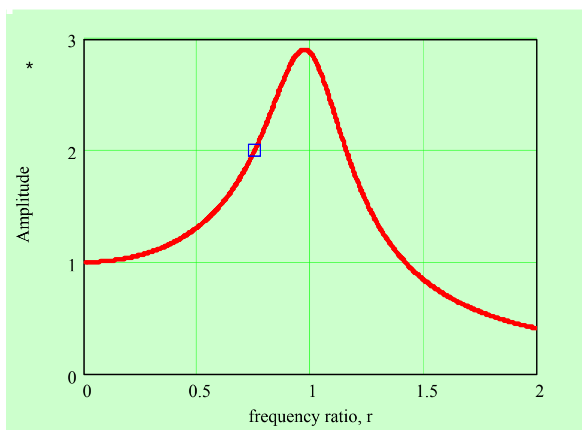
b) With the result of (a), determine the transmitted force to ground if the excitation frequency coincides with the system natural frequency. Is this the maximum transmissibility ever?

c) Provide a value of frequency such that the transmitted force is less than the applied force, irrespective of the damping in the system.

Solution: From the fundamental FRF amplitude for a base force excitation $F_{transmitted} = K\,X + D\,\dot{X}$

$$\left|\frac{F_{transmitted}}{F_o}\right| = A_{T(r)} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

Set A_T=2 and r=0.75, and find the damping ratio ζ.



$$\Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{1-A_T^2\left(1-r_a^2\right)^2}{A_T^2-1}\right]^{½}$$

=0.186

Finally, calculate the viscous damping coefficient $D = \zeta D_c$

At resonance, $r=1$, $A_T = \frac{\left[1+(2\zeta)^2\right]^{.5}}{2\zeta}$. Then calculate the magnitude of the transmitted force.

Again, the maximum transmissibility occurs at a frequency f_* which satisfies $\left(\partial A_T / \partial r \right) = 0$. Perform the derivation and find a closed form solution.

Recall that operation at frequencies $r \geq \sqrt{2}$, i.e. for $\Omega \geq 1.414\,\omega_n$, (41 % above the natural frequency) determines transmitted forces that are lower than the applied force (i.e. an **effective structural isolation** is achieved).

PROBLEM TYPE 4

Consider a system excited by a periodic force with magnitude $F_o = M\,a_{cc}$ (for example) and frequency Ω.

a) Determine the damping ratio ζ needed such that the maximum acceleration in the system does not exceed (say) **4 g**'s for operation at a frequency (say) 30% above the natural frequency of the system ($\Omega = 1.3\omega_n$).

b) With the result of (a), determine the system acceleration for operation with an excitation frequency coinciding with the system natural frequency. Explain your result

Recall the periodic response is $X(t) = X_s H_{(r)} \cos(\Omega t + \psi)$, then the acceleration of the system is

$$\ddot{X}(t) = -\Omega^2 X_s H_{(r)} \cos(\Omega t + \psi) = -\Omega^2 X(t)$$

Solution: From the amplitude of FRF

$$\left|\frac{\ddot{X}}{F_o/K}\right| = \frac{\omega_n^2 r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}} \rightarrow \left|\frac{\ddot{X}}{F_o/M}\right| = \frac{r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}}$$

Follow a similar procedure as in other problems above.

OTHER PROBLEMS

Think of similar problems and questions related to system dynamic performance.

In particular, you may also "cook up" similar questions related to the dynamic response of **first-order systems** (mechanical, thermal, electrical, etc). $M\dot{V} + DV = F_o \cos(\Omega t)$

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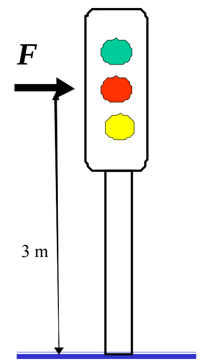
The following worked problems should teach you how to apply the frequency response function to resolve issues and to design many mechanical systems

P2. Periodic forced response of a SDOF mechanical system. DESIGN COMPONENT

The signal lights for a rail may be modeled as a 176 lb mass mounted 3 m above the ground of an elastic post. The natural frequency of the system is measured to be 12.2 Hz. Wind buffet generates a horizontal harmonic force at 12 Hz. The light filaments will break if their peak accelerations exceed *15g*. Determine the maximum acceptable force amplitude $|F|$ when the damping ratio ζ=0.0 and 0.01.

Full grade requires you to explain the solution procedure with due attention to physical details



The excitation force is periodic, say F(t)=Fo sin(ωt). then the system response will also be periodic, Y(t), with same frequency as excitation. Assuming steady state conditions:

STEADY RESPONSE of M-K-C system to PERIODIC Force with frequency ω

Case: periodic force of constant magnitude Define operating frequency ratio: $r = \frac{\omega}{\omega_n}$

$$F(t) = F_o \cdot \sin(\omega \cdot t)$$

System periodic response: $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t + \Psi)$ (1)

where:

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

From (1), the acceleration is $a(t) = -\omega^2 \cdot Y(t) = A \cdot \sin(\omega(t + \Psi - 180))$

the magnitude of acceleration is $A = \frac{F_o}{K_e} \cdot \frac{\omega^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$ or $A = \frac{F_o}{M_e} \cdot \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$

hence, define $A_{max} := 10 \cdot g$ maximum allowed acceleration of filament

$$HZ := 2 \cdot \pi \cdot \frac{1}{s}$$

system mass $M_e := 150 \cdot lb$

$f_n := 12 \cdot HZ$ natural frequency

$f := 11.5 \cdot HZ$ excitation frequency due to wind buffets

Let $r_o := \frac{f}{f_n}$ $\quad r_o = 0.958$ close to natural frequency

The maximum force allowed equals

$$F_{max}(r, \zeta) := A_{max} \cdot M_e \cdot \frac{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}{r^2}$$

without any damping $F_{max}(r_o, 0) = 133.27\,\text{lbf}$ Note the importance of damping that leads to a substantial increase in force allowed

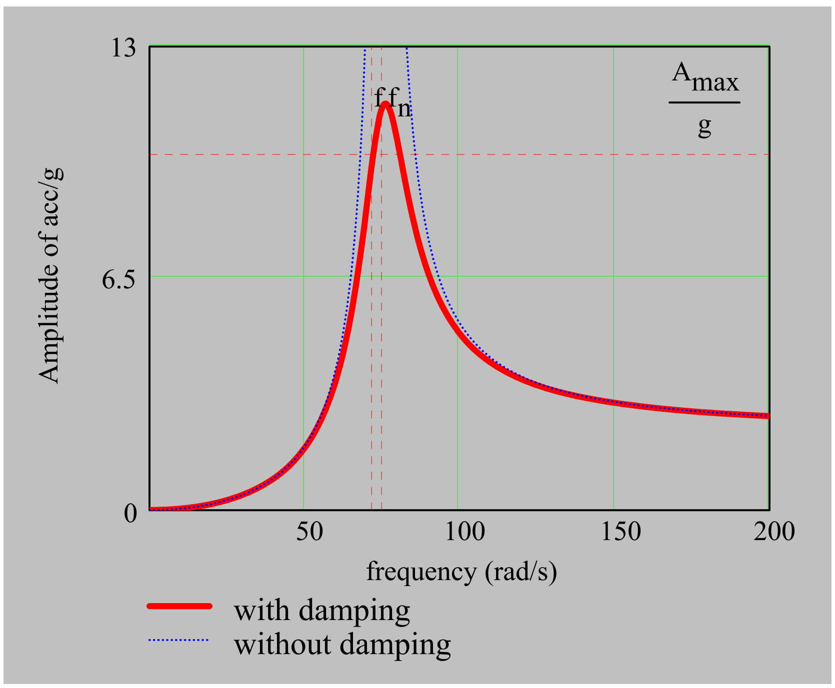
with damping $\xi := 0.1$ $F_{max}(r_o, \xi) = 340.231\,\text{lbf}$

$$\frac{F_{max}(r_o,\xi)}{F_{max}(r_o,0)} = 2.553$$

For the force found the amplitude of acceleration is

$F_o := F_{max}(r_o, \xi)$

$$A(r,\zeta) := \frac{F_o}{M_e} \cdot \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$$



$f_n = 75.398\,\frac{1}{s}$

$\frac{1}{2 \cdot \xi} = 5$

GRAPH NOT FOR EXAM

SInce r_o~1, a simpler enginering formula gives $A_{max} \cdot M_e \cdot 2 \cdot \xi = 300\,\text{lbf}$

which gives a very good estimation of the maximum wind force allowed

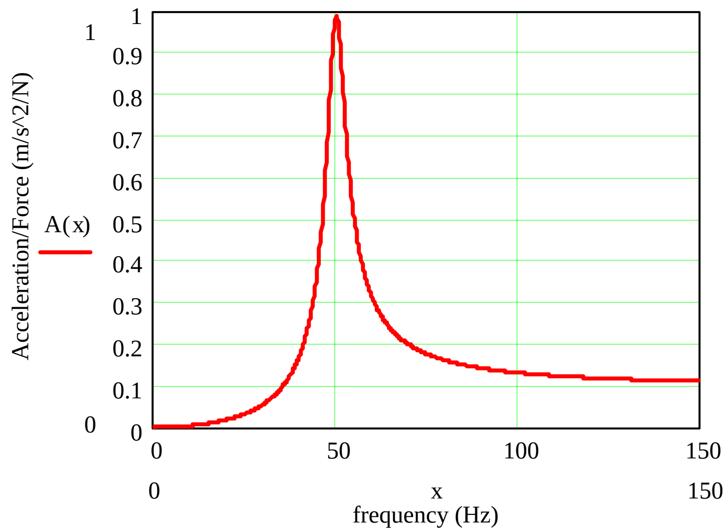
b) a system with damping ξ=0.1 will produce a 255 % increase in allowable force
Hence, the rail lightsystem will be more reliable, lasting longer.

$$\frac{\quad}{F_{max}(r_o,0)} = 2.553$$

c) Posts are usually hollow for the cables to be routed. These posts have layers of elastomeric material (~rubber-like) inside to increase their structural damping. Modern posts are wound up fro composites that integrate damping layers. Clearly, adding a "true" dashpot is not cost-effective

EXAMPLE - EXAM 2 TYPE:

Dynamic measurements were conducted on a mechanical system to determine its FRF (frequency response function). Forcing functions with multiple frequencies were exerted on the system and a digital signal analyzer (FFT) recorded the magnitude of the ACCELERATION/FORCE ([m/s2]/N) Frequency Response Function, as shown below. From the recorded data determine the system parameters, i.e. natural frequency (wn:rad/s) and damping ratio (z), and system stiffness (K:N/m), mass (M:kg), and viscous damping coefficient (C:N.s/m).
Explain procedure of ANALYSIS/INTERPRETATION of test data for full credit.



Test data showing amplitude of (acceleration/force)

Magnitude of FRF for mechanical system

Solution:

Recall that for an imposed external force of periodic form:

$$F(t) = F_o \cdot \sin(\omega t) \qquad [1]$$

the system response Y(t) is given by:

$$Y(t) = Y_{op} \cdot \sin(\omega t + \psi) \qquad [2]$$

where the amplitude of motion (Yop) and phase angle (Y) are defined as:

$$Y_{op}(r) = \frac{\frac{F_o}{K}}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad [3a] \qquad \Psi = -\operatorname{atan}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right) \qquad [3b]$$

with $r = \frac{\omega}{\omega_n}$ [4]

from [2], we find that the acceleration is given by:

$$a_Y(t) = -\omega^2 \cdot Y_{op} \cdot \sin(\omega t + \psi) = a_{op} \cdot \sin(\omega t + \Psi - 180) \qquad [5]$$

where:

$$a_{op}(r) = \frac{\frac{F_o}{M} \cdot r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad [6]$$

since: $\frac{F_o}{K} \cdot \omega^2 = \frac{F_o}{M} \cdot \frac{\omega^2}{\omega_n^2} = \frac{F_o}{K} \cdot r^2$

thus, the magnitude of amplitude of acceleration over force amplitude follows as:

$$\frac{a_{op}(r)}{F_o} = \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}\cdot\frac{1}{M} \qquad [7]$$

The units of this expression are 1/kg = $\dfrac{\frac{m}{s^2}}{N}$

For excitation at very high frequencies, r>>1.0 $\quad \dfrac{1}{M} \leftarrow \dfrac{a_{op}(r)}{F_o}$

From the graph (test data): $\quad \dfrac{1}{M} = 0.1\cdot\left(\dfrac{m}{s^2\cdot N}\right)$ $\quad$ Thus $\quad M := 10\cdot kg$

The system appears to have little damping, i.e. amplitude of FRF around a frequency of 50 Hz is rather large and varying rapidly over a narrow frequency range.

Thus, take the natural frequency as $\quad f_n := 50\cdot Hz$

expressed in rad/s as: $\quad \omega_n := f_n\cdot 2\cdot\pi \qquad \omega_n = 314.159\,\dfrac{rad}{s}$

We can estimate the stiffness (K) from the fundamental relationship:

$$K := \omega_n^{\,2}\cdot M \qquad K = 9.87\times 10^5\,\frac{N}{m}$$

for excitation at the natural frequency (r=1), the ratio of amplitude of acceleration to force reduces to

$$\frac{a_{op}(1)}{f_o} = \frac{1}{2\cdot M\cdot\zeta}$$

from the graph (test data), the ratio is approximately equal to one (1/kg). Thus. the damping ratio is determined as

$$\zeta := \frac{1}{2\cdot M\cdot\left(\frac{1.0}{kg}\right)} \qquad \zeta = 0.05$$

That is, the system has a damping ratio equal to 5%. This result could have also been easily obtained by studying the ratio of (amplitude at the natural frequency divided by the amplitude at very high frequency, i.e.)

$$\frac{1}{2\cdot\zeta} = \frac{1}{0.1} = 10$$

Once the damping ratio is obtained, the damping coefficient can be easily determined from the formula:

$$C := \zeta\cdot 2\cdot M\cdot\omega_n \qquad C = 314.159\,N\cdot\frac{s}{m}$$

The number of calculations is minimal. One needs to interpret correctly the test data results, however.