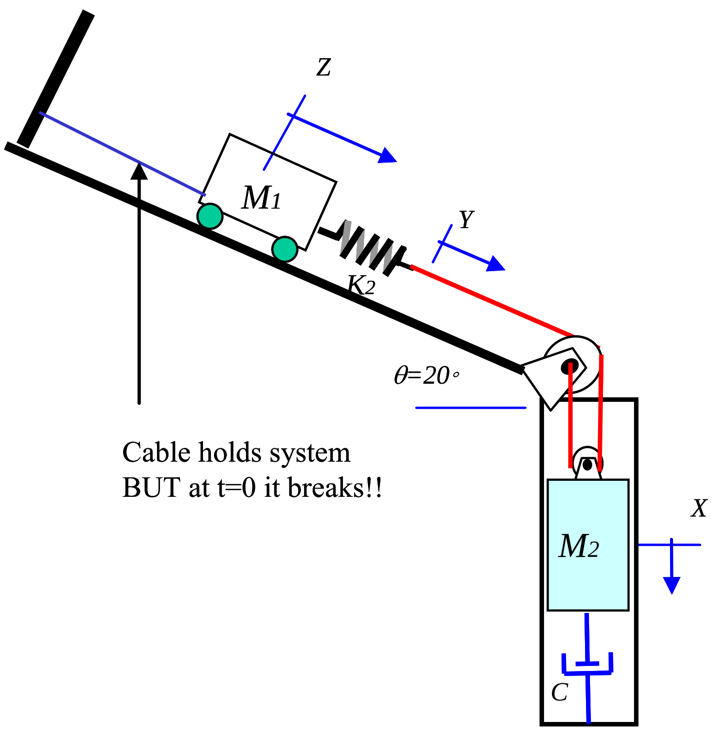
EXAMPLE #2 for MEEN 363 – FALL 2010

Objectives:

a) To derive EOMS of a 2DOF system
b) To understand concept of static equilibrium
c) To learn the correct usage of physical units (US system)
d) To calculate natural frequencies and natural mode shapes – **RIGID BODY MODE**
e) To predict the response of a system using modal coordinates. Case: constant amplitude load. Explain what a rigid body means.
f) To learn how to combine mathematical statements with explanatory sentences.



Car 1 must pull a heavy block stuck in a hollow and deep mining shaft. The front end of car 1 is tied to a big tree with a cable. A flexible cable of stiffness K_2 is connected to an inextensible cable that in turn, with a pulley system, is connected to the block. The damping coefficient *(C)* represents the viscous drag between the block and shaft walls. In the figure, ***Y=X=Z=0*** denote the **static equilibrium position** (SEP) of the system.

SEP means no motion of car and block. Thus, at the SEP spring K_2 is already deflected since it must support 50% of the block weight (W_2) as easily seen from the cable & pulley constraint. The top cable connected to a fixed point also holds the system with a force = 50% of W_2 plus a fraction of the car 1 weight, i.e. $W_1\sin(20°)$. **This knowledge is BASIC, does not require of elaborate thinking or deriving lengthy equations.**

At time t=0 sec, the cable BREAKS! and both the car and block start falling. (Notice change of direction of coordinates X,Y,Z from prior example). Furthermore, the engine in car is off ! Oopps, forgot to set the hand brake!

a) Identify the kinematical constraint relating motions *Y* and *X*. The cable does NOT slip on the pulley.
b) Determine the **static** deflection (δ_s) of spring element as well as top cable force before it breaks.
c) For t>0, after top cable breaks, draw free body diagrams for the car and block, label all forces and show their constitutive relation in terms of the motion coordinates, if applicable.
d) For t>0, Derive EOMs for the car and block motion in terms of coordinates *Z* & *X*.
e) Find the system (undamped) natural frequencies and mode shapes, i.e. solve for the system eigenvalues and eigenvectors. DISCUSS Significance of rigid body modes.
f) Find the (undamped) response of the system using modal analysis. Graph responses (modal and physical) versus time.
g) Find the spring-2 force and graph it vs. time. What knowledge can be gained from this force?

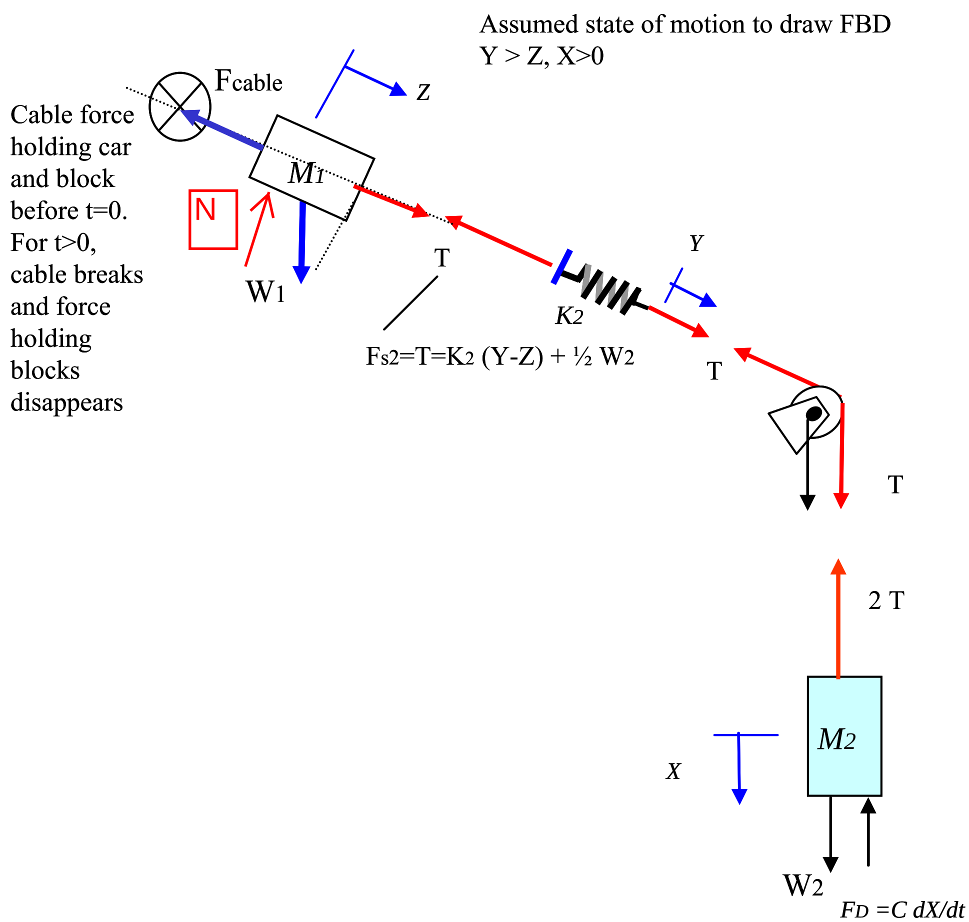
FREE BODY diagrams and kinematic constraints

Definitions:

F_{cable} = force from elastic cable connected to fixed point (valid for t<0 only)
F_{s2} = force from elastic cable connecting car to cable on pulley = T = Tension on cable
F_D = viscous drag force

δ_{s2} static deflection for spring 2

FREE BODY DIAGRAMS



MEEN 363 EXAMPLE 2DOF- ANALYSIS: block pulls car - RIGID BODY MOTION ORIGIN := 1

$K_2 := 10^5 \cdot \frac{lb}{in}$ $W_2 := 5000 \cdot lb$ $C := 1500 \cdot lb \cdot \frac{sec}{in}$ $M_2 := \frac{W_2}{g}$ $M_2 = 155.405 \frac{lb\,sec^2}{ft}$

$W_1 := 1000 \cdot lb$ $\theta := 20 \cdot \frac{\pi}{180}$ $M_1 := \frac{W_1}{g}$ $M_1 = 31.081 \frac{lb\,sec^2}{ft}$

masses need be expressed in lb.sec^2/ft for consistency in EOM

(a) kinematic constraint - inextensible cable

The cable length is constant, thus $l_c = l_c + 2 \cdot X - Y$ and the kinematic constraint follows as $Y = 2 \cdot X$

(b) Find the top cable "holding force" and **spring-2 static deflection.** For statics, assume NO motion

For static equilibrium:
force from spring2 = tension $T := \frac{W_2}{2}$ and $T = K_2 \cdot \delta_{s2} = F_{s2}$ $\delta_{s2} := \frac{W_2}{2 \cdot K_2}$

$\delta_{s2} = 0.025\ in$

Cable attached to fiex point holds fraction of car weight and also balances tension

and $F_{cable} := W_1 \cdot \sin(\theta) + \frac{W_2}{2}$ $F_{cable} = 2.842 \times 10^3\ lb$

(c) Derive EOMS: Cable breaks, for t>0 Assume a state of motion with Y-Z>0, X>0 **block pulls car (engine off)**

Block of mass M2: From the FBD diagram, with X>0, and apply Newton's 2nd law to obtain:

$$M_2 \cdot \frac{d^2}{dt^2} X = W_2 - F_{Damper} - 2 \cdot T \quad (1)$$

where $F_{Damper} = C \cdot \frac{d}{dt} X$ (2) is the viscous drag force

$$T = \left[K_2 \cdot (Y - Z) + K_2 \cdot \delta_{s2}\right] = F_{s2} \quad (3)$$

T is the cable tension = Force from spring 2.
(Y-Z)>0, and δs2 is the static deflection for spring 2

Car of mass M1: From the FBD diagram, with Y>Z, Y=2X, apply Newton's 2nd law to obtain:

$$M_1 \cdot \frac{d^2}{dt^2} Z = W_1 \cdot \sin(\theta) + T - F_{cable} \quad (4)$$

But $F_{cable} := 0 \cdot lb$ since it broke!

substitute T and Fdamper into (1) to obtain (1)

$$M_2 \cdot \frac{d^2}{dt^2} X = W_2 - C \cdot \frac{d}{dt} X - 2 \cdot \left[K_2 \cdot (Y - Z) + K_2 \cdot \delta_{s2}\right]$$

and since $2 \cdot K_2 \cdot \delta_{s2} = W_2$ and $Y = 2 \cdot X$

$$M_2 \cdot \frac{d^2}{dt^2} X = W_2 - C \cdot \frac{d}{dt} X + 2 \cdot \left[K_2 \cdot (Z - 2 \cdot X)\right] - \frac{2 \cdot W_2}{2}$$

BLOCK of MASS M2:

$$M_2 \cdot \frac{d^2}{dt^2} X + C \cdot \frac{d}{dt} X + 4 \cdot K_2 \cdot X - 2 \cdot K_2 \cdot Z = 0 \quad (5)$$

substitute and T into (4)

$$M_1 \cdot \frac{d^2}{dt^2} Z = W_1 \cdot \sin(\theta) + \left[K_2 \cdot (Y - Z) + \frac{W_2}{2} \right]$$ Sub Y=2X

$$M_1 \cdot \frac{d^2}{dt^2} Z = W_1 \cdot \sin(\theta) + \left[K_2 \cdot (2 \cdot X - Z) + \frac{W_2}{2} \right]$$

CAR of MASS M1

$$M_1 \cdot \frac{d^2}{dt^2} Z + K_2 \cdot Z - 2 \cdot K_2 \cdot X = W_1 \cdot \sin(\theta) + \frac{W_2}{2} \qquad (6)$$

Eqns. (5) and (6) are the desired equations of motion for the car and block. Please recall that motions are from the static equilibrium position

In matrix form, the EOMs are:

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} Z \\ X \end{pmatrix} + \begin{pmatrix} K_2 & -2K_2 \\ -2 \cdot K_2 & 4 \cdot K_2 \end{pmatrix} \cdot \begin{pmatrix} Z \\ X \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} W_1 \cdot \sin(\theta) + \frac{W_2}{2} \\ 0 \end{pmatrix} \qquad (7)$$

Notes

* for t>0, weights of block and (fraction of) car PULL the system downwards!
* mass & stiffness matrices are symmetric.The damping matrix is NOT
* Stiffness matrix is singular, i.e. its determinant equals zero, A RIGID BODY MODE EXPECTED

$$\Delta(K) = \left(4 \cdot K_2^{\,2} - 4 \cdot K_2^{\,2}\right) = 0$$

(d) Find natural frequencies and natural mode shapes of UNDAMPED system.

Disregading damping, and letting the force RHS=0, eq. (7) becomes

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} Z \\ X \end{pmatrix} + \begin{pmatrix} K_2 & -2K_2 \\ -2 \cdot K_2 & 4 \cdot K_2 \end{pmatrix} \cdot \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (8)$$

Let $Z = a_1 \cdot \cos(\omega \cdot t)$ $\quad X = a_2 \cdot \cos(\omega \cdot t)$ (9)

Substitution of (9) into (8) renders the homogeneous system of eqns

$$\begin{pmatrix} K_2 - M_1 \cdot \omega^2 & -2K_2 \\ -2 \cdot K_2 & 4 \cdot K_2 - M_2 \cdot \omega^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (10)$$

eqn (10) has a non-trivial solution if the determinant of the system of equations equals zero, i.e.:

$$\Delta(\omega) = \left(K_2 - M_1 \cdot \omega^2\right) \cdot \left(4 \cdot K_2 - M_2 \cdot \omega^2\right) - 4 \cdot K_2^{\,2} = 0$$ Let $\lambda = \omega^2$

and expanding the products in the eqn. above $0 = \lambda^2 \cdot M_1 \cdot M_2 - \lambda \cdot \left(K_2 \cdot M_2 + 4 \cdot K_2 \cdot M_1\right) + K_2 \cdot 4K_2 - 4 \cdot K_2^{\,2}$

Let: $0 = \left(a \cdot \lambda^2 + b \cdot \lambda + c\right)$ with: $a := M_1 \cdot M_2$

$b := -\left(K_2 \cdot M_2 + 4 \cdot K_2 \cdot M_1\right)$ $\quad c := K_2 \cdot 4 \cdot K_2 - 4 \cdot K_2^{\,2}$ Note c=0 !!!

$$a = 4.83 \times 10^{3} \frac{lb^2 \, sec^4}{ft^2} \qquad b = -3.357 \times 10^{8} \frac{lb^2 \, sec^2}{ft^2} \qquad c = 0 \frac{lb^2}{ft^2}$$

NOTE That the coefficients a,b,c have consistent physical units. That is, since the physical unit for λ is (1/sec^2), the physical units of the determinant are (lbf/ft)^2

The roots (eigenvalues) of the characteristic equation are

$$\lambda_1 := \frac{-b - \left(b^2 - 4 \cdot a \cdot c\right)^{0.5}}{2 \cdot a} \qquad \lambda_2 := \frac{-b + \left(b^2 - 4 \cdot a \cdot c\right)^{0.5}}{2 \cdot a} \qquad \lambda = \begin{pmatrix} 0 \\ 6.95 \times 10^{4} \end{pmatrix} \frac{1}{sec^2}$$

and the natural frequencies are:

$$\omega_1 := \left(\lambda_1\right)^{0.5} \qquad \omega_2 := \left(\lambda_2\right)^{0.5} \qquad \omega = \begin{pmatrix} 0 \\ 263.621 \end{pmatrix} \frac{rad}{sec} \qquad (11)$$

Now, find the eigenvectors

The two equations in (10) are linearly dependent. Thus, one cannot solve for a1 and a2. Set $a_1 := 1$ arbitrarily; and from the first equation

for ω_1

$$a_2 := \frac{\left[K_2 - M_1 \cdot \left(\omega_1\right)^2\right] \cdot a_1}{2 \cdot K_2} \qquad \phi_1 := a \qquad \phi_1 = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$$

is the first eigenvector (natural mode) (12)

DOF1 (Z) and DOF2 (X) move in phase, with Z=Y, X=0.5 Y
(**recall kinematic constraint**)
RIGID BODY MOTION

for ω_2

$$a_2 := \frac{\left[K_2 - M_1 \cdot \left(\omega_2\right)^2\right] \cdot a_1}{2 \cdot K_2} \qquad \phi_2 := a \qquad \phi_2 = \begin{pmatrix} 1 \\ -0.4 \end{pmatrix}$$

is the 2nd eigenvector (natural mode) (13)

DOF1 (Z) and DOF2 (X) move 180 deg OUT of phase, with |Z|>|X|

(e) Response in MODAL coordinates:

Use the MODAL transformation $x = A \cdot q$

Build the physical matrices:

$$M := \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \qquad K := \begin{pmatrix} K_2 & -2K_2 \\ -2 \cdot K_2 & 4 \cdot K_2 \end{pmatrix}$$

Disregard DAMPING

Make modal matrix using eigenvectors

$$A := \text{augment}\left(\phi_1, \phi_2\right) \qquad A = \begin{pmatrix} 1 & 1 \\ 0.5 & -0.4 \end{pmatrix} \qquad (14)$$

check orthogonality property of natural modes

$$M_M := A^T \cdot M \cdot A \qquad K_M := A^T \cdot K \cdot A$$

$$M_M = \begin{pmatrix} 69.932 & -5.279 \times 10^{-15} \\ -3.553 \times 10^{-15} & 55.946 \end{pmatrix} \frac{lb \, sec^2}{ft} \qquad K_M = \begin{pmatrix} 0 & 0 \\ 0 & 3.888 \times 10^{6} \end{pmatrix} \frac{lb}{ft}$$

non-diagonal elements are very small= non zero b/c of roundoff with computer define modal masses and stiffnesses:

$M_{m_1} := M_{M_{1,1}}$ $M_{m_2} := M_{M_{2,2}}$

$K_{m_1} := K_{M_{1,1}}$ $K_{m_2} := K_{M_{2,2}}$ $\omega = \begin{pmatrix} 0 \\ 263.621 \end{pmatrix} \frac{rad}{sec}$

check $\left(\frac{K_{m_1}}{M_{m_1}}\right)^{0.5} = 0 \frac{rad}{sec}$ $\left(\frac{K_{m_2}}{M_{m_2}}\right)^{0.5} = 263.621 \frac{rad}{sec}$ OK

define initial conditions: displacements and velocities in modal coordinates

At time t=0s, the system is at its static equilibrium position, hence the initial conditions are null displacements and null velocities. Of course, the same applies to modal space, i.e. null initial displacements and velocities

for generality, define: $X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot ft$ Z X $V_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \frac{ft}{sec}$

Calculate inverse of A matrix $A_{inv} := A^{-1}$ $A = \begin{pmatrix} 1 & 1 \\ 0.5 & -0.4 \end{pmatrix}$

and in modal coordinates $q_o := A_{inv} \cdot X_o$ $q_{o\_dot} := A_{inv} \cdot V_o$

$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ft$ $q_{o\_dot} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \frac{ft}{sec}$ (15) as expected - Do not perform this step unless Initial conditions are different from null

Define modal force $F := \begin{pmatrix} W_1 \cdot \sin(\theta) + \frac{W_2}{2} \\ 0 \end{pmatrix}$ Physical force vector $F = \begin{pmatrix} 2.842 \times 10^3 \\ 0 \end{pmatrix} lb$

$Q := A^T \cdot F$

$Q = \begin{pmatrix} 2.842 \times 10^3 \\ 2.842 \times 10^3 \end{pmatrix} lb$ Both natural modes will be excited (16)

Find modal responses

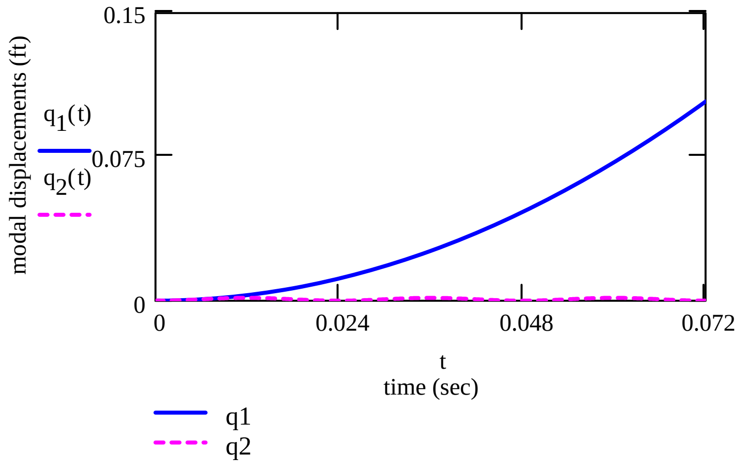
first mode: **Rigid Body Mode** $M_{m_1}\left(\frac{d^2}{dt^2} q_1\right) = Q_1$ since $K_{m_1} = 0 \frac{lb}{ft}$ (17a)

second mode: **vibratory mode** $M_{m_2}\left(\frac{d^2}{dt^2} q_2\right) + K_{m_2} \cdot q_2 = Q_2$ (17b)

and since the initial conditions are null - using cheat sheet:

$q_1(t) := \frac{Q_1}{M_{m_1}} \cdot \frac{t^2}{2}$ and $q_2(t) := \frac{Q_2}{K_{m_2}} \cdot (1 - \cos(\omega_2 \cdot t))$ 18)

GRAPH modal responses $T_{large} := \frac{2 \cdot \pi}{\omega_2} \cdot 3$ arbitrary scale for plot



$T_{large} = 0.072\ \text{sec}$

Modal response 1 shows quadratic increase in time - RIGID BODY MODE

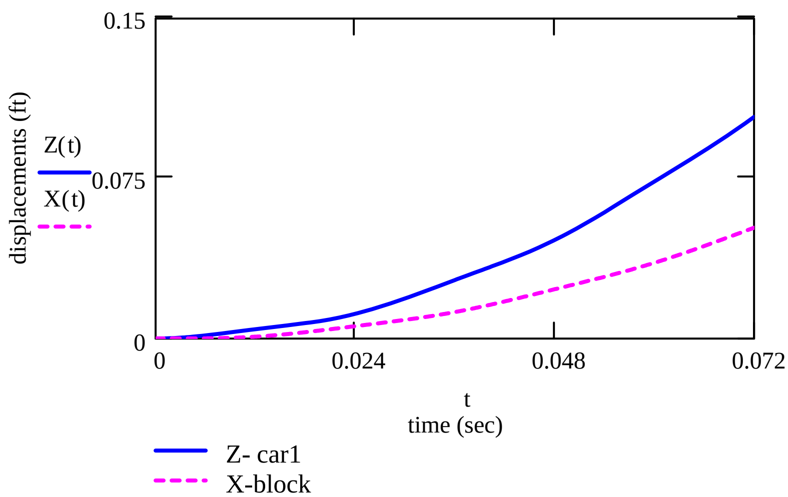
Modal response 2 is too small to be seen (will discuss later)

The response in physical coordinates, Z and X, equals (from transformation x=Aq)

$$A = \begin{pmatrix} 1 & 1 \\ 0.5 & -0.4 \end{pmatrix}$$

$$Z(t) := q_1(t) + q_2(t)$$

$$X(t) := 0.5 \cdot q_1(t) - 0.4 \cdot q_2(t) \qquad (19)$$



$Z(T_{large}) = 0.104\ \text{ft}$

$X(T_{large}) = 0.052\ \text{ft}$

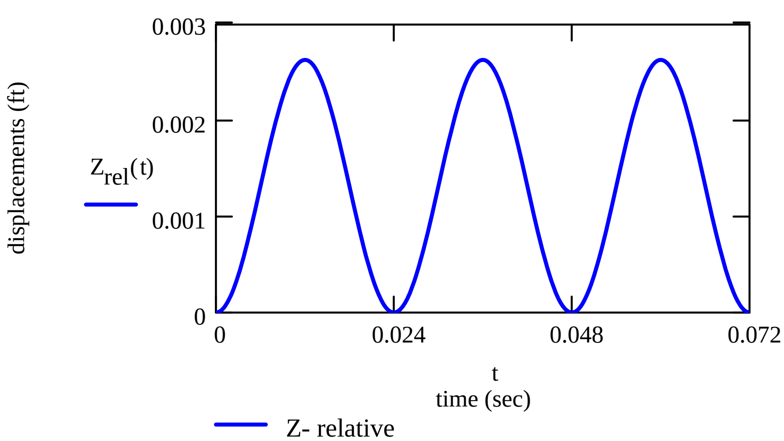
Note that for t>>0, Z=Y= 2X

Motion shows car (Z) and block (X) falling by the pull of gravity, i.e. their weights

One question remains: Where is the second mode? i.e oscillatory behavior with period equal to $T_2 := \dfrac{(2 \cdot \pi)}{\omega_2}$

$T_2 = 0.024\ \text{sec}$

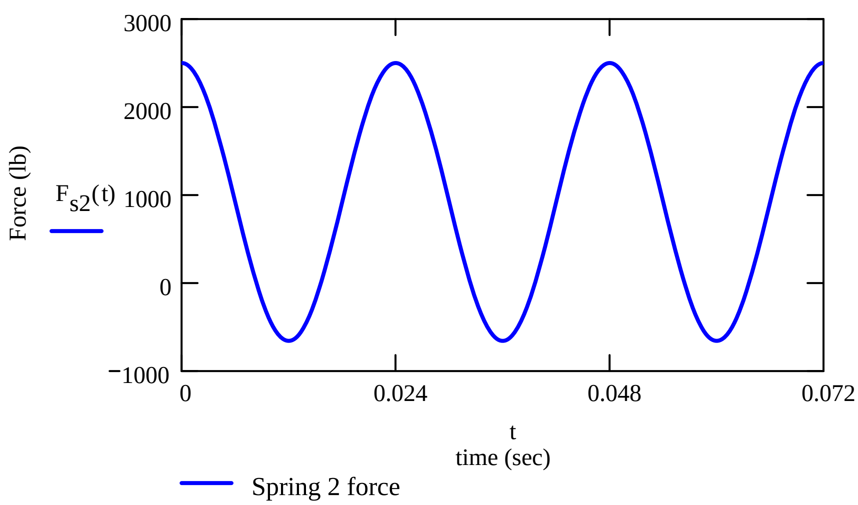
Let's graph the relative motion Z-Y $\quad Z_{rel}(t) := Z(t) - 2 \cdot X(t)$



which clearly shows the oscillatory response of the second mode.

So why is this important? Recall that the connecting spring force is, eqn. (3)

$$F_{s2}(t) := \left[K_2 \cdot (2 \cdot X(t) - Z(t)) + \frac{W_2}{2}\right] \qquad t>0$$



$$\frac{W_2}{2} = 2.5 \times 10^3 \text{ lb}$$

thus, although the dynamic deflections in spring2 maybe small, the spring force is rather large!!!

Spring2 force varies from $F_{s2}(0 \cdot sec) = 2.5 \times 10^3 \text{ lb}$ to $F_{s2}\left(\frac{T_2}{2}\right) = -657.8 \text{ lb}$