In the figure, two cables of stiffness $K$ connect a wheel of mass $M_c$ to ground. The wheel with radius $r_c$, has mass moment of inertia $I_C$. The pendulum, attached to the wheel center, has mass $M_B$ and mass moment of inertia, $I_B$, with $L_B$ as the distance $OG$ from the pin connection to the pendulum $CG$. The two cables of stiffness $K$ are initially stretched with force $F_{asy}$. At time $t=0$ s, the pendulum is displaced angle $\Theta_0$ and released, motion of the whole system follows. Assume there is no friction between the wheels and the horizontal surface.

a) Define DOFs, establish kinematic constraints and select independent coordinates for the motion of the wheel and pendulum.

b) draw FBDs and use Newton’s Laws (Forces & Moments) to derive the system EOMs of the wheel and the pendulum for $t>0$ s. OR

b) Write system energies and using Lagrangian derive the system EOMs.
Forces:
\( F_{asy} \): assembly force for springs (extension or stretched)

Parameters:
\( M_B, M_C \): masses pendulum & wheel
\( K \): stiffness coefficients
\( I_B = \) pendulum mass moment of inertia
\( I_C = \) wheel mass moment of inertia
\( r_c = \) radius of wheel
\( L_B = \) distance O-G

Coordinates (Variables): 5 DOF
\( X_C \): translation of wheel
\( \alpha \): rotation of wheel
\( X_B, Y_B \): translation of pendulum cg
\( \theta \): rotation of pendulum,

(Absolute frames of reference, with origin at SEP)

DEFINITIONS:
SEP: \( X_C = X_B = 0 \)

\( X_B = X_C + L_B \sin \theta \)
\( Y_B = -L_B \cos \theta \)

\( X_C = r_c \alpha \)

Rolling w/o slipping

Kinematic constraints:
\( X_B = X_C + L_B \sin \theta \)
\( Y_B = -L_B \cos \theta \)
**Forces:**
- $F_L, F_L$: forces from springs
- $Ox, Oy$: connection pin forces
- $W = Mg$: weight, $N$: normal force
- $F_C$: contact force

**Parameters:**
- $M_B, M_C$: masses pendulum & wheel
- $K$: stiffness coefficients
- $I_B = \text{pendulum mass moment of inertia}$
- $I_C = \text{wheel mass moment of inertia}$
- $r_c = \text{radius of wheel}$, $L_B = \text{distance O-G}$

**Coordinates (Variables):** 5 DOF
- $X_C$: translation of wheel
- $\alpha$: rotation of wheel
- $X_B, Y_B$: translation of pendulum cg
- $\theta$: rotation of pendulum

**5 DOF – 3 Eqns. Of constraint = 2 independent DOFS**

\[
\begin{align*}
X_B &= X_C + L_B \sin \theta \\
Y_B &= -L_B \cos \theta \\
X_C &= r_c \alpha 
\end{align*}
\]

\[X_B = \dot{X}_C + L_B \dot{\theta} \cos \theta \tag{2}\]

\[Y_B = L_B \dot{\theta} \sin \theta \tag{2}\]

\[
\begin{align*}
\ddot{X}_B &= \ddot{X}_C + L_B \ddot{\theta} \cos \theta - L_B \dot{\theta}^2 \sin \theta \\
\ddot{Y}_B &= L_B \ddot{\theta} \sin \theta + L_B \dot{\theta}^2 \cos \theta \tag{3}
\end{align*}
\]
DEFINITIONS:

Forces:
- $F_L, F_R$: forces from springs
- $O_x, O_y$: connection pin forces
- $W=Mg$: weight, $N$: normal force
- $F_C$: contact force

Parameters:
- $M_B, M_C$: masses pendulum & wheel
- $K$: stiffness coefficients
- $I_B$: pendulum mass moment of inertia
- $I_C$: wheel mass moment of inertia
- $r_c$: radius of wheel, $L_B$: distance O-G

Coordinates (Variables): 5 DOF
- $X_C$: translation of wheel
- $\alpha$: rotation of wheel
- $X_B$: translation of pendulum cg
- $\theta$: rotation of pendulum

Select $X_C$ and $\theta$ as independent variables
Equations of importance are (4a) and (5c): translation of wheel and rotation of pendulum.

In Eqn (4a): substitute spring forces (2), contact force from (4c), and pin force Ox from eqn (5a)

Derive Equations of Motion:

Equations of importance are (4a) and (5c): translation of wheel and rotation of pendulum.

In Eqn (4a): substitute spring forces (2),

contact force from (4c),

and pin force Ox from eqn (5a)

This is the first EOM describing translation of wheel, rolling w/o slipping, and connected to a swinging pendulum
In eqn (5c) for rotation of pendulum:

substitute pin reaction forces $O_x$, $O_y$ from eqns (5a,b)

Derive Equations of Motion:

In eqn (5c) for rotation of pendulum:

$$I_B \ddot{\theta} = O_x L_B \cos \theta + O_y L_B \sin \theta$$

$$= \left( -M_B \left( \dot{X}_C + L_B \dot{\theta} \cos \theta - L_B \dot{\theta}^2 \sin \theta \right) \right) L_B \cos \theta$$

$$+ \left( -M_B \left( L_B \dot{\theta} \sin \theta + L_B \dot{\theta}^2 \cos \theta \right) - W_B \right) L_B \sin \theta$$

$$= \left( I_B + M_B L_B^2 \right) \ddot{\theta} + M_B L_B \cos \theta \dot{X}_C + W_B L_B \sin \theta =$$

$$= \left( M_B L_B \dot{\theta}^2 \sin \dot{\theta} \right) L_B \cos \theta + \left( -M_B L_B \dot{\theta}^2 \cos \theta \right) L_B \sin \theta$$

Cancel equal terms

$$\left( I_B + M_B L_B^2 \right) \ddot{\theta} + M_B L_B \cos \theta \dot{X}_C + W_B L_B \sin \theta = 0$$

This is the 2nd EOM describing rotation of pendulum, connected to a wheel, rolling w/o slipping
**ENERGIES for system components**

**Kinetic energy**

\[ T = T_{\text{wheel translation}} + T_{\text{wheel rotation}} + T_{\text{pend - translation}} + T_{\text{pend rotation}} \]

\[
T = \frac{1}{2} M_C \dot{X}_C^2 + \frac{1}{2} I_C \ddot{\alpha}^2 + \frac{1}{2} M_B \left( \dot{X}_B^2 + \dot{Y}_B^2 \right) + \frac{1}{2} I_B \dot{\theta}^2 = \\
= \frac{1}{2} \left( M_C + \frac{I_C}{r_C^2} \right) \dot{X}_C^2 + \frac{1}{2} M_B \left( 2 \dot{X}_C \dot{X}_B \dot{\theta} \cos \theta + L_B \dot{\theta} \right) + \frac{1}{2} \left( I_B + M_B L_B^2 \right) \dot{\theta}^2
\]  

(7)

**Potential energy**

\[
V = V_{\text{strain energy in cables}} + V_{\text{gravitational from pendulum}}
\]

\[
V = \frac{1}{2} K \left( \delta - X_C \right)^2 + \frac{1}{2} K \left( \delta + X_C \right)^2 + W_B L_B \left( 1 - \cos \theta \right)
\]  

(8)

Spring on right, spring on left

\[ \delta \text{ is static deflection (stretching from assembly)} = \frac{F_{\text{asy}}}{K} \]

**Viscous dissipated power**

\[ \delta \varphi_v = 0 \]  

(9)

**External work**

\[ Q = 0 \]  

(10)

No external forces applied.

Contact force (rolling w/o slipping) does not perform work

Wheel:

\[ M_C, I_C, r_C \]

Spring on right, spring on left

\[ \delta = \frac{F_{\text{asy}}}{K} \]

Rolling w/o slipping

\[ X_B = X_C + L_B \sin \theta \]

\[ Y_B = -L_B \cos \theta \]

\[ X_C = r_C \alpha \]
Derive Equations of Motion:

**STEP 2: Lagrange’s equations**

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \frac{1}{2} \frac{\partial^2 V}{\partial \dot{q}_k^2} = Q_k \quad k=1,2,...n
\]  

(11)

**STEP 3: Derivatives of potential energies & kinetic energies**

\[
V = \frac{1}{2} K(\delta - X_C)^2 + \frac{1}{2} K(\delta + X_C)^2 + W_B L_B (1 - \cos \theta)
\]

(12)

\[
\frac{\partial V}{\partial X_C} = K(\delta - X_C)(-1) + K(\delta + X_C) = K(-\delta + X_C + \delta + X_C) = 2K X_C;
\]

\[
\frac{\partial V}{\partial \theta} = W_B L_B \sin \theta
\]

\[
T = \frac{1}{2} \left( M_C + \frac{I_C}{r_C^2} + M_B \right) \dot{X}_C^2 + \frac{1}{2} M_B \left( 2 \dot{X}_C L_B \theta \cos \theta + \frac{1}{2} \left( I_B + M_B L_B^2 \right) \dot{\theta}^2 \right)
\]

\[
\left( \frac{\partial T}{\partial \dot{X}_C} \right) = \left( M_C + \frac{I_C}{r_C^2} + M_B \right) \ddot{X}_C + M_B \left( L_B \dot{\theta} \cos \theta \right);
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{X}_C} \right) = \left( M_C + \frac{I_C}{r_C^2} + M_B \right) \dot{X}_C + M_B L_B \left( \dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta \right)
\]

(13)

\[
\left( \frac{\partial T}{\partial \dot{\theta}} \right) = M_B L_B \left( \dot{X}_C \cos \theta \right) + \left( I_B + M_B L_B^2 \right) \dot{\theta};
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = \left( I_B + M_B L_B^2 \right) \ddot{\theta} + M_B L_B \left( \ddot{X}_C \cos \theta - \dot{X}_C \dot{\theta} \sin \theta \right)
\]
Derive Equations of Motion:

\[
V = \frac{1}{2} K (\delta - X_C)^2 + \frac{1}{2} K (\delta + X_C)^2 + W_B L_B (1 - \cos \theta)
\]

From:

\[
T = \frac{1}{2} \left( M_C + \frac{I_C}{r_C^2} + M_B \right) \ddot{X}_C + \frac{1}{2} M_B \left( 2 \dot{X}_C L_B \dot{\theta} \cos \theta \right) + \frac{1}{2} \left( I_B + M_B L_B^2 \right) \dot{\theta}^2
\]

**Equation for translation of wheel**

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \ddot{X}_C} \right) - \frac{\partial T}{\partial X_C} + \frac{\partial V}{\partial X_C} + 0 = 0
\]

\[
\left( M_C + \frac{I_C}{r_C^2} + M_B \right) \dddot{X}_C + M_B \left( L_B \ddot{\theta} \cos \theta - L_B \dot{\theta}^2 \sin \theta \right) + 2 K X_C = 0
\]  \(\text{(14)}=\text{(6)}\)

**Equation for rotation of pendulum**

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} + 0 = 0
\]

\[
\left( I_B + M_B L_B^2 \right) \ddot{\theta} + M_B L_B \left( \dddot{X}_C \cos \theta - \dot{X}_C \dot{\theta} \sin \theta \right) + M_B L_B \left( \dot{X}_C \dot{\theta} \sin \theta \right) + W_B L_B \sin \theta = 0
\]

\[
\left( I_B + M_B L_B^2 \right) \dddot{\theta} + M_B L_B \cos \theta \dddot{X}_C + W_B L_B \sin \theta = 0
\]  \(\text{(15)}=\text{(7)}\)

Equations of motion (14) and (15) = equations (6) and (7)