MEEN 617 - Handout 4a

ELEMENTS OF ANALYTICAL MECHANICS

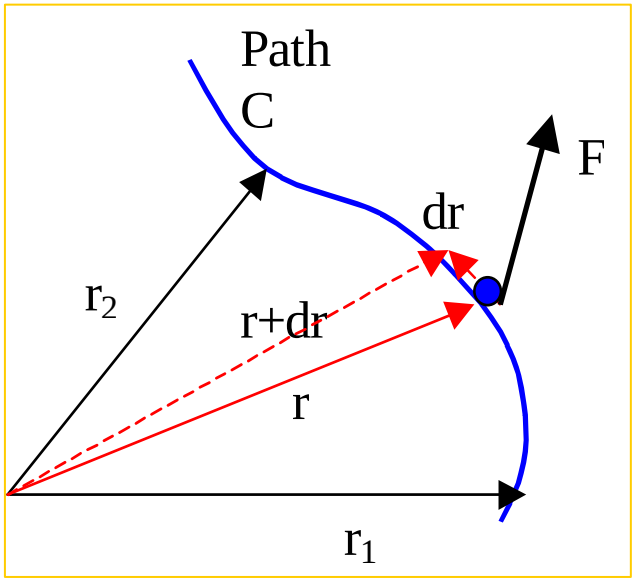
Newton's laws (Euler's fundamental principles of motion) are formulated for a single particle and easily extended to systems of particles and rigid bodies. In describing the motions, physical coordinates and forces are employed in terms of vectorial quantities (**vectorial mechanics**). The major drawback is that Newton's Laws consider the individual components of a system separately, thus requiring the calculation of **interacting forces** resulting from **kinematical constraints**. The calculation of these forces is many times of no consequence or interest in the final formulation of the equations of motion.

A different approach known as **ANALYTICAL MECHANICS** considers the system as a whole and it is more general than the simple Newtonian formulation. The motion of a system is formulated in terms of two scalar quantities, **WORK and KINETIC ENERGY**. The mathematical formulation is independent of any special system of coordinates and relies on **the principle of virtual displacements.**

When dealing with multi-degree of freedom systems (MDOF), it is often more expedient to derive the equations of motion by using the analytical mechanics approach. The method is also valid for **continuous systems**. In this case

not only the equations of motion are obtained but also the associated (natural) boundary conditions.

WORK AND ENERGY FOR A SINGLE PARTICLE



Consider a particle (point mass) moving along the curve C under the action of a force $\vec{F}$. The position of the particle at any time is given by the position vector $\vec{r}$.

If the particle moves over an element of distance $d\vec{r}$, the work (dW) is the scalar product

$$dW = \vec{F} \cdot d\vec{r} \qquad (1)$$

If the particle moves from position 1 to 2, along the path *C*, the work performed is

$$W_{12} = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \qquad (2)$$

For a particle of constant mass, **Newton's second law** establishes

$$\vec{F} = \frac{d(m\dot{\vec{r}})}{dt} = m\frac{d\ddot{\vec{r}}}{dt} \qquad (3)$$

Since $d\vec{r} = \dot{\vec{r}}dt$, with $\dot{\vec{r}}$ as the velocity of the particle, then Eq. (2) is rewritten as

$$W_{12} = \int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} m\left(d\dot{\vec{r}}/dt\right)\cdot\dot{\vec{r}}\,dt = \frac{1}{2}\int_{t_1}^{t_2} m\,d\left(\frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{dt}\right)dt$$

$$W_{12} = \frac{1}{2}m\,\dot{r}_2^2 - \frac{1}{2}m\,\dot{r}_1^2 = T_2 - T_1 \qquad (4)$$

where T is the kinetic energy, $T = \frac{1}{2}m\dot{r}^2$ (5)

Eq. (4) shows that W_{12} is the work to change the kinetic energy of the system from T_1 to T_2.

In many physical systems, the force field depends on the position alone and is independent on the path followed, i.e., the force field is a **conservative field**, and therefore, it can be derived from a potential function V. To this end, introduce the definition,

$$dW_c = \vec{F}\cdot d\vec{r} = -dV(\vec{r}) = -\vec{\nabla}V\cdot d\vec{r} \qquad (6)$$

where $\vec{\nabla}$ is the gradient vector operator. In the Cartesian coordinate system,

$$\vec{\nabla} = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/d\,y + \vec{k}\,\partial/\partial z \qquad (7)$$

From (6), it is easily inferred that

$$-W_{12c} = -\int_{r_1}^{r_2}\vec{F}\cdot d\vec{r} = \int_{r_1}^{r_2}\vec{\nabla}V\cdot d\vec{r} = \left[V(\vec{r}_2) - V(\vec{r}_1)\right] = V_2 - V_1 \qquad (8)$$

i.e., the **work performed by conservative forces is equal to the change in the potential energy function *V***. Note that Eq. (8) makes evident the conservative nature of the field, i.e., the change in potential depends only on the beginning and ending positions and not on the path followed.

In general, there are both conservative and non-conservative forces acting upon a particle. The **non-conservative** forces are **energy dissipating** forces such as friction forces (drag type), or forces imparting (inputting) energy into the system, i.e., external forces. **Non-conservative forces usually do not depend just on position alone and can not be derived from a potential function.**

Thus the work can be divided into conservative and non-conservative parts,

$$W_{12} = W_{12c} + W_{12nc} = -(V_2 - V_1) + W_{12nc} \qquad (9)$$

And from Eq. (4), $W_{12} = T_2 - T_1$, then

$$\begin{aligned} W_{12nc} &= (T_2 - T_1) + (V_2 - V_1) \\ &= (T_2 + V_2) - (T_1 + V_1) \end{aligned}$$

or

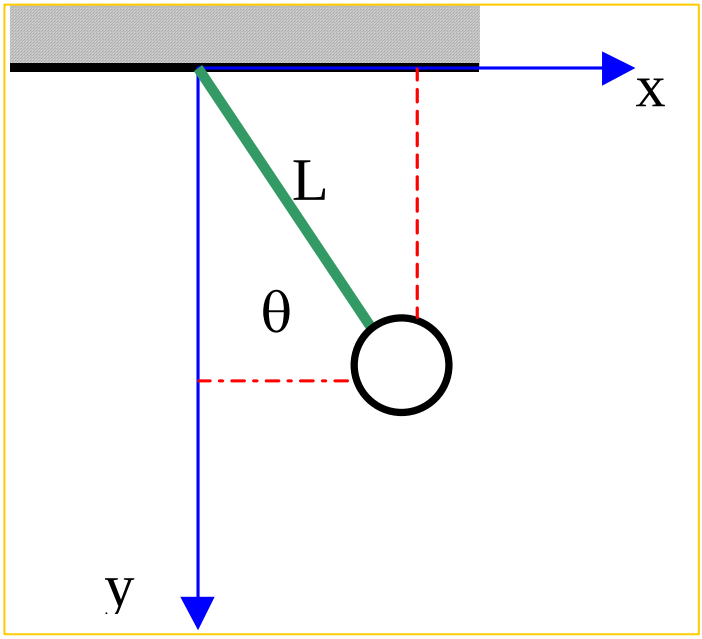
$$W_{12nc} = E_2 - E_1 \qquad (10)$$

where $E = T + V$ is the **mechanical energy** of the particle, equal to the addition of its kinetic energy plus potential energy. Eq. (10) indicates that the work performed by non-conservative forces is responsible for the change in the mechanical energy of the particle.

For a **purely conservative force field and in the absence of external forces,** $W_{12nc} = 0$. Then it follows that the mechanical energy is constant (invariant) for all times.

$$W_{12nc} = 0 \rightarrow E_2 = E_1 \quad \forall t, \quad \Rightarrow \frac{d}{dt}(T + V) = 0$$

Degrees of freedom for systems with constraints



Consider a simple pendulum consisting of a mass *m* suspended by a non-extensible string of length *L* and free to oscillate in the *xy* plane.

The coordinates *x* and *y* define the position of the mass. Note that these coordinates are not independent since

$$x^2 + y^2 = L^2$$

which denotes a constraint relationship. Thus only one coordinate is needed to express the position of *m*. It can be either *x* or *y*. However, it is **more natural** to use the angle θ, since

$$x = L\cos\theta; \quad y = L\sin\theta$$

The minimum number of independent coordinates needed to describe the motion of a system is called the **degree of freedom of the system**. Thus, the pendulum is a single degree of freedom (SDOF) system.

Three (3) coordinates describe the position of a particle free to move in three dimensions. If a system of N particles must satisfy c constraint equations, then the number of independent coordinates to describe the system is,

$$n = 3N - c$$

because each constraint equation reduces the degree of freedom of the system by one. The system is categorized as an n-degree of freedom system (nDOF).

The constraint equation for the pendulum is reworded as

$$f(x,y,t) = f(x,y) = c$$

A system in which the constraints are functions of the coordinates or coordinates and time (but not velocities) is called **holonomic**.

GENERALIZED COORDINATES

The minimum number of coordinates necessary to fully describe a system is a set of generalized coordinates. These are independent and each coordinate represents one of the degrees of freedom of the system.

Generalized coordinates, usually denoted as $\{q_1, q_2, \ldots q_n\}$, are not necessarily Cartesian coordinates.

The selection of the set of generalized coordinates $\{q_k\}_{k=1,..,n}$ is evident in some problems.

In other cases where the coordinates are related by constraint equations, coordinate transformations are required to arrive to an independent set of generalized coordinates.

THE PRINCIPLE OF VIRTUAL WORK: STATICS CASE

The principle of virtual work is essentially a statement of an equilibrium state of a mechanical system. Several definitions are needed at the outset:

A **displacement coordinate** is a quantity used to specify the change of configuration in a system.

A **constraint** is a kinematic (usually geometrical) restriction on the **physically possible** configuration the system may assume.

A **virtual displacement** is an infinitesimally small and arbitrary change of configuration of a system **CONSISTENT OR COMPATIBLE** with its constraints. Virtual displacements are not actual displacements since there is no time change associated with them.

If the actual coordinates for a system are related by the constraint equation,

$$f\left(x_1,.....x_n,\ y_1,.....y_n,\ z_1,.....z_n,\ t\right)=c \qquad (11)$$

then, the virtual displacement or variation δ must be such that

$$f\left(x_1+\delta x_1,....., y_1+\delta y_1.....,z_1+\delta z_1,.....,z_n+\delta z_n,t\right)=c \qquad (12)$$

Note that **time is held constant** in Eq. (12).

The operations concerning the **variation** δ follow the rules of **elementary calculus**. Expanding Eq. (12) as a Taylor series and keeping first-order terms (neglecting higher order terms) in $\delta x_1 \ldots\ldots \delta z_n$ leads to

$$f(x_1,.....x_n,\ y_1,.....y_n,\ z_1,.....z_n,\ t)+ \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\delta x_i+\frac{\partial f}{\partial y_i}\delta y_i+\frac{\partial f}{\partial z_i}\delta z_i\right)=c \qquad (13)$$

However, from Eq. (11), $f = c$, and thus

$$\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\delta x_i+\frac{\partial f}{\partial y_i}\delta y_i+\frac{\partial f}{\partial z_i}\delta z_i\right)=0 \qquad (14)$$

This is the relation the **virtual displacements** $(\delta x_i,\ \delta y_i,\ \delta z_i)_{i=1,..n}$ must satisfy to be compatible with the system constraint $f=c$

For the simple pendulum, $f = x^2 + y^2 = L^2$, or

$x = L\cos(\theta)$ and $y = L\sin(\theta)$;

then a **variation** in configuration leads to

$$y + \delta y = L\sin(\theta + \delta\theta) = L(\sin\theta\ \cos\delta\theta + \sin\delta\theta\ \cos\theta)$$
$$= L\sin\theta + L\cos\theta\ \delta\theta$$

since $\cos(\delta\theta) \approx 1$ and $\sin(\delta\theta) \approx \delta\theta$ because $\delta\theta \sim 0$

Thus, for small $\delta\theta$, $\quad y + \delta y = y + x\ \delta\theta$

and $\quad \delta y = x\ \delta\theta = L\cos\theta\ \delta\theta$

similarly, $\quad \delta x = -y\ \delta\theta = -L\sin\theta\ \delta\theta$

Consider a particle (*i*) acted upon by some forces with resultant vector $\vec{R}_i$. If the system is in **STATIC EQUILIBRIUM**, the resultant force is zero and therefore, the work performed over the virtual displacement $\delta\vec{r}_i$ must also be zero, i.e.,

$$\delta W_i = \vec{R}_i \cdot \delta\vec{r}_i = 0 \qquad (15)$$

If there are constraints in the system, then

$$\vec{R}_i = \vec{F}_i + \vec{f}_i \qquad (16)$$

where $\vec{F}_i$ is the resultant vector of external forces applied on the particle and $\vec{f}_i$ is the resultant of the constraint forces. Hence equation (15) becomes

$$\delta W_i = \vec{R}_i \cdot \delta\vec{r}_i = \vec{F}_i \cdot \delta\vec{r}_i + \vec{f}_i \cdot \delta\vec{r}_i = 0 \qquad (17)$$

However, **constraint forces do not perform work since (by definition) the displacements do not have any components in the direction of the constraint forces ($\vec{f}_i \cdot \delta\vec{r}_i = 0$).** As an example, consider a particle moving on a smooth surface. The constraint force is normal to the surface and the displacements are parallel to the surface. Another example is rolling without sleeping.

Hence, it follows that,

$$\delta W_i = \vec{F}_i \cdot \delta\vec{r}_i = F_{x_i}\delta x_i + F_{y_i}\delta y_i + F_{z_i}\delta z_i = 0 \qquad (18)$$

In general, for a system of *N* particles, the sum of the virtual works over all particles must be zero, or

$$\delta W = \sum_i \delta W_i = \sum_i \left(\vec{F}_i \cdot \delta\vec{r}_i\right) = 0 \qquad i=1,2....N \qquad (19)$$

This expression also includes the cancellation of the virtual work done by internal forces on rigid bodies (action and reaction principle).

Equation (19) is the expression of the **PRINCIPLE OF VIRTUAL WORK**, and stated as:

If a system of forces is in equilibrium, the work done by the externally applied forces through virtual displacements compatible with the constraints of the system is zero.

D'ALEMBERT'S PRINCIPLE: DYNAMICS CASE

The principle of virtual work can also be extended to the state of dynamics (motion), i.e. dynamic equilibrium.

If there are some unbalanced forces acting upon a particle m_i, then according to Newton's 2nd law, the force resultant vector must be equal to the rate of change of the linear momentum, i.e,

$$\vec{F}_i + \vec{f}_i = \dot{\vec{p}}_i = \frac{d}{dt}\left(m\dot{\vec{r}}_i\right) \qquad (20)$$

Think of an "**inertia force" (reversed acceleration)** as a force whose magnitude equals the time rate of change of the momentum vector $\vec{p}$, is collinear with it, but acts in the opposite direction. If such a force is applied to the particle, then one can express the dynamic equilibrium condition as

$$\vec{F}_i + \vec{f}_i - \dot{\vec{p}}_i = 0 \qquad (21)$$

D'Alembert's principle states that the resultant force is in equilibrium with the inertia force. Following prior reasoning, the **virtual work** of the external and inertia forces must also be nil, i.e.

$$\left(\vec{F}_i - \dot{\vec{p}}_i\right)\cdot\delta\vec{r}_i = 0 \qquad (22)$$

and for a system of N particles,

$$\sum_{i=1}^{N}\left(\vec{F}_i - \dot{\vec{p}}_i\right)\cdot\delta\vec{r} = \delta W = 0$$

$$\sum_i \left(F_{x_i} - m_i\ddot{x}_i\right)\delta x_i + \sum_i \left(F_{y_i} - m_i\,\ddot{y}_i\right)\delta y_i + \sum_i \left(F_{z_i} - m_i\,\ddot{z}_i\right)\delta z_i = 0$$

(23)

Thus, the principle of virtual work for any system of particles is expressed as

$$\delta W = \delta W_{\text{external forces}} + \delta W_{\text{inertia forces}} = 0 \qquad (24)$$

HAMILTON'S PRINCIPLE

Hamilton's principle is (perhaps) the most advanced **variational principle** of mechanics. The principle considers the **motion of a whole system between two instants of time, t_1 and t_2,** and is therefore an integral principle. One remarkable advantage of this formulation is that it is **invariant to the coordinate system used** (Principle of material frame indifference)[1].

Consider a system of *N* particles of constant mass. The system may be subject to kinematical (constraint) conditions.

The virtual work expression in conjunction with D'Alembert's principle establishes

[1] The stuff from which great theories are made of.

$$-\delta W_{total} = \sum_{i=1}^{N} \left(m_i \ddot{\vec{r}}_i - \vec{F}_i \right) \cdot \delta \vec{r}_i = 0 \qquad (25)$$

Let $$\delta W = \sum_{i=1}^{N} \left(\vec{F}_i \cdot \delta \vec{r}_i \right) \qquad (26)$$

be the **virtual work done by the external forces** on the system.

The operations *d/dt* and δ are interchangeable (linear operators). Then

$$\frac{d\left(\dot{\vec{r}}_i \cdot \delta \vec{r}_i\right)}{dt} = \ddot{\vec{r}}_i \, \delta \vec{r}_i + \dot{\vec{r}}_i \, \frac{d\,\delta \vec{r}}{dt} = \ddot{\vec{r}}_i \, \delta \vec{r}_i + \dot{\vec{r}}_i \, \delta\left(\frac{d\,\vec{r}}{dt}\right) =$$

$$= \ddot{\vec{r}}_i \, \delta \vec{r}_i + \dot{\vec{r}}_i \, \delta\left(\dot{\vec{r}}_i\right) = \ddot{\vec{r}}_i \, \delta \vec{r}_i + \frac{1}{2} \delta\left(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right)$$

Hence, $$\ddot{\vec{r}}_i \, \delta \vec{r}_i = \frac{d\left(\dot{\vec{r}}_i \cdot \delta \vec{r}_i\right)}{dt} - \frac{1}{2} \delta\left(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right) \qquad (27)$$

Multiply Eq. (27) by m_i and sum over the whole set of particles to obtain

$$\sum_{i=1}^{N}\left(m_i\,\ddot{\vec{r}}_i\cdot\delta\vec{r}_i\right)=\sum_{i=1}^{N}\left(m_i\,\frac{d\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)}{dt}\right)-\frac{1}{2}\sum_{i=1}^{N}\left(m_i\,\delta\left(\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i\right)\right)$$

$$=\sum_{i=1}^{N}\left(m_i\,\frac{d\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)}{dt}\right)-\delta\left(\sum_{i}^{N}\frac{1}{2}m_i\,\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i\right) \qquad (28)$$

$$=\sum_{i=1}^{N}\left(m_i\,\frac{d\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)}{dt}\right)-\delta T$$

where T is the kinetic energy of the system (whole set of particles), i.e.

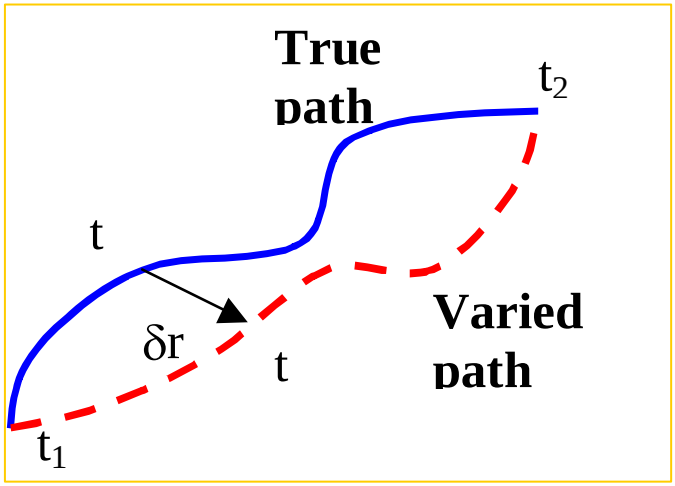
$$T=\sum_{i}^{N}T_i=\sum_{i}^{N}\frac{1}{2}m_i\left(\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i\right)$$

Inserting Eqs. (26) and (28) into Eq. (25) renders,

$$\delta T+\delta W=\sum_{i=1}^{N}\left[m_i\,\frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)\right] \qquad (29)$$

The instantaneous configuration of a system is given by the values **of *n* generalized coordinates**. These values correspond to a point in *n*-dimensional space known as the **configuration space**.

The system changes with time tracing a path known as "true" in the configuration space. A slightly different path, known as the **varied path**, is obtained **if** at any given instant a small variation in position $\delta \vec{r}_i$ is allowed without an associated change in time, i.e. $\delta t = 0$.



The stipulation is made, however, that at two instants t_1 and t_2 the **true** and **varied** paths coincide[2]. That is,

$$\delta \vec{r}_i = 0 \quad \Leftarrow \text{ at } t = t_1 \text{ and at } t = t_2$$

Multiply Eq. (29) by *dt* and integrate it between t_1 and t_2 to obtain

[2] This means that varied path must still satisfy the spatial constraints

$$\int_{t_1}^{t_2}(\delta T+\delta W)\,dt=\int_{t_1}^{t_2}\sum_{i=1}^{N}\left[m_i\frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)\right]dt$$

$$=\sum_{i=1}^{N}\left[\int_{t_1}^{t_2}\left[m_i\frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)\right]dt\right]$$

$$=\sum_{i=1}^{N}\left(m_i\frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\delta\vec{r}_i\right)\right]_{t_1}^{t_2}=0$$

since $\delta\vec{r}_i(t_1)=0=\delta\vec{r}_i(t_2)$

Thus,

$$\int_{t_1}^{t_2}(\delta T+\delta W)\,dt=0 \qquad (31)$$

If the applied forces are divided into conservative and non-conservative, then a **potential energy function** exists such that,

$$\delta W=\delta W_c+\delta W_{nc}=-\delta V+\delta W_{nc} \qquad (32)$$

Introduce the **Lagrangian function**, $L=T-V$ (33)

Then equation (31) for a **conservative system** ($\delta W_{nc}=0$) shows

$$\int_{t_1}^{t_2}(\delta L)\,dt=0 \qquad (34)$$

while for both conservative and non-conservative forces we have

$$\int_{t_1}^{t_2} \left(\delta L + \delta W_{nc} \right) dt = 0 \qquad (35)$$

If the system has only **holonomic constraints**, i.e., geometrical or depending only on the coordinates but not their time derivatives, then one can interchange the δ (variation) and integral in Eq. (34). Thus, for a conservative system,

$$\delta \left(\int_{t_1}^{t_2} L \, dt \right) = 0 \qquad (36)$$

This is the mathematical statement of **HAMILTON'S PRINCIPLE.**

In words it can be explained as follows:

Only the **true path** renders the value of the integral stationary (a minimum) with respect to all possible **neighboring paths** that the system may be imagined to take between two instants of time.

Note that Eq. (35) corresponds to a **principle of least action**. **HAMILTON'S PRINCIPLE IS A FORMULATION AND NOT A SOLUTION OF A DYNAMICS PROBLEM.**

LAGRANGE'S EQUATIONS OF MOTION FOR HOLONOMIC SYSTEMS

In a system of N particles with c **holonomic** constraints, the dependent variables $\vec{r}_i$ in terms of $n=(3)N-c$ generalized coordinates (q_k) and time (t) are expressed as,

$$\vec{r}_i = \vec{r}_i\left(q_1, q_2, \ldots q_n, t\right); \quad i=1,\ldots N, \quad n=3N-c \tag{36}$$

where ***n*** is the **number of independent degrees of freedom (DOF)** of the system. Velocities are obtained by differentiation of Eq (36) as

$$\dot{\vec{r}}_i = \frac{d\,\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t} = \sum_{k=1}^{n}\left(\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right) + \frac{\partial \vec{r}_i}{\partial t} \tag{37}$$

and the system **kinetic energy *T*** is

$$T = \tfrac{1}{2}\sum_{i=1}^{N} m_i\,\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i =$$

$$= \tfrac{1}{2}\sum_{i=1}^{N} m_i\left(\sum_{r=1}^{n}\left(\frac{\partial \vec{r}_i}{\partial q_r}\dot{q}_r\right) + \frac{\partial \vec{r}_i}{\partial t}\right)\left(\sum_{s=1}^{n}\left(\frac{\partial \vec{r}_i}{\partial q_s}\dot{q}_s\right) + \frac{\partial \vec{r}_i}{\partial t}\right) =$$

$$T = \tfrac{1}{2}\sum_{i=1}^{N} m_i\left(\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\partial \vec{r}_i}{\partial q_r}\frac{\partial \vec{r}_i}{\partial q_s}\dot{q}_r\,\dot{q}_s + 2\frac{\partial \vec{r}_i}{\partial t}\sum_{s=1}^{n}\left(\frac{\partial \vec{r}_i}{\partial q_s}\dot{q}_s\right) + \frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial t}\right) \tag{38}$$

Note that $$T = T\left(q_1, q_2, \ldots., q_n, \dot{q}_1, \dot{q}_2, \ldots., \dot{q}_n, t\right) \tag{39}$$

i.e., the system kinetic energy depends on the generalized displacements and velocities as well as in time.

In addition, for conservative force fields, the **potential energy function** V depends only on position and time, i.e.

$$V = V\left(q_1, q_2, \ldots., q_n, t\right) \qquad (40)$$

The **virtual work performed by non-conservative forces** (both dissipative and external) is the product of "**generalized forces**" Q_k acting over n generalized displacements, δq_k. The directions of the generalized forces **coincide** with the directions of the generalized displacements, thus

$$\delta W_{nc} = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots. Q_n\,\delta q_n = \sum_{i=1}^{n}\left(Q_i\,\delta q_i\right) \qquad (41)$$

Note that the generalized force Q_k **may NOT need to actually represent a force or a moment**. **However, the product $Q_k\ \delta q_k$ MUST have physical units of work (= energy).**

Substitute Eqs. (39), (40) and (41) into the generalized Hamilton's principle, Eq. (35)= $\int_{t_1}^{t_2}\left(\delta L + \delta W_{nc}\right)dt = 0$, to obtain

$$\int_{t_1}^{t_2}\left(\delta T - \delta V + \delta W_{nc}\right)dt = 0 \qquad (35)$$

Since,

$$\delta V = \sum_{k=1}^{n}\left(\frac{\partial V}{\partial q_k}\delta q_k\right);\quad \delta T = \sum_{k=1}^{n}\left(\frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial T}{\partial q_k}\delta q_k\right)$$

And recall that $\delta t = 0$, *i.e.* time does not vary while obtaining the virtual changes in energy. Then Eq. (35), $\int_{t_1}^{t_2}(\delta T - \delta V + \delta W_{nc})\,dt = 0$, becomes

$$0 = \int_{t_1}^{t_2}\left\{\sum_{k=1}^{n}\left(\frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial T}{\partial q_k}\delta q_k - \frac{\partial V}{\partial q_k}\delta q_k + Q_k\,\delta q_k\right)\right\}dt \quad (42)$$

The terms involving $\delta \dot{q}_k$ are integrated by parts, and using $\delta \dot{q} = \frac{d}{dt}(\delta q)$, to obtain

$$\int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k\,dt = \int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{q}_k}\frac{d(\delta q_k)}{dt}\,dt = \left.\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\delta q_k\,dt$$

(43)

The first term on the RHS vanishes since (using Hamilton's Principle) the initial and final configuration of the system (at t_1 and t_2, respectively) are known, i.e.,

$$\delta q_k(t_1) = 0,\quad \delta q_k(t_2) = 0$$

Thus, Eq. (42) is rewritten as,

$$0=\int_{t_1}^{t_2}\left\{\sum_{k=1}^{n}\left(-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\delta q_k+\frac{\partial T}{\partial q_k}\delta q_k-\frac{\partial V}{\partial q_k}\delta q_k+Q_k\,\delta q_k\right)\right\}dt$$

or

$$0=\int_{t_1}^{t_2}\left\{\sum_{k=1}^{n}\delta q_k\left(-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)+\frac{\partial T}{\partial q_k}-\frac{\partial V}{\partial q_k}+Q_k\right)\right\}dt$$

(44)

The variations $\{\delta q_k\}_{k=1,\ldots n}$, are independent (corresponding to the *n* degrees of freedom in the system).

Hence, Eq. (44) is **true** (satisfied at all times) only when the bracketed expression vanishes for <u>each</u> degree of freedom, i.e.,

$$\left(-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)+\frac{\partial T}{\partial q_k}-\frac{\partial V}{\partial q_k}+Q_k\right)=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}=Q_k \qquad k=1,2\ldots n \qquad (45)$$

Eqs. (45) are known as **LAGRANGE'S EQUATIONS OF MOTION**. **The solution of these equations is equivalent to the statement that Hamilton's Principle is also satisfied.**

If $V = V(q_k)$ only, define the **Lagrangian function**, $\boldsymbol{L=T-V}$, and since $\frac{\partial V}{\partial \dot{q}_k} = \emptyset$, then Eq. (45) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \qquad k=1,2...n \tag{46}$$

MODIFIED LAGRANGE'S EQUATION FOR SYSTEMS WITH VISCOUS DAMPING

If some of the external non-conservative forces are of viscous type, i.e. proportional to the velocity, then the **viscous dissipated power** ($\wp_v$) is a general function of the velocities, i.e.

$$\wp_v = \wp_v\left(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\right) \tag{47}$$

The *n*-equations of motion using the Lagrangian approach are,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \frac{1}{2}\frac{\partial \wp_v}{\partial \dot{q}_k} = Q_k \qquad k=1,2...n \tag{48}$$

REFERENCES

Meirovitch L., ANALYTICAL METHODS IN VIBRATIONS, pp. 30-50.
Craig R., STRUCTURAL MECHANICS, pp. 25-26, 243-247.
Ginsberg, G., MECHANICAL AND STRUCTURAL VIBRATIONS, Appendix A.

A THOUGHT:

"Those who have meditated on the beauty and utility of the general method of Lagrange - who have felt the power and dignity of that central dynamic problem which he deduced from a combination of the principle of virtual velocities with the principle of D'Alembert - and who have appreciated the simplicity and harmony which he introduced by the idea of the variation of parameters, must feel the unfolding of a central idea.

Lagrange has perhaps done more than any other analyst to give extent and harmony to such deductive researches, by showing the most varied consequences may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem."

W.R. Hamilton