Handout 8

Modal Analysis of MDOF Systems with Proportional Damping

The governing equations of motion for a *n*-DOF linear mechanical system with viscous damping are:

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{D}\dot{\mathbf{U}}+\mathbf{K}\mathbf{U}_{(t)}=\mathbf{F}_{(t)} \qquad (1)$$

where $\mathbf{U}, \dot{\mathbf{U}}, \text{and } \ddot{\mathbf{U}}$ are the vectors of generalized displacement, velocity and acceleration, respectively; and $\mathbf{F}_{(t)}$ is the vector of generalized (external forces) acting on the system. $\mathbf{M}, \mathbf{D}, \mathbf{K}$ represent the matrices of inertia, viscous damping and stiffness coefficients, respectively[1].

The solution of Eq. (1) is uniquely determined once initial conditions are specified. That is,

$$\text{at } t=0 \rightarrow \mathbf{U}_{(0)}=\mathbf{U}_o,\ \dot{\mathbf{U}}_{(0)}=\dot{\mathbf{U}}_o \qquad (2)$$

Consider the case in which the damping matrix $\mathbf{D}$ is of the form

$$\mathbf{D}=\alpha\mathbf{M}+\beta\mathbf{K} \qquad (3)$$

where α, β are constants[2], usually empirical. This type of damping is known as **PROPORTIONAL,** i.e., proportional to either the mass **M** of the system, or the stiffness **K** of the system, or both.

[1] The matrices are square with *n*-rows = *n* columns, while the vectors are n-rows.

[2] These constants have physical units, α is given in [1/s] and β in [s]

Proportional damping is rather unique, since only one or two parameters (at most), α and β, **appear to fully describe the complexity of damping**, irrespective of the system number of DOFs, *n*. This is clearly not realistic. Hence, **proportional damping is not a rule but rather the exception.**

Nonetheless the approximation of proportional damping is useful since, most times, damping is quite an elusive phenomenon, i.e., difficult to model (predict) and hard to measure but for a few DOFs.

Next, consider one already has found the natural frequencies and natural modes (eigenvectors) for the UNDAMPED case, i.e. given $\mathbf{M}\ddot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}=\mathbf{0}$,

$$\left\{\omega_i, \boldsymbol{\varphi}_{(i)}\right\}_{i=1,2...n} \text{ satisfying } \left[-\mathbf{M}\,\omega_i^2+\mathbf{K}\right]\boldsymbol{\varphi}_{(i)}=\mathbf{0},\ _{i=1,...n}. \qquad (4)$$

with properties $\boldsymbol{\Phi}^T\mathbf{M}\,\boldsymbol{\Phi}=\left[M\right]; \quad \boldsymbol{\Phi}^T\mathbf{K}\,\boldsymbol{\Phi}=\left[K\right]$ (5)

As in the undamped modal analysis, consider the **modal transformation**

$$\mathbf{U}_{(t)}=\boldsymbol{\Phi}\,\mathbf{q}_{(t)} \qquad (6)$$

And with $\dot{\mathbf{U}}_{(t)}=\boldsymbol{\Phi}\,\dot{\mathbf{q}}_{(t)}$; $\ddot{\mathbf{U}}_{(t)}=\boldsymbol{\Phi}\,\ddot{\mathbf{q}}_{(t)}$, then EOM (1) becomes:

$$\mathbf{M}\,\boldsymbol{\Phi}\,\ddot{\mathbf{q}}+\mathbf{D}\,\boldsymbol{\Phi}\,\dot{\mathbf{q}}+\mathbf{K}\,\boldsymbol{\Phi}\,\mathbf{q}=\mathbf{F}_{(t)} \qquad (7)$$

which offers no advantage in the analysis. However, premultiply the equation above by $\boldsymbol{\Phi}^T$ to obtain

$$\left(\boldsymbol{\Phi}^T\mathbf{M}\,\boldsymbol{\Phi}\right)\ddot{\mathbf{q}}+\left(\boldsymbol{\Phi}^T\mathbf{D}\,\boldsymbol{\Phi}\right)\dot{\mathbf{q}}+\left(\boldsymbol{\Phi}^T\mathbf{K}\,\boldsymbol{\Phi}\right)\mathbf{q}=\boldsymbol{\Phi}^T\mathbf{F}_{(t)} \qquad (8)$$

And using the modal properties, Eq. (5), and

$$\mathbf{\Phi}^T \mathbf{D} \mathbf{\Phi} = \mathbf{\Phi}^T \left(\alpha \mathbf{M} + \beta \mathbf{K} \right) \mathbf{\Phi} = \alpha \, \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} + \beta \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi}$$

$$\mathbf{\Phi}^T \mathbf{D} \mathbf{\Phi} = \alpha \left[M \right] + \beta \left[K \right] \rightarrow \left[D \right] \tag{9}$$

i.e., $[D]$ is a diagonal matrix known as **proportional modal damping**. Then Eq. (7) becomes

$$\left[M \right] \ddot{\mathbf{q}} + \left[D \right] \dot{\mathbf{q}} + \left[K \right] \mathbf{q} = \mathbf{Q} = \mathbf{\Phi}^T \mathbf{F}_{(t)} \tag{10}$$

Thus, the equations of motion are **uncoupled in modal space**, since $[M]$, $[D]$, and $[K]$ are diagonal matrices. Eq. (10) is just a set of n-uncoupled ODEs. That is,

$$\begin{aligned} &M_1 \ddot{q}_1 + D_1 \dot{q}_1 + K_1 q_1 = Q_1 \\ &M_2 \ddot{q}_2 + D_2 \dot{q}_2 + K_2 q_2 = Q_2 \\ &\ldots\ldots \\ &M_n \ddot{q}_n + D_n \dot{q}_n + K_n q_n = Q_n \end{aligned} \tag{11}$$

Or $$M_j \ddot{q}_j + D_j \dot{q}_j + K_j q_j = Q_j \quad , \quad {}_{j=1,2\ldots n} \tag{12}$$

Where $\omega_{n_j} = \sqrt{K_j / M_j}$ and $D_j = \alpha M_j + \beta K_j$. Modal damping ratios are also easily defined as

$$\zeta_j = \frac{D_j}{2\sqrt{K_j M_j}} = \frac{\alpha M_j + \beta K_j}{2\sqrt{K_j M_j}} \quad ; j=1,2,\ldots.n \tag{13}$$

For **damping proportional to mass only**, $\beta = 0$, and

$$\zeta_j = \frac{\alpha M_j}{2\sqrt{K_j M_j}} = \frac{\alpha}{2\,\omega_{n_j}} \qquad (13a)$$

i.e., the *j*-modal damping ratio **decreases** as the natural frequency increases.

For **damping proportional to stiffness only**, $\alpha = 0$, **(structural damping)** and

damping ratio is dimensionless

$$\zeta_j = \frac{\beta K_j}{2\sqrt{K_j M_j}} = \frac{\beta\,\omega_{n_j}}{2} \qquad (13b)$$

i.e., the *j*-modal damping ratio **increases** as the natural frequency increases. In other words, higher modes are increasingly more damped than lower modes.

The response for each modal coordinate satisfying the modal Eqn. $M_j \ddot{q}_j + D_j \dot{q}_j + K_j q_j = Q_j, \quad {}_{j=1,2\ldots n}$ is obtained in the same way as for a single DOF system (See **Handout 2**).

First, find initial values in modal space $\left\{ q_{o_j}, \dot{q}_{o_j} \right\}$. These follow from either

$$\mathbf{q}_o = \mathbf{\Phi}^{-1}\mathbf{U}_o \;\; ; \;\; \dot{\mathbf{q}}_o = \mathbf{\Phi}^{-1}\dot{\mathbf{U}}_o \qquad (14)$$

or

$$\mathbf{q}_o = \left[M\right]^{-1}\mathbf{\Phi}^T\mathbf{M}\,\mathbf{U}_o\,, \qquad \dot{\mathbf{q}}_o = \left[M\right]^{-1}\mathbf{\Phi}^T\mathbf{M}\,\dot{\mathbf{U}}_o \qquad (15a)$$

$$q_{o_k} = \frac{1}{M_k}\boldsymbol{\varphi}_{(k)}^T\left(\mathbf{M}\,\mathbf{U}_o\right),\; \dot{q}_{o_k} = \frac{1}{M_k}\boldsymbol{\varphi}_{(k)}^T\left(\mathbf{M}\,\dot{\mathbf{U}}_o\right) \qquad (15b)$$

$k=1,\ldots.n$

Free response in modal coordinates

Without modal forces, $Q=0$, the modal EOM is

$$M_j \ddot{q}_{H_j} + D_j \dot{q}_{H_j} + K_j q_{H_j} = 0 = Q_j \qquad (16)$$

with solution, for **an elastic underdamped mode** $\zeta_j < 1$

$$q_{H_j} = e^{-\zeta_j \omega_{d_j} t}\left(C_j \cos\left(\omega_{d_j} t\right) + S_j \sin\left(\omega_{d_j} t\right)\right) \quad \text{if } \omega_{n_j} \neq 0 \quad (17a)$$

where $\omega_{d_j} = \omega_{n_j}\sqrt{1-\zeta_j^2}$, $\omega_{n_j} = \sqrt{K_j / M_j}$ and

$$C_j = q_{o_j};\; S_j = \frac{\dot{q}_{o_j} + \zeta_j \omega_{n_j} q_{o_j}}{\omega_{d_j}} \qquad (17b)$$

cheat sheet too!

See **Handout (2a)** for formulas for responses corresponding to overdamped and critically damped SDOF systems.

Forced response in modal coordinates

For step-loads, $Q_{S\,j}$, the modal equations are

$$M_j \ddot{q}_j + D_j \dot{q}_j + K_j q_j = Q_{S\,j} \qquad (18)$$

with solution, for **an elastic underdamped mode** $\zeta_j < 1$

$$q_j = e^{-\zeta_j \omega_{d_j} t}\left(C_j \cos\left(\omega_{d_j} t\right) + S_j \sin\left(\omega_{d_j} t\right)\right) + q_{S_j} \quad \omega_{n_j} \neq 0 \quad (19a)$$

where $\omega_{d_j} = \omega_{n_j} \sqrt{1-\zeta_j^2}$, $\omega_{n_j} = \sqrt{K_j / M_j}$ and

$$q_{S_j} = \frac{Q_{S_j}}{K_j}; \; C_j = \left(q_{o_j} - q_{S_j}\right); \; S_j = \frac{\dot{q}_{o_j} + \zeta_j \omega_{n_j} C_j}{\omega_{d_j}} \tag{19b}$$

cheat sheet too!

See **Handout (2a)** for formulas for physical responses corresponding to overdamped and critically damped SDOF systems.

For periodic-loads,

Consider the case of force excitation with frequency $\Omega \neq \omega_{n_j}$ and acting for very long times. The EOMs in physical space are

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{D}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F_P}\cos\left(\Omega t\right)$$

The modal equations are

$$M_j\,\ddot{q}_j + D_j\,\dot{q}_j + K_j\,q_j = Q_{P_j}\cos(\Omega t) \tag{20}$$

with solutions

for **an elastic mode,** $\omega_{n_j} \neq 0$

$$q_j = q_{transient} + q_{ss(t)} =$$

$$e^{-\zeta_j \omega_{n_j} t}\left(C_j \cos\left(\omega_{d_j} t\right) + S_j \sin\left(\omega_{d_j} t\right)\right) + C_{c_j}\cos\left(\Omega t\right) + C_{s_j}\sin\left(\Omega t\right) \tag{21}$$

The **steady state or periodic response** is of importance, since the transient response will disappear because of the dissipative effects of damping. Hence, the *j*-mode response is:

$$q_{PS_j} = \left(\frac{Q_{P_j}}{K_j}\right) A_j \cos\left(\Omega t - \psi_j\right) \qquad (22)$$

Let $f_j = \frac{\Omega}{\omega_{n_j}}$ be a j_{th}-*mode* excitation frequency ratio. Then, define

$$A_j = \frac{1}{\sqrt{\left(1-f_j^2\right)^2 + \left(2\zeta_j f_j\right)^2}} \quad \text{and} \quad \tan\left(\psi_j\right) = \frac{2\,\zeta_j\, f_j}{\left(1-f_j^2\right)} \qquad (23)$$

Recall that ψ_j is a **phase angle** and A_j is an **amplitude ratio** for the j_{th}-mode.

Note that depending on the magnitude of the excitation frequency Ω, the frequency ratio for a particular mode, say *k,* determines the regime of operation, i.e**., below, above or around the natural frequency.**

Using the **mode displacement method**, the response in physical coordinates is

$$\mathbf{U} \approx \sum_{j=1}^{m}\left(\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j} A_j \cos\left(\Omega t - \psi_j\right)\right) \qquad (24)$$

and recall that $K_j = \omega_{n_j}^2 M_j = \boldsymbol{\varphi}_{(j)}^T \mathbf{K}\, \boldsymbol{\varphi}_{(j)}$ and $Q_{P_j} = \boldsymbol{\varphi}_{(j)}^T \mathbf{F}_\mathbf{P}$.

A **mode acceleration method** can also be easily developed to give (* read addendum)

$$\mathbf{U} \approx \mathbf{U}_{\mathbf{SP}} \cos\left(\Omega t\right) - \sum_{j=1}^{m} \frac{2\zeta_j}{\omega_j} \boldsymbol{\varphi}_j\, \dot{q}_{PS_j} - \sum_{j=1}^{m} \frac{\boldsymbol{\varphi}_j}{\omega_j^2} \ddot{q}_{PS_j} \qquad (25)$$

where $\mathbf{U_{SP}} = \mathbf{K}^{-1}\mathbf{F_p}$. Note that the mode acceleration method cannot be applied if there are any rigid body modes (**K** is singular)

Frequency response functions for damped MDOF systems.

The **steady state or periodic modal response** for the *j*-mode is:

$$q_{PS_j} = \left(\frac{Q_{P_j}}{K_j}\right) A_j \cos\left(\Omega t - \psi_j\right) \tag{22}$$

Or, taking the real part of the following complex number expression

$$q_{PS_j} = \left(\frac{Q_{P_j}}{K_j}\right) H_j e^{i\Omega t} \tag{26}$$

where

$$H_j = \frac{1}{\left(1 - f_j^2\right) + i\left(2\zeta_j f_j\right)} \tag{27}$$

with $i = \sqrt{-1}$ is the imaginary unit, and where $f_j = \frac{\Omega}{\omega_{n_j}}$ is the j_{th}-*mode* excitation frequency ratio. Then, recall from Eqs. (23)

$$A_j = \left|H_j\right| \frac{1}{\sqrt{\left(1 - f_j^2\right)^2 + \left(2\zeta_j f_j\right)^2}} \quad \text{and } \psi_j = \arg\left(H_j\right) \tag{28}$$

Using the **modal transformation,** the periodic response $\mathbf{U_P}$ in physical coordinates is

$$\mathbf{U_P} = \sum_{j=1}^{n} \left(\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j} A_j \cos\left(\Omega t - \psi_j\right) \right) \tag{24}$$

or take the real part of the equation below

$$\mathbf{U_P} = \boldsymbol{\Phi}\,\mathbf{q} = \sum_{j=1}^{n} \left(\boldsymbol{\varphi}_j q_j \right) = \sum_{j=1}^{n} \left(\boldsymbol{\varphi}_j \frac{\boldsymbol{\varphi}_j^T \mathbf{F_P}}{K_j} H_j e^{i\Omega t} \right)$$

$$= \left\{ \sum_{j=1}^{n} \left(\boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^T \frac{H_j}{K_j} \mathbf{F_P} \right) \right\} e^{i\Omega t} \tag{29}$$

Now, the product $\boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^T = \mathbf{matrix}(n \times n)$. That is, define the elements of the complex – frequency response matrix $\mathbf{H}$ as

$$H_{p,q} = \left(\frac{\boldsymbol{\varphi}_{j_p} \boldsymbol{\varphi}_{j_q}^T}{K_j} \left(\frac{1}{\left(1 - f_j^2\right) + i\left(2\zeta_j f_j\right)} \right) \right) \tag{30}$$

p,q =1,2…. n. The response in physical coordinates thus becomes:

$$\mathbf{U_P} = \mathbf{H}\,\mathbf{F_P}\,e^{i\Omega t} \tag{31}$$

or in component form,

$$U_{P_j} = \left(\sum_{r=1}^{n} H_{j,r} F_{P_r} \right) e^{i\Omega t} \;; \quad {}_{j=1,2..n} \tag{32}$$

This matrix is a function of the frequency ratio fj

The components of the frequency response matrix $\mathbf{H}$ are determined numerically or experimentally. In any case, the components of $\mathbf{H}$ depend on the excitation frequency (Ω).

Determining the elements of **H** seems laborious and (perhaps) its physical meaning remains elusive.

Direct Method to Find Frequency Responses in MDOF Systems

Nowadays, with fast computing power at our fingertips, the young engineer prefers to pursue a more direct approach, one known as **brute force or direct approach**. Recall that the equation of motion is

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{D}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F_P}\cos(\Omega t)$$

Or

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{D}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathrm{Re}\left(\mathbf{F_P}\,e^{i\Omega t}\right) \tag{33}$$

Assume a periodic solution of the form $\mathbf{U} = \mathbf{V_P}\,e^{i\Omega t}$ (34)

where $\mathbf{V_P}$ is a vector in the complex domain. Substitution of Eq. (34) into Eq. (33) gives

$$\left[\mathbf{K} + i\Omega\mathbf{D} - \Omega^2\,\mathbf{M}\right]\mathbf{V_P} = \mathbf{F_P} \tag{35}$$

Define at each excitation frequency the complex impedance (dynamic stiffness) matrix as:

$$\mathbf{K_{D(\Omega)}} = \left[\mathbf{K} + i\Omega\mathbf{D} - \Omega^2\,\mathbf{M}\right] \tag{36}$$

And find the **vector** of physical responses (amplitude and phase) as

$$\mathbf{V_P} = \left[\mathbf{K_{D(\Omega)}}\right]^{-1}\mathbf{F_P} \tag{37}$$

Since $\mathbf{V_P} = \mathbf{V_{P_{real}}} + i\,\mathbf{V_{P_{imaginary}}}$, the physical response for each DOF follows as:

$$U_r = V_{P_r}\cos(\Omega t - \gamma_r); \quad r=1,2\ldots n$$
$$V_{P_r} = \sqrt{V^2_{P_r-real} + V^2_{P_r-imaginary}}; \ \tan(\gamma_r) = -\left(V_{P_r-imaginary} \big/ V_{P_r-real} \right) \tag{38}$$

The direct method requires calculating the inverse of the dynamic stiffness matrix at each excitation frequency. The computational effort to perform this task could be excessive but for systems with a few DOFs (*n* small).

ORIGIN := 1 PropDamping_STEPMDOF.mcd

FORCED RESPONSE of MDOF Linear system with proportional damping

Original by Dr. Luis San Andres for MEEN 617 class /SP 2012, FA2001

The equations of motion are: **M** $d^2X dt^2$ + C dX/dt+ **K X = F**(t) (1)

where **M,K,C** are **nxn** matrices of inertia, stiffness, and damping force coefficients, and **X, V**=d**X**/dt, d^2**X**/dt^2,and are the **nx1** vectors of displacements, velocity and accelerations, respectively.
F(t) is a vector of nodal forces. At t=0, **Xo,Vo=**d**X/**dt are known.

For proportional damping, **C** = a **M** + b **K,** so the undamped mode shapes are still valid. a & b are physical constants usually determined from measurements.

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1. Define elements of inertia, damping & stiffness matrices:

$n := 3$ # of DOF

$M := 10^2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $K := 10^7 \cdot \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

example

$a := 0$

$b := .001$

$C := a \cdot M + b \cdot K$

2. Calculate the undamped naural frequencies and natural mode shapes from the fundamental relationship:

$\left(-\omega^2 \cdot M + K\right) \cdot \phi = 0$ (2) Let: $\lambda = \omega^2$ (3)

If M is invertable, then define $A := M^{-1} \cdot K$ (4)

and write Eq. (2) as: $\lambda \cdot \phi = A \cdot \phi$ (5)

$\lambda := \text{sort}(\text{eigenvals}(A))$ <---- find eigenvalues

$j := 1 .. n$

$\omega_{n_j} := \sqrt{\lambda_j}$

$f_{n_j} := \frac{\omega_{n_j}}{2 \cdot \pi}$

The undamped natural frequencies are:

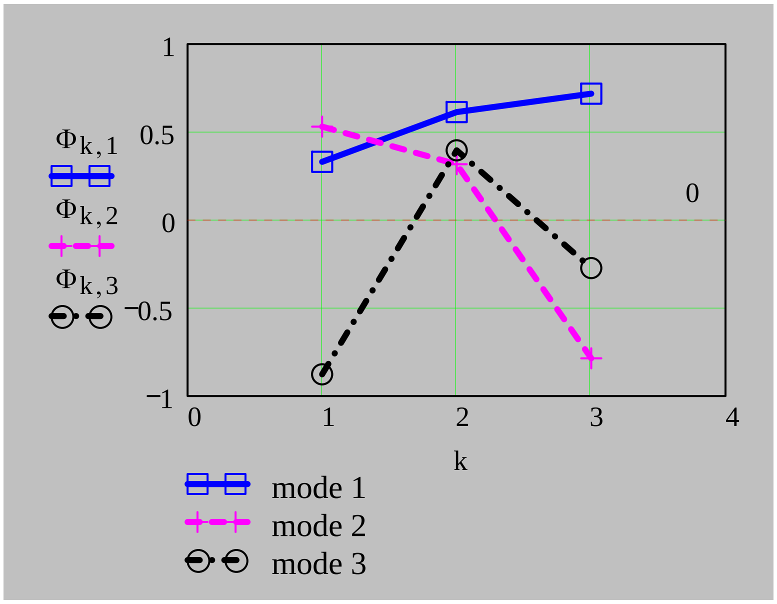
$\omega_n^T = (120.57 \quad 374.57 \quad 495.14)$ rad/s

$f_n^T = (19.19 \quad 59.61 \quad 78.8)$ Hz

and **undamped natural modes**: $\phi_j := \text{eigenvec}\left(A, \lambda_j\right)$

$\Phi^{\langle j \rangle} := \phi_j$

3. Plot the natural mode shapes:



This is the matrix of undamped modal vectors

$$\Phi = \begin{pmatrix} 0.33 & 0.53 & -0.88 \\ 0.61 & 0.32 & 0.4 \\ 0.72 & -0.79 & -0.27 \end{pmatrix}$$

Modal matrices

4. Modal transformation of physical equations to (natural) modal coordinates

Using the transformation $X = \Phi \cdot q$, the equations of motion (1) become uncoupled in the modal space (principal coordinates):

$$M_m \cdot \frac{d^2 q}{dt^2} + C_m \cdot \frac{dq}{dt} + K_m \cdot q = Q \qquad (6)$$

where: $M_m := \Phi^T \cdot M \cdot \Phi$ Modal matrices

$K_m := \Phi^T \cdot K \cdot \Phi$ (7)

$C_m := \Phi^T \cdot C \cdot \Phi$ $C_m = a \cdot M_m + b \cdot K_m$

and **modal forces**: $Q = \Phi^T \cdot F$ (8)

and with the **initial conditions**:

$$q_o = M_m^{-1} \cdot \left(\Phi^T \cdot M \cdot X_o \right) \qquad \frac{dq_o}{dt} = M_m^{-1} \cdot \left(\Phi^T \cdot M \cdot V_o \right) \qquad (9)$$

Define the damping ratios and damped natural freqs. in modal space: $k := 1..n$

$$\zeta_k := \frac{C_{m_{k,k}}}{2\cdot M_{m_{k,k}}\cdot\omega_{n_k}} \quad \text{(10a)}$$

$$\zeta^T = (0.06 \quad 0.19 \quad 0.25)$$

note all damping ratios < 1

$$\omega_{d_k} := \omega_{n_k}\cdot\left[1-(\zeta_k)^2\right]^{.5} \quad (10b)$$

UNDERDAMPED CASE

The **modal responses** for arbitrary excitations are easily obtained for each natural mode (based on response of simple 1DOF system).

And, the response in the physical coordinates is given by the superposition of the modal responses, i.e.

$$X(t) = \Phi\cdot q(t) \quad (11)$$

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Modal matrices

Response to a STEP Load

Set STEP load vector:

$$F := \begin{pmatrix} 2000 \\ -3000 \\ 1000 \end{pmatrix}$$

modal force is: → $Q := \Phi^T\cdot F$

$$Q = \begin{pmatrix} -460.81 \\ -675.13 \\ -3.21\times 10^3 \end{pmatrix}$$

and set **vectors of initial conditions**

$$X_o := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad V_o := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Calculate response

calculate modal initial displs. & vels.

$$q_o := M_m^{-1}\cdot\left(\Phi^T\cdot M\cdot X_o\right) \qquad q\_dot_o := M_m^{-1}\cdot\left(\Phi^T\cdot M\cdot V_o\right)$$

Evaluate response in modal coordinates:

$j := 1..n$

$$q_{s_j} := \frac{Q_j}{K_{m_{j,j}}}$$: static displacement in modal space

$$A_{c_j} := \left(q_{o_j} - q_{s_j}\right) \qquad A_{s_j} := \frac{\left(q\_dot_{o_j} - \zeta_j\cdot\omega_{n_j}\cdot A_{c_j}\right)}{\omega_{d_j}}$$

$p := 1..N_{points}$

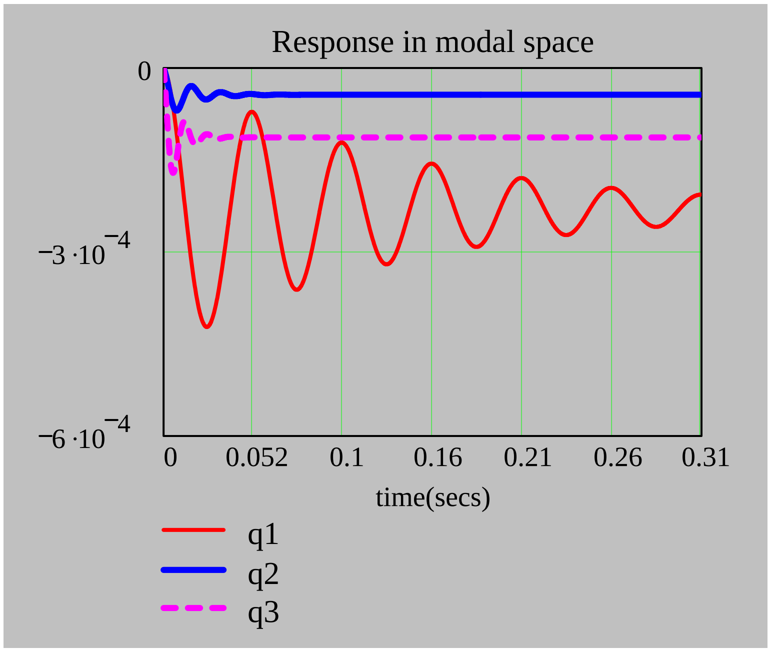
$$t_p := (p-1)\cdot\Delta t$$

$$q_{j,p} := e^{-\zeta_j\cdot\omega_{n_j}\cdot t_p}\cdot\left(A_{c_j}\cdot\cos\left(\omega_{d_j}\cdot t_p\right) + A_{s_j}\cdot\sin\left(\omega_{d_j}\cdot t_p\right)\right) + q_{s_j}$$

see your cheat-sheet
(1 DOF response)

Calculate response

Calculate response



$$q_o = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

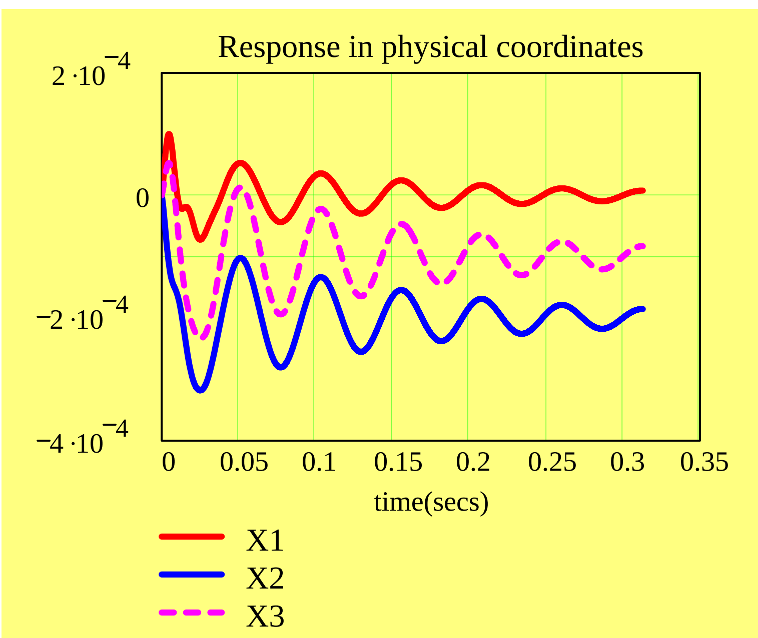
$$q\_dot_o = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$q_s = \begin{pmatrix} -2.3 \times 10^{-4} \\ -4.37 \times 10^{-5} \\ -1.13 \times 10^{-4} \end{pmatrix}$$

The response in physical coordinates is: $X(t) = \Phi \cdot q(t)$ (11)

Let: $m := 3$ <------ select number of modes for physical response

Static response as t-> inf.



Recall I.C's

$$X_o = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad V_o = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Steady state response (long times)

$$X_s = \begin{pmatrix} 0 \\ -2 \times 10^{-4} \\ -1 \times 10^{-4} \end{pmatrix}$$

Check steady-state (t infinite) response:

$$X_{s,N_{points}} = \begin{pmatrix} 7.86 \times 10^{-6} \\ -1.85 \times 10^{-4} \\ -8.29 \times 10^{-5} \end{pmatrix} \qquad X_s = \begin{pmatrix} 0 \\ -2 \times 10^{-4} \\ -1 \times 10^{-4} \end{pmatrix}$$

ORIGIN := 1 PropDamping_STEPMDOF.mcd

PERIODIC RESPONSE of MDOF system with proportional damping

Original by Dr. Luis San Andres for MEEN 617 class /SP 2012

The equations of motion are: **M** d²Xdt² + C dX/dt+ **K X = Fo cos(Ωt)** (1)

where **M,K,C** are **nxn** matrices of inertia, stiffness, and damping force coefficients, and **X, V**=d**X**/dt, d²**X**/dt²,and are the **nx1** vectors of displacements, velocity and accelerations, respectively.
F(t) is a vector of nodal forces - periodic. At t=0, **Xo,Vo=**d**X/**dt are known.

For proportional damping, **C** = a **M** + b **K,** so the undamped mode shapes are still valid. a & b are physical constants usually determined from measurements.

1. Define elements of inertia, damping & stiffness matrices:

$n := 3$ # of DOF

$$M := 10^2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K := 10^7 \cdot \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

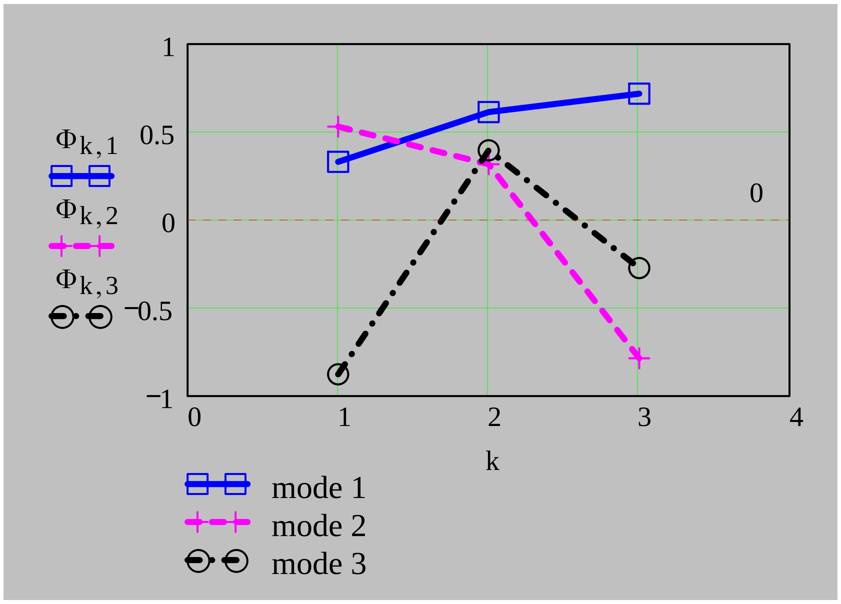
example

$a := 0$

$b := .00025$

$C := a \cdot M + b \cdot K$

find natural freqs.



$$\Phi = \begin{pmatrix} 0.33 & 0.53 & -0.88 \\ 0.61 & 0.32 & 0.4 \\ 0.72 & -0.79 & -0.27 \end{pmatrix}$$

$$f_n^T = (19.19 \quad 59.61 \quad 78.8) \text{ Hz}$$

Modal matrices

damped natural freqs. and damping ratios

$$f_d^T = (19.19 \quad 59.55 \quad 78.65) \text{ Hz} \qquad \zeta^T = (0.02 \quad 0.05 \quad 0.06)$$

Response to a PERIODIC Load

Set amplitude load vector: $F_o := \begin{pmatrix} 2000 \\ -4000 \\ 6000 \end{pmatrix}$

modal force is:

$Q_o := \Phi^T \cdot F_o$

$Q_o = \begin{pmatrix} 2.51 \times 10^3 \\ -4.92 \times 10^3 \\ -4.97 \times 10^3 \end{pmatrix}$

Assume effect of **initial conditions has vanished since periodic load acts for very long time.**

Calculate response

Vary the excitation frequency Ω to determine the amplitude and phase of the FRF for each mode:

set maximum frequency (rad/sec) to display calculations $\Omega_{max} := 4 \cdot \omega_{n_n}$ (larger than highest natural freq)

$k := 1..\log(\Omega_{max}) \cdot 100$

$\omega_{n_n} = 495.14$ rad/s

excitation Frequency $\Omega_k := 10^{\frac{k}{100}}$

$\Omega_{max} = 1.98 \times 10^3$

$i := 1..n$

$freq_k := \frac{\Omega_k}{2 \cdot \pi}$ in Hz

Evaluate response in modal coordinates:

$r_{i,k} := \frac{\Omega_k}{\omega_{n_i}}$ **frequency ratio**

$q_{s_i} := \frac{Q_{o_i}}{Km_{i,i}}$

$q_s^T = \begin{pmatrix} 1.26 \times 10^{-3} & -3.19 \times 10^{-4} & -1.75 \times 10^{-4} \end{pmatrix}$

The MODAL **complex** amplitudes are:

$$q_{i,k} := \frac{q_{s_i}}{\left[\left[1 - (r_{i,k})^2\right] + j \cdot \left[(2 \cdot \zeta_i) r_{i,k}\right]\right]}$$

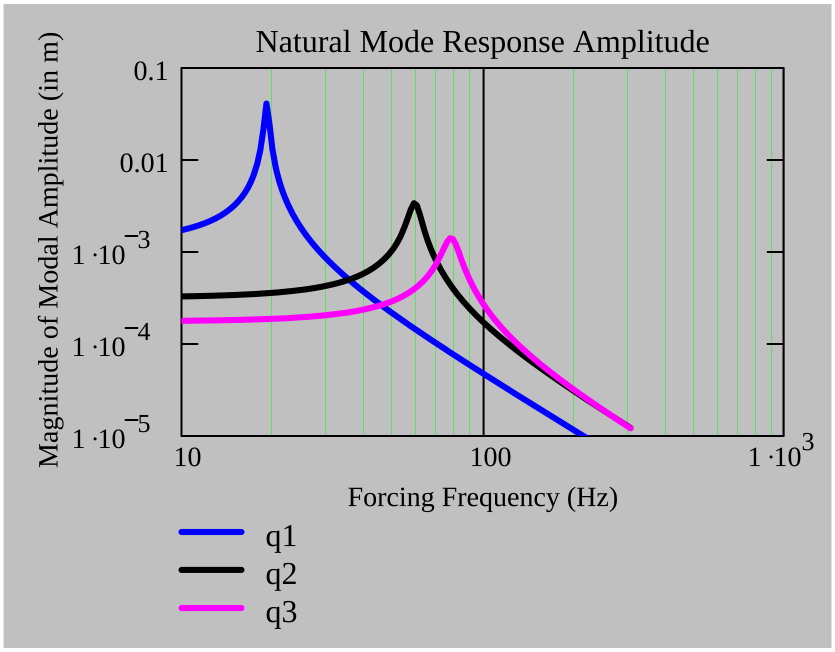
and MODAL Phase Angles:

$$\alpha_{i,k} := \begin{cases} \operatorname{atan}\left[\frac{(2 \cdot \zeta_i) r_{i,k}}{1 - (r_{i,k})^2}\right] \cdot \frac{180}{\pi} & \text{if } (\omega_{n_i}) - (\Omega_k) \geq 0 \\ 180 + \operatorname{atan}\left[\frac{(2 \cdot \zeta_i) r_{i,k}}{1 - (r_{i,k})^2}\right] \cdot \frac{180}{\pi} & \text{otherwise} \end{cases}$$

See your cheat-sheet (1 DOF response)

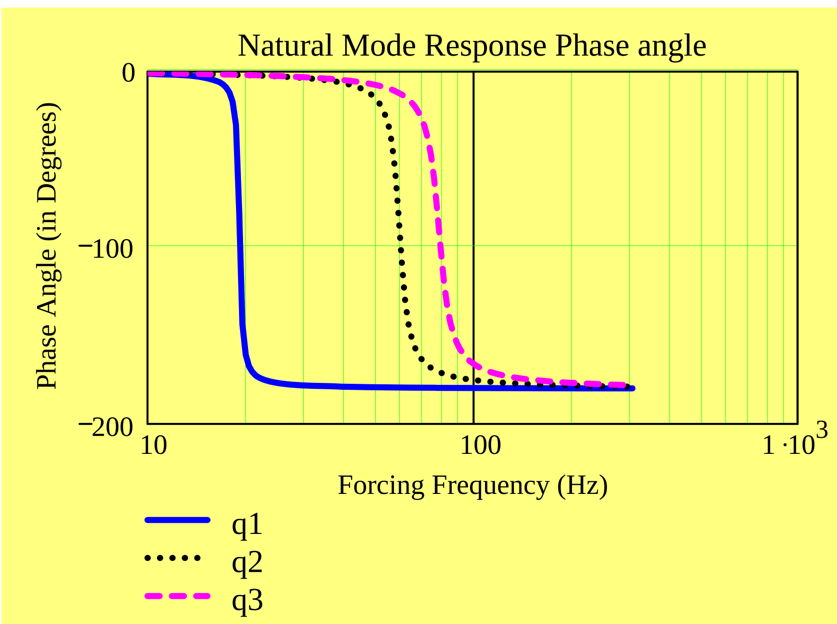
$\alpha := -\alpha$

Calculate response



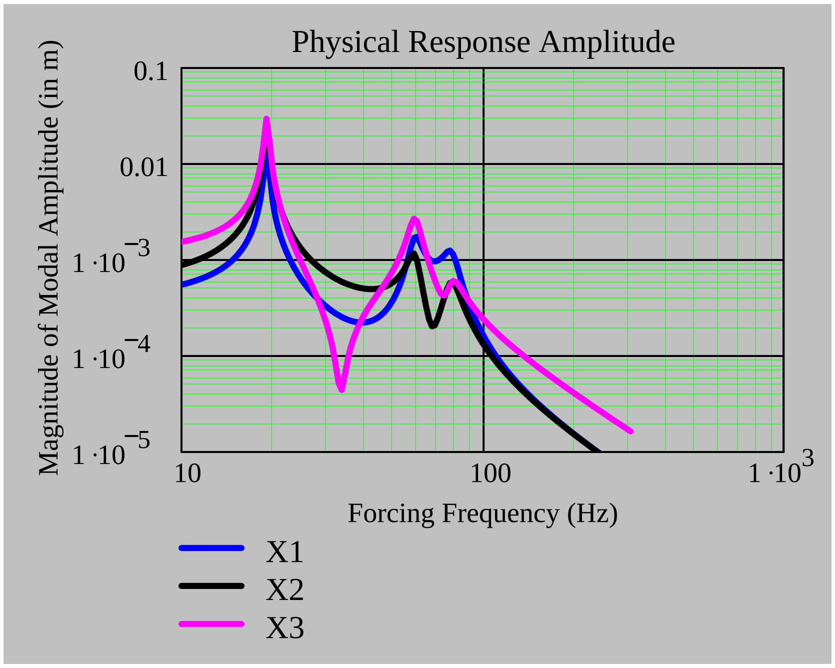
$$f_d^T = (19.19 \quad 59.55 \quad 78.65) \quad \text{Hz}$$

$$\zeta^T = (0.02 \quad 0.05 \quad 0.06)$$



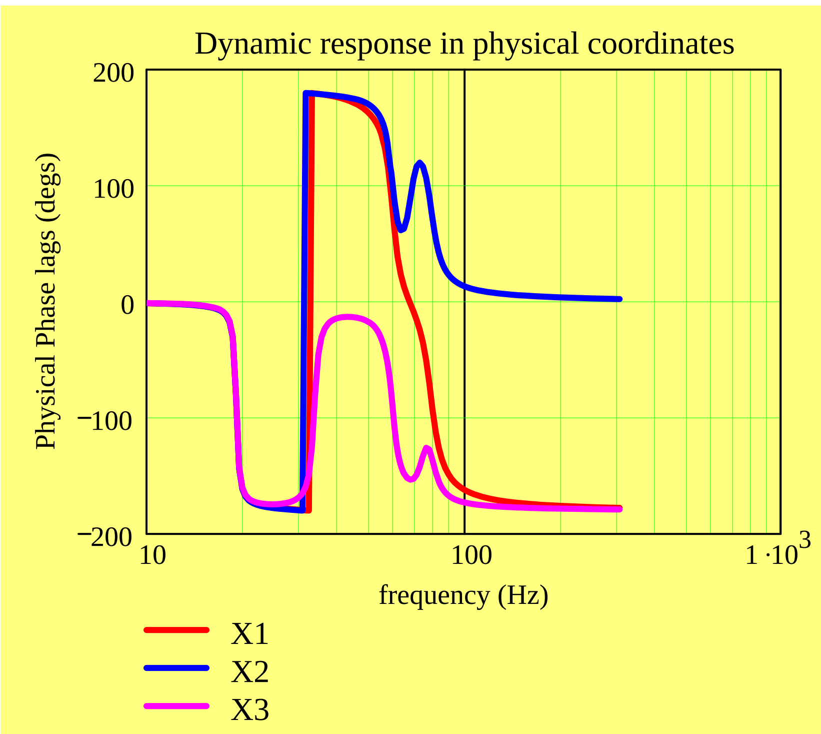
The response in physical coordinates is: $X(t) = \Phi \cdot q(t)$

Let: $m := 3$ <------ select number of modes for physical response



$\zeta^T = (0.02 \quad 0.05 \quad 0.06)$

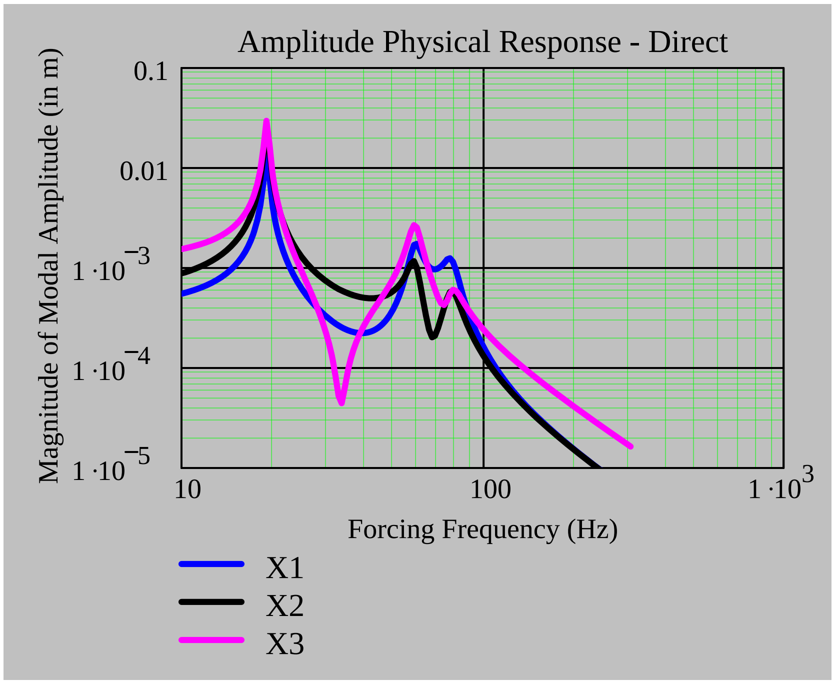
$f_d^T = (19.19 \quad 59.55 \quad 78.65)$ Hz



DIRECT METHOD

make $K_{D_k} := K - M \cdot (\Omega_k)^2 + i \cdot \Omega_k \cdot C$ for each Ω_k

and solve $Z_k := \left(K_{D_k}\right)^{-1} \cdot F_o$



$f_d^T = (19.19 \quad 59.55 \quad 78.65)$ Hz

$\zeta^T = (0.02 \quad 0.05 \quad 0.06)$

Addendum

Derivation of Mode Acceleration Method for MDOF systems (proportional damping or light damping)

(Luis San Andrés, Lecturer. Based on homework delivered by Mr. Rahul Kar)

Problem Statement

Determine the system response of a MDOF system with proportional damping using the **Mode Acceleration method**.

Solution

The differential equation governing the motion of a *n*-DOF linear system is:

$$[M]\ddot{X}+[C]\dot{X}+[K]X=P(t) \tag{1}$$

where [*M*], [*K*], [*C*] are the (*n*x*n)* matrices of (constant) mass, stiffness and damping coefficients. P(t) is a vector of *n*-external forces, time dependent, and *X(t)* is the vector of system displacements (physical responses). The physical damping is of proportional type, i.e. [*C*] = *a* [*M*] + *b* [*K*]

The system described by (1) has a set of natural frequencies $(\omega_i)_{i=i,..n}$ and associated modal (eigen) vectors $({}^i\phi)_{i=i,..n}$. Each pair $(\omega_i\ {}^i\phi)$ satisfies the fundamental relationship

$$[K]\ {}^i\phi=\omega_i^2\ [M]\ \ {}^i\phi, \qquad i=1,2,\ldots n \tag{2}$$

The physical response *X(t)* or solution to (1) can be found using modal analysis, i.e.

$$X(t)=[\Phi]\eta(t)=\sum_{i=1}^{n}{}^i\phi\ \eta_i \tag{3}$$

where $[\Phi]=\left\{{}^1\phi\ \ {}^2\phi\ldots\ldots{}^n\phi\right\}$ is the modal matrix. Each of the components of the modal response vector $\eta(t)$ is obtained from solution of the (uncoupled) equations:

$$M_{m\,i}\,\ddot{\eta}_i+C_{m\,i}\,\dot{\eta}_i+K_{m\,i}\,\eta_i=Q_i \qquad \text{i=1,2,\ldots n} \tag{4}$$

where $Q=[\Phi]^T\ P$, and $(K_m, M_m, C_m)_i$ are the *i*-th **modal** stiffness, mass and damping coefficients obtained from:

$$[M_m]=[\Phi]^T[M][\Phi];\ [K_m]=[\Phi]^T[K][\Phi];[C_m]=[\Phi]^T[C][\Phi]; \tag{5}$$

In (3), using a number of modes *m* less than the *n*-DOF is known as **the mode displacement method, i.e.**

$$X(t)\cong\sum_{i=1}^{m}{}^i\phi\ \eta_i\ \ ;\ \ _{m<n} \tag{6}$$

The **mode acceleration method** aims to find exactly the system static response should *P* be a vector of constant generalized forces. In this case, the mode displacement method does poorly when just a few modes, $m<<n$, are used

To derive the appropriate equations, pre-multiply (1) by $[K]^{-1}$, i.e. the *flexibility matrix* (obviously this operation precludes any rigid body motion), to obtain:

$$[K]^{-1}[M]\ddot{X} + [K]^{-1}[C]\dot{X} + X = [K]^{-1}P(t)$$

and

$$X = [K]^{-1}P(t) - [K]^{-1}[M]\ddot{X} - [K]^{-1}[C]\dot{X} \qquad (7)$$

from (6) it follows that $\dot{X} \cong \sum_{i=1}^{m} \phi_i \dot{\eta}_i(t)$ and $\ddot{X} \cong \sum_{i=1}^{m} \phi_i \ddot{\eta}_i(t)$. Replacing these relationships into (7) gives:

$$X(t) \cong [K]^{-1} P(t) - \sum_{i=1}^{m} [K]^{-1}[M]\,{}^i\phi\, \ddot{\eta}_i(t) - \sum_{i=1}^{m} [K]^{-1}[C]\,{}^i\phi\, \dot{\eta}_i(t); \quad m<n \qquad (8)$$

Let's work with the terms: $[K]^{-1}[M]\,{}^i\phi$ *and* $[K]^{-1}[C]\,{}^i\phi$. Since each pair $(\omega_i, {}^i\phi)$ satisfies the fundamental relationship

$$[K]\,{}^i\phi = \omega_i^2\,[M]\,{}^i\phi \qquad (2)$$

then

$$[K]^{-1}[M]\,{}^i\phi = (1/\omega_i^2)\,{}^i\phi \qquad (9.a)$$

and similarly,

$$[K]^{-1}[C]\,{}^i\phi = (2\xi_i/\omega_i)\,{}^i\phi \qquad (9.b)$$

where ξ_i is the *i*-th **modal** damping ratio defined as

$$\xi_i = \frac{C_{m\,i}}{C_{cr_i}} \quad ; \; [C_m] = [\Phi]^T [C][\Phi],\; C_{cr_i} = 2\left(K_{m_i} M_{m_i}\right)^{1/2} \qquad (10)$$

Note that in the equation above, $(K_m, M_m)_i$ are the *i*-th **moda**l stiffness and mass coefficients satisfying $\left(\frac{K_{m_i}}{M_{m_i}}\right)^{1/2} = \omega_i$

Replacing (9) into (8) gives the physical response of the system as:

$$X(t) \cong [K]^{-1} P(t) - \sum_{i=1}^{m} \frac{{}^i\phi}{\omega_i^2} \ddot{\eta}_i(t) - \sum_{i=1}^{m} \frac{2\xi_i\,{}^i\phi}{\omega_i} \dot{\eta}_i(t); \quad m<n \qquad (10)$$

which is known as the **mode acceleration response method**. The first term in the response $[K]^{-1}P(t)$ corresponds to a "pseudostatic" static displacement due to *P(t)*.

Note that for $P = P_s$ *(constant)*, $X=X_s=[K]^{-1}P_s$ since all $\eta_i=0$. This simple check certifies the accuracy of the mode acceleration method even when using few modes ($m<n$).

Reference:

MEEN 617 Handout #8 Modal Analysis of MDOF Systems with Proportional Damping, L. SanAndrés, 2008.