MODAL ANALYSIS OF MDOF Systems with VISCOUS DAMPING

The motion of a $n$-DOF linear system is described by the set of 2nd order differential equations

$$\mathbf{M} \ddot{\mathbf{U}} + \mathbf{C} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} = \mathbf{F}(t) \quad (1)$$

where $\mathbf{U}(t)$ and $\mathbf{F}(t)$ are $n$ rows vectors of displacements and external forces, respectively. $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ are the system $(nxn)$ matrices of mass, stiffness, and viscous damping coefficients. These matrices are symmetric, i.e. $\mathbf{M} = \mathbf{M}^T$, $\mathbf{K} = \mathbf{K}^T$, $\mathbf{C} = \mathbf{C}^T$.

The solution to Eq. (1) is determined uniquely if vectors of initial displacements $\mathbf{U}_0$ and initial velocities $\mathbf{V}_0 = \left( \frac{d \mathbf{U}}{dt} \right)_{t=0}$ are specified.

For free vibrations, the force vector $\mathbf{F}(t) = 0$, and Eq. (1) is

$$\mathbf{M} \ddot{\mathbf{U}} + \mathbf{C} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} = 0 \quad (2)$$

A solution to Eq. (2) is of the form

$$\mathbf{U} = e^{\alpha t} \psi \quad (3)$$

where in general $\alpha$ is a complex number. Substitution of Eq. (3) into Eq. (2) leads to the following characteristic equation:

$$\left( \alpha^2 \mathbf{M} + \alpha \mathbf{C} + \mathbf{K} \right) \psi = \left[ \mathbf{f}(\alpha) \right] \psi = 0 \quad (4)$$
where \([f(\alpha)]\) is a \(nxn\) square matrix. The system of homogeneous equations (4) has a nontrivial solution only if the determinant of the system of equation equals zero, i.e.

\[
\Delta(\alpha) = \det(f(\alpha)) = 0 = c_0 + c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3 + \ldots c_{2n} \alpha^{2n} \tag{5}
\]

The roots of the characteristic polynomial \(\Delta(\alpha)\) given by Eq. (5) can be of three types:

a) **Real and negative**, \(\alpha < 0\), corresponding to over damped modes.

b) **Purely imaginary**, \(\alpha = \pm i \omega\), for undamped modes.

c) **Complex conjugate pairs**\(^1\) of the form, \(\alpha = \zeta \omega \pm i \omega_d\), for under damped modes.

Clearly if the real part of any \(\alpha > 0\), it means the system is unstable.

The constituent solution, Eq. (3), \(U = e^{\alpha t} \Psi\) can be written as the **superposition of the solution roots** \(e^{\alpha_r}\) and its associated vectors \(\Psi_r\) satisfying Eq. (4), i.e.,

\[
U(t) = \sum_{1}^{2n} C_r \Psi_r e^{\alpha_r t} \tag{6}
\]

or letting

\[
[\Psi]_{n\times2n} = [\Psi_1 \Psi_2 \ldots \Psi_{2n}] \tag{7}
\]

write Eq. (6) as

\(^1\) Only if the system is defined by symmetric matrices. Otherwise, the complex roots may **NOT BE** complex conjugate pairs.
\[ \mathbf{U}(t) = [\Psi] \{ C_r e^{\alpha_r t} \} \] (8)

However, a transformation of the form,
\[ \mathbf{U}_{nx1} = [\Psi] \mathbf{q}_{(t)_{2nx1}} \] (9)
is not possible since this implies the existence of 2n-modal coordinates which is not physically apparent when the number of physical coordinates is only \( n \).

To overcome this apparent difficulty, reformulate the problem in a slightly different form. Let \( \mathbf{Y} \) be a 2n-rows vector composed of the physical velocities and displacements, i.e.
\[
\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix}, \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 0 \\ \mathbf{F}(t) \end{bmatrix}
\] (10)
be a modified force vector. Then write \( M \ddot{\mathbf{U}} + C \dot{\mathbf{U}} + K \mathbf{U} = \mathbf{F}(t) \) as
\[
\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}} \\ \dot{\mathbf{U}} \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F} \end{bmatrix}
\] (11.a)
or
\[
\mathbf{A} \ddot{\mathbf{Y}} + \mathbf{B} \mathbf{Y} = \mathbf{Q}
\] (11.b)
where
\[
\mathbf{A} = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}
\] (12)
\[ A \text{ and } B \text{ are } 2n \times 2n \text{ matrices, symmetric if the } M, C, K \text{ matrices are also symmetric.} \]

For free vibrations, \( Q = 0 \), and a solution to Eq. (11.b) is sought of the form:

\[
\begin{bmatrix} \dot{U} \\ U \end{bmatrix} = Y = \Phi e^{\alpha t}
\]

Substitution of Eq. (13) into Eq. (11.b) gives:

\[
[\alpha A + B] \Phi = 0
\]

which can be written in the familiar form:

\[
D \Phi = \frac{1}{\alpha} \Phi
\]

where

\[
D = -B^{-1}A = \begin{pmatrix} M^{-1} & 0 \\ 0 & -K^{-1} \end{pmatrix} \begin{pmatrix} 0 & M \\ M & C \end{pmatrix}, \quad \text{or} \quad D = \begin{pmatrix} 0 & I \\ -K^{-1}M & -K^{-1}C \end{pmatrix}
\]

with \( I \) as the \( n \times n \) identity matrix. From Eq. (15) write

\[
\left[ D - \frac{1}{\alpha} I \right] \Phi = \begin{bmatrix} f(\alpha) \end{bmatrix} \Phi = 0
\]

The eigenvalue problem has a nontrivial solution if

\[
\Delta(\alpha) = \left| \begin{bmatrix} f(\alpha) \end{bmatrix} \right| = 0
\]
From Eq. (18) determine $2n$ eigenvalues $\{\alpha_r\}$, $r=1, 2, \ldots, 2n$ and associated eigenvectors $\{\Phi_r\}$. Each eigenvector must satisfy the relationship:

$$D \Phi_r = \frac{1}{\alpha_r} \Phi_r$$

and can be written as $\Phi_r = \begin{bmatrix} \Psi_r^1 \\ \Psi_r^2 \end{bmatrix}$ where $\Psi_r$ is an $n \times 1$ vector satisfying:

$$
\begin{pmatrix}
0 & I \\
-K^{-1}M & -K^{-1}C
\end{pmatrix}
\begin{bmatrix}
\Psi_r^1 \\
\Psi_r^2
\end{bmatrix} = \frac{1}{\alpha_r}
\begin{bmatrix}
\Psi_r^1 \\
\Psi_r^2
\end{bmatrix}
$$

from the first row of Eq. (20) determine that:

$$I \Psi_r^2 = \frac{1}{\alpha_r} \Psi_r^1$$

or

$$\Psi_r^1 = \alpha_r \Psi_r^2$$

(21)

and from the second row of Eq. (20), with substitution of the relationship in Eq. (21), obtain

$$-K^{-1}M \Psi_r^1 - K^{-1}C \Psi_r^2 = \frac{1}{\alpha_r} \Psi_r^2$$

or

$$
\begin{bmatrix}
(-K^{-1}M) \alpha_r - (K^{-1}C) - I \frac{1}{\alpha_r}
\end{bmatrix}
\Psi_r^2 = 0
$$

(23)

for $r=1, 2, \ldots, 2n$. Note that multiplying Eq. (23) by $(-\alpha, K)$ gives

$$
\begin{bmatrix}
M \alpha_r^2 + C \alpha_r + K
\end{bmatrix}
\Psi_r^2 = 0
$$

(4)

i.e., the original eigenvalue problem.
Solution of Eq. (19), \( D \Phi_r = \frac{1}{\alpha_r} \Phi_r \), delivers the **2n-eigenpairs**

\[
\left( \alpha_r; \Phi_r = \begin{bmatrix} \alpha_r \\ \Psi_r \end{bmatrix} \right)_{r=1,2,..2n}
\]  

(24)

In general, the \( j \)-components of the eigenvectors \( \Psi_r \) are complex numbers written as

\[
\Psi_{rj} = a_{rj} + i b_{rj} = \delta_{rj} e^{i \phi_{rj}}
\]

Where \( \delta \) and \( \phi \) denote the magnitude and the phase angle.

**Note:** for viscous damped systems, not only the amplitudes but also the phase angles are arbitrary. However, the ratios of amplitudes and differences in phase angles are constant for each of the elements in the eigenvector \( \Psi_r \). That is,

\[
\left( \frac{\delta_j}{\delta_k} \right) = \text{const}_{jk} \quad \text{and} \quad \left( \phi_j - \phi_k \right) = \text{const}_{jk} \quad \text{for} \quad j, k = 1, 2, ..N
\]

A constituent solution of the homogeneous equation (free vibration problem) is then given as:

\[
Y = \begin{bmatrix} \ddot{U} \\ U \end{bmatrix} = \sum_{r=1}^{2n} C_r e^{\alpha_r t} \Phi_r
\]  

(25)

Let the (roots) \( \alpha_r \) be written in the form (when under-damped)

\[
\alpha_r = \zeta_r \omega_r + i \omega_{dr}
\]

(26)

and write Eq. (25) as
\[ Y = \begin{bmatrix} \dot{U} \\ U \end{bmatrix} = \sum_{r=1}^{2n} C_r \Phi_r e^{(-\zeta_r \omega_r + i \omega_d r) t} \]  

and since \( \Phi_r = \begin{bmatrix} \alpha_r \\ \psi_r \end{bmatrix} \), the vector of displacements is just

\[ U = \sum_{r=1}^{2n} C_r \Psi_r e^{(-\zeta_r \omega_r + i \omega_d r) t} \]  

\[ (27) \]

**ORTHOGONALITY OF DAMPED MODES**

Each eigenvalue \( \alpha_r \) and its corresponding eigenvector \( \Phi_r \) satisfy the equation:

\[ \alpha_r A \Phi_r + B \Phi_r = 0 \]  

\[ (28) \]

Consider two different eigenvalues (not complex conjugates):

\( \{ \alpha_s; \Phi_s \} \) and \( \{ \alpha_q; \Phi_q \} \), then if \( A = A^T \) and \( B = B^T \) (a symmetric system), it is easy to demonstrate that:

\[ (\alpha_s - \alpha_q) \Phi_s^T A \Phi_q = 0 \]

and infer

\[ \Phi_s^T A \Phi_q = 0 \quad ; \quad \Phi_s^T B \Phi_q = 0 \quad \text{for} \quad \alpha_s \neq \alpha_q \]  

\[ (29) \]

Now, construct a modal damped matrix \( \Phi \) \((2nx2n)\) formed by the columns of the modal vectors\( \Phi_r \), i.e.

\[ \Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_n & \ldots & \Phi_{2n-1} & \Phi_{2n} \end{bmatrix} \]  

\[ (30) \]
And write the **orthogonality property** as:

\[
\Phi^T A \Phi = \sigma \quad \Phi^T B \Phi = \beta
\]  

(31)

Where \( \sigma \) and \( \beta \) are \((2n \times 2n)\) diagonal matrices.

Now, recall that the equations of motion in physical coordinates are:

\[
M \dddot{U} + C \ddot{U} + K U = F(t)
\]  

(1)

With the definition \( Y = \begin{bmatrix} \ddot{U}^T & U^T \end{bmatrix} \), Eqs. (1) are converted into 2\(n\) first order differential equations:

\[
A \dot{Y} + B Y = Q = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}
\]  

(32)

where

\[
A = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \quad B = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}
\]  

(12)

To uncouple the set of 2\(n\) first-order Eqs. (32), a solution of the following form is assumed:

\[
Y(t) = \sum_{r=1}^{2n} \Phi_r Z_r(t) = \Phi Z(t)
\]  

(33)

Substitution of Eq. (33) into Eq. (32) gives:

\[
A \Phi \dot{Z} + B \Phi Z = Q
\]  

(34)
Premultiply this equation by $\Phi^T$ and use the orthogonality property\(^2\) of the damped modes to get:

$$
(\Phi^T A \Phi) \dot{Z} + (\Phi^T B \Phi) Z = \Phi^T Q \tag{35}
$$

or

$$
\sigma \dot{Z} + \beta Z = G = \Phi^T Q \tag{36}
$$

Eq. (36) represents a set of 2n uncoupled first order equations:

$$
\sigma_1 \dot{z}_1 + \beta_1 z_1 = g_{1(t)}
$$

$$
\sigma_2 \dot{z}_2 + \beta_2 z_2 = g_{2(t)}
$$

$$
\vdots
$$

$$
\sigma_{2N} \dot{z}_{2N} + \beta_{2N} z_{2N} = g_{2N(t)} \tag{37}
$$

where

$$
\sigma_r = \Phi_r^T A \Phi_r ; \beta_r = \Phi_r^T B \Phi_r = -\alpha_r \sigma_r, \quad r=1, 2..2N
$$

$$
\alpha_r = -\beta_r / \sigma_r \tag{38}
$$

since $\alpha_r A \Phi_r + B \Phi_r = 0$. In addition,

$$
g_{r(t)} = \Phi_r^T Q(t) \tag{39}
$$

Initial conditions are also determined from $Y_o = \begin{bmatrix} \dot{u}_o \\ u_o \end{bmatrix}$ with the transformation $Y_{(0)} = \Phi Z_{(t=0)}$

$$
\sigma Z_0 = \Phi^T A Y_0 \tag{40.a}
$$

\(^2\) The result below is only valid for symmetric systems, i.e. with $M$, $K$ and $C$ as symmetric matrices. For the more general case (non symmetric system), see the textbook of Meirovitch to find a discussion on LEFT and RIGHT eigenvectors.
or
\[ Z_{o_r} = \frac{1}{\sigma_r} \Phi_r^T A Y_o \quad r=1, 2, \ldots 2n \quad (40.b) \]

The general solution of the first order equation
\[ \sigma_r \ddot{z}_r + \beta_r z_r = g_r(t) \]
with initial condition \( z_{r(t=0)} = z_{o_r} \), is derived from the Convolution integral
\[ z_r = z_{o_r} e^{\alpha_r t} \frac{1}{\sigma_r} \int_0^t g_r(\tau) e^{\alpha_r (t-\tau)} d\tau \quad (41) \]
with \( \alpha_r = - \beta_r / \sigma_r \)

Once each of the \( z_{r(t)} \) solutions is obtained, then return to the physical coordinates to obtain:
\[ Y(t) = \begin{bmatrix} \dot{U} \\ U \end{bmatrix} = \sum_{r=1}^{2n} \Phi_r z_{r(t)} = \Phi Z(t) \quad (33=43) \]
and since \( \Phi_r = \begin{bmatrix} \alpha_r \\ \Psi_r \end{bmatrix} \), the physical displacement dynamic response is given by:
\[ \dot{U}(t) = \sum_{r=1}^{2n} \alpha_r \Psi_r z_{r(t)} \quad (44) \]
and the velocity vector is correspondingly equal to:
\[ \dot{U}(t) = \sum_{r=1}^{2n} \alpha_r \Psi_r z_{r(t)} \quad (45) \]
Read/study the accompanying MATHCAD® worksheet with a detailed example for discussion in class.
MODAL ANALYSIS of MDOF linear systems with viscous damping

Original by Dr. Luis San Andres for MEEN 617 class / SP 08, 12

The equations of motion are:

\[ M \frac{d^2U}{dt^2} + C \frac{dU}{dt} + K U = F(t) \]  \hspace{1cm} (1)

where \( M, C, K \) are nxn SYMMETRIC matrices of inertia, viscous damping and stiffness coefficients, and \( U, \frac{dU}{dt}, \frac{d^2U}{dt^2} \) are the nx1 vectors of displacements, velocity and accelerations. \( F(t) \) is the nx1 vector of generalized forces. Eq (1) is solved with appropriate initial conditions, at \( t=0, Uo,Vo=dU/dt \)

====================================================================
Define elements of inertia, stiffness, and damping matrices:

\[
\begin{align*}
\text{m}_1 & := 100 & \text{k}_1 & := 1.0 \times 10^7 & \text{c}_1 & := 5000 \\
\text{m}_2 & := 100 & \text{k}_2 & := 1.0 \times 10^7 & \text{c}_2 & := 2000 \\
\text{m}_3 & := 50 & \text{k}_3 & := 2.0 \times 10^7 & \text{c}_3 & := 1000 \\
\end{align*}
\]

\[ n := 3 \# \text{ of DOF} \]

Make matrices:

\[
M := \begin{pmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{pmatrix}, \quad
K := \begin{pmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3
\end{pmatrix}, \quad
C := \begin{pmatrix}
c_1 + c_2 & -c_2 & 0 \\
-c_2 & c_2 + c_3 & -c_3 \\
0 & -c_3 & c_3
\end{pmatrix}
\]

Initial conditions in displacement and velocity:

\[
Uo := \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
Vo := \begin{pmatrix}
0 \\
0.1 \\
0
\end{pmatrix}
\]

PLOTs - only \( N := 1024 \) for time steps

natural freqs - undamped

damped eigenvals

====================================================================
2. Evaluate the damped eigenvalues: Rewrite Eq (1), as

\[ A \frac{dY}{dt} + B Y = Q(t), \]

where \( Y = [ \frac{dU}{dt}, U ]^T \), and \( Q = [0, F(t) ]^T \) are 2n row vectors of (velocity, displacements ) and generalized forces; and initial conditions \( Y_0 = [ Vo, Uo ] \)

\[
A = \begin{pmatrix}
0 & M \\
M & C
\end{pmatrix}, \quad
B = \begin{pmatrix}
-M & 0 \\
0 & K
\end{pmatrix}
\]

are 2nx2n Symmetric matrices
2.1 define the A & B augmented matrices:
zero := identity(n) - identity(n)  the (nxn) null matrix
A := stack(augment(zero, M), augment(M, C))
B := stack(augment(-M, zero), augment(zero, K))

2.2 Use MATHCAD function to calculate eigenvectors and eigenvalues of the
generalized eigenvalue problem, M X = α N X.

In vibrations problems we set the problem as: α A φ + B φ = 0,
 hence M=-B, N=A to use properly the MATHCAD functions genvecs & genvals
α := genvals(B, -A)_rad/s
ΦD := genvecs(B, -A)

\[
α = \begin{pmatrix}
-5.33 + 165.43i \\
-5.33 - 165.43i \\
-33.31 + 474.21i \\
-33.31 - 474.21i \\
-21.36 + 803.53i \\
-21.36 - 803.53i
\end{pmatrix}
\]

Recall the undamped natural frequencies
ω = \begin{pmatrix}
165.46 \\
475.33 \\
804.17
\end{pmatrix}

note that the (damped) eigenvalues are complex conjugates, with the same real part
and +/- imaginary parts

\[
ΦD = \begin{pmatrix}
-0.35 - 0.12i & -0.35 + 0.12i & -0.38 + 0.75i \\
-0.6 - 0.21i & -0.6 + 0.21i & 0.07 - 0.21i \\
-0.64 - 0.23i & -0.64 + 0.23i & 0.22 - 0.44i \\
-6.34 \times 10^{-4} + 2.13i \times 10^{-3} & -6.34 \times 10^{-4} - 2.13i \times 10^{-3} & 1.62 \times 10^{-3} + 6.97i \times 10^{-4} \\
-1.18 \times 10^{-3} + 3.64i \times 10^{-3} & -1.18 \times 10^{-3} - 3.64i \times 10^{-3} & -4.48 \times 10^{-4} - 1.14i \times 10^{-4} \\
-1.28 \times 10^{-3} + 3.91i \times 10^{-3} & -1.28 \times 10^{-3} - 3.91i \times 10^{-3} & -9.63 \times 10^{-4} - 4.05i \times 10^{-4}
\end{pmatrix}
\]

Note that the eigenvectors are conjugate pairs, i.e. they show the same real part and
 +/- imaginary part. In addition the first n-rows of an eigenvector are proportional to
the 2nd n-rows. The proportionality constant is the damped eingenvalue.
2.2. Form "damped" modal matrices using the orthogonality properties:

\[ \sigma := \Phi_D^T \cdot A \cdot \Phi_D \]

\[ \beta := \Phi_D^T \cdot B \cdot \Phi_D \]

\[
\sigma = \begin{pmatrix}
0.54 - 0.75i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.54 + 0.75i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.26 + 0.26i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.26 - 0.26i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.12 + 0.11i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.12 - 0.11i & 0
\end{pmatrix}
\]

\[
\beta = \begin{pmatrix}
-121.82 - 93.33i & 6.53 \times 10^{-15} - 1.82i \times 10^{-15} & 9.12i \times 10^{-15} \\
6.53 \times 10^{-15} + 1.82i \times 10^{-15} & -121.82 + 93.33i & 2.54 \times 10^{-15} + 1.29i \times 10^{-15} \\
3.17 \times 10^{-15} + 4.86i \times 10^{-15} & 2.57 \times 10^{-15} + 1.99i \times 10^{-14} & 113.59 + 132.39i \\
2.57 \times 10^{-15} - 1.99i \times 10^{-14} & 3.17 \times 10^{-15} - 4.86i \times 10^{-15} & -1.3 \times 10^{-14} \\
4.71 \times 10^{-15} - 2.16i \times 10^{-14} & 8.43 \times 10^{-15} + 2.27i \times 10^{-14} & -2.45 \times 10^{-14} + 1.33i \times 10^{-14} \\
8.43 \times 10^{-15} - 2.27i \times 10^{-14} & 4.71 \times 10^{-15} + 2.16i \times 10^{-14} & 2.34 \times 10^{-15} + 1.56i \times 10^{-14}
\end{pmatrix}
\]

Note off-diagonal terms are small - but NOT zero (as they should)

Check the orthogonality property: compare \( \beta/\sigma \) ratios to eigenvalues

**undamped:**

\[
\omega = \begin{pmatrix}
165.46 \\
475.33 \\
804.17
\end{pmatrix}
\]
for underdamped systems only

\[ \omega_d = \omega_n \left( 1 - \zeta^2 \right)^{0.5} \]

\[ \omega_n = \frac{\text{Re}(\alpha_j)}{-\xi_j} \]

\[ \omega_d = \text{Im}(\alpha_j) \]

\[ \xi_j := \left[ \frac{1}{\left( \frac{\text{Im}(\alpha_j)}{\text{Re}(\alpha_j)} \right)^2 + 1} \right]^{0.5} \]

\[ j := 1 \ldots 2 \cdot n \]

\[ \tilde{\xi}_j = \begin{bmatrix} 0.03 & 0.03 & 0.07 & 0.07 & 0.03 & 0.03 \end{bmatrix} \]

\[ \omega_n^T = \begin{bmatrix} 165.52 & 165.52 & 475.37 & 475.37 & 803.81 & 803.81 \end{bmatrix} \]

\[ \omega_d^T = \begin{bmatrix} 165.43 & -165.43 & 474.21 & -474.21 & 803.53 & -803.53 \end{bmatrix} \]

\[ \omega^T = \begin{bmatrix} 165.46 & 475.33 & 804.17 \end{bmatrix} \]
3. The 2n first order equations \( A \frac{dY}{dt} + B Y = Q(t) \) with the transformation \( Y=\Phi_d Z \) **become** 2n equations of the form:

\[
\sigma_i \frac{dZ_i}{dt} + \beta_i Z_i = G_i, \quad i=1,2,3,...,2n
\]

where \( G=\Phi_d^T Q(t) \) and initial conditions \( Z_0=\sigma^{-1} \Phi_d^T A Y_0 \):

3.1 solve the 2n-first order differential equations for the **Free Response** to initial conditions, \( F(t)=0 \):

From initial conditions in displacement and velocity, set \( Y_0 := \text{stack}(V_0, U_0) \)

and in **damped modal space**:

\[
p := 1..N \quad \text{number of data points}
\]

\[
t_p := (p - 1) \cdot \Delta t \quad \text{time sequence}
\]

get the modal response:

\[
Z_{j,p} := Z_{0j} \cdot e^{\alpha_j t_p}
\]

and back into the **physical coordinates**:

\[
Y = \Phi_d Z
\]

FREE RESPONSE using DAMPED MODES

\[
Y_{s,p} := \sum_{q=1}^{m} \Phi_{D_{s,q}} \cdot Z_{q,p}
\]

\( j := 1..2\cdot n \)
Recall:

\[ \mathbf{U}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

**Response: U1**

![Graph showing damped and undamped responses for U1](image)

**Response: U2**

![Graph showing damped and undamped responses for U2](image)

**Response: U3**

![Graph showing damped and undamped responses for U3](image)

\[ \mathbf{T} = \begin{pmatrix} 0.03 & 0.03 & 0.07 & 0.07 & 0.03 & 0.03 \end{pmatrix} \]
4. Forced response to step load $F(t) = F_0$ and initial conditions

Set initial conditions in displacement and velocity:

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\quad \begin{pmatrix}
30000 \\
10^5 \\
-10^5
\end{pmatrix}
\quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

STEP FORCE

**STEP RESPONSE**

Then set:

\[
Y_0 := \text{stack}(V_0, U_0)
\]

and transform to damped modal space:

\[
Z_0 := \sigma^{-1} \Phi_D^T A \cdot Y_0
\]

\[
T_{\text{max}} := \frac{8 \text{secs}}{f_1}, \quad \Delta t := \frac{T_{\text{max}}}{N}
\]

\[
p := 1 \ldots N
\]

\[
t_p := (p - 1) \cdot \Delta t
\]

Then:

\[
Z_{j, p} := Z_{0j} \cdot e^{\alpha_j t_p} + \frac{G_{0j}}{\beta_{j, j}} \cdot (1 - e^{\alpha_j t_p})
\]

and in physical coordinates:

\[
Y = \Phi_d Z
\]

**STEP RESPONSE**

using DAMPED MODES

6. For completeness, obtain also the undamped forced response:

\[
\eta_0 := Mm^{-1} \Phi^T M \cdot U_0
\]

\[
u_0 := Mm^{-1} \Phi^T M \cdot V_p := \Phi^T F_0
\]

\[
j := 1 \ldots n
\]

\[
\eta_{j, p} := \eta_{0j} \cdot \cos(\omega_j \cdot t_p) + \frac{\nu_{0j}}{\omega_j} \cdot \sin(\omega_j \cdot t_p) + \frac{p_j}{Km_j, j} \cdot (1 - \cos(\omega_j \cdot t_p))
\]

and into physical coordinates:

\[
Y_s, p := \sum_{q = 1}^{m} \Phi_{D_s, q} \cdot Z_{q, p}
\]

**STEP RESPONSE**

using UNDAMPED MODES

\[
U := \Phi \cdot \eta
\]

**STEP RESPONSE**

using UNDAMPED MODES

\[
U_s, p := \sum_{q = 1}^{n} \Phi_{s, q} \cdot \eta_{q, p}
\]

**STEP RESPONSE**

using UNDAMPED MODES

\[
U_s := K^{-1} \cdot F_0
\]

**Plot the displacements (last $n$ rows of $Y$ vector):**

\[ U_s^T = \begin{pmatrix} 3 \times 10^{-3} & 3 \times 10^{-3} & -2 \times 10^{-3} \end{pmatrix} \]

\[ \xi^T = (0.03 \quad 0.03 \quad 0.07 \quad 0.07 \quad 0.03 \quad 0.03) \]
5. **Periodic response to loading**, \( F(t) = F_0 \sin(\Omega t) \):

Response due to initial conditions vanishes after a long time because of damping:

Freq. of excitation

\[
F_0 = \begin{pmatrix} 3 \times 10^4 \\ 1 \times 10^5 \\ -1 \times 10^5 \end{pmatrix}
\]

\[
f\text{Hz} = 80
\]

\[
\Omega = 2 \cdot \pi \cdot f\text{Hz}
\]

\[
\Omega = 502.65
\]

\[
\omega = \begin{pmatrix} 165.46 \\ 475.33 \\ 804.17 \end{pmatrix}
\]

Recall the undamped frequencies

Plot the displacements (last n rows of Y vector):

**RESPONSE: U1**

![Graph showing damped and undamped responses for U1](image)

**RESPONSE: U2**

![Graph showing damped and undamped responses for U2](image)

**RESPONSE: U3**

![Graph showing damped and undamped responses for U3](image)
6) FRF Response to periodic loading, $F = F_0 \cos(\Omega t)$

**UNDAMPED CASE**

Set:

$$\dot{\xi} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{p} := \Phi^T \cdot F_0$

Modal force magnitude:

$$\omega_j = \begin{bmatrix} 165.46 \\ 475.33 \\ 804.17 \end{bmatrix}$$

And the modal "S-S" response is modes.

$$j := 1 \ldots n$$

$$\Omega_{\text{max}} := 3 \cdot \omega_n$$

$k_{\text{min}} := 1$ \hspace{1cm} $k_{\text{max}} := 200$

$$k := k_{\text{min}} \ldots k_{\text{max}}$$

$$\Omega_k := \frac{k}{k_{\text{max}}} \cdot \Omega_{\text{max}}$$

Modal response:

$$q_{j,k} := \frac{p_j}{K_{mj,j}} \cdot \frac{1}{1.0 - \left( \frac{\Omega_k}{\omega_j} \right)^2 + \left( 2 \cdot \xi_j \cdot \frac{\Omega_k}{\omega_j} \right) i} \times \cos(\Omega t)$$

$s := 1 \ldots n$

FRF in physical plane:

$$U_{s,k} := \sum_{i = 1}^{n} \Phi_{s,i} \cdot q_{i,k} \times \cos(\Omega t)$$

**DAMPED CASE**

Set:

$Q_0 := \text{stack}(O, F_0)$

Forcing vector:

$$\mathbf{G}_0 := \Phi_D^T \cdot Q_0$$

$$k := k_{\text{min}} \ldots k_{\text{max}}$$

$$\Omega_k := \frac{k}{k_{\text{max}}} \cdot \Omega_{\text{max}}$$

$$j := 1 \ldots 2 \cdot n$$

Modal response:

$$Z_{j,k} := \frac{G_0_j}{\beta_{j,j}} \cdot \frac{1}{1 - i \cdot \frac{\Omega_k}{\alpha_j}} \times \cos(\Omega t)$$

And back into the physical plane:
\( s := 1 \ldots 2n \quad m := 2n \)

**PERIODIC RESPONSE using DAMPED MODES**

\[
Y_{s,k} := \sum_{q=1}^{m} \phi_{D_{s,q}} Z_{q,k} \cdot x \cos(\Omega t)
\]

Plot magnitude of response in physical space
DAMPED Amplitude FRF

Damped physical response

Amplitude vs. frequency (rad/s)

UNDAMPED Amplitude FRF

UnDamped physical response

Amplitude vs. frequency (rad/s)
select coordinate to displace physical response - damped & undamped

\[ j := 2 \quad n = 3 \quad \text{DOFs} \]

LINEAR vertical scale

undamped:

\[ \omega = (165.46 \quad 475.33 \quad 804.17) \]