Handout #2c (pp. 56-71)

Dynamic Response of Second Order Mechanical Systems with Viscous Dissipation forces

$$M\,\ddot{X} + D\,\dot{X} + K\,X = F_{(t)}$$

Interpretation of Forces for the Periodic Forced Response of a 2nd Order Mechanical System.

Transmissibility: Analysis of forces transmitted to base or foundation

Frequency Response Function for Support or Ground Motion

Interpretation of Forces for the Periodic Forced Response of a 2nd Order Mechanical System

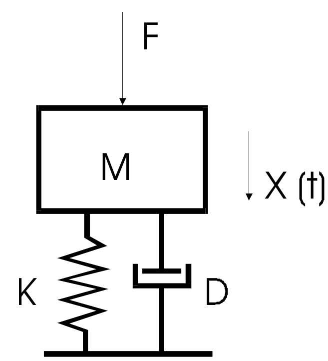
Recall that the equation of motion for a 2nd order system forced into motion by a harmonic force is:

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=F_o\sin(\Omega t) \quad (61)$$

or as a balance of forces: $\boxed{F(t)+F_D+F_K+F_I=0}$ (62)

with solution $X(t)=X_{ss}\,A\sin(\Omega t-\varphi)$ (63)

and $X_{ss}=\dfrac{F_o}{K}$: $\dfrac{\text{magnitude of external force}}{\text{stiffness}}$



where: Damping Force, $F_D=-D\dot{X}$

Elastic Force, $F_K=-KX$ (64)

Inertia Force, $F_I=-M\ddot{X}$

From Eq. (63)

DISPLACEMENT $\Rightarrow$ $X(t)=X_{ss}\,A\sin(\Omega t-\varphi)$

VELOCITY $\Rightarrow$ $\dot{X}(t)=X_{ss}\Omega A\cos(\Omega t-\varphi)$ (65)

ACCELERATION $\Rightarrow$ $\ddot{X}(t)=-X_{ss}\Omega^2A\sin(\Omega t-\varphi)=-\Omega^2X(t)$

Then $F_K(t)=-F_o\,A\sin(\Omega t-\varphi)$

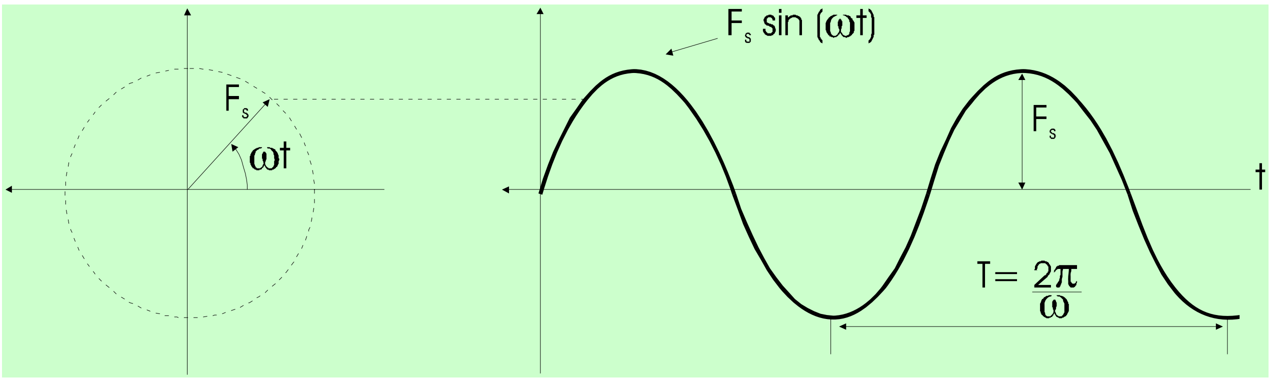
$\boxed{f=\Omega/\omega_n}$

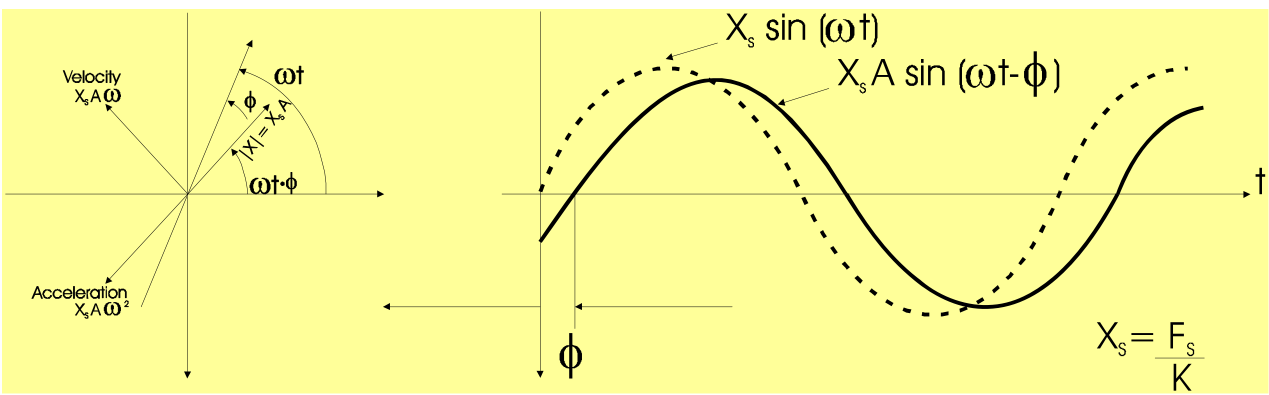
$$F_D=-D\dot{X}(t)=-F_o(2\zeta f)A\cos(\Omega t-\varphi) \quad (66)$$

$$F_I(t) = -M\,\ddot{X} = F_o\ f^2\, A \sin\left(\Omega t - \varphi\right)$$

$$\boxed{\zeta = \frac{D}{2\sqrt{K\,M}}}$$

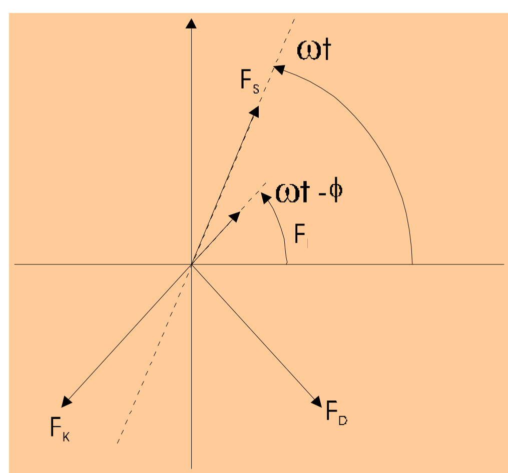
Let's plot the external force and ($X, \dot{X}, \ddot{X}$) as rotating **phasors**.





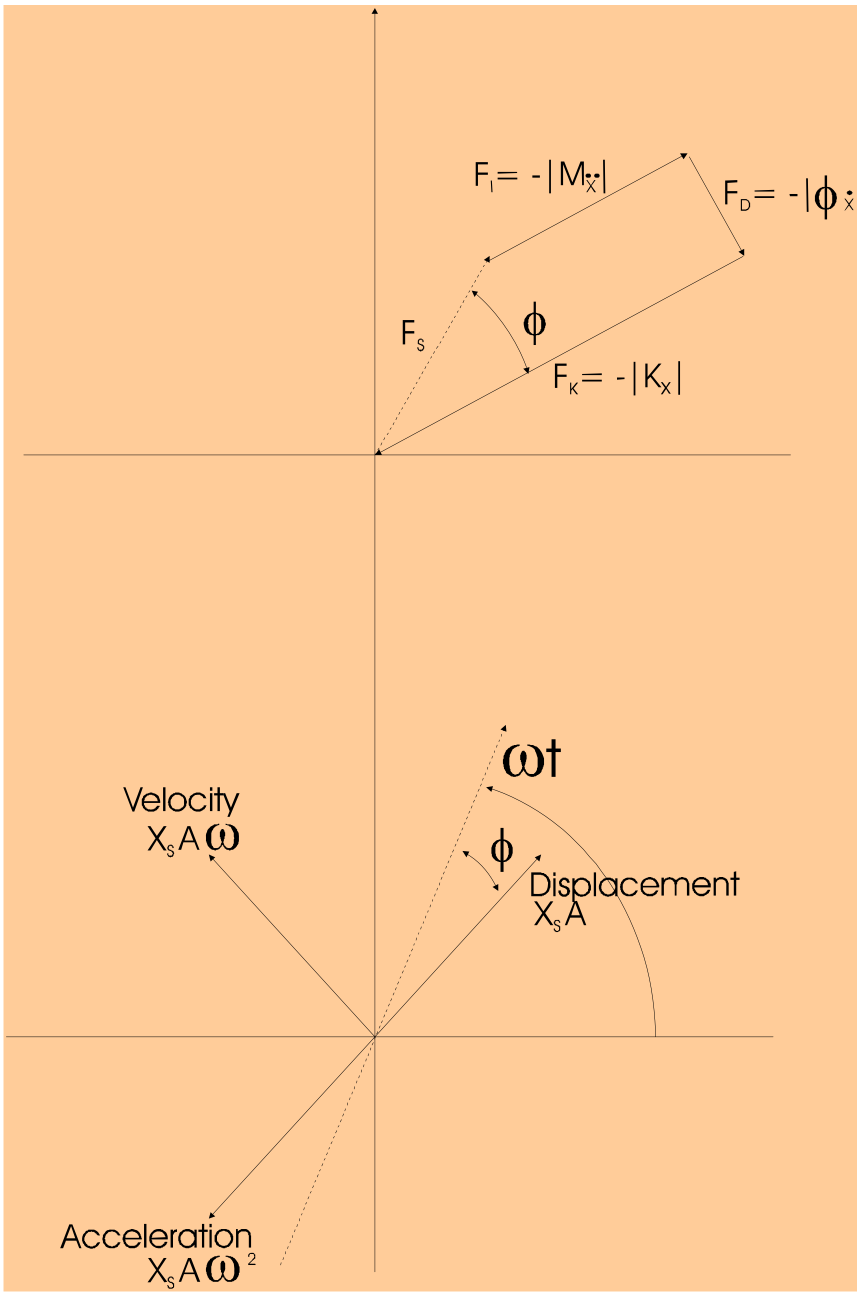
Note that the **velocity** $\dot{X}$ leads by 90° the displacement $X(t)$, and the **acceleration** $\ddot{X}$ leads by 180° the displacement $X(t)$.

The elastic, damping and inertia forces are graphed as:

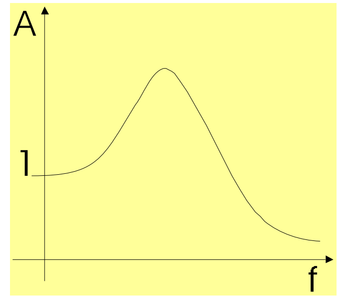


$$M\frac{d^2 X}{dt^2}+D\frac{dX}{dt}+K\,X=F_o\sin\left(\Omega t\right) \qquad (61)$$

Balance of Forces for 2nd Order System: $\boxed{F(t)+F_D+F_K+F_I=0}$



Let's study the forced response at various excitation frequencies of:



At low frequencies:

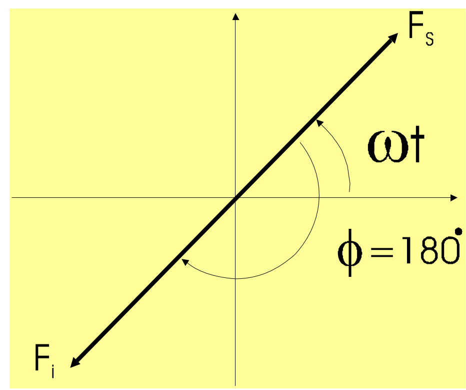
$$\Omega \ll \omega_n \rightarrow f \ll 1$$ Then: $$\Rightarrow A \rightarrow 1 \;\; \varphi \rightarrow 0$$

$$F_K(t) = -F_o\, A \sin\left(\Omega t - \varphi\right) \approx -F_o\, 1 \sin\left(\Omega t - 0\right) \approx -F(t)$$

$$F_D \approx 0, \; F_I \approx 0$$

At low frequencies the elastic force F_K **balances** the **external force** $F_{(t)}$.

At high frequencies:



$$\Omega \gg \omega_n \rightarrow f \gg 1$$ then

$$as\; f \rightarrow \infty \Rightarrow A \rightarrow {}^{1}\!/_{f^2} \sim 0 \;\;,\;\; \varphi \rightarrow \pi \;\; \left(180^\circ\right)$$

$$A\, f^2 \rightarrow 1 \;,\; A\, f^2 \rightarrow 1/f \rightarrow 0$$

and

$$F_K(t) \approx 0 = -F_o\, A \sin\left(\Omega t - \varphi\right) \approx F_o\, 0 \sin\left(\Omega t - 0\right) \approx 0$$

$$F_D \approx 0$$

$$F_I(t) = F_o\, \left(f^2 A\right) \sin\left(\Omega t - \varphi\right) \approx F_o\; 1 \sin\left(\Omega t - \pi\right) =$$

$$= -F_o\;\sin\left(\Omega t\right) = -F(t)$$

At high frequencies the **inertia force F_I balances** the external force $F_{(t)}$

At resonant conditions, i.e. excitation with a frequency close to the system natural frequency,

$$\Omega = \omega_n \rightarrow f = 1 \qquad \Rightarrow A \rightarrow \frac{1}{2\zeta} \; ; \; \varphi \rightarrow \frac{\pi}{2} \; \left(90^\circ\right)$$

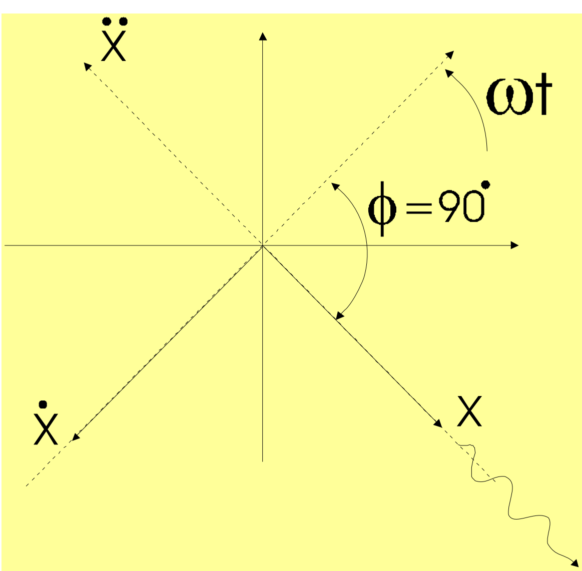
$$F_K(t) = -F_o\, A \sin\left(\Omega t - \pi/2\right) = F_o\, A \cos\left(\Omega t\right)$$

$$F_I(t) = F_o\, A\, f^2 \sin\left(\Omega t - \pi/2\right) = -F_K$$

$$F_D = -F_o\left(2\zeta\, f\right)\frac{1}{2\zeta} \cos\left(\Omega t - \pi/2\right) =$$

$$= -F_o \cos\left(\Omega t - \pi/2\right) = -F_o \sin\left(\Omega t\right) = -F(t)$$

i.e., the **viscous damping force** balances the **external force.**

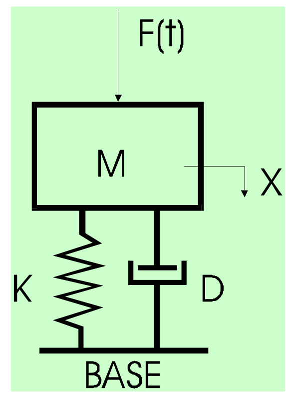


If no damping is present, then the equilibrium of forces is not possible and the system develops amplitudes of motion increasing and leading to a catastrophic failure.

TRANSMISSIBILITY: Analysis of forces transmitted to base or foundation

The analysis of transmitted forces is important to determine the maximum stresses on the structural supports as well as to verify the isolation characteristics of the system from the base.

The equation of motion of a (*M,K,D*) system for a periodic force of constant magnitude F_o and frequency (Ω) is:



$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+K\,X=F_o\sin\left(\Omega t\right) \tag{61}$$

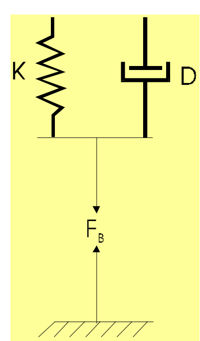
with solution $$X(t)=X_{ss}\,A\sin\left(\Omega t-\varphi\right) \tag{62}$$

where

$$X_{ss}=\frac{F_o}{K}\quad ;\ A\ =\ \frac{1}{\left[\left(1-f^2\right)+\left(2\,\zeta\,f\right)^2\right]^{1/2}}\qquad \varphi=\tan^{-1}\left(\frac{2\,\zeta\,f}{1-f^2}\right)$$

and $f=\Omega/\omega_n$ as the ratio of the excitation frequency to the system natural frequency.

The **dynamic force transmitted to the base** or foundation is:



$$F_B=F_D+F_K=D\,\dot{X}+K\,X \tag{64}$$

Substitution of Eq. (62) into Eq. (64) gives,

$$F_B = K\, X_{ss}\, A \;\sin(\Omega t-\varphi) + D\, X_{ss}\, A\,\Omega\; \cos(\Omega t-\varphi) \qquad (65)$$

$$F_B = F_o\, A\left[\;\sin(\Omega t-\varphi) + \frac{D\,\Omega}{K}\;\cos(\Omega t-\varphi)\right]$$

with $\frac{D}{K} = \frac{2\zeta}{\omega_n}$ and $f = \Omega/\omega_n$, then

$$F_B = F_o\, A\left[\;\sin(\Omega t-\varphi) + 2\zeta\, f\;\cos(\Omega t-\varphi)\right] \qquad (66)$$

Define: $\cos(\alpha) = \frac{1}{\sqrt{1+(2\zeta\, f\,)^2}}$; $\sin(\alpha) = \frac{2\zeta\, f}{\sqrt{1+(2\zeta\, f\,)^2}}$

And write Eq. (66) as:

$$F_B = F_o\, A\sqrt{1+(2\zeta\, f\,)^2}\;\left[\;\cos(\alpha)\,\sin(\Omega t-\varphi) + \sin(\alpha)\;\cos(\Omega t-\varphi)\right]$$

$$F_B = F_o\, A\sqrt{1+(2\zeta\, f\,)^2}\;\sin(\Omega t-\varphi+\alpha)$$

$$F_B = F_o\; A_T\;\sin\left(\Omega t - \phi_T\right) \qquad (68)$$

where

$$A_T = \frac{\left[1+(2\,\zeta\, f)^2\,\right]^{1/2}}{\left[\left(1\text{-}\, f^2\right)+\left(2\,\zeta\, f\right)^2\right]^{1/2}};\quad \varphi_T = \varphi+\alpha;$$

$$\varphi = \tan^{-1}\left(\frac{2\,\zeta\, f}{1-f^2}\right);\quad \alpha = \tan^{-1}\left(2\,\zeta\, f\right) \qquad (69)$$

Define the **transmissibility (T)** as the ratio of force transmitted to base or foundation $|F_B|$ to the (input) excitation force $|F_0 \sin(\Omega t)|$, i.e.,

$$T = \left|\frac{F_B}{F_o}\right| \equiv A_T = \frac{\sqrt{1+(2\zeta f)^2}}{\sqrt{(1-f^2)^2+(2\zeta f)^2}} \qquad (71)$$

Regimes of operation:

at low frequencies:

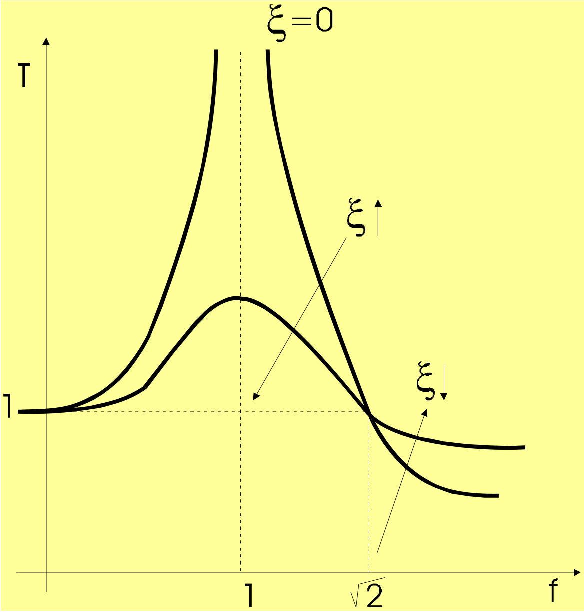
$$\Omega \ll \omega_n \rightarrow f \rightarrow 0 \Rightarrow A_T = 1$$

at high frequencies:

$$\Omega \gg \omega_n \rightarrow f \rightarrow \infty \Rightarrow A_T = 2\zeta / f$$

at resonance:

$$\Omega = \omega_n \rightarrow f = 1 \Rightarrow A_T = \frac{\sqrt{1+4\zeta^2}}{2\zeta}$$



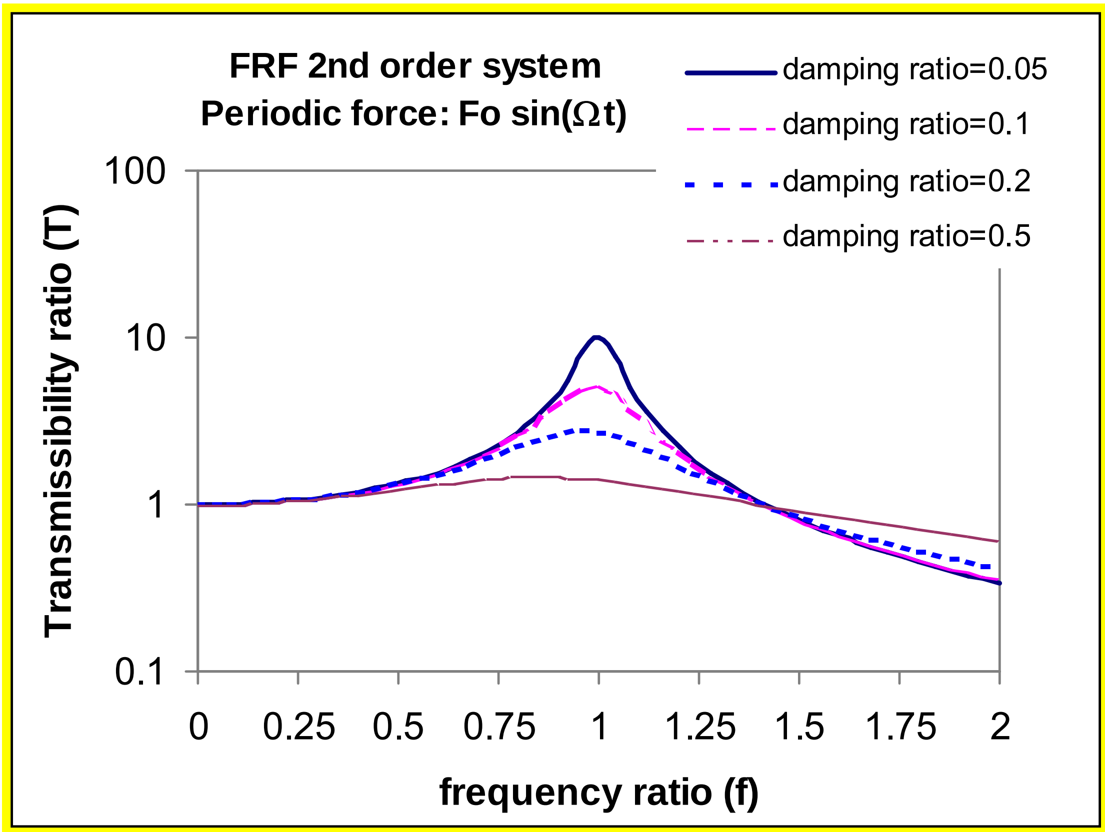
NOTES:

At low frequencies, $f < \sqrt{2}$, the transmitted force (to base) is larger than external force, i.e. *T>1*

At $f = \sqrt{2}$, the system shows the same transmissibility regardless of the damping value.

Operation above $f > \sqrt{2}$ determines the lowest transmitted forces, i.e. mechanical system is **ISOLATED** from base (foundation). A desirable operating condition

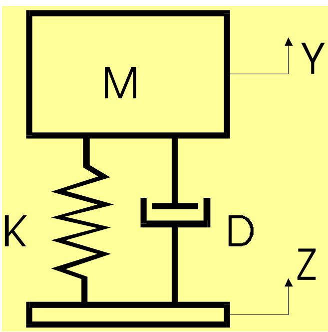
When operation at large frequencies, $f > \sqrt{2}$, viscous damping causes transmitted forces to be larger than w/o damping. Damping is NOT desirable for operation at high frequencies.



Frequency Response Function for Support or Ground Motion

Consider the motion of a (*M, K, D*) system with its base (or support) moving with known or specified periodic displacement $Z_{(t)} = b \cos(\Omega t)$.

The dynamic response of this system is of particular importance for the correct design and performance of vehicle suspension systems. Response to earthquake excitations as well.



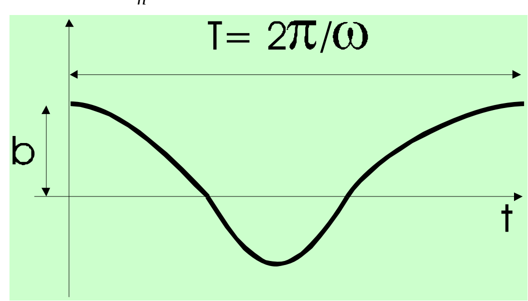
The EOM is:

$$M\ddot{Y} + K(Y - Z) + D(\dot{Y} - \dot{Z}) = 0 \qquad (71)$$

or

$$M\ddot{Y} + D\dot{Y} + KY = KZ + D\dot{Z} \qquad (72)$$

Recall $\omega_n = \sqrt{\frac{K}{M}}$; $\zeta = \frac{D}{2\,\omega_n M}$.



Since $Z = b\cos(\Omega t)$ is prescribed, then

$$\dot{Z} = -b\,\Omega \sin(\Omega t)$$

Substitution of *Z* and d*Z*/dt into eqn. (72) gives

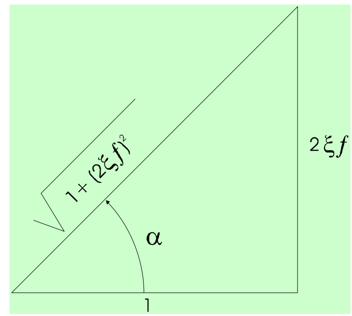
$$\begin{aligned} M\ddot{Y} + D\dot{Y} + KY &= Kb\cos(\Omega t) - D\Omega\, b\sin(\Omega t) \\ &= Kb\left[\cos(\Omega t) - \frac{D}{K}\Omega\sin(\Omega t)\right] \\ &= Kb\left[\cos(\Omega t) - 2\zeta f\sin(\Omega t)\right] \end{aligned} \qquad (73a)$$

where $\frac{D}{K}=\frac{2\zeta}{\omega_n}$ and $f=\Omega/\omega_n$

The equation of motion is rewritten as:

$$M\,\ddot{Y}+D\,\dot{Y}+K\,Y=\;K\,b\left[\cos(\Omega t)-2\zeta f\,\sin(\Omega t)\right] \quad (73b)$$

Let : $\cos(\alpha)=\frac{1}{\sqrt{1+(2\zeta f)^2}}$; $\sin(\alpha)=\frac{2\zeta f}{\sqrt{1+(2\zeta f)^2}}$



and write Eq.(73b) as:

$$M\,\ddot{Y}+D\dot{Y}+K\,Y=\;K\,b\sqrt{1+(2\zeta f)^2}\;\cos(\Omega t-\alpha)=F_o\cos(\Omega t-\alpha)$$

(74)

After all transients die out due to damping, the system periodic steady-state response (or FRF) is:

$$Y(t)=\frac{K}{K}\,b\,\sqrt{1+(2\,\zeta\,f)^2}\;A\;\cos(\omega t-\;\alpha-\varphi) \quad (75)$$

or

FRF of Base Motion

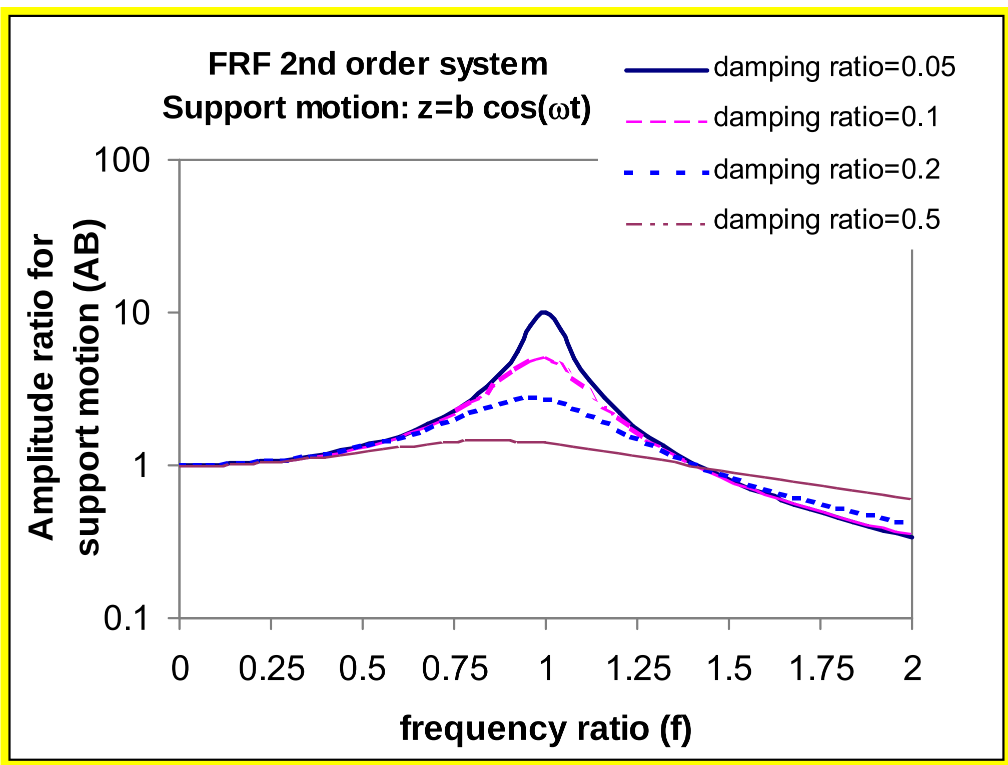
$$Y(t) = b\,A_B\ \cos\left(\omega t - \varphi_B\right) \tag{76}$$

$$A_B = \frac{\left[1+\left(2\,\zeta\,f\right)^2\right]^{1/2}}{\left[\left(1-f^2\right)+\left(2\,\zeta\,f\right)^2\right]^{1/2}}; \quad \varphi_B = \varphi + \alpha;$$

with (77)

$$\varphi = \tan^{-1}\left(\frac{2\,\zeta\,f}{1-f^2}\right); \quad \alpha = \tan^{-1}\left(2\,\zeta\,f\right)$$

NOTE that A_B is identical to the amplitude of FRF for transmitted force, i.e. the transmissibility ratio



EXAMPLE:

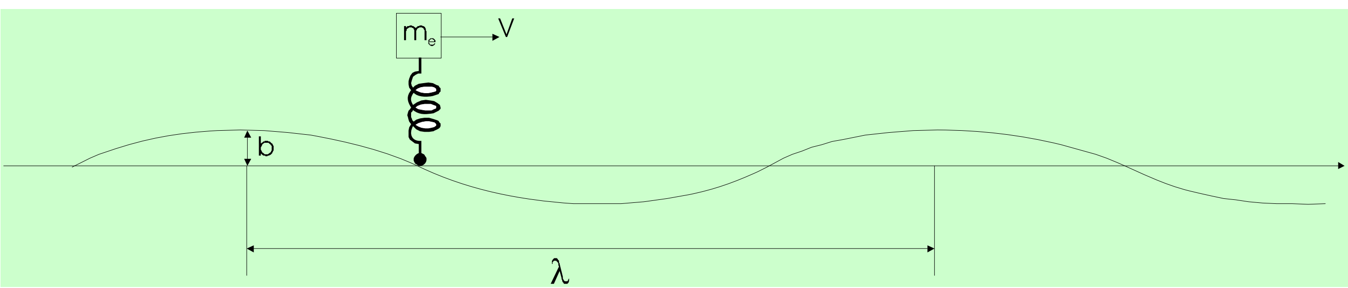
A 3000 lb (empty) automobile with a 10' wheel-base has wheels which weigh 70 lb each (with tires). Each tire has an effective stiffness (contact patch to ground) of 1000 lb/in. A static test is done in which 5 passengers of total weight 800 lb climb inside and the car is found to sag (depress toward the ground) by 2".

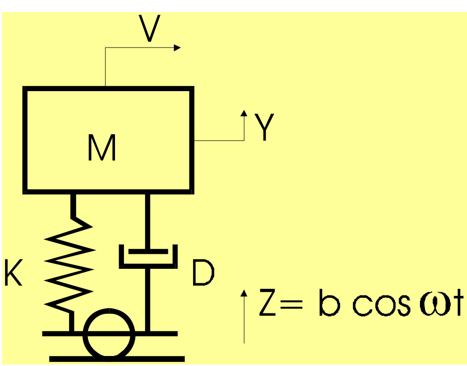
(a) From the standpoint of the passenger comfort, what is the worst wavelength (in feet) [sine wave road] which the car (with all 5 passengers) could encounter at 65 mph?

(b) For the worst case in (a) above, what percent of critical damping is required to keep the absolute amplitude of the vertical heaving oscillations less than ½ of the amplitude of the undulated road?

(c) What is the viscous damping coefficient required for the shock absorber on each wheel (assume they are all the same) to produce the damping calculated in (b) above? Give the physical units of your answer.

(d) State which modes of vibration you have neglected in this analysis and give justifications for doing so.





The wavelength is equal to $\lambda = v\,T$, with T as the period of motion. And the frequency **(ω)** of the forced motion is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi\, v}{\lambda}$$

The system mass is:

$$M_{eq} = \frac{W}{g} = \frac{(3000 + 800 - 4x70)}{g}$$

$$K = \frac{800 \text{ lb}}{2 \text{ in}} = 400 \frac{\text{lbf}}{\text{in}}; \quad M_{eq} = \frac{3{,}520 \text{ lbf}}{386.4 \text{ in/sec}^2} = 9.12 \frac{lbf \text{ sec}^2}{in}$$

$$\omega_n = \left(\frac{K}{M_{eq}}\right)^{1/2} = 6.2\frac{rad}{sec}\ (1.05\ Hz)$$

(a) For passenger comfort, the worst wavelength (in feet) which the car could encounter at 65 mph is when the excitation frequency coincides with the system natural frequency, i.e. $\omega = \omega_n$. Thus from

$\omega = \frac{2\pi}{T} = \frac{2\pi v}{\lambda} = \omega_n$. Then

$$\lambda = \frac{2\pi v}{\omega_n} = \frac{2\pi\ 65\ mph \cdot \left(\frac{5,280\ ft/mile}{3,600\ \sec/hour}\right)}{6.62\ \frac{rad}{\sec}} = \lambda = 90.47\ \text{ft} = (0.0171\ \text{miles})$$

(b) For the worst case what percent of critical damping is required to keep the absolute amplitude of the vertical heaving oscillations less than ½ of the amplitude of the undulated road? i.e. What value of damping ratio (ζ) makes $\frac{|Y|}{b} = \frac{1}{2}$ at $\omega = \omega_n$?

Recall that at $\omega = \omega_n$, the amplitude of the support FRF is from eqn. (77):

$$A_B = \left[\frac{1+(2\zeta)^2}{(2\zeta)^2}\right]^{1/2} \Rightarrow \frac{1}{2} \quad ? \quad \frac{1}{4\zeta^2} = -\frac{3}{4}$$

The solution indicates that the damping ratio (ζ) is imaginary! This is clearly impossible. Note that the amplification ratio $\boldsymbol{A}_B > 1$ at $\boldsymbol{f} = 1$, i.e. the amplitude of motion **|Y|** for the system will always be larger than the amplitude of the base excitation (***b***), regardless of the amount of damping.

(c) What is the viscous damping coefficient required for the shock absorber on each wheel (assume they are all the same) to produce the damping calculated in (b) above?

No value of viscous damping ratio (ζ) is available to reduce the amplitude of motion. However, if there should be one value, then

$$\zeta = \frac{D_{eq}}{2\,M_{eq}\,\omega_n} = \frac{4D}{2\,M_{eq}\,\omega_n}\ ;\ \text{then} \rightarrow D = \frac{1}{2}\,\zeta\,M_{eq}\,\omega_n \quad \left[\frac{\mathrm{lb}_f}{\mathrm{in/sec}}\right]$$

(d) State which modes of vibration you have neglected in this analysis and give justifications for doing so.

Heaving (up & down) motion is the most important mode and the one we have studied. In this example, pitching motion is not important because the road wavelength λ is large. We have also neglected yawing which is not important if the car cg is low.

One important mode to consider is the one related to "**tire bouncing**", i.e. the tires have a mass and spring coefficient of their own, and therefore, its natural frequency is given by

$$\omega_{n_{tire}} = \sqrt{\frac{1,000}{70/386.4}} = 74.25\frac{rad}{\sec}$$

However, the car bouncing natural frequency is 6.62 rad/sec is much lower than the tire natural frequency, i.e.

$$\omega_{n\,\mathrm{car}} = 6.62 \text{ rad/s} << \omega_{n\,\mathrm{tire}} = 74.25 \text{ rad/sec}$$

Therefore, it is reasonable to neglect the "tire" bouncing mode since its frequency is so high that it can not be excited by the road wavelength specified.