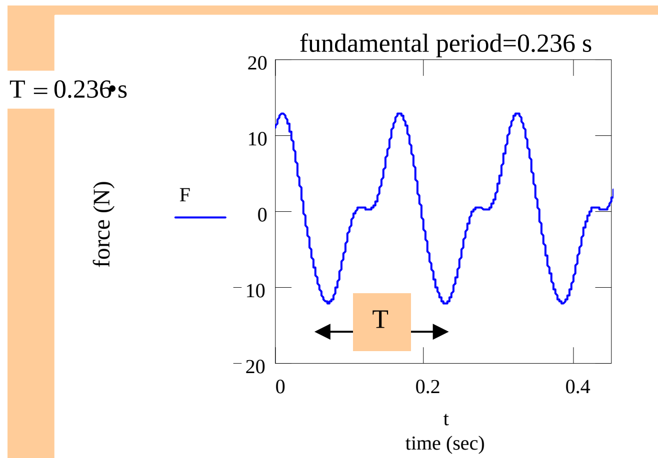
Handout #2d (pp. 72-82)

DYNAMIC RESPONSE OF A SDOF SYSTEM TO ARBITRARY PERIODIC LOADS

Fourier Series

Forces acting on structures are frequently periodic or can be approximated closely by superposition of periodic loads. As illustrated, the **function *F(t)* is periodic but not harmonic**.



Any periodic function, however, can be represented by a convergent series of harmonic functions whose frequencies are integer multiples of a certain fundamental frequency Ω.

The integer multiples are called harmonics. The series of harmonic functions is known as a **FOURIER SERIES**, and written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t) \qquad \textbf{(1)}$$

with $F(t+T) = F(t)$ and where $T = 2\pi/\Omega$ is the fundamental period. a_n , b_n are the coefficients of the n_{th} harmonic, and calculated from

$$a_n = \frac{2}{T}\int_t^{t+T} F(t)\cos(n\Omega t)dt, \quad n = 0,1,2,\ldots\ldots\infty$$

(2)

$$b_n = \frac{2}{T}\int_t^{t+T} F(t)\sin(n\Omega t)dt, \quad n = 1,2,\ldots\ldots\infty$$

each representing a measure of the participation of the harmonic content of *cos(nΩt)* and *sin(nΩt)*, respectively. All the a_0, b_n, c_n have the units of a generalized force.

Note that $(\frac{1}{2} a_0)$ is the time averaged value of the function *F(t)*.

In practice *F(t)* may be approximated by a relatively small number of terms. Some useful simplifications arise

If *F(t)* is an EVEN function, i.e., *F(t) = F(-t)* then, $b_n = 0$ for all *n*

If *F(t)* is an ODD function, i.e., *F(t) = -F(-t)* then, $a_n = 0$ *for* all *n*

The Fourier series representation, Eq. (1), can also be written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\Omega t - \beta_n) \qquad \textbf{(3)}$$

where $c_n = \left(a_n^2 + b_n^2\right)^{1/2}$ and $\beta_n = \tan^{-1}\left(b_n / a_n\right), \quad n = 1,2,\ldots\ldots\infty$

are the magnitude and phase angle respectively of each harmonic content.

PERIODIC FORCED RESPONSE OF AN UNDAMPED SDOF

In an undamped SDOF system, the **steady state response** (w/o the transient solution) produced by each sine and cosine term in the harmonic loading series is given by as

$$Xs_m(t) = \frac{b_m/K}{1-f_m^{\,2}} \sin\left(m\Omega t\right) \quad \textbf{(4a)}$$

$$Xc_m(t) = \frac{a_m/K}{1-f_m^{\,2}} \cos\left(m\Omega t\right) \qquad m=1,2,\ldots \quad \textbf{(4b)}$$

where $f_m = {m\Omega}/{\omega_n}$, $\omega_n = \sqrt{K/M}$.

For the constant force a_0 , the s-s response is simply

$$X_0(t) = \frac{a_0}{K} \quad \textbf{(4c)}$$

Using the **principle of superposition**, then the total periodic response is expressed as the sum of the individual component as follows,

$$X(t) = \frac{1}{K}\left(\frac{a_0}{2} + \sum_{m=1}^{\infty} \frac{1}{\left(1-f_m^{\,2}\right)} \left[a_m \cos(m\Omega t) + b_m \sin(m\Omega t) \right] \right) \quad \textbf{(5)}$$

Note that for the undamped case if $m\Omega = \omega_n$, i.e. there is a harmonic frequency equal to the natural frequency of the system, then the system response will become UNBOUNDED (system failure).

PERIODIC FORCED RESPONSE OF A DAMPED SDOF

In a damped SDOF system, the **steady-state response** produced by each sine and cosine term in the harmonic load series is

$$Xc_m(t)=\frac{a_m}{K}\frac{\left[\left(1-f_m^{\,2}\right)\cos(m\Omega t)+\left(2\zeta f_m\right)\sin(m\Omega t)\right]}{\left[\left(1-f_m^{\,2}\right)^2+\left(2\zeta f_m\right)^2\right]} \qquad \textbf{(6a)}$$

$$Xs_m(t)=\frac{b_m}{K}\frac{\left[-\left(2\zeta f_m\right)\cos(m\Omega t)+\left(1-f_m^{\,2}\right)\sin(m\Omega t)\right]}{\left[\left(1-f_m^{\,2}\right)^2+\left(2\zeta f_m\right)^2\right]} \qquad \textbf{(6b)}$$

and for the constant term a_0 , the s-s response is

$$X_0=\frac{a_0}{2K} \qquad \textbf{(6c)}$$

Superposition gives the total system response as

$$X(t)=\frac{a_0}{2K}+\frac{1}{K}\sum_{m=1}^{\infty}\left[\frac{a_m\left(1-f_m^2\right)-2\zeta f_m b_m}{\left(1-f_m^2\right)^2+\left(2\zeta f_m\right)^2}\cos(m\Omega t)\right] + \frac{1}{K}\sum_{m=1}^{\infty}\left[\frac{b_m\left(1-f_m^2\right)+2\zeta f_m a_m}{\left(1-f_m^2\right)^2+\left(2\zeta f_m\right)^2}\sin(m\Omega t)\right] \qquad \textbf{(7)}$$

or,

$$X(t) = \frac{a_0}{2K} + \frac{1}{K} \sum_{m=1}^{\infty} \left[\frac{c_m}{\left(1 - f_m^2\right)^2 + \left(2\zeta f_m\right)^2} \cos(m\Omega t - \gamma_m) \right]$$

(8)

where, $f_m = {}^{m\Omega}\!/\!_{\omega_n}$; $\omega_n = \sqrt{K/M}$

and $c_m = \sqrt{a_m{}^2 + b_m{}^2}$;

$$\gamma_m = \tan^{-1} \frac{\left\{ b_m \left(1 - f_m{}^2\right) + 2\zeta f_m a_m \right\}}{\left\{ a_m \left(1 - f_m{}^2\right) - 2\zeta f_m b_m \right\}}, \quad m = 1, 2, \ldots . \infty$$

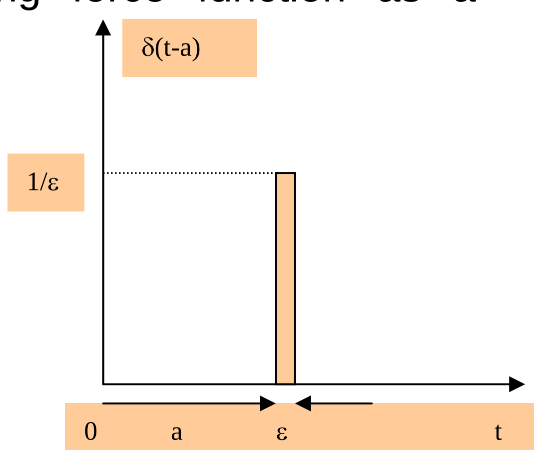
RESPONSE OF A SDOF SYSTEM TO NON-PERIODIC (ARBITRARY) FORCE EXCITATIONS

The steady-state response of a SDOF system to a periodic excitation of fundamental period T is also periodic with a fundamental period $T=2\pi/\Omega$.

Consider now an **arbitrary external force** (transient, non-periodic, etc.) Clearly, in this case there is no steady-state response and the entire system response may be regarded as transient.

Various approaches can be used to obtain the system dynamic response, including direct numerical integration. A natural extension would be to use **Fourier Integral transforms** obtained from the Fourier series in the limit as the period $T \to \infty$. **This topic is discussed at length in your textbook.**

Another way to find the dynamic response to arbitrary load excitations is to regard the acting force function as a **superposition of impulses** of very short duration.



First, introduce the concept of **unit impulse** or **Direct Delta Function** as shown in the Figure. The mathematical definition of a unit impulse is

$$\delta_{(t-a)} = 0 \;\; for\; t \neq a \qquad \textbf{(9)}$$

and such that $\int_{-\infty}^{+\infty} \delta_{(t-a)}\, dt = 1$

The time over which the function δ is different from zero is infinitesimally small, $\varepsilon \to 0$. Note that the physical units of δ are 1/sec.

An impulsive force of arbitrary magnitude *F* applied at time *t=a* is written conveniently as

$$F(t) = \hat{F}\,\delta_{(t-a)} \qquad \textbf{(10)}$$

where $\hat{F}$ has the units of impulse, i.e. N-sec or lb-sec.

Now, consider a damped SDOF mechanical system and find the response to the impulsive force applied at time *a=0* , i.e.

$$M\,\ddot{X} + D\,\dot{X} + K\,X(t) = \hat{F}\;\delta_{(t-a)} \qquad \textbf{(11)}$$

Recall that the force *F(t)* acts over a very short time, $\varepsilon \to 0$. Now, integrate Eq. (11) over time in the interval $\Delta t = \varepsilon \to 0$,

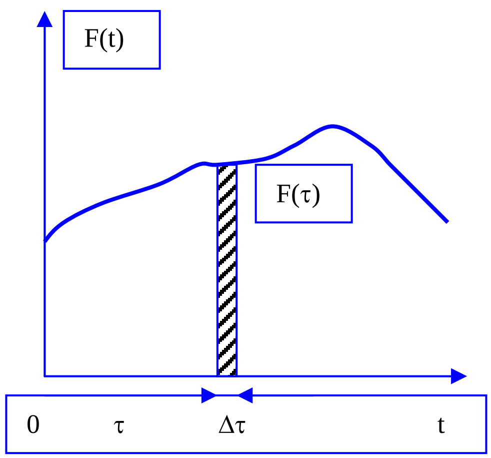
$$\int_0^{\varepsilon} \left(M\,\ddot{X} + D\,\dot{X} + K\,X \right) dt = \int_0^{\varepsilon} \hat{F}\,\delta_{(t)}\,dt \;= \hat{F} \int_0^{\varepsilon} \delta_{(t)}\,dt = \hat{F} \qquad \textbf{(12)}$$

Then,

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon} M\;\ddot{X}\;dt = M\;\dot{X}\Big|_0^{\varepsilon} = M\left[\dot{X}(\varepsilon) - \dot{X}(0)\right]$$
$$= M\left[\dot{X}(0_+) - \dot{X}(0)\right]$$

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon} D\;\dot{X}\;dt = D\;\lim_{\varepsilon \to 0} X\Big|_0^{\varepsilon} = D\;\lim_{\varepsilon \to 0}\left[X(0_+) - X(0)\right] \simeq 0$$

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon} K\ X\ dt = K \lim_{\varepsilon \to 0} X_{ave}\ \Delta t \approx 0$$



The notation $\dot{X}(0+)$ is interpreted as a change in velocity that occurs shortly after the time $\Delta t = \varepsilon \to 0$ elapses. **Note that there is no instantaneous change in displacement** $X(0+) \approx X(0)$ **because *Δt* is too short for displacements to happen (no displacement jump!)**. Thus,

$$\dot{X}(0+) = \hat{F}/M \qquad \textbf{(13)}$$

when $\dot{X}(0) = 0$ (initial null velocity).

As a physical interpretation, the **impulsive force produces an instantaneous change in velocity.** Hence, one can regard the effect of the impulse applied at *t=0* as the equivalent of an initial velocity equal to ($\hat{F}/M$).

Recall that the **free vibration** response of an underdamped *(*$(\zeta < 1)$*<1)* SDOF system to an **initial velocity** is,

$$X(t) = e^{-\zeta \omega_n t} \frac{1}{\omega_d} \frac{\hat{F}}{M} \sin(\omega_d t) \qquad t > 0, \quad \zeta < 1 \qquad \textbf{(14)}$$

where $\omega_n = \sqrt{(K/M)}$ and $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

The **unit impulse response** *h(t)* is simply obtained by letting $\hat{F} = 1$, so that

$$h(t) = e^{-\zeta\omega_n t}\frac{1}{M\omega_d}\sin(\omega_d t); \qquad t > 0 \tag{15}$$

Let's find the system response to an arbitrary force excitation function *F(t)*. **Interpret *F(t)* as a train of (short time) impulses of varying amplitude**.

As shown in the Figure, at arbitrary time $t=\tau$ and corresponding to the time increment $\Delta\tau$, there is an impulse of magnitude $F_{(\tau)}\Delta\tau$, and expressed as $F_{(\tau)}\Delta\tau\,\delta_{(t-\tau)}$,

The response to a unit load impulse at $t=\tau$ is $h_{(t-\tau)}$. Then the contribution of $F_{(\tau)}\Delta\tau\,\delta_{(t-\tau)}$ to the total response at time *t* is

$$\Delta X_{(t,\tau)} = F_{(\tau)}\Delta\tau\; h_{(t-\tau)} \tag{16}$$

Thus, the total response is

$$X_{(t)} = \sum F_{(\tau)}\, h_{(t-\tau)}\,\Delta\tau \tag{17}$$

As $\Delta\tau \rightarrow 0$, the summation becomes an integral, i.e.

$$X_{(t)} = \int_0^t F_{(\tau)}\, h_{(t-\tau)}\, d\tau \tag{18}$$

This is known as the **Convolution or Duhamel's integral** and expresses the system response as the superposition of individual responses to impulse loads.

Replacing $h_{(t-\tau)}$ from Eq. (15) into Eq. (18), gives the system response as

$$X(t)=\frac{1}{M\,\omega_d}\int_0^t F_{(\tau)}\, e^{-\zeta\omega_n(t-\tau)}\,\sin\left(\omega_d\left[t-\tau\right]\right)d\tau \qquad \textbf{(19)}$$

for all $t>0$, with $\omega_n=\sqrt{K/M}$, $\omega_d=\omega_n\sqrt{1-\xi^2}$

For **non-zero initial conditions in displacement and velocity**, the **superposition principle** allows to express the total response of the underdamped ($\zeta<1$) SDOF to an arbitrary excitation force $F(t)$, i.e.

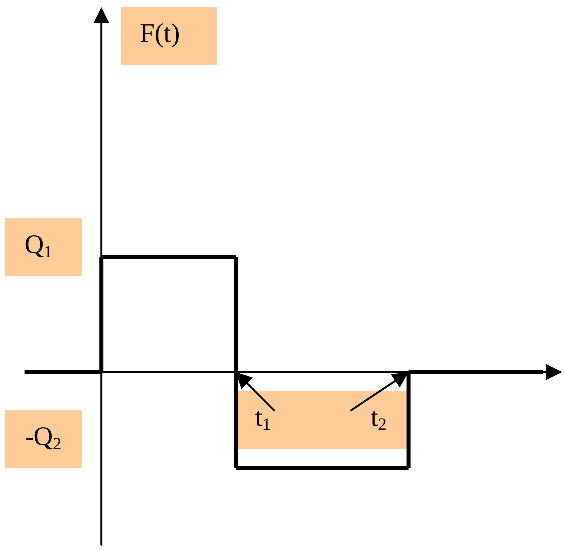
$$X(t)=e^{-\zeta\,\omega_n t}\left(X_0\cos\left(\omega_d t\right)+\left(\frac{\dot{X}_0+\zeta\,\omega_n X_0}{\omega_d}\right)\sin\left(\omega_d t\right)\right)+\frac{1}{M\,\omega_d}\int_0^t F_{(\tau)}e^{-\zeta\,\omega_n(t-\tau)}\sin\left[\omega_d\,(t-\tau)\right]\,d\tau \qquad (20)$$

Note that for a SDOF system without any viscous damping, ζ ***=0***, Eq. (20) simplifies to

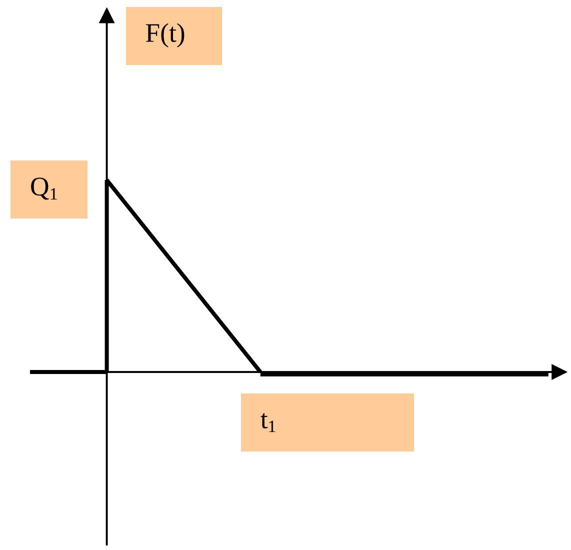
$$X(t)=X_0\cos(\omega_n t)+\frac{\dot{X}_0}{\omega_n}\sin(\omega_n t)+\frac{1}{M\omega_n}\int_0^t F_{(\tau)}\sin\left[\omega(t-\tau)\right]d\tau \qquad \textbf{(21)}$$

Homework exercises:

Determine ANALITICALLY the time response of an undamped (ζ=0) SDOF system to the forcing functions depicted below.



(a)



(b)

Answers:

a)

$$X(t)=\frac{Q_1}{K}\left[1\text{-}\cos\left(\omega_n t\right)\right] \qquad 0\le t\le t_1$$

$$X(t)=\frac{Q_1}{K}\left[\cos\left(\omega_n\left(t-t_1\right)\right)-\cos\left(\omega_n t\right)\right]-\frac{Q_2}{K}\left[1-\cos\left(\omega_n\left(t-t_1\right)\right)\right] \qquad t_1\le t\le t_2$$

$$X(t)=\frac{Q_1}{K}\left[\cos\left(\omega_n\left(t-t_1\right)\right)-\cos\left(\omega_n t\right)\right]-\frac{Q_2}{K}\left[\cos\left(\omega_n\left(t-t_2\right)\right)\right] \qquad t_2\le t$$

b)

$$X(t)=\frac{Q_1}{K}\left(1-\cos\left(\omega_n t\right)-\frac{\left\{t-\frac{1}{\omega_n}\sin\left(\omega_n t\right)\right\}}{\omega_n t_1}\right) \qquad 0\le t\le t_1$$

$$X(t)=\frac{Q_1}{K}\left(-\cos\left(\omega_n t\right)+\frac{\left\{\sin\left(\omega_n t\right)-\sin\left(\omega_n\left(t-t_1\right)\right)\right\}}{\omega_n t_1}\right) \qquad t_1\le t$$

where $\omega_n=\sqrt{K/M}$