EXAMPLES for PREDICTION OF THE TRANSIENT RESPONSE OF 1DOF MECHANICAL SYSTEMS

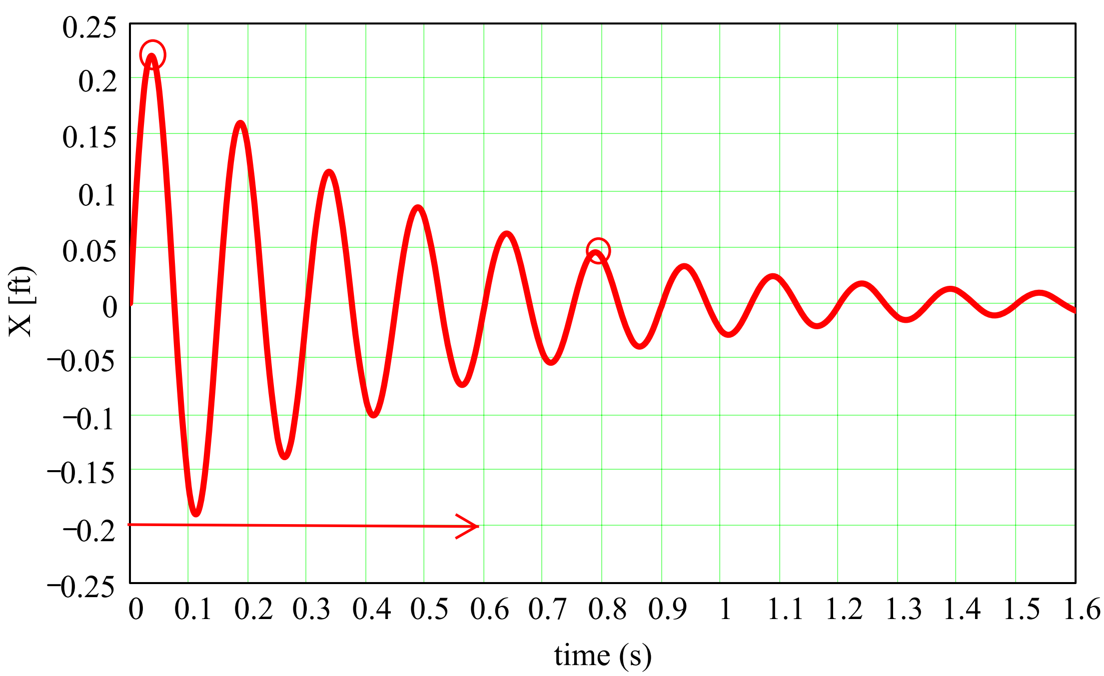
Identification of parameters

Use LOGDEC

The figure shows the dynamic free response (amplitude [ft] versus time [sec]) of a simple mechanical structure. Static load measurements determined the structure stiffness K=1000 lb_f/in. From the measurements, determine

a) damped period of motion T_d (sec)
b) damped natural frequency ω_d [rad/s],
c) Using the concept of log-dec (δ), if applicable, determine the system damping ratio ξ. Explain your method
d) Undamped natural frequency ω_n [rad/s],
e) Estimate the system equivalent mass, M_e [lb]
f) Estimate the system damping coefficient, C_e [lb_f s/in]

DISPLACEMENT (ft) vs time (sec)



(a) Determine damped period of motion: $T_d := \frac{0.6 \cdot sec}{4}$ from 4 periods of damped motion $T_d = 0.15\,sec$

(b) Determine damped natural frequency: $\omega_d := \frac{2 \cdot \pi}{T_d}$ $\omega_d = 41.888\,\frac{rad}{sec}$

(c) Determine damping ratio from log-dec:

Select two amplitudes of motion (well spaced) and count number od periods in between

$X_o := 0.23 \cdot ft$ after $n := 5$ periods $X_n := 0.05 \cdot ft$

Log-dec is derived from ratio: $\delta := \frac{1}{n} \cdot \ln\left(\frac{X_o}{X_n}\right)$ $\delta = 0.305$

from log-dec formula $\delta = \frac{2 \cdot \pi \cdot \xi}{\left(1 - \xi^2\right)^{0.5}}$ $\xi := \frac{\delta}{\left(4 \cdot \pi^2 + \delta^2\right)^{.5}}$ $\xi = 0.049$

Note that approximate formula: $\frac{\delta}{2 \cdot \pi} = 0.049$ is a very good **estimation** of damping ratio

(d) Determine damped natural frequency: $\omega_n := \frac{\omega_d}{\left(1 - \xi^2\right)^{0.5}}$ $\omega_n = 41.937 \frac{rad}{sec}$

a little higher than the damped frequency (recall damping ratio is small)

(e) Determine system mass:

Static tests conducted on the structure show its stiffness to be $K = 1 \times 10^3 \frac{lbf}{in}$

and from the equation for natural frequency, the equivalent system mass is

$M_e := \frac{K}{\omega_n^2}$ $M_e = 219.526\,lb$

(f) Determine system damping coefficient: $C_e := \xi \cdot 2 \cdot \left(K \cdot M_e\right)^{0.5}$ $C_e = 2.314\,lbf \cdot \frac{sec}{in}$

Note:
Actual values of parameters are $M = 220\,lb$ $C = 2.4\,lbf \cdot \frac{sec}{in}$ $\zeta = 0.05$

TRANSIENT RESPONSE - COLLISION

The car of mass *M*=500 kg is traveling at constant speed V_o= 50 kilometer/hour when it hits a rigid wall. A spring (*K*) and a viscous dashpot (*C)* represent the car **front bumper system**. The system natural frequency f_n=3 Hz, and the dashpot provides **critical damping**. Disregard friction on the car wheels and ground.

a) Derive the EOM for the car **after the collision** using the coordinate *Y*(t) that has its origin at the car location when the bumper first touches the wall. Provide initial conditions in speed and displacement. [10]

b) **State** the solution to the EOM, i.e. give the system response *Y(t)* as a function of the system parameters [natural frequency, damping ratio, etc] and initial conditions. [10]

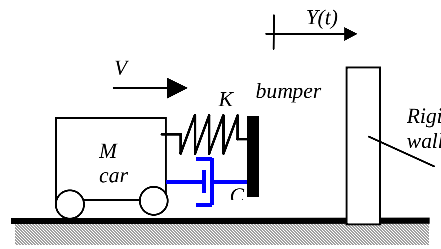
c) **Find** the maximum bumper deflection and the time, after collision, when this event occurs. Compare the calculated deflection with the maximum deflection for a system without damping. Comment on your findings. [5]

d) **Sketch** the motion *Y(t) vs.* time (to 0.5 sec) labeling the axes with physical units and noting important parameters. Will the car bounce from the wall? Explain your answer. [5]

$kph := \frac{1000 \cdot m}{3600 \cdot s}$

a) EQN of motion for car after collision, use Y coordinate: (Y=0, no bumper deflection)

$$M \cdot \frac{d^2}{dt^2} Y = -F_{damper} - F_{spring} \qquad F_{damper} = C \cdot \frac{d}{dt} Y \qquad F_{spring} = K \cdot (Y - 0)$$



Thus the EOM is:

$$M \cdot \frac{d^2}{dt^2} Y + C \cdot \frac{d}{dt} Y + K \cdot Y = 0 \qquad \textbf{(1)}$$

with I.C's at t=0 $\quad Y(0) = Y_0 = 0 \cdot m$ no bumper deflection, and

and $\frac{d}{dt} Y = V_0 \quad V_0 := 50 \cdot kph = \quad V_0 = 13.889 \frac{m}{s}$ car speed at instant of collision with wall.

Known: natural frequency and mass of car $\quad f_n := 3 \cdot Hz \quad \omega_n := f_n \cdot 2 \cdot \pi \quad M := 500 \cdot kg$

calculate stiffness: $\quad K := M \cdot \omega_n^{\ 2} \quad K = 1.777 \times 10^5 \frac{N}{m}$

natural period of motion $\quad T_n := \frac{1}{f_n}$

Transient response of **critically damped system**, step force Fo=0 $\qquad \zeta=1$

$$M \cdot \frac{d^2}{dt^2} Y + C \cdot \frac{d}{dt} Y + K \cdot Y = F_o$$

at t=0 + initial conditions $\quad Y_0 := 0 \cdot m \quad V_0 = 13.889 \frac{m}{s}$ and $\quad F_o := 0 \cdot N \quad Y_{ss} := \frac{F_o}{K}$

From cheat sheet, soln is: $\quad Y(t) = e^{ss \cdot t} \cdot (A_1 + t \cdot A_2) + Y_{ss} \quad$ **(2)** where $\quad ss := -\omega_n \quad$ root of characteristic eqn.

$A_1 := Y_0 - Y_{ss} \qquad (A_1 \cdot s + A_2) = V_0 \qquad ss = -18.85 \frac{1}{s}$

$A_2 := V_0 - A_1 \cdot ss \qquad A_1 = 0\,m \qquad A_2 = 13.889 \frac{m}{s}$ =Vo

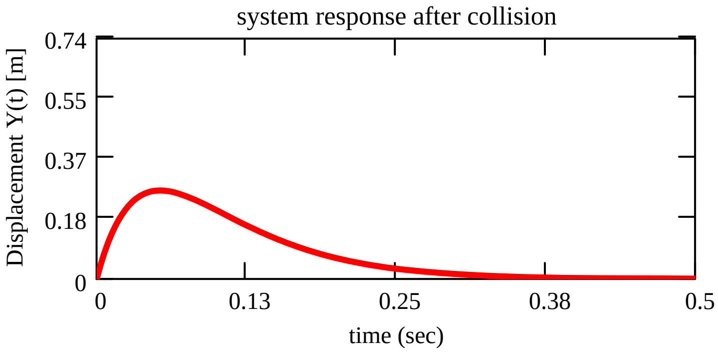
The **dynamic response of the car Y(t)** is given by: $\quad Y(t) := e^{ss \cdot t} \cdot t \cdot A_2 \quad$ **(3)**

The graph below displays the car motion for a time equal to 3 periods of natural motion (undamped). Note the overshoot and largely damped response w/o oscillations. The car does not bounce from the wall.

$T_{max} := 1.5 \cdot T_n$

without damping, max bumper deflection is

Based on PCME $\quad \frac{1}{2} \cdot M \cdot V_0^{\ 2} = \frac{1}{2} K X_{max}^{\ 2} \qquad X_{max} := \frac{V_0}{\omega_n} \qquad X_{max} = 0.737\,m$



$f_n = 3\,Hz$

$T_n = 0.333\,s$

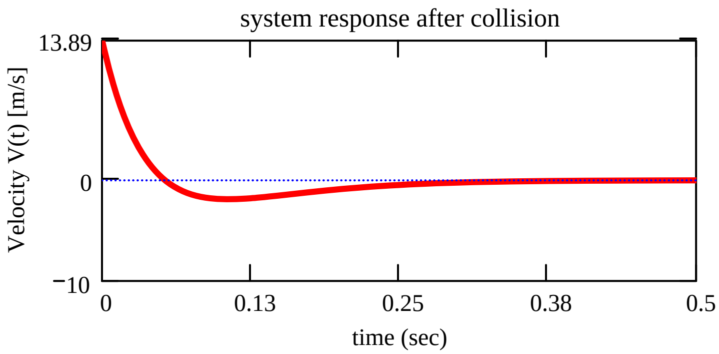
Calculation of maximum bumper deflection

It occurs when velocity is 0 m/s

(Take time derivative of soln, Eq. (3), to obtain)

$$V(t) := A_2 \cdot (1 + ss \cdot t) \cdot e^{ss \cdot t}$$

Graph not for exam



Bumper velocity is null at time t_a

$$\left(1 + ss \cdot t_a\right) = 0$$

Hence $\quad t_a := \frac{-1}{ss}$ $\quad t_a = 0.053\,s$

Maximum deflection: $Y(t_a) = 0.271\,m$

compare to undamped system: $X_{max} = 0.737\,m$

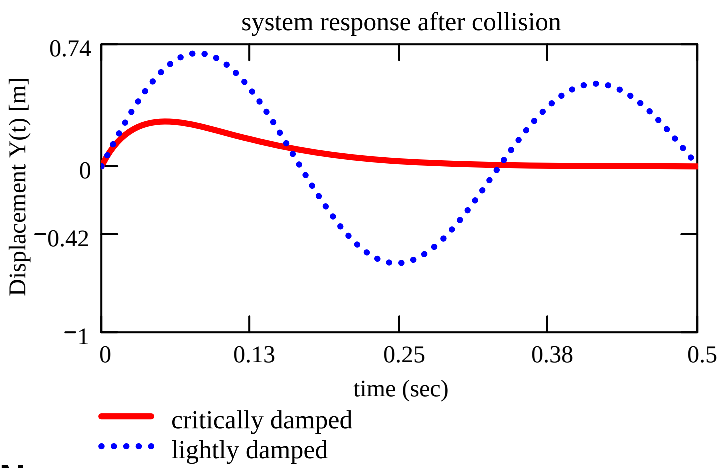
$$\frac{X_{max}}{Y(t_a)} = 2.718$$

A significant reduction in deflection. Note how quickly the dashpot dissipates the initial kinetic energy.

Comparison not for exam for completeness, compare the response with that of a lightly UNDERDAMPED system

$\zeta := 0.05$ $\quad \omega_d := \omega_n \cdot \sqrt{1 - \zeta^2}$ $\quad X(t) := \frac{V_0}{\omega_d} \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t)$

Note that model with little damping predicts the car will bounce from wall.



Let's find the forces transmitted to the wall - certainly same as force "felt" by car

$$F_w(t) := K \cdot Y(t) + C \cdot V(t)$$

$$F_{elas\_max} := K \cdot Y(t_a)$$

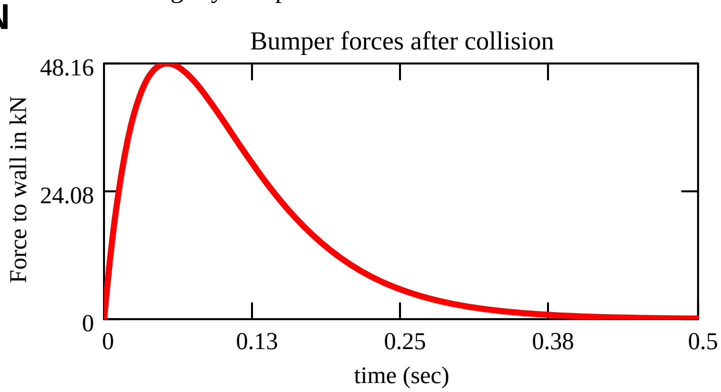
$$F_{elas\_no\_damping} := K \cdot X_{max}$$

$$F_{elas\_max} = 4.816 \times 10^4\,N$$

$$F_{elas\_no\_damping} = 1.309 \times 10^5\,N$$

$$K = 1.777 \times 10^5\,\frac{N}{m}$$

kN



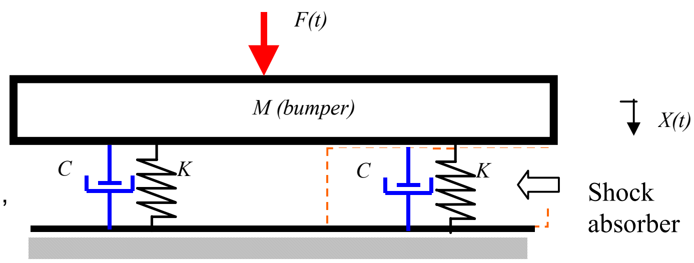
Note how large is the force in the bumper. No wonder why a car crash is always a disastrous event!

CAR BUMPER SYSTEM RESPONSE TO AN IMPULSE

The sketch shows a test stand for automobile bumpers. Each of the two shock absorbers consists of a spring having a stiffness *K*=3000 N/m that acts concentrically with a dashpot (*C*). The bumper mass *(M)* is 40 kg. The system is at rest in the static equilibrium position when a force *F* having an impulse of 2000 N-s acting over a short interval is applied at the centerline of the bumper.

(a) determine the value *C* that will bring the bumper to rest in the shortest possible time without rebound
(b) If *C*= 1500 Ns/m determine the displacement *x(t)* of the bumper.
(c) Find the bumper's maximum displacement from the equilibrium position and its time of occurrence. Is the result reasonable? Explain.

Assume:
bumper is rigid - undeformable
X=0 denotes SEP.



For motions from the static equilibrium position,
The equation of motion is:

$$M \cdot \frac{d^2}{dt^2}X + 2 \cdot C \cdot \frac{d}{dt}X + 2 \cdot K \cdot X = F$$

where $M := 40 \cdot kg$ bumper mass, regarded as rigid - non deformable

$K := 3000 \cdot \frac{N}{m}$ absorber stiffness and damping coefficients

C (damping coefficient to be determined)

The natural frequency equals $\omega_n := \left(\frac{2 \cdot K}{M}\right)^{.5}$ $\omega_n = 12.247 \frac{rad}{s}$

$f_n := \frac{\omega_n}{2 \cdot \pi}$ $f_n = 1.949\,Hz$

(a) **The system must be critically damped** to bring the damper to rest in the shortest possible time. Thus, the amount of physical damping C must equal (for each shock absorber)

$C_{crit} := 2 \cdot (2 \cdot K \cdot M)^{.5}$ $C_{crit} = 979.796\,N \cdot \frac{s}{m}$

$C := \frac{1}{2} \cdot C_{crit}$ $C = 489.898 \frac{N \cdot s}{m}$ Amount of damping on **each** absorber

(b) If $C := 1500 \cdot \frac{N \cdot s}{m}$, the damping ratio is $\xi := \frac{2 \cdot C}{C_{crit}}$ $\xi = 3.062$ i.e. an overdamped system

The solution of the EOM, with RHS = 0, is:

$$x(t) = A \cdot e^{s_1 \cdot t} + B \cdot e^{s_2 \cdot t} \qquad \frac{d}{dt}x = A \cdot s_1 \cdot e^{s_1 \cdot t} + B \cdot s_2 \cdot e^{s_2 \cdot t}$$

The roots of the characteristic equation are

$$s_1 := -\xi \cdot \omega_n + \omega_n \cdot \left(\xi^2 - 1\right)^{.5}$$

$$s_1 = -2.056 \frac{1}{s}$$

$$s_2 := -\xi \cdot \omega_n - \omega_n \cdot \left(\xi^2 - 1\right)^{.5}$$

$$s_2 = -72.944 \frac{1}{s}$$

satisfying the initial conditions. At t=0 s $\quad x(0) = 0 = A + B \quad$ Thus, $\quad B = -A \quad$ no initial displacement

The **impulse** produces an initial velocity equal to

$$Imp := 2000 \cdot N \cdot s \qquad v(0) = v_0 = \frac{Imp}{M} = A \cdot \left(s_1 - s_2\right)$$

as explained in class

$$v_o := \frac{Imp}{M} \qquad v_o = 50 \frac{m}{s}$$

initial velocity is quite large!
(180 km/hour!)

$$A := \frac{v_o}{\left(s_1 - s_2\right)} \qquad A = 0.705\,m$$

for graphing

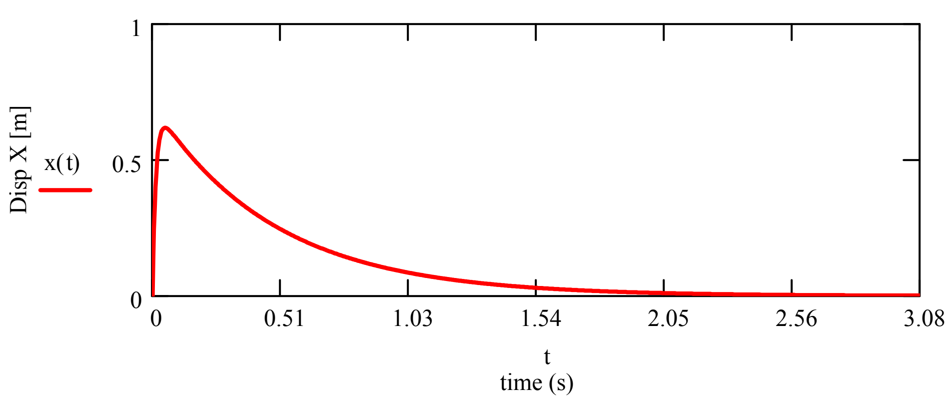
$$T_n := \frac{1}{f_n}$$

$$T_{max} := 3 \cdot T_n$$

thus, the dynamic response of the bumper after the impuse acts is:

DISPLACEMENT

$$x(t) := A \cdot \left(e^{s_1 \cdot t} - e^{s_2 \cdot t}\right)$$



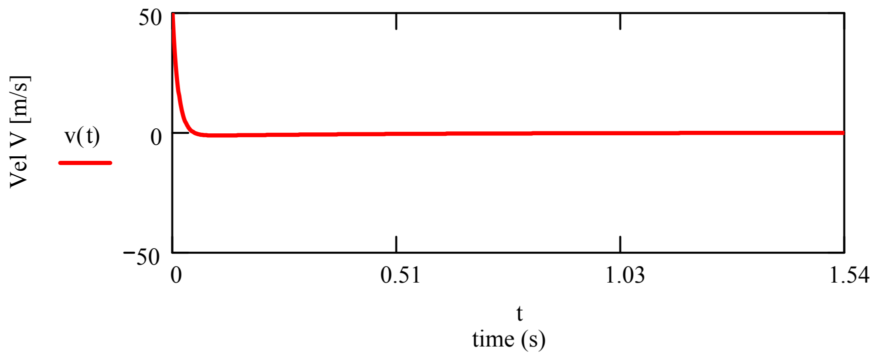
Note large overshoot
(max deflection)
and return
to equilibrium with no
oscillations.

VELOCITY

$$v(t) := A \cdot \left(e^{s_1 \cdot t} \cdot s_1 - e^{s_2 \cdot t} \cdot s_2\right)$$

$$v(0 \cdot s) = 50 \frac{m}{s}$$

s



Max bumper displacement (deflection) occurs when velocity equals 0, i.e

$$v = 0 = A\cdot\left(e^{s_1\cdot t_M}\cdot s_1 - e^{s_2\cdot t_M}\cdot s_2\right)$$

$$\frac{e^{s_1\cdot t_M}}{e^{s_2\cdot t_M}} = \frac{s_2}{s_1}$$

take natural log of both sides to find

$$\left(s_1 - s_2\right)\cdot t_M = \ln\left(\frac{s_2}{s_1}\right)$$

$$t_M := \frac{\ln\left(\frac{s_2}{s_1}\right)}{s_1 - s_2}$$

$$t_M = 0.05\,s$$

and the maximum bumper deflection is $x\left(t_M\right) = 0.618\,m$

This is a large deflection! probably bumper will deform plastically before this occurs.

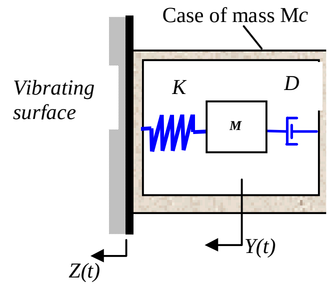
Example Problem: Seismic instrument

MEEN 363 FALL 02
kinetics of M-K-C system
L San Andres

The figure shows a seismic instrument with its case rigidly attached to a vibrating surface. The instrument displays the motion *Y(t)* **RELATIVE** to the case motion *Z(t)*.

a) Derive the equation of motion for the instrument using the **relative motion *Y(t)*** as the output variable. Show all assumptions and modeling steps for full credit.
b) Determine the instrument natural frequency ω_n [rad/s]and critical damping D_c [lb.s/in] for M=0.02 lb and K=13 lb/in.
c) When will the instrument show $Y=0$, i.e. no motion?
d) If the vibrating surface moves with a constant acceleration of **3 g,** what will the instrument display at s-s? Give value in inches.

Note: the sensor works under all attachment configurations, i.e. vertical, horizontal, top, bottom, etc.



a) EQN of motion using Y (relative) coordinate:

The relationship between the absolute motion X(t) of the sensor mass and the case motion is

$$X(t) = Z(t) + Y(t) \qquad (1)$$

PCLM (Newton's Law) refers to inertial (absolute) references:

$$M\cdot\frac{d^2}{dt^2}X = -F_{damper} - F_{spring} \qquad (2)$$

No effect of gravity since sensor is positioned horizontal. Actually, not important for any dynamic motion.

with

$$F_{damper} = D\cdot\frac{d}{dt}Y \qquad F_{spring} = K\cdot Y \qquad (3)$$

Substitution of (3) and (1) into (2) gives the EOM as:

$$M\cdot\frac{d^2}{dt^2}Y + D\cdot\frac{d}{dt}Y + K\cdot Y = -M\cdot\frac{d^2}{dt^2}Z \qquad (4)$$

b) Determine system parameters:

$$W := 0.02\cdot lbf \qquad g = 386.089\frac{in}{s^2}$$

$$M := \frac{W}{g}$$

$$K := 13\frac{lbf}{in} \qquad D = D\cdot\frac{lb_s}{in}$$

Damping coeffic.not specified

natural frequency $\omega_n := \left(\frac{K}{M}\right)^{.5}$ $\qquad \omega_n = 500.957\frac{rad}{s}$

Critical damping $D_c := 2(K\cdot M)^{.5}$ $\qquad D_c = 0.052\frac{lbf\cdot s}{in}$

Damping ratio. $\zeta := \frac{D}{2(K \cdot M)^{.5}}$

c) Instrument will record NO motion Y=0, if Z(t)= cte or dZ/dt=cte, i.e. the "vibrating surface" remains stationary or moves at constant speed.

d) if the vibrating surface moves with a constant acceleration equal to 3 g's. Then, the instrument will display, at steady state (after transients have disappeared due to damping):

$$Y_s := -3 \cdot g \cdot \frac{M}{K}$$

$$Y_s = -4.615 \times 10^{-3}\,\text{in}$$

Let's calculate the instrument dynamic response:

For example, let $\zeta := 0.10$

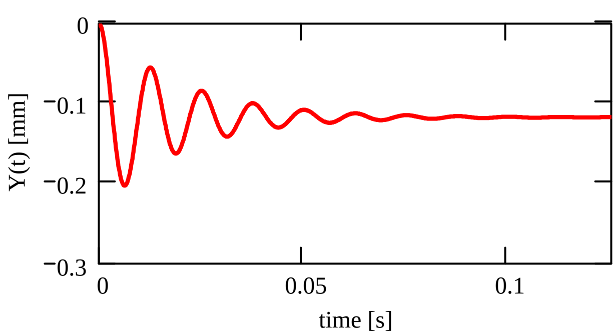
Then the sensor response for a sudden (3g) acceleration becomes with **zero initial conditions**:

$$Y_0 := 0 \cdot m \qquad V_0 := 0 \cdot \frac{m}{s} \qquad \omega_d := \omega_n \cdot \left(1 - \zeta^2\right)^{.5}$$

$$C_1 := \left(Y_0 - Y_s\right) \qquad C_2 := \frac{\left[V_0 + \zeta \cdot \omega_n \cdot \left(Y_0 - Y_s\right)\right]}{\omega_d}$$

$$Y(t) := \left(e^{-\zeta \cdot \omega_n \cdot t}\right) \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + Y_s$$

$$T_d := \left(\frac{2\pi}{\omega_d}\right)$$



$$Y_s = -0.117\,\text{mm}$$

Response of seismic instrument to sudden acceleration

SEE Video_seismic_instrument

The spring-mass-damper system represents a package cushioning an electronic component. The package rests on a hard floor. A pulse force F(t) of very, very short duration is exerted on the package as shown. The component vibrates without rebounding. Let K=60 lb/in, M= 5.9 lb, and C=0.04 lb-sec/in,

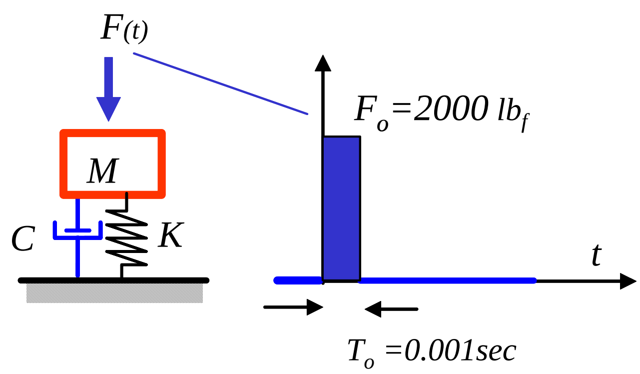
a) Calculate the system natural frequency (Hz) and damping ratio [8].

b) Provide an engineering estimation (value) for the maximum system velocity (ft/sec).

c) The maximum deflection (ft) of the spring (displacement of system) and the time when it occurs.

d) Draw a graph of the system displacement x vs. time. Label all physical coordinates.

TRANSIENT RESPONSE - EFFECT OF AN IMPULSE



KEY: KNOWLEDGE base: DOES APPLIED FORCE ACT OVER A LONG TIME OR NOT? IS time To very long?

Pulse: force magnitude $F := 2000 \cdot \text{lbf}$ applied during $T_o := 0.001 \cdot \text{sec}$

(a) Find the natural frequency, natural period of motion and viscous damping ratio:

$$K := 60 \cdot \frac{\text{lbf}}{\text{in}} \qquad M := 5.9 \cdot \text{lb} \qquad C := 0.04 \cdot \text{lbf} \cdot \frac{\text{sec}}{\text{in}}$$

NATURAL FREQUENCY: $\omega_n := \sqrt{\frac{K}{M}}$ $\quad f_n := \frac{\omega_n}{2 \cdot \pi}$ $\quad f_n = 9.973\,\text{Hz}$ $\quad \omega_n = 62.66 \frac{\text{rad}}{\text{sec}}$

$T_n := \frac{1}{f_n}$ $\quad T_n = 0.1\,\text{sec}$ natural period of motion

DAMPING RATIO:

$\zeta := \frac{C}{2 \cdot M \cdot \omega_n}$ $\quad \zeta = 0.021$ a small value

DAMPED NATURAL FREQUENCY: $\omega_d := \omega_n \cdot \left(1 - \zeta^2\right)^{.5}$ $\quad \omega_d = 62.647 \frac{\text{rad}}{\text{sec}}$

$T_d := \frac{2 \cdot \pi}{\omega_d}$ $\quad T_d = 0.1\,\text{sec}$

The damping ratio is very small, thus the system is very **lightly damped**. $\frac{T_n}{T_o} = 100.274$

(b) engineering judgment: The (damped) natural period of motion is very large compared to the time the load acts. Hence, the pulse can be treated as an **impulse,** which changes "instantaneously" the **initial velocity** of the system. This initial velocity equals

$\text{Impulse} := F \cdot T_o$ $\quad V_o := \frac{\text{Impulse}}{M}$ (1) $\quad V_o = 130.877 \frac{\text{in}}{\text{sec}}$

This **initial velocity is (of course) the maximum** ever, since damping will remove the initial kinetic energy due to the impact until the system returns to rest.

The motion starts from rest, X(0)=0.m. Thus, the response of the system becomes (cheat sheet):

$$X(t) := \frac{V_o}{\omega_d} \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t) \qquad (2)$$

However, since damping is so small ζ~0, the **engineering approximation** leads to:

$$X_a(t) := \frac{V_o}{\omega_n} \cdot \sin(\omega_n \cdot t) \qquad \text{with velocity} \qquad V_a(t) := V_o \cdot \cos(\omega_n \cdot t) \qquad (3)$$

The **maximum displacement** is:

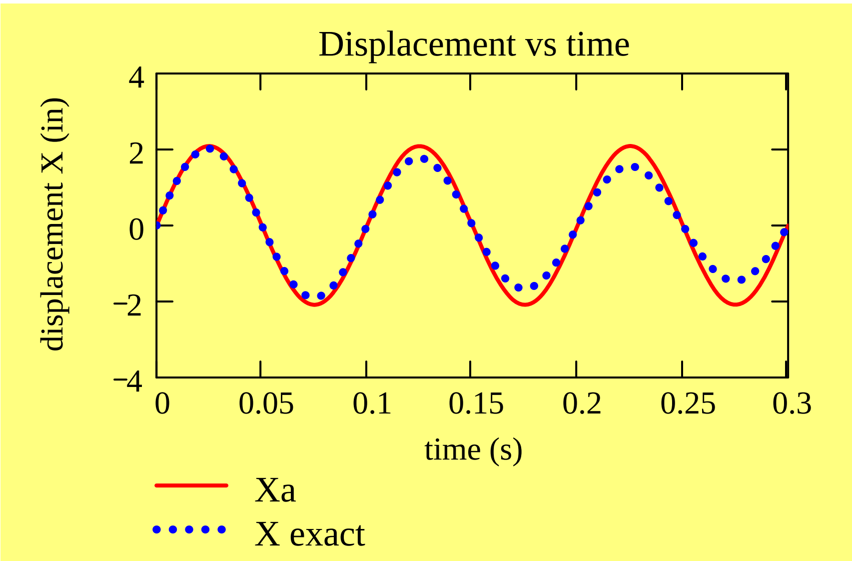
$$X_{max} := \frac{V_o}{\omega_n}$$

$$X_{max} = 2.089\,in$$

which occurs at a time ~ 1/4 natural period of motion, i.e

$$\frac{T_n}{4} = 0.025\,sec$$

Let's graph the responses (displacement and velocity):



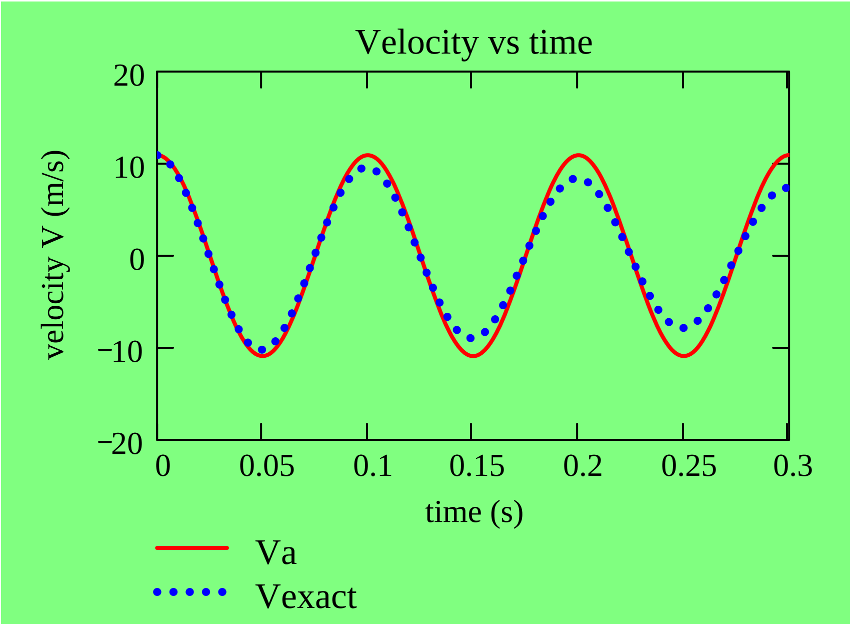
Note how close the engineering approximation is with respect to the exact solution, in particular during the first period of motion.

$$X_{max} = 2.089\,in$$

Note that Xmax can also be easily determined from PCME: T=V

$$\frac{1}{2} \cdot M \cdot V_o^2 = \frac{1}{2} \cdot K \cdot X_{max}^2$$

$$V_o = 10.906\,\frac{ft}{sec}$$



Maxiimum displacement happens when speed =0, i.e. $\frac{d}{dt}\left(\frac{V_o}{\omega_d}\cdot e^{-\zeta\cdot\omega_n\cdot t}\cdot\sin(\omega_d\cdot t)\right) = 0$

Take time derivative of X(t) and obtain time t_

$$-\zeta\cdot\omega_n\cdot\sin(\omega_d\cdot t\_) + \omega_d\cdot\cos(\omega_d\cdot t\_) = 0$$

$$\tan(\omega_d\cdot t\_) = \frac{\omega_d}{\omega_n\cdot\zeta} = \frac{\left(1-\zeta^2\right)^{0.5}}{\zeta}$$

Let: $\theta\_ := \operatorname{atan}\left[\frac{\left(1-\zeta^2\right)^{0.5}}{\zeta}\right]$ $\theta\_\cdot\frac{180}{\pi} = 88.803$

and time for maximum displacement is $t\_ := \frac{\theta\_}{\omega_d}$ $\frac{t\_}{T_n} = 0.247$ $t\_ = 0.025\,\text{sec}$

and the maximum deflection is $X(t\_) = 2.022\,\text{in}$

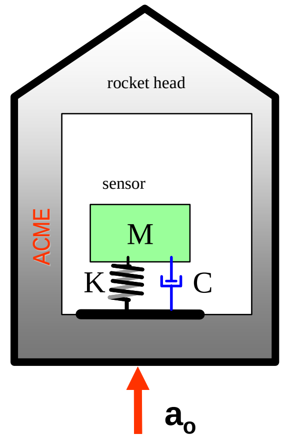
compare it to engineering approx: $X_{max} = 2.089\,\text{in}$ $\frac{X(t\_)}{X_{max}} = 0.968$

excellent agreement!

Q3: Description of motion in a moving reference frame

Luis San Andres, MEEN 363 (c) FALL 2010

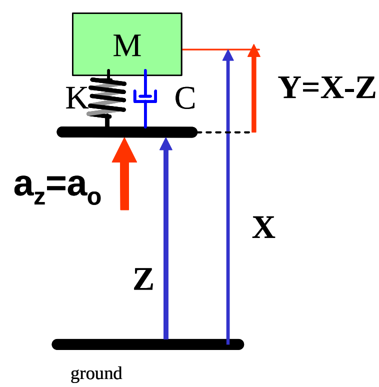
An instrument package installed in the nose of a rocket is cushioned against vibration with a soft spring-damper. The rocket, fired vertically from rest, has a constant acceleration *ao*. The instrument mass is *M*, and the support stiffness is *K* with damping *C*. The instrument-support system is underdamped. The **relative motion** of the instrument **with respect to the rocket** is of importance.
a) Derive the equation of relative motion for the instrument
b) Give or find an analytical expression for the relative displacement of the instrument vs. time. Express your answer with well defined parameters and variables.
c) Given *M*=1 kg, *K*=1 N/mm, and damping ratio ζ=0.10. Find the natural frequency & damping coefficient of the system
d) For ao=3g, find the steady-state displacement of the instrument, relative to rocket and absolute.



Definitions: coordinate systems

$X(t)$ Absolute displacement of instrument recorded from ground

$Z(t)$ Absolute displacement of rocket from ground.



$Y = X - Z$ displacement of instrument relative to rocket

$a_Z = \frac{d^2}{dt^2}Z = a_o$ acceleration of rocket fired from REST

sensor parametes: $M := 1\cdot kg$ $K := 1000\cdot\frac{N}{m}$

$a_o := 3\cdot g$ $\zeta := 0.10$

Equation of motion for instrument

Newton's Laws are applicable to inertial CS

from **free body diagram**, let Y=(X-Z)>0

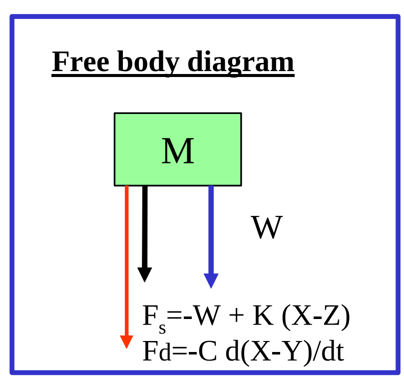
$$M\cdot\frac{d^2}{dt^2}X = -W - F_s - F_d \quad (1)$$

where

$$F_s = -W + K\cdot(X - Z) = -W + K\cdot Y \quad (2a)$$

is the spring force supporting instrument. The dashpot force is

$$F_d = C\cdot\frac{d}{dt}(X - Z) \quad (2b)$$



Substitute Eqs. (2) into Eq.(1):

$$M\cdot\frac{d^2}{dt^2}X = -W + W - K\cdot Y - C\cdot\frac{d}{dt}Y$$

$$M\cdot\frac{d^2}{dt^2}X = -K\cdot Y - C\cdot\frac{d}{dt}Y$$

But interest is in the **relative motion Y;** hence, substitute X=Y+Z

$$M\cdot\frac{d^2}{dt^2}(Y+Z) + K\cdot Y + C\cdot\frac{d}{dt}Y = 0$$

$$M\cdot\left(\frac{d^2}{dt^2}Y + C\cdot\frac{d}{dt}Y\right) + K\cdot Y = -M\cdot a_Z = -M\cdot a_o$$

(3) is the desired EOM.

Find natural frequency and damping coefficient

natural frequency of sensor is:

$$\omega_n := \left(\frac{K}{M}\right)^{.5} \qquad \omega_n = 31.623\,\frac{rad}{s}$$

Natural period: $T_n := \frac{2\cdot\pi}{\omega_n}$

damping coefficient.

$$C := \zeta\cdot 2\cdot(K\cdot M)^{0.5} \qquad C = 6.325\,N\cdot\frac{s}{m}$$

$$T_n = 0.199\,s$$

damped natural frequency

$$\omega_d := \omega_n\cdot\left(1-\zeta^2\right)^{0.5} \qquad \omega_d = 31.464\,\frac{rad}{s}$$

Solution of ODE - prediction of relative motion

The solution of ODE Eq. (3) with null initial conditions since motion starts from rest is (Use cheat sheet)

Motion starts from rest

$$Y_o := 0\cdot m \qquad V_o := 0\cdot\frac{m}{s}$$

$$Y_s := \frac{-M\cdot a_o}{K}$$

$$Y = Y_s + e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos\left(\omega_d\cdot t\right) + C_2\cdot\sin\left(\omega_d\cdot t\right)\right)$$

$$C_1 := Y_o - Y_s \qquad C_2 := \frac{\left(V_o + \zeta\cdot\omega_n\cdot C_1\right)}{\omega_d}$$

is the formula describing the motion of instrument relative to rocket

$$C_1 = 0.029\,m \qquad C_2 = 2.957\times 10^{-3}\,m$$

and, after long time Y approaches: $Y_s = -0.029\,m$

To find the instrument absolute displacement, first determine the absolute motion of the rocket, i.e.

velocity $V_z(t) := a_o\cdot t$ and displacement $Z(t) := a_o\cdot\frac{t^2}{2}$

The absolute displacement of the sensor is $X = Y + Z$

after very-long tines, times removes the homogenous (transient) response; and the senso reaches its steady state motion

$$X_{SS}(t) := Z(t) + Y_s$$

not for Quiz Lets graph the relative and absolute displacements of the sensor

without damping

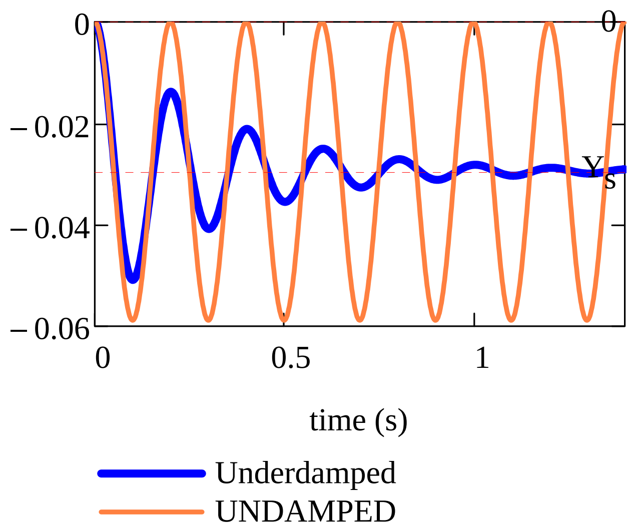
$Y(t) := Y_s + e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right)$ $Y\_(t) := Y_s \cdot \left(1 - \cos\left(\omega_n \cdot t\right)\right)$

for plots, set $T_{max} := 7 \cdot T_n$

Relative displacement of sensor w/r to rocket

Y

[m]



$Y\left(5 \cdot T_n\right) = -0.028\,m$

note the effect of damping

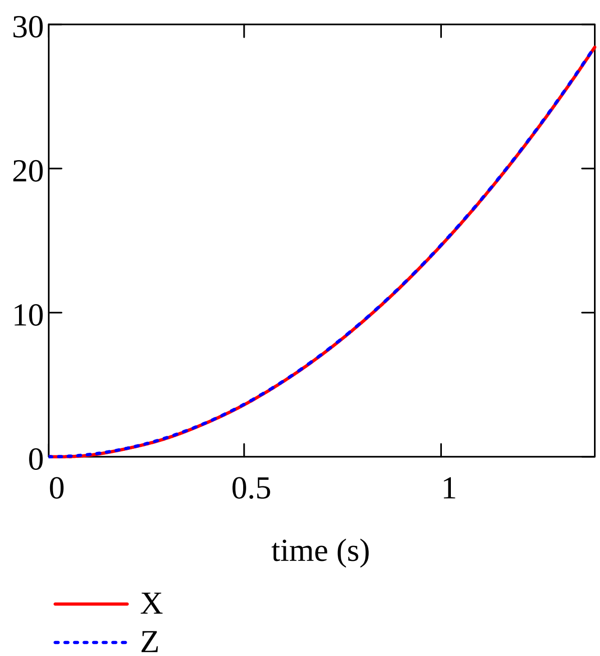
$Y_s = -0.029\,m$

The absolute displacement of the sensor is $X = Y + Z$ where

Displacements - Sensor (X) and Rocket (Z)

[m]

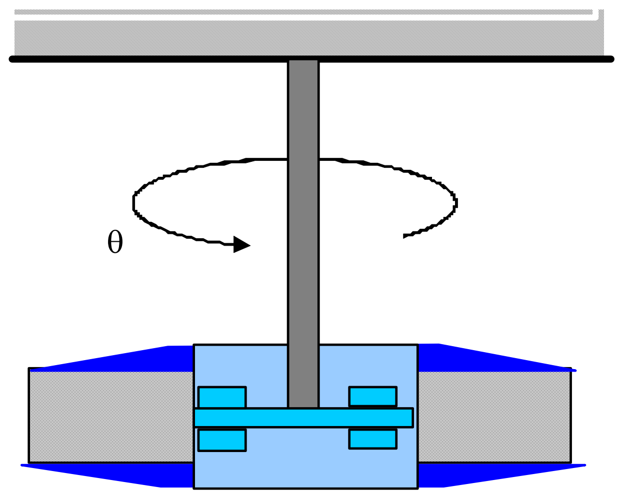
X, Z



$kmh := \frac{1}{3.6} \cdot \frac{m}{s}$

Application: a torsional pendulum

A device designed to determine the moment of inertia of a **wheel-tire assembly** consists of a 2 mm diameter steel suspension wire, 2 m long, and a mounting plate, to which it is attached the wheel-tire assembly. The suspension wire is fixed at its upper end and hangs vertically. When the system oscillates as a **torsional pendulum**, the period of oscillation without the wheel tire assembly is 4 seconds. With the wheel tire assembly mounted to the support plate, the period of oscillation is 25 seconds. Determine the mass moment of inertia of the wheel tire assembly. Recall that the shear modulus for steel is G= 82.7×10^9 N/m^2 and the torsional stiffness of the cable is $K_\theta=(\pi d^4/32)G/L$.



Consider the system does not have any damping - and there is no external moment acting'

The general EOM for rotations θ(t) about a fixed axis (o-o) is

$$I_o \cdot \frac{d^2}{dt^2}\theta + k_\theta \cdot \theta = 0 \qquad (1)$$

where Io is a mass moment of inertia and Kθ is a torsional stiffness

The natural frequency of the system is

$$\omega_n = \left(\frac{K_\theta}{I_o}\right)^{\frac{1}{2}} = \left(\frac{2\cdot\pi}{T_n}\right) \qquad (2)$$

where Tn is the natural period of motion

Let:

$G := 82.7\cdot 10^9 \cdot \frac{N}{m^2}$ Shear modulus for steel

$L := 2\cdot m$ cable length

$d := 2\cdot mm$ cable diameter

$$K_\theta := \frac{G}{L}\cdot\left(\frac{\pi\cdot d^4}{32}\right)$$

$$K_\theta = 0.065\,N\cdot\frac{m}{rad}$$

From (2)

$$I_o = \left(\frac{T_n}{2\cdot\pi}\right)^2\cdot K_\theta$$

first case: period of oscillation of assembly alone (w/o tire)

$$T_{n1} := 4\cdot s$$

Find moment of inertia of support plate

$$I_{plate} := \left(\frac{T_{n1}}{2\cdot\pi}\right)^2\cdot K_\theta$$

$$I_{plate} = 0.026\,kg\,m^2$$

second case: period of oscillation of assembly with wheel included

$$T_{n2} := 25\cdot s$$

Find moment of inertia of system (plate+wheel)

$$I_{system} := \left(\frac{T_{n2}}{2\cdot\pi}\right)^2\cdot K_\theta$$

$$I_{system} = 1.028\,kg\,m^2$$

Solve: find mass moment of inertia of wheel-tire

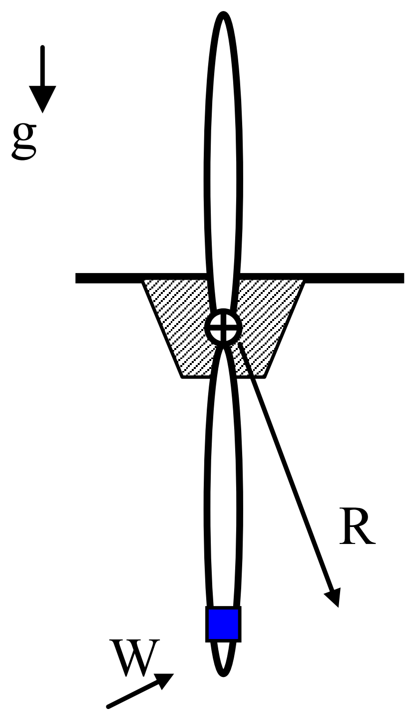
$$I_{wheel\_tire} := I_{system} - I_{plate}$$

$$I_{wheel\_tire} = 1.002\,kg\,m^2$$

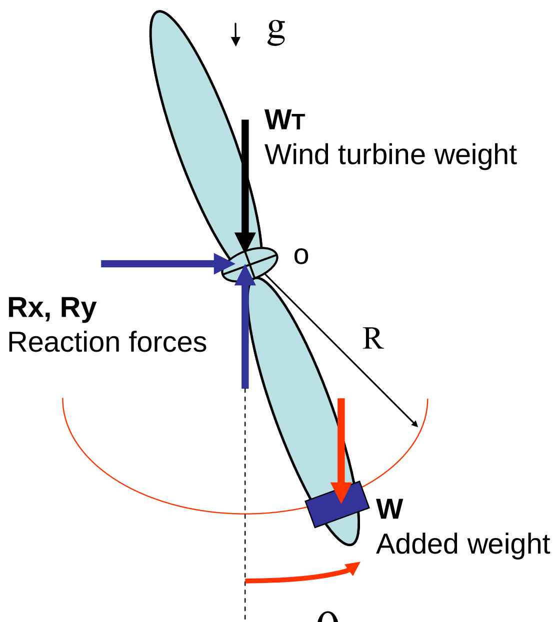
Application example

A two-blade composite-aluminum of a wind turbine is supported on a shaft & bearing system so that it is free to rotate about its centroidal axis. A weight **W**=100 kg is taped to one of the blades at a distance R=20 m from the axis of rotation as shown. When a blade is pushed a small angle from the vertical position and released, the wind turbine is found to oscillate 2 periods of natural motion in 1 minute. Assume there is no friction at the support bearings.

* Determine the wind turbine centroidal mass moment of inertia (I_T) in (kg.m^2).
* The turbine weighs 2500 kg. Determine the radius of gyration.



Make a free body diagram of turbine swinging (oscillating) about pivot O with angle θ(t)



FREE BODY DIAGRAM

Assume:

- no drag (frictionless bearings & no air drag)
- center of mass of turbine = center of rotation O

Let I_T: mass moment of inertia of turbine; hence EOM (moments) about O is

$$(I_T + M R^2)\ddot{\theta} = I_O \ddot{\theta} = -W R \sin\theta \quad (1)$$

For small θ angles, Eq. (1) reduces to the linear form

$$I_O \ddot{\theta} + M g R \theta = 0 \quad (2)$$

i.e. the typical equation of a pendulum with natural period given as

$$T_n = \frac{2\pi}{\omega_n} = 2\pi \left(\frac{I_o}{M g R} \right)^{1/2} \quad (3)$$

Hence, the wind turbine mass moment of inertia can be easily determined from the measured period of oscillation

$$I_T = -M R^2 + \left(\frac{T_n}{2\pi} \right)^2 M g R \quad (4)$$

Given

W=100 kg_f taped to one of the blades at a distance R=20 m from the axis of rotation. Natural period T_n= 60 seconds / 2 periods of natural motion = 30 sec.   Note **M=W/g =** 100 kg

$$I_T = -M R^2 + \left(\frac{T_n}{2\pi}\right)^2 M\, g\, R \quad : \quad I_T = 4.071\times 10^5\, kg\, m^2 \quad *$$

Since the turbine weighs W_T=2500 kg_f, and from $I_T = M_T\, r_k^2$, the radius of gyration is

$$r_k = \left[\frac{I_T}{M_T}\right]^{1/2} = \quad r_k = 12.761 m \quad *$$

Example - Cushioning of package

EXAMPLE to assist you in your assignment

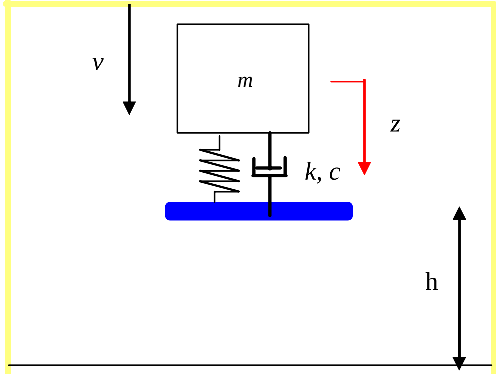
The EOM **after** the package first contacts hard ground is:

$$M \cdot \frac{d^2}{dt^2} z + c \cdot \frac{d}{dt} z + k \cdot z = m \cdot g \quad [1]$$

$M := 2 \cdot kg$

$k := 50 \cdot \frac{N}{mm}$

k represent the stiffness of the packaging material. M is the component mass.



Schematic view of package falling and impacting ground (hard surface).

Assume no damping, c=0

with I.C's: $z(0) = 0 \quad \frac{d}{dt} z = v_o$

Equation [1] is valid for z>0, i.e. as long as packaging material is being compressed.

Note that if package rebounds, then $F_{spring}=0$

$h := 250 \cdot mm$ height of drop

$$\omega_n := \left(\frac{k}{M}\right)^{0.5} \quad \omega_n = 158.114 \frac{rad}{s} \quad f_n := \frac{\omega_n}{2 \cdot \pi} \quad f_n = 25.165\, Hz$$ Natural frequency of sytem

Natural period: $T_n := 2 \cdot \frac{\pi}{\omega_n} \quad T_n = 0.04\, s$

$$z_{ss} := \frac{(M \cdot g)}{k} \quad [2] \quad z_{ss} = 0.392\, mm$$ s-s response - after transients die out

from **free fall**, assume no air drag. the package velocity when touching ground is $v_o := (2 \cdot g \cdot h)^{0.5}$

The EOM of motion w/o damping is: $$M \cdot \frac{d^2}{dt^2} z + k \cdot z = W \quad [3]$$

with IC. $z_o := 0 \cdot m$ $\quad v_o = 2.214 \frac{m}{s}$

Dynamic response for undamped system,

$\xi := 0$

The solution of this ODE is very simple, i.e. the superposition of the particular solution (z_{ss}) and the homogenous solution (periodic with natural frequency).

$$z(t) = z_{ss} + A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t) \quad [4a]$$ displacement of package

$$v(t) = \frac{d}{dt} z = \omega_n \cdot \left(-A_c \cdot \sin(\omega_n \cdot t) + A_s \cdot \cos(\omega_n \cdot t)\right) \quad [4b]$$ velocity of package

$$a(t) = \frac{d}{dt} v = -\omega_n^{\,2} \cdot \left(-A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t)\right) \quad [4c]$$ acceleration of package

The constants Ac and As are determined from the initial conditions. At time t=0 s, the pakage materi is not (yet) deflected z(0)=0 and the pakage initial velocity is that of the free fall, v(0)=vo

From Eq [4a]: $z(0) = z_o = 0 = z_{ss} + A_c$ $A_c := -z_{ss}$ [5a]

From Eq [4b] $v(o) = v_o = A_s \cdot \omega_n$ $A_s := \frac{v_o}{\omega_n}$ [5b]

$z(t) := z_{ss} + A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t)$ [4a]

NOTE that package REBOUNDS when it crosses z=0 on its upward motion (remember when you jump on a trampolin.)

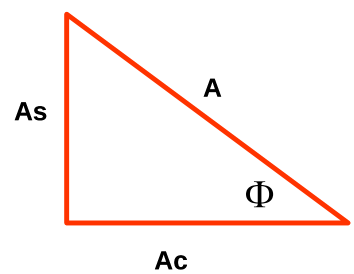
What is the maximum dynamic displacement (deflection)?

Prior to obtaining this deflection, let's recall some trigonometry:

Let: $A = \left(A_c^{\,2} + A_s^{\,2}\right)^{0.5}$ $A := \left[z_{ss}^{\,2} + \left(\frac{v_o}{\omega_n}\right)^2\right]^{.5}$ amplitude of dyanmic motion

and in the formula $A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t)$, multiply by **A** and divide by **A**

$A \cdot \left(\frac{A_c}{A} \cdot \cos(\omega_n \cdot t) + \frac{A_s}{A} \cdot \sin(\omega_n \cdot t)\right)$ [*]



Now, define a phase angle Φ such that

$\cos(\Phi) = \frac{A_c}{A}$ $\sin(\Phi) = \frac{A_s}{A}$

and write expression [*] above as with $\tan(\Phi) = \frac{A_s}{A_c} = \frac{v_o}{\omega_n \cdot (-z_{ss})}$

$A \cdot (\cos(\Phi) \cdot \cos(\omega_n \cdot t) + \sin(\Phi) \cdot \sin(\omega_n \cdot t))$

or

$A \cdot (\cos(\omega_n \cdot t - \Phi))$

where $\Phi := \operatorname{atan}\left[\frac{v_o}{\omega_n \cdot (-z_{ss})}\right] + \pi$ $\Phi = 1.599$ radians

Hence, with the aid of the simple trigonometry "trick" we write the displacement z(t), Eqn [4a] as

$z(t) := z_{ss} + A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t)$ [4a]

$z(t) := z_{ss} + A \cdot (\cos(\omega_n \cdot t - \Phi))$ [6a] displacement of package

for graphs

$T_{nn} := 0.04 \cdot s$

Let's return to the question, **what is the maximum displacement?**

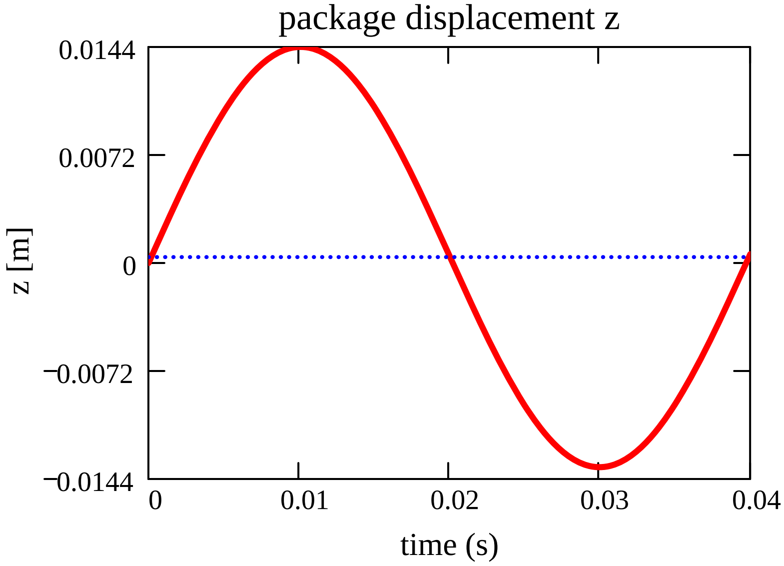
Clearly, the trig function cos(x) can be at most +1 or -1, hence the

$z(0 \cdot s) = 0\,m$

Max dynamic displacement is

$z_{max} := z_{ss} + A$ [7a]

$$z_{max} := z_{ss} + \left[z_{ss}^2 + \left(\frac{v_o}{\omega_n} \right)^2 \right]^{.5}$$



$z_{max} = 14.403\,mm$

$z_{ss} = 0.392\,mm$

z>0 means travel downwards, i.e. compression of pakaking material.

Motion does not die since there is no damping.

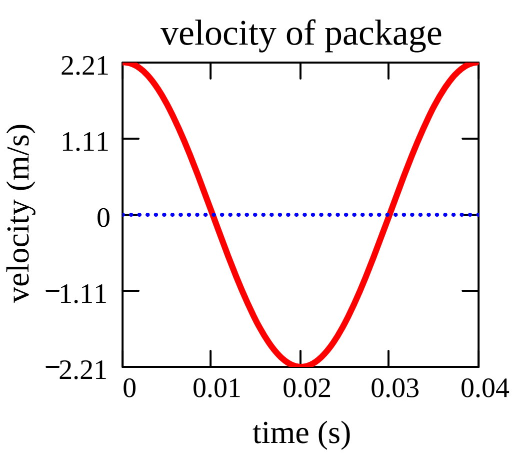
The response found is strictly valid for z>0, i.e. when packaging material "spring" is compressed. "Spring" of package can not be stretched.

Velocity [m/s], v(t)=dz/dt

$v(t) := -A \cdot \omega_n \cdot \left(\sin\left(\omega_n \cdot t - \Phi \right) \right)$ [6b]

what is maximum speed? $v_{max} := A \cdot \omega_n = \left[1 + \left(z_{ss} \cdot \frac{\omega_n}{v_o} \right)^2 \right]^{.5} \cdot v_o = 2.215 \frac{m}{s}$ [7b]

[m/s]



$v_{max} = 2.215 \frac{m}{s}$

$\frac{v_{max}}{v_o} = 1$

$v(0 \cdot s) = 2.214 \frac{m}{s}$

$v_o = 2.214 \frac{m}{s}$

acceleration [m/s2], a(t)=dv/dt,

$$a(t) := -A\cdot\omega_n^2\cdot(\cos(\omega_n\cdot t - \Phi))$$ [6c]

what is maximum accel? $a_{max} := A\cdot\omega_n^2$ $a_{max} = 350.256\frac{m}{s^2}$

since $$A := \left[z_{ss}^2 + \left(\frac{v_o}{\omega_n}\right)^2\right]^{.5}$$

then $$a_{max} = A\cdot\omega_n^2 = \left[z_{ss}^2\cdot\omega_n^4 + (v_o\cdot\omega_n)^2\right]^{.5}$$

but $$z_{ss}\cdot\omega_n^2 = \left(\frac{M\cdot g}{k}\right)\cdot\frac{k}{M} = g$$

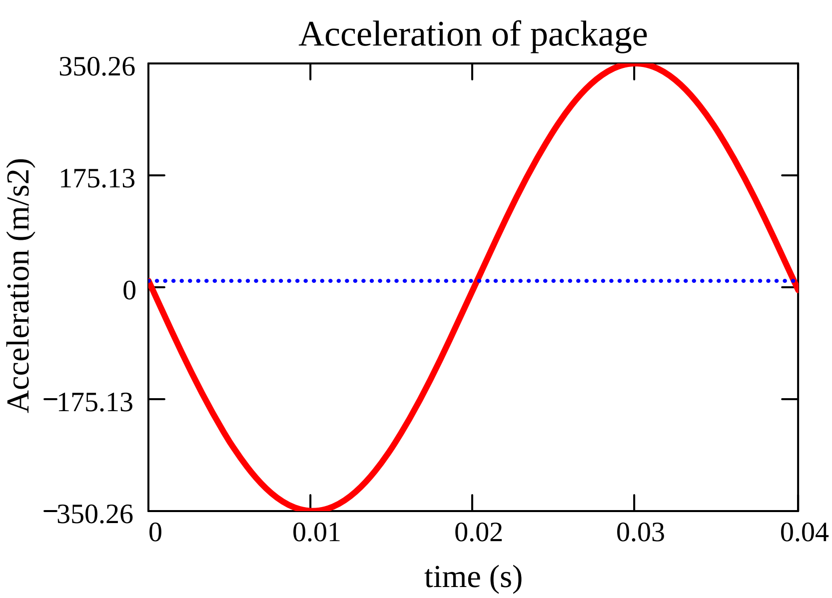
$$\left(\frac{v_o\cdot\omega_n}{g}\right)^2 = 2\cdot g\cdot h\cdot\frac{k}{M\cdot g}\cdot\frac{1}{g} = \frac{2\cdot h}{z_{ss}}$$

Hence:

$$a_{max} := g\cdot\sqrt{\left(1 + \frac{2\cdot h}{z_{ss}}\right)}$$ [7c] where $z_{ss} = \frac{W}{k}$

Note that if packaging stiffness (k) is very large then z_{ss} is very small, and hence, the maximum (peak) acceleration can be quite large!!!

Max acceleration if no damping: $a_{max} = 350.256\frac{m}{s^2}$ $\frac{a_{max}}{g} = 35.716$



peak decelerations can be much larger than 1g!

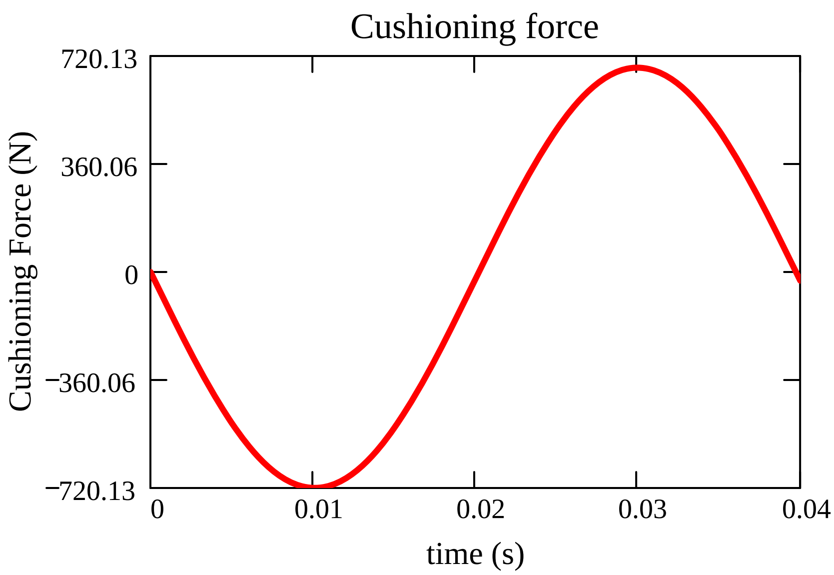
$a(0\cdot s) = 9.807\frac{m}{s^2}$

negative (peak) acceleration denotes upward force

The force from the cushioning into the package is **F = -k z - c v = - W + M acc**

$$F(t) := (-g + a(t))\cdot M$$

$$F_{max} := M\cdot(a_{max} + g)$$



$$F_{max} = 720.125\,N$$

$$M\cdot g = 19.613\,N$$

F<0 means upwards force - spring being compressed

$$\frac{F_{max}}{M\cdot g} = 36.716$$

Qute large!! Much larger than component weight (W)

The package will REBOUND when z=0 (z<0) and dz/dt=V < 0.

Approximately at $t_{rebound} := \frac{T_n}{2}$ $\quad t_{rebound} = 0.02\,s$