MEEN 617 - Cheat Sheet for SDOF motion

Luis San Andres - January 2013 Pages 1 and 2 (only) allowed in exams

EOM: $M_e\ddot{y}+K_e y+C_e\dot{y}=F(t)$

Given a mechanical system with equivalent parameters (Me) mass, (Ke) stiffness, (Ce) viscous damping coefficient.

define natural frequency and damping ratio as: $\omega_n = \sqrt{\frac{K_e}{M_e}}$ $\zeta = \frac{C_e}{2\cdot M_e\cdot\omega_n} = \frac{C_e}{2\cdot\sqrt{K_e\cdot M_e}} = \frac{C_e\cdot\omega_n}{2\cdot K_e}$

TRANSIENT RESPONSE system to STEP Force F(t)=Fo

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = F_o$$

Underdamped system only, $\zeta<1$

+ initial conditions $Y_0 = Y(0)$ $V_0 = \frac{d}{dt}Y$ at t=0

system response is: $Y(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right) + Y_{ss}$

where $Y_{ss} = \frac{F_o}{K_e}$ $C_1 = Y_0 - Y_{ss}$ $C_2 = \frac{\left[V_0 + \zeta\cdot\omega_n\cdot(Y_0 - Y_{ss})\right]}{\omega_d}$ and $\omega_d = \omega_n\cdot\sqrt{1-\zeta^2}$

damped natural frequency

for ζ>=1: see page 2)

LOG DEC (δ): formula to estimate damping ratio (ζ) from a free response $\delta = \frac{1}{n}\cdot\ln\left(\frac{Ao}{An}\right) = \frac{2\pi\cdot\zeta}{\sqrt{1-\zeta^2}}$ $\zeta = \frac{\delta}{\sqrt{4\cdot\pi^2 + \delta^2}}$

Ao and An are peak motion amplitudes separated by n periods

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TRANSIENT RESPONSE system to Force F(t)=A+Bt

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = A + B\cdot t$$

$Y = a + b\cdot t + e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right)$

for $\zeta < 1$

$a = \left(A - C_e\cdot\frac{B}{K_e}\right)\cdot\frac{1}{K_e}$; $b = \frac{B}{K_e}$;

$C_1 = Y_0 - a$

$C_2\cdot\omega_d = V_0 - b + \zeta\cdot\omega_n\cdot(Y_0 - a)$

Transient response of **overdamped system**, step force Fo=constant $\zeta>1$

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = F_o$$

+ initial conditions $Y_0 = Y(0)$ $V_0 = \frac{d}{dt}Y$ at t=0

$Y(t) = A_1\cdot e^{s_1\cdot t} + A_2\cdot e^{s_2\cdot t} + Y_{ss}$

where: $s_1 = \omega_n\cdot\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$ $s_2 = \omega_n\cdot\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$

$Y_{ss} = \frac{F_o}{K_e}$ $A_1 + A_2 = Y_0 - Y_{ss}$

$(A_1\cdot s_1 + A_2\cdot s_2) = V_0$ **Solve for A1 and A2**

s1, s2 < 0

Transient response of **critically damped system**, step force Fo=constant $\zeta=1$

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = F_o$$

+ initial conditions $Y_0 = Y(0)$ $V_0 = \frac{d}{dt}Y$ at t=0

$Y(t) = e^{s\cdot t}\cdot(A_1 + t\cdot A_2) + Y_{ss}$

where: $s = -\omega_n$

$Y_{ss} = \frac{F_o}{K_e}$ $A_1 = Y_0 - Y_{ss}$ $(A_1\cdot s + A_2) = V_0$

STEADY RESPONSE of system to PERIODIC LOADS with frequency ω

Case: periodic force of constant magnitude

$$F(t) = F_o \cdot \sin(\omega \cdot t)$$

Define operating frequency ratio:

$$r = \frac{\omega}{\omega_n}$$

System periodic response: $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t + \Psi)$

where:

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

Case: base motion of constant amplitude

$$M_e \cdot \frac{d^2}{dt^2} Y + C_e \cdot \frac{d}{dt} Y + K_e \cdot Y = K \cdot Y_B + C_e \cdot \frac{d}{dt} Y_B \qquad Y_B(t) = A \cdot \cos(\omega \cdot t)$$

System periodic response: $Y(t) = A \cdot G(r) \cdot \sin(\omega \cdot t + \Psi + \phi)$

where:

$$G(r) = \left[\frac{1 + (2 \cdot \zeta \cdot r)^2}{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2}\right]^{.5} \qquad \tan(\Psi + \phi) = \frac{-2 \cdot \zeta \cdot r^3}{1 + 4 \cdot \zeta^2 - r^2}$$

Case: response to mass imbalance

$$F(t) = m \cdot e \cdot \omega^2 \cdot \sin(\omega \cdot t)$$

u=imbalance (offset center of mass) displacement

System periodic response: $Y(t) = e \cdot \frac{m}{M_e} J(r) \cdot \sin(\omega \cdot t + \Psi)$

$$M_e = M + m \qquad m \cdot e = M_e \cdot u$$

$$J(r) = \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

OTHER USEFUL formulas: (program them in a calculator)

Underdamped system ζ<1: step force response

constants in formulas

Given

$$Y(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos(\omega_d \cdot t) + C_2 \cdot \sin(\omega_d \cdot t)\right) + Y_{ss}$$

find velocity $V(t) = \frac{d}{dt} Y$

$$V(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos(\omega_d \cdot t) + D_2 \cdot \sin(\omega_d \cdot t)\right)$$

where:

$$D_1 = (-\zeta \cdot \omega_n \cdot C_1) + C_2 \cdot \omega_d \qquad D_2 = (-\zeta \cdot \omega_n \cdot C_2) - C_1 \cdot \omega_d$$

find acceleration

$a(t) = \frac{d}{dt} V$

$$a(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(E_1 \cdot \cos(\omega_d \cdot t) + E_2 \cdot \sin(\omega_d \cdot t)\right)$$

where:

$$E_1 = (-\zeta \cdot \omega_n \cdot D_1) + D_2 \cdot \omega_d \qquad E_2 = (-\zeta \cdot \omega_n \cdot D_2) - D_1 \cdot \omega_d$$

A case your memory should retain forever

NO DAMPING, C=0 Ns/m

TRANSIENT RESPONSE of M-K system to STEP Force F(t)=Fo

Undamped system, $\zeta=0$

$$M_e \cdot \frac{d^2}{dt^2}Y + K_e \cdot Y = F_o$$

+ initial conditions $Y_0 = Y(0)$ $V_0 = \frac{d}{dt}Y$ at t=0

response is:

$$Y(t) = \left(C_1 \cdot \cos(\omega_n \cdot t) + C_2 \cdot \sin(\omega_n \cdot t)\right) + Y_{ss}$$

where $Y_{ss} = \frac{F_o}{K_e}$ $C_1 = Y_0 - Y_{ss}$ $C_2 = \frac{V_0}{\omega_n}$

MOTION never dies since there is no dissipation action (no damping)

and velociy and acceleration:

$$V(t) = \frac{d}{dt}Y \qquad V(t) = \left(D_1 \cdot \cos(\omega_n \cdot t) + D_2 \cdot \sin(\omega_n \cdot t)\right)$$

$D_1 = C_2 \cdot \omega_n$ $D_1 = -C_1 \cdot \omega_n$

$$a(t) = \frac{d}{dt}V \qquad a(t) = \left(E_1 \cdot \cos(\omega_n \cdot t) + E_2 \cdot \sin(\omega_n \cdot t)\right)$$

$E_1 = -C_1 \cdot \omega_n^2$ $E_2 = -C_2 \cdot \omega_n^2$

Note that the velocity and acceleration superimpose a cos & a sin functions. Thus, the maximum values of velocity and acceleration equal

$$V_{max} = \left(D_1^2 + D_2^2\right)^{0.5} \qquad V_{max} = \omega_n \cdot \sqrt{C_1^2 + C_2^2}$$

$$a_{max} = \left(D_1^2 + D_2^2\right)^{0.5} \qquad a_{max} = \omega_n^2 \cdot \sqrt{C_1^2 + C_2^2} = \omega_n \cdot V_{max}$$

since

$$Y_{ss} = \frac{F_o}{K_e} \qquad C_1 = Y_0 - Y_{ss} \qquad C_2 = \frac{V_0}{\omega_n}$$

$$a_{max} = \omega_n^2 \cdot \sqrt{(Y_0 - Y_{ss})^2 + \left(\frac{V_0}{\omega_n}\right)^2}$$

Note: the function $x(t) = a \cdot \cos(\omega \cdot t) + b \cdot \sin(\omega \cdot t)$ can be written as

$$x(t) = c \cdot \cos(\omega \cdot t - \phi)$$

where $c = \sqrt{a^2 + b^2}$ $\tan(\phi) = \frac{b}{a}$

OTHER important information

given a function $f(t)$ find its maximum value

The maximum or mimimum values are obtained from $\frac{d}{dt}f = 0$

For example, for the underdamped response, ζ<1, the system response for a step load is

$$Y(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right) + Y_{ss}$$

where $Y_{ss} = \frac{F_o}{K_e}$ $\quad C_1 = Y_0 - Y_{ss}$ $\quad C_2 = \frac{\left[V_0 + \zeta\cdot\omega_n\cdot(Y_0 - Y_{ss})\right]}{\omega_d}$ and $\omega_d = \omega_n\cdot\sqrt{1-\zeta^2}$

damped natural frequency

when does Y(t) peak (max or min) ?

from the formulas sheet

$$V(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(D_1\cdot\cos(\omega_d\cdot t) + D_2\cdot\sin(\omega_d\cdot t)\right)$$

$$D_1 = (-\zeta\cdot\omega_n\cdot C_1) + C_2\cdot\omega_d \qquad D_2 = (-\zeta\cdot\omega_n\cdot C_2) - C_1\cdot\omega_d$$

A peak value occurs at time t=τ when dY/dt=V=0, i.e.

$$0 = e^{-\zeta\cdot\omega_n\cdot\tau}\cdot\left(D_1\cdot\cos(\omega_d\cdot\tau) + D_2\cdot\sin(\omega_d\cdot\tau)\right)$$

$e^{-\zeta\cdot\omega_n\cdot\tau} \neq 0$ for most times; hence $\quad 0 = D_1\cdot\cos(\omega_d\cdot\tau) + D_2\cdot\sin(\omega_d\cdot\tau)$

$$\tan(\omega_d\cdot\tau) = \frac{-D_1}{D_2}$$

solve this equation to find τ

there are an infinite # of time values (τ) satisfying the equation above. Select the lowest τ as this will probaly will give you the largest peak.

Example:

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = A + B\cdot t$$

Obtain constants **C1** and **C2** for case of force F(t)=A+Bt - underdamped response

Given $Y(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right) + a + b\cdot t$

$V(t) = \frac{d}{dt}Y$ $\qquad V(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(D_1\cdot\cos(\omega_d\cdot t) + D_2\cdot\sin(\omega_d\cdot t)\right) + b$

$$D_1 = (-\zeta\cdot\omega_n\cdot C_1) + C_2\cdot\omega_d \qquad D_2 = (-\zeta\cdot\omega_n\cdot C_2) - C_1\cdot\omega_d$$

at t=0, $\quad Y_0 = Y(0) \quad V_0 = \frac{d}{dt}Y$ at t=0

satisfy initial conditions: $\quad Y_0 = Y(0) \qquad Y_0 = C_1 + a \qquad C_1 = Y_0 - a$

$V_0 = \frac{d}{dt}Y \qquad V_0 = D_1 + b = (-\zeta\cdot\omega_n\cdot C_1) + C_2\cdot\omega_d + b$

$V_0 = \left[-\zeta\cdot\omega_n\cdot(Y_0 - a)\right] + C_2\cdot\omega_d + b \qquad C_2 = \frac{V_0 - b + \zeta\cdot\omega_n\cdot(Y_0 - a)}{\omega_d}$

FREQUENCY RESPONSE FUNCTIONS for PERIODIC LOAD with frequency ω

Case: periodic force of constant magnitude

$$M_e \cdot \frac{d^2}{dt^2} Y + C_e \cdot \frac{d}{dt} Y + K_e \cdot Y = F_o \cdot \sin(\omega \cdot t)$$

Define operating frequency ratio: $r = \frac{\omega}{\omega_n}$

System periodic responses:

Displacement: $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t + \Psi)$

velocity: $V(t) = \frac{d}{dt} Y = \delta s \cdot \omega \cdot H(r) \cdot \sin(\omega \cdot t + \Psi)$

acceleration: $a(t) = \frac{d^2}{dt^2} Y = -\delta s \cdot \omega^2 \cdot H(r) \cdot \sin(\omega \cdot t + \Psi) = -\omega^2 \cdot Y$

where: harmonic response

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

Define **dimensionless amplitudes of frequency response function for**

displacement $\frac{|Y|}{\delta_s} = H(r)$

velocity $\frac{|V| \cdot \omega_n}{\delta_s} = r \cdot H(r)$

acceleration $\frac{|a| \cdot \omega_n^2}{\delta_s} = r^2 \cdot H(r) = J(r)$

Disclaimer:

The list of formulas listed above (solutions to ODE) WILL AID you in the analysis of system response
The list is NOT exhaustive.

The formulas do NOT answer the question: Which formula do I use? One must use judgement (knowledge of the problem - how it works) to select the appropriate formula (response)

Dr. San Andres does NOT guarantee that all formulae given above are type CORRECTLY. It is the student's responsibility to verify the formulas above.

ONLY PAGES ONE & TWO of this DOCUMENT ALLOWED in ME617 exams

GRAPHS for AMPLITUDE OF TRANSFER functions

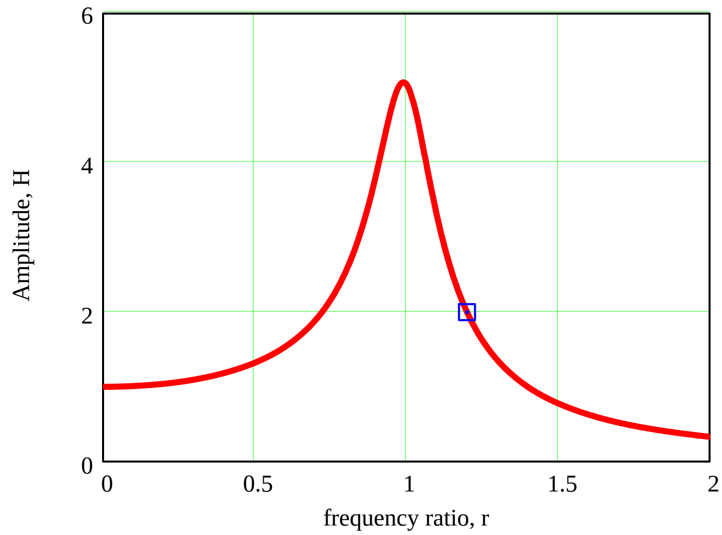
Constant amplitude force

Given desired H and freq r

$$H(r,\zeta) := \frac{1}{\left[\left(1-r^2\right)^2+(2\cdot\zeta\cdot r)^2\right]^{.5}}$$

formula for damping ratio $H_a := 2$ $r_a := 1.2$

$$\zeta := \frac{1}{r_a\cdot 2}\cdot\left[\frac{1}{H_a^{\,2}}-\left(1-r_a^{\,2}\right)^2\right]^{.5} \qquad \zeta = 0.099$$



$H(r_a,\zeta) = 2$

$\frac{1}{2\cdot\zeta} = 5.053$

$H(1,\zeta) = 5.053$

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Acceleration or Imbalance

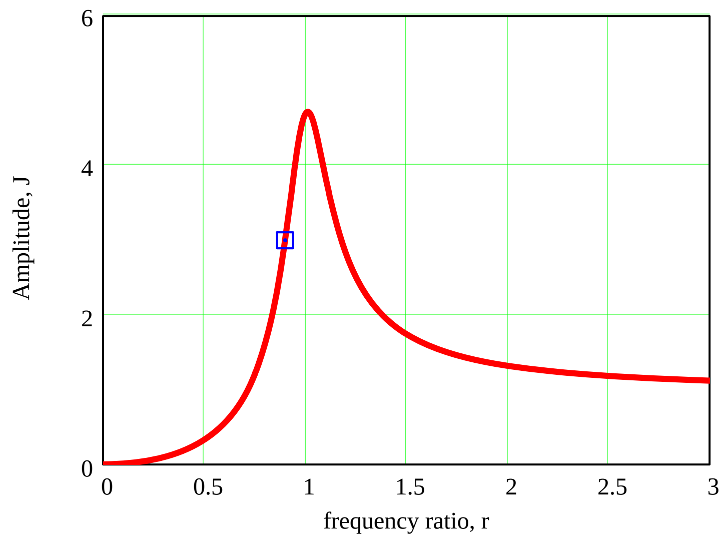
Given desired J and freq r

$$J(r,\zeta) := \frac{r^2}{\left[\left(1-r^2\right)^2+(2\cdot\zeta\cdot r)^2\right]^{.5}}$$

formula for damping ratio $J_a := 3$ $r_a := 0.9$

$$\zeta := \frac{1}{r_a\cdot 2}\cdot\left[\frac{r_a^{\,4}}{J_a^{\,2}}-\left(1-r_a^{\,2}\right)^2\right]^{.5}$$

$\zeta = 0.107$



$J(r_a,\zeta) = 3$

$\frac{1}{2\cdot\zeta} = 4.692$

$J(1,\zeta) = 4.692$

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Transmitted force to base or foundation

Given desired T and freq r

$r_a := 0.75 \quad T_a := 2.0$

$$T(r,\zeta) := \frac{\left[1+(2\cdot\zeta\cdot r)^2\right]^{0.5}}{\left[\left(1-r^2\right)^2+(2\cdot\zeta\cdot r)^2\right]^{.5}}$$

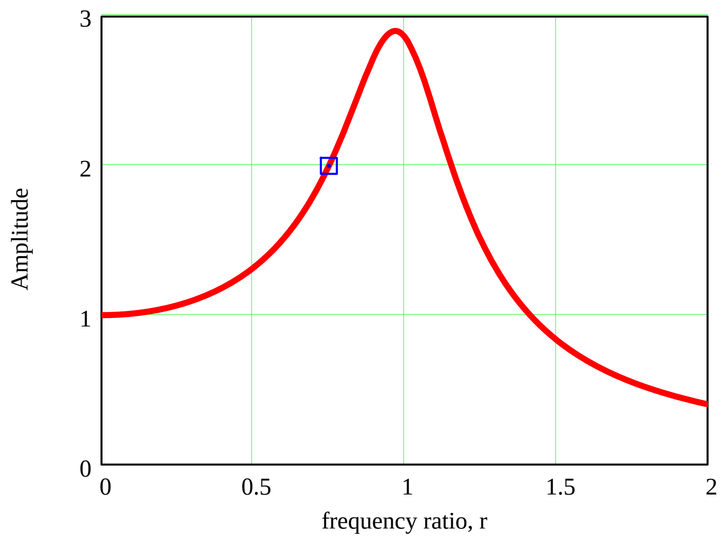
$$T^2\cdot\left[\left(1-r^2\right)^2+(2\cdot\zeta\cdot r)^2\right] = 1+(2\cdot\zeta\cdot r)^2$$

$$\left[\left(1-r^2\right)^2\cdot T^2-1\right] = (2\cdot\zeta\cdot r)^2\cdot\left(1-T^2\right)$$

$T(r_a,0) = 2.286$

$$\zeta := \frac{1}{r_a\cdot 2}\cdot\left[\frac{\left[\left(1-r_a^{\,2}\right)^2\cdot T_a^{\,2}-1\right]}{1-T_a^{\,2}}\right]^{.5}$$

$\zeta = 0.186$



$T(r_a,\zeta) = 2$

$T(1,\zeta) = 2.864$

$$\frac{\left[1+(2\cdot\zeta)^2\right]^{.5}}{2\cdot\zeta} = 2.864$$