Handout #2b (pp. 40-55)

Dynamic Response of Second Order Mechanical Systems with Viscous Dissipation forces

$$M\,\ddot{X} + D\,\dot{X} + K\,X = F_{(t)}$$

Periodic Forced Response to

$F_{(t)} = F_o \sin(\Omega\, t)$* and *$F_{(t)} = M\, u\, \Omega^2 \sin(\Omega t)$

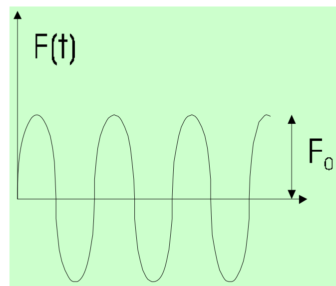
Frequency Response Function of Second Order Systems

(c) Forced response of 2nd order mechanical system to a periodic force excitation

Let the external force be PERIODIC of frequency Ω (period $T=2\pi/\Omega$) and consider the system to have initial displacement X_0 and velocity V_0. The equation of motion for a system with viscous dissipation mechanism, is:

$$M \frac{d^2 X}{d t^2} + D \frac{d X}{d t} + K X = F_o \sin(\Omega t) \tag{41}$$

with initial conditions $V(0) = V_o$ and $X(0) = X_o$



The external force $F_{(t)}$ has amplitude F_o and frequency Ω. This type of forced excitation is known as **PERIODIC LOADING.**

The solution of the non-homogeneous ODE (41) is of the form:

$$X(t) = X_H + X_P = A_1 e^{s_1 t} + A_2 e^{s_2 t} + C_c \cos(\Omega t) + C_s \sin(\Omega t) \tag{42}$$

where X_H is the solution to the homogeneous form of (1) and such that (s_1, s_2) satisfy the characteristic equation of the system:

$$\left(s^2 + 2\zeta\, \omega_n\, s + \omega_n^2 \right) = 0 \tag{43}$$

The roots of this 2nd order polynomial are:

$$s_{1,2} = -\zeta\, \omega_n \mp \omega_n \left(\zeta^2 - 1 \right)^{1/2} \tag{44}$$

where $\omega_n = \sqrt{K/M}$ is the natural frequency, $\zeta = \dfrac{D}{D_{cr}}$ is the viscous damping ratio, and $D_{cr} = 2\sqrt{K M}$ is the critical damping coefficient.

The value of damping ratio determines whether the system is underdamped (ζ *<1*), critically damped (ζ *=1*), overdamped (ζ *>1*).

Response of 2nd Order Mechanical System to a Periodic Loading:

and

$$X_P = C_c \cos(\Omega t) + C_s \sin(\Omega t) \tag{45}$$

is the particular solution due to the periodic loading, $F_o \sin(\Omega t)$.

Substitution of Eq. (42) into Eq. (41) gives

$$\begin{aligned}&\cos(\Omega t)\left[\left\{K - \Omega^2 M\right\} C_c + \Omega D C_s\right] + \\ &\sin(\Omega t)\left[\left\{K - \Omega^2 M\right\} C_s - \Omega D C_s\right] \quad = F_o \sin(\Omega t)\end{aligned} \tag{46}$$

since the ***sin()*** and ***cos()*** functions are linearly independent, it follows that

$$\begin{aligned}&\left[\left\{K - \Omega^2 M\right\} C_c + \Omega D C_s\right] = 0 \\ &\left[\left\{K - \Omega^2 M\right\} C_s - \Omega D C_s\right] \quad = F_o\end{aligned} \tag{47}$$

$$\begin{bmatrix} \left\{K - \Omega^2 M\right\} & \Omega D \\ -\Omega D & \left\{K - \Omega^2 M\right\} \end{bmatrix} \begin{Bmatrix} C_c \\ C_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_o \end{Bmatrix}$$

i.e. a system of 2 algebraic equations with two unknowns, C_c and C_s. Divide Eq. (47) by K and obtain:

$$\begin{bmatrix} \left\{1 - \Omega^2 M/K\right\} & \Omega D/K \\ -\Omega D/K & \left\{1 - \Omega^2 M/K\right\} \end{bmatrix} \begin{Bmatrix} C_c \\ C_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_o/K \end{Bmatrix} \tag{48}$$

since $\omega_n = \sqrt{K/M}$; and with $\zeta = \dfrac{D}{D_{cr}}$ then

$$\frac{D}{K}=\frac{D}{K}\frac{\zeta\sqrt{K M}}{D}=\frac{\zeta}{}\sqrt{\frac{M}{K}}=\frac{\zeta}{\omega_n}$$

and $X_{ss} = F_o/K$ **is a "pseudo" static displacement**

Define a <u>frequency</u> <u>ratio</u> as

$$f=\frac{\Omega}{\omega_n} \qquad (49)$$

relating the (external) excitation frequency *(Ω)* to the *natural frequency of the system (ω_n);* i.e. when

$\Omega \ll \omega_n \rightarrow f \ll 1$, the system operates ***below*** its natural frequency

$\Omega \gg \omega_n \rightarrow f \gg 1$, the system is said to operate ***above*** its natural frequency

With this definition, write Eq. (48) as:

$$\begin{bmatrix} \left\{1-\Omega^2/\omega_n^2\right\} & \Omega\,\zeta/\omega_n \\ -\Omega\,\zeta/\omega_n & \left\{1-\Omega^2/\omega_n^2\right\} \end{bmatrix}\begin{Bmatrix} C_c \\ C_s \end{Bmatrix}=\begin{Bmatrix} 0 \\ X_{ss} \end{Bmatrix}$$

$$\begin{bmatrix} \left\{1-f^2\right\} & f\,\zeta \\ -f\,\zeta & \left\{1-f^2\right\} \end{bmatrix}\begin{Bmatrix} C_c \\ C_s \end{Bmatrix}=\begin{Bmatrix} 0 \\ X_{ss} \end{Bmatrix} \qquad (50)$$

Solve Eq. (50) using Cramer's rule to obtain the coefficients C_s and C_c:

$$C_c = X_{ss}\frac{-2\zeta f}{\left(1-f^2\right)^2+\left(2\zeta f\right)^2};\; C_s = X_{ss}\frac{\left(1-f^2\right)^2}{\left(1-f^2\right)^2+\left(2\zeta f\right)^2} \qquad (51)$$

Response of 2nd Order Mechanical System to a Periodic Load:

For an **underdamped** system, $0 < \zeta < 1$, the roots of the characteristic eqn. have a real and imaginary part, i.e.

$$s_{1,2} = -\zeta\,\omega_n \mp i\,\omega_n \left(1-\zeta^2\right)^{1/2} \tag{52}$$

where $i=\sqrt{-1}$ is the imaginary unit. The homogeneous solution is

$$X_{H(t)} = e^{-\zeta\,\omega_n t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) \tag{53}$$

where $\omega_d = \omega_n \left(1-\zeta^2\right)^{1/2}$ is the damped natural frequency of the system.

Thus, the total response is $X(t) = X_H + X_P$ =

$$X_{(t)} = e^{-\zeta\,\omega_n t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) + C_c \cos\left(\Omega t\right) + C_s \sin\left(\Omega t\right) \tag{54}$$

where C_s and C_c are given by Eq. (51).

At time $t = 0$, the initial conditions are $V(0) = V_o$ and $X(0) = X_o$. Then

$$C_1 = \left(X_0 - C_c\right) \text{ and } C_2 = \frac{V_0 + \zeta\,\omega_n C_1 - \Omega C_s}{\omega_d} \tag{55}$$

Now, provided $\zeta > 0$, the homogeneous solution (also known as the **TRANSIENT or Free response**) will die out as time elapse. Thus, after all transients have passed, the dynamic response of the system is just the particular response $X_{P(t)}$

Steady State – Periodic Forced Response of Underdamped 2nd Order System

As long as the system has some damping *(ζ > 0)*, the transient response (homogeneous solution) will die out and **cease** to influence the behavior of the system. Then, the **steady-state** (or **quasi-stationary**) response is given by:

$$X_{(t)} \approx C_c \cos(\Omega t) + C_s \sin(\Omega t) = C \sin(\Omega t - \varphi) \quad (56.a)$$

where *(C_s, C_c)* are given by equation (51) as:

$$C_c = X_{ss} \frac{-2\zeta f}{\left(1-f^2\right)^2 + \left(2\zeta f\right)^2}; \quad C_s = X_{ss} \frac{\left(1-f^2\right)^2}{\left(1-f^2\right)^2 + \left(2\zeta f\right)^2}$$

and $X_{ss} = F_o/K$. Define $C_s = C\cos(\varphi)$; $C_c = -C\sin(\varphi)$; then

$$\tan(\varphi) = \frac{-C_c}{C_s} = \frac{2\zeta f}{\left(1-f^2\right)}$$

where φ is a **phase angle**, and $C = \sqrt{C_S^2 + C_C^2} = X_{ss}\, A$

with $$A = \frac{1}{\sqrt{\left(1-f^2\right)^2 + \left(2\zeta f\right)^2}};$$ as the ***amplitude ratio***

where $f = \dfrac{\Omega}{\omega_n}$ is the frequency ratio. Thus, the system response is:

$$X_{(t)} = X_{ss}\, A\, \sin(\Omega t - \varphi) \quad (56.b)$$

Regimes of Dynamic Operation:

$\Omega \ll \omega_n \rightarrow f \ll 1$, the system operates ***below*** its natural frequency

$$\left(1-f^2\right) \rightarrow 1; \left(2\zeta f\right) \rightarrow 0 \quad \Rightarrow A \rightarrow 1 \;\; \varphi \rightarrow 0$$

$X(t) \rightarrow X_{ss} \sin\left(\Omega t\right)$ i.e. similar to the "static" response

$\Omega = \omega_n \rightarrow f = 1$, the system is excited ***at*** its natural frequency

$$\left(1-f^2\right) \rightarrow 0; \quad \Rightarrow A \rightarrow \frac{1}{2\zeta} \;;\; \varphi \rightarrow \frac{\pi}{2} \;\; \left(90^{\circ}\right)$$

$$X(t) \rightarrow \frac{X_{ss}}{2\zeta} \sin\left(\Omega t - \frac{\pi}{2}\right)$$

if $\zeta < 0.5$, the amplitude ratio $A > 1$ and a resonance is said to occur.

$\Omega \gg \omega_n \rightarrow f \gg 1$, the system is said to operate ***above*** its natural frequency

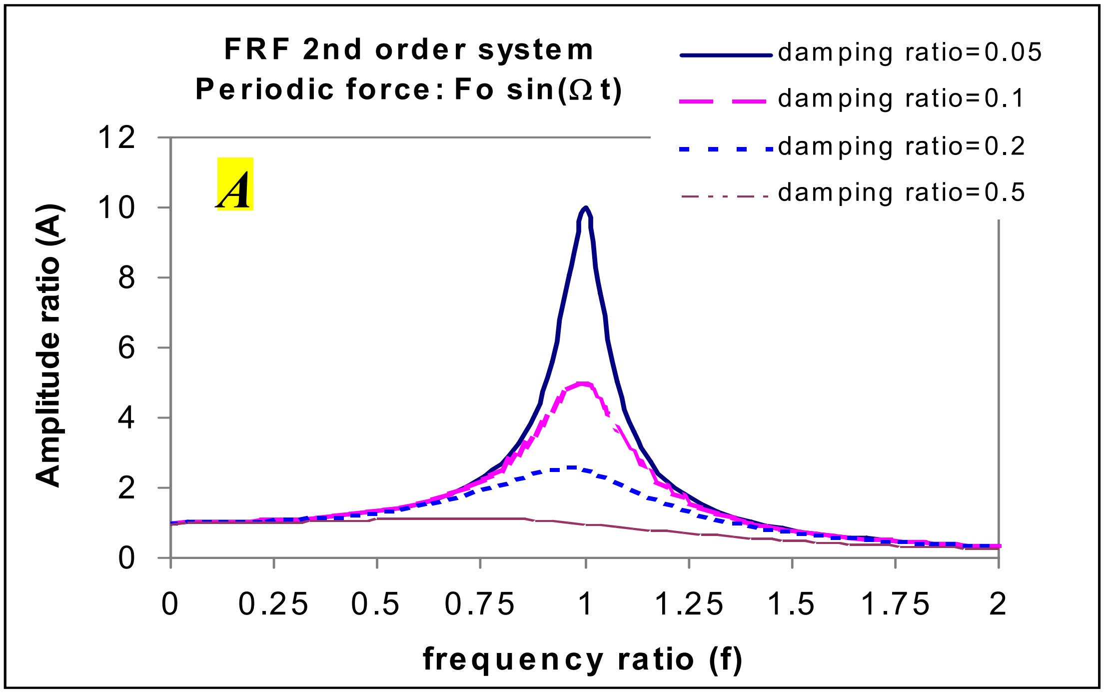
$$\left(1-f^2\right) << 1; \left(2\zeta f\right) \gg 0 \quad \Rightarrow A \rightarrow 0 \;\; \varphi \rightarrow \pi \;\; \left(180^{\circ}\right)$$

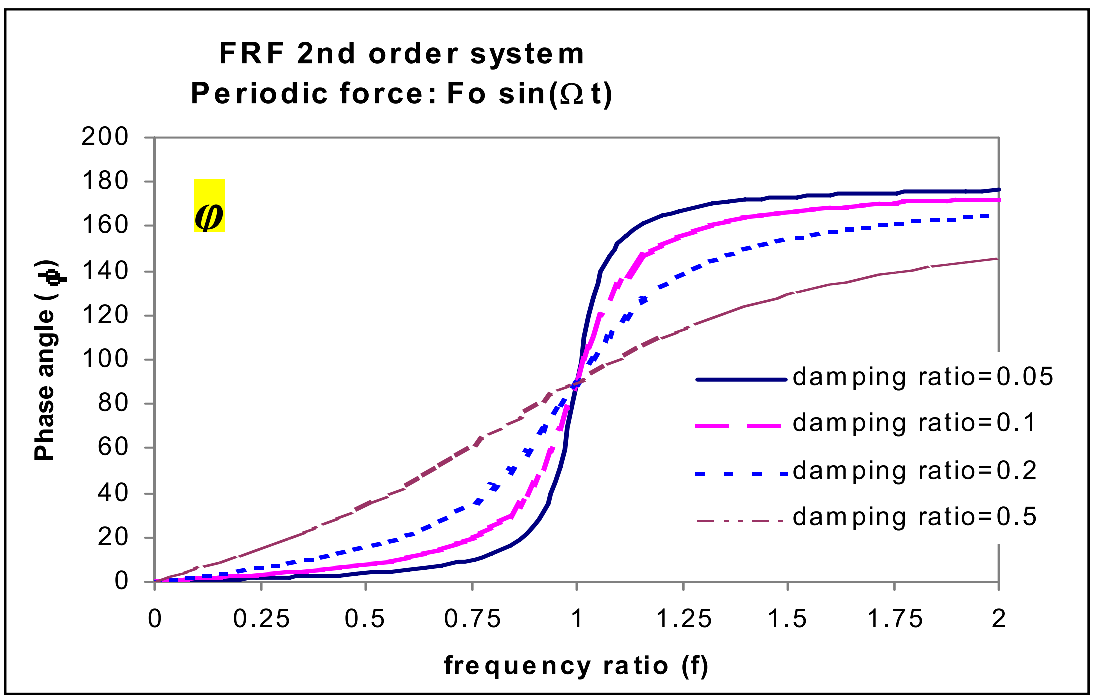
$$X(t) \rightarrow X_{ss} A \sin\left(\Omega t - \pi\right) = -X_{ss} A \sin\left(\Omega t\right)$$

A <<< 1, i.e. very small,

Frequency Response of Second Order Mechanical System

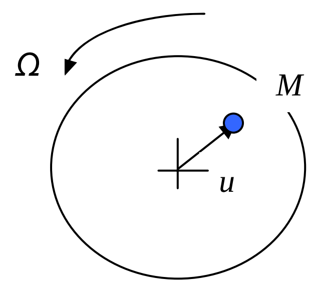
$$X_{(t)} = X_{ss}\, A \sin\left(\Omega t - \varphi\right) \qquad \text{for} \qquad F_{(t)} = F_o \sin\left(\Omega t\right)$$





Steady State – Periodic Forced Response of 2nd Order system: Imbalance Load

Imbalance loads are typically found in rotating machinery. In operation, due to inevitable wear, material build ups or assembly faults, the center of mass of the rotating machine does not coincide with the center of rotation (spin). Let the center of mass be located a distance (***u***) from the spin center, and thus, the imbalance load is a centrifugal "force" of magnitude $F_o = M\,u\,\Omega^2$ and rotating with the same frequency as the rotor speed (***Ω***). This force excites the system and induces vibration[1]. Note that the imbalance force is proportional to the frequency^2 and grows rapidly with speed.



In practice the offset distance (***u***) is very small (a few thousands of an inch).

For example if the rotating shaft & disk has a small imbalance mass (***m***) located at a radius (***r***) from the spin center, then it is easy to determine that the center of mass offset (***u***) is equal to ***(m r/M)***. Note that ***u<<r.***

$$u\left(M+m\right)=r\,m \quad \rightarrow \; u \simeq \frac{r\,m}{M}$$

The equation of motion for the 2nd order system is

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+K\,X=F_o\sin\left(\Omega t\right)=M\,u\,\Omega^2\sin\left(\Omega t\right)$$

---

[1] The current analysis only describes vibration along direction *X*. In actuality, the imbalance force induces vibrations in two planes (*X*,*Y*) and the rotor whirls in an orbit around the center of rotation. For isotropic systems, the motion in the *X* plane is identical to that in the *Y* plane but out of phase by 90 degrees.

then $$X_{ss} = \frac{F_o}{K} = \frac{M\,u\,\Omega^2}{K} = u\frac{\Omega^2}{\omega_n^2} = u\,f$$

with ($f = \Omega/\omega_n$). The system response at "steady-state" is

$$X_{(t)} = X_{ss}\,A\,\sin\left(\Omega t - \varphi\right) = u\,\left(\Omega^2 A\right)\,\sin\left(\Omega t - \varphi\right)$$

$$\boxed{X_{(t)} = u\;B\;\sin\left(\Omega t - \varphi\right)} \quad (59)$$

where φ is a **phase angle**, $\tan\left(\varphi\right) = \dfrac{2\,\zeta\,f}{\left(1 - f^2\right)}$, and

$$B = \frac{f^2}{\sqrt{\left(1 - f^2\right)^2 + \left(2\zeta\,f\right)^2}}$$

(60) is an ***amplitude ratio***

recall $f = \dfrac{\Omega}{\omega_n}$ is the frequency ratio.

Regimes of Dynamic Operation:

$\Omega \ll \omega_n \rightarrow f \ll 1$, excitation ***below*** its natural frequency

$$\left(1 - f^2\right) \rightarrow 1;\ \left(2\zeta\,f\right) \rightarrow 0 \quad \Rightarrow B \rightarrow f^2 \approx 0 \quad \text{and} \quad \varphi \rightarrow 0$$

$$X(t) \rightarrow u\, f^{2} \sin(\Omega t) \rightarrow 0$$ i.e. little motion

$\Omega = \omega_n \rightarrow f = 1$, the system is excited ***at*** its natural frequency

$$\left(1 - f^{2}\right) \rightarrow 0; \quad \Rightarrow B \rightarrow \frac{1}{2\zeta} \; ; \; \varphi \rightarrow \frac{\pi}{2} \quad \left(90^{\circ}\right)$$

$$X(t) \rightarrow \frac{u}{2\zeta} \sin\left(\Omega t - \frac{\pi}{2}\right) = -\frac{u}{2\zeta} \cos(\Omega t)$$

if ζ < *0.5*, the amplitude ratio *B* > 1 and a resonance is said to occur.

$\Omega \gg \omega_n \rightarrow f \gg 1$, the system operates ***above*** its natural frequency

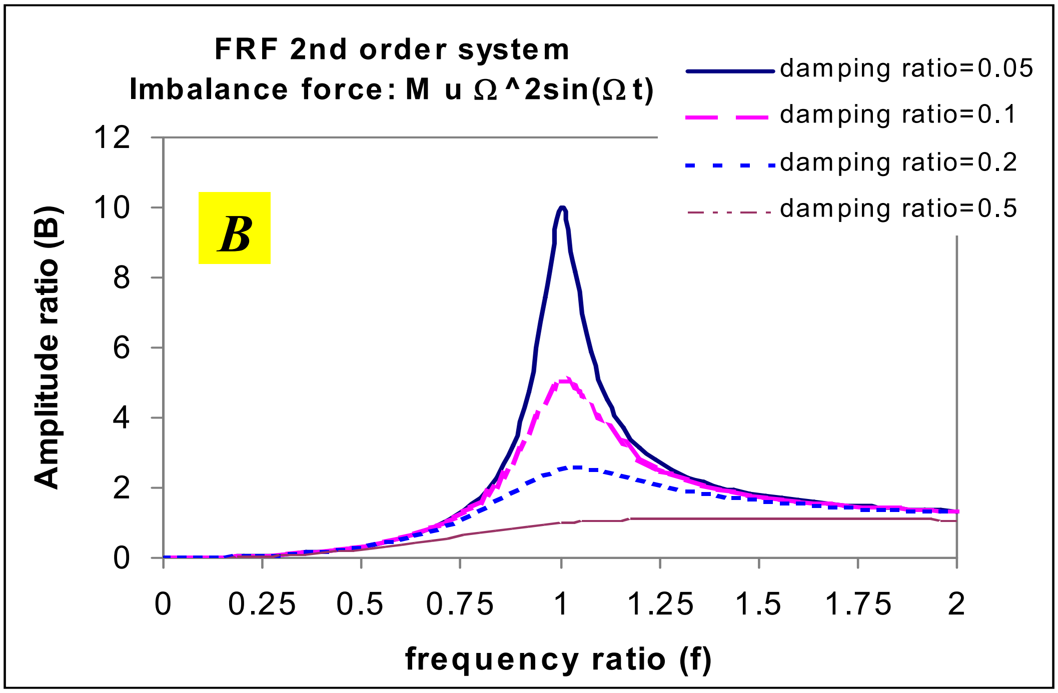
$$\frac{\left(1 - f^{2}\right)}{f^{2}} \rightarrow -1 \; ; \left(\frac{2\zeta f}{f^{2}}\right) \rightarrow 0 \; \Rightarrow B \rightarrow 1 \quad \varphi \rightarrow \pi \quad \left(180^{\circ}\right)$$

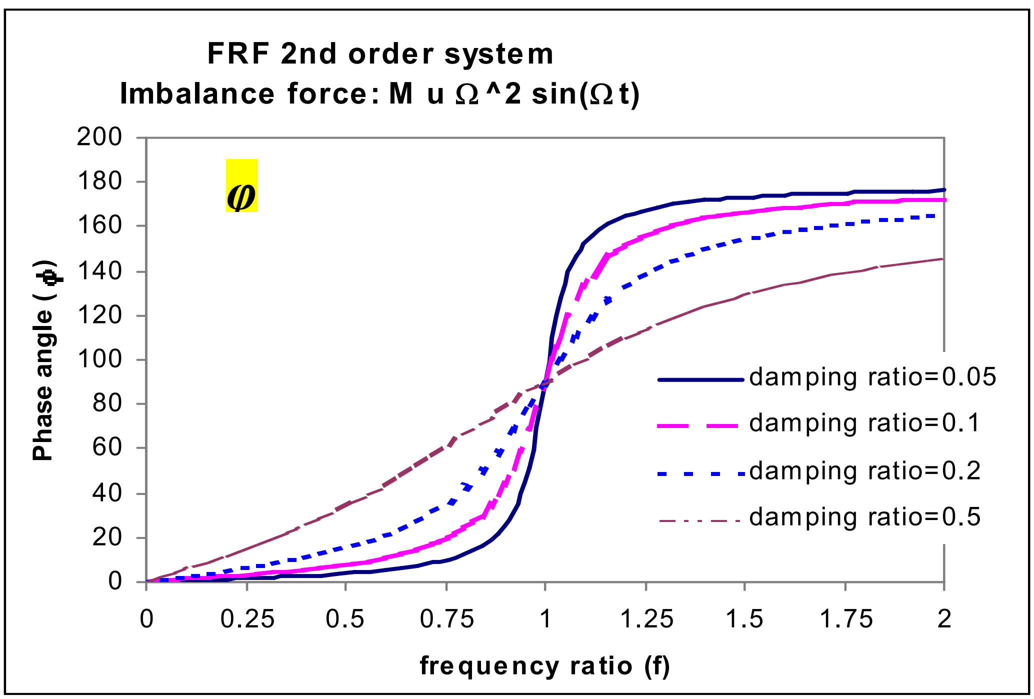
$$X(t) \rightarrow u \sin(\Omega t - \pi) = -u \sin(\Omega t)$$

B ~ 1, at high frequency operation, the maximum amplitude of vibration (X_{max}) equals the unbalance displacement (*u)*

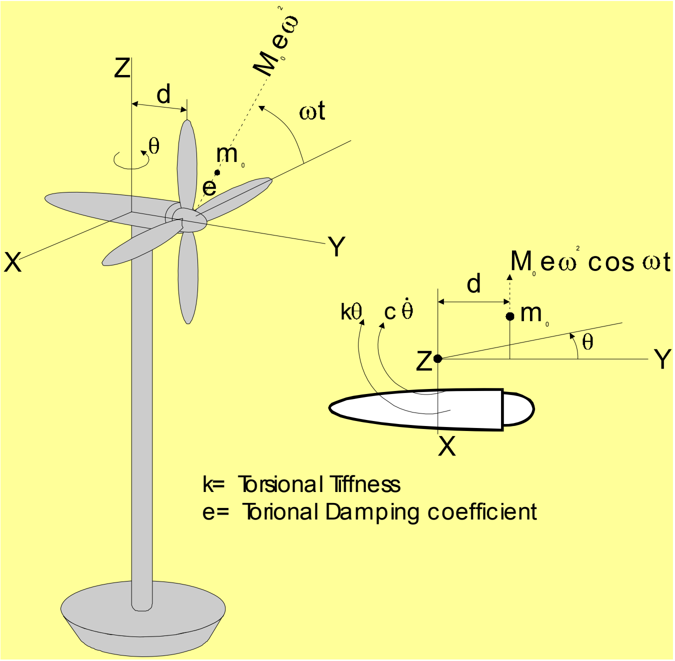
Frequency Response of Second Order Mechanical System due to an Imbalance Load

$$X_{(t)} = u\,B \sin(\omega t - \varphi) \quad \text{for} \quad F_{(t)} = M\,u\,\omega^2 \sin(\omega t)$$





EXAMPLE:



A cantilevered steel pole supports a small wind turbine. The pole torsional stiffness is **K** (N.m/rad) with a rotational damping coefficient **C** (N.m.s/rad).

The four-blade turbine rotating assembly has mass $\mathbf{m_o}$, and its center of gravity is displaced distance **e** [m] from the axis of rotation of the assembly.

$\mathbf{I_z}$ (kg.m^2) is the mass moment of inertia about the *z* axis of the complete turbine, including rotor assembly, housing pod, and contents.

The total mass of the system is **m** (kg). The plane in which the blades rotate is located a distance **d** (m) from the *z* axis as shown.

For a complete analysis of the vibration characteristics of the turbine system, determine:

a) Equation of motion of torsional vibration system about z axis.
b) The steady-state torsional response $\theta_{(t)}$ (after all transients die out).
c) For system parameter values of k=98,670 N.m/rad, I_z=25 kg.m^2, C = 157 N.m.s/rad, and m_O = 8 kg, e = 1 cm, d = 30 cm, present graphs showing the response amplitude (in rads) and phase angle as the turbine speed (due to wind power variations) changes from 100 rpm to 1,200 rpm.
d) From the results in (c), at what turbine speed should the largest vibration occur and what is its magnitude?
e) Provide a design recommendation or change so as to reduce this maximum vibration amplitude value to half the original value.

Neglect any effect of the mass and bending of the pole on the torsional response, as well as any gyroscopic effects.

Note: the torque or moment induced by the mass imbalance is

$$T_{(t)} = d \times F_u = \underbrace{m_O\, e\, d\, \omega^2}_{T(\omega)} \cos(\omega t),$$ i.e., a function of frequency

The equation describing torsional motions of the turbine-pole system is:

$$I_z\ddot{\theta} + C\dot{\theta} + K\theta = m_o e\, d\, \omega^2 \cos \omega t = T_{(\omega)} \cos \omega t \qquad \text{(e.1)}$$

Note that all terms in the EOM represent moments or torques.

(b) After all transients die out, the periodic forced response of the system:

$$\theta(t) = \frac{\theta_{ss}}{\left[\left(1-f^2\right)^2 + \left(2\,\zeta\, f\right)^2\right]^{1/2}} \cos\left(\omega t - \phi\right) \qquad \text{(e.2)}$$

but $\theta_{ss} = \dfrac{T_{(\omega)}}{K} = \dfrac{m_o\, ed\, \omega^2}{K}\left(\dfrac{I_z}{I_z}\right) = \dfrac{m_o\, ed}{I_z}\, f^2$ (e.3)

with $f = \dfrac{\omega}{\omega_n};\ \omega_n = \sqrt{\dfrac{K}{I_z}}\ ;\ \zeta = \dfrac{C}{2\sqrt{K\, I_z}}$, and $\phi = \tan^{-1}\left(\dfrac{2\zeta\, f}{1-f^2}\right)$ (e.4)

(e.3) in (**e.2**) leads to

$$\theta(t) = \frac{m_o\, ed}{I_z}\ \frac{f^2}{\left[\left(1-f^2\right)^2 + \left(2\,\zeta\, f\right)^2\right]^{1/2}} \cos\ (\omega t - \phi) \qquad \text{(e.5)}$$

Let $\theta_\infty = \dfrac{m_o\, ed}{I_z}$ (e.6) $\qquad B = \dfrac{f^2}{\left[\left(1-f^2\right)^2 + \left(2\zeta\, f\right)^2\right]^{1/2}}$ (e.7)

and rewrite (e.5) as: $\theta(t) = \theta_\infty\ B \cos\ (\omega t - \phi)$ (e.8)

(c) for the given physical values of the system parameters:

K = 98,670 N.m/rad

I_z = 25 kg · m^2

C = 157. N.m.s/rad

$$\omega_n = \sqrt{\frac{K}{I_Z}} = 62.82 \ \frac{\text{rad}}{\text{sec}}$$

$$\zeta = \frac{C}{2\sqrt{K\, I_Z}} = 0.05$$

$$\theta_\infty = \frac{m_o e\, d}{I_Z} = \frac{8 \cdot 0.01 \cdot 0.3}{25} = \frac{0.024}{25} = 96\ 10^{-5}\text{rad}$$

And the turbine speed varies from 100 rpm to 1,200 rpm, i.e.

ω= rpm π/30 = 10.47 rad/s to 125.66 rad/s, i.e.

$$f = \frac{\omega}{\omega_n} = \ 0.167 \ \text{to} \ 2.00,$$

thus indicating the system will operate through resonance.

Hence, the angular response is $\theta(t) = \left(96.4\ 10^{-5}\ \text{rad}\right) \cdot B \cos\ (\omega t - \phi)$

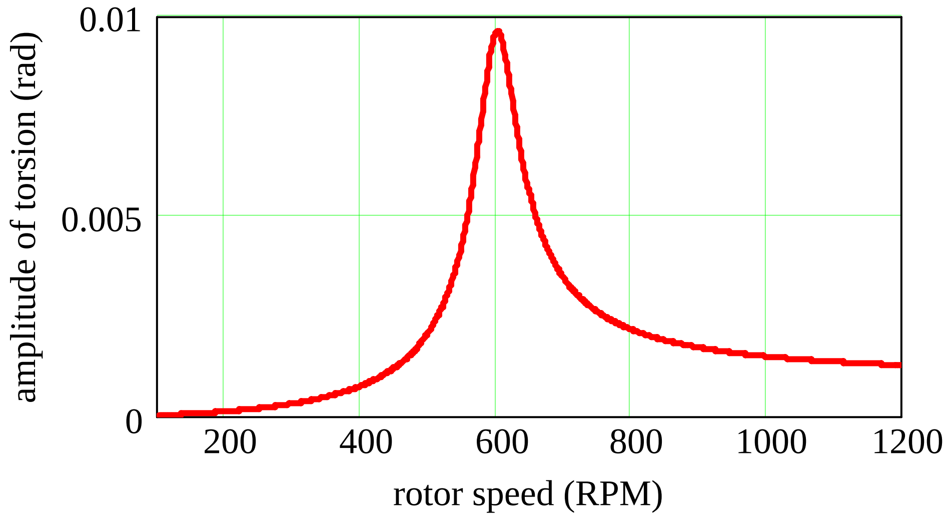
(d) Maximum amplitude of response:
since ζ << 1, the maximum amplitude of motion will occur when the turbine speed coincides with the natural frequency of the torsional system, i.e.

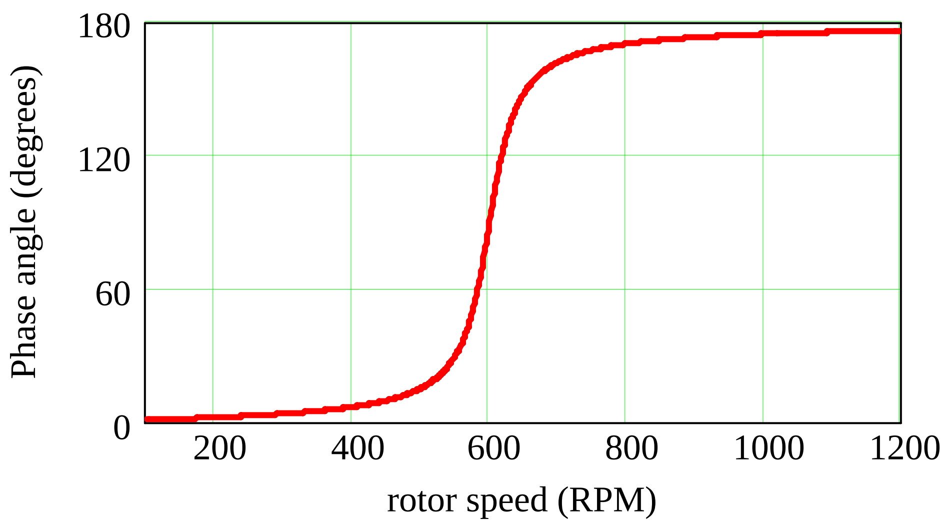
at f = 1, $B \approx {}^{1}\!/_{2\zeta}$ and

$$\theta(t) = \theta_\infty \cdot \frac{1}{2\zeta} \cos\left(\omega t - \frac{\pi}{2}\right)$$

the magnitude is $\boldsymbol{\theta_{MAX}}$ **=** $\theta_{\max} = \frac{\theta_\infty}{2\xi} = 0.964 \text{ x } 10^{-2}$ rad**, i.e. 10 times larger than** θ_∞.

The **amplitude** (degrees) and **phase angle** (degrees) of the polo twist are shown as a function of the turbine rotational speed (**RPM**)





(e) Design change: DOUBLE DAMPING but first BALANCE ROTOR!