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Calculation of the Dynamic Coefficients of a Journal Bearing, Using a Variational Approach

The dynamic bearing coefficients are obtained from a solution to the variational equivalent of Reynolds equation. A perturbation method is applied to find the individual dynamic coefficients. The Finite Element Method is used in the numerical evaluation of the equations. The flow is assumed to be laminar, the lubricant is Newtonian. Allowance is made for viscosity-temperature dependency in circumferential and axial directions.

Introduction

Calculation of the dynamic bearing coefficients has mainly been performed in two different ways in the past. The classical method is based on numerical differentiation of Reynolds equation, and is still widely used although it is rather inaccurate. The second method, which is based on a mathematical differentiation of Reynolds equation, before the numerical evaluation is performed, was developed recently [1]. This second method has several advantages compared to the classical. The dynamic coefficients are found directly by solving the equations and the method is thus as accurate as the numerical method allows for. In addition, there are considerable savings in computational time.

In the past decade the Finite Element Method has, to a still increasing extent, been used in the numerical evaluation of Reynolds equation. This interest in the Finite Element Method is mainly caused by the extensive use of this method in structural analysis and the ability of the method to cope with complex geometries.

This paper presents the use of the Finite Element Method in calculating the dynamic coefficients. A variational equivalent to each of the equations for the static as well as the dynamic behavior is employed. It is demonstrated, by comparisons with results obtained by the Finite Difference Method [1], that the method is working satisfactorily.

It is assumed that sleeve and shaft are aligned, i.e., that no dynamic coefficients caused by shaft tilting appears, the theory can, however, easily cope with this situation.

Governing Equations

Using the conventional assumptions of thin film lubrication and allowing for viscosity-temperature dependency is circumferential and axial direction, Reynolds equation in nondimensional variables is:

$$\frac{\partial}{\partial\theta}\left(\frac{\bar{h}^3}{\mu}\frac{\partial\bar{p}}{\partial\theta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{\bar{h}^3}{\bar{\mu}}\frac{\partial\bar{p}}{\partial\zeta}\right)=\frac{\partial\bar{h}}{\partial\theta}+2\frac{\nu}{\omega}\frac{\partial\bar{h}}{\partial\tau} \tag{1}$$

The film thickness is:

$$\bar{h}=1+\epsilon\cos\theta=1+\epsilon\cos(\theta'-\varphi)=1+\bar{x}\cos\theta'+\bar{y}\sin\theta' \tag{2}$$

In the equilibrium position the coordinate for the journal center is:

$$\begin{aligned}\bar{x}_0&=\epsilon_0\cos\varphi_0\\ \bar{y}_0&=\epsilon_0\sin\varphi_0\end{aligned} \tag{3}$$

Under dynamic conditions the motion of the journal axis is described by the lateral amplitudes $\Delta\bar{x}$ and $\Delta\bar{y}$, measured in the centerplane of the bearing from the static equilibrium position:

$$\begin{aligned}\bar{x}&=\bar{x}_0+\Delta\bar{x}\\ \bar{y}&=\bar{y}_0+\Delta\bar{y}\\ \dot{\bar{x}}&=\dot{\bar{x}}_0+\Delta\dot{\bar{x}}\\ \dot{\bar{y}}&=\dot{\bar{y}}_0+\Delta\dot{\bar{y}}\qquad,\ \Delta\bar{x},\ \Delta\bar{y},\ \Delta\dot{\bar{x}},\ \Delta\dot{\bar{y}}<<1\end{aligned} \tag{4}$$

The "dot" denotes differentiation with respect to time. It is assumed that the bearing is operating in its equilibrium position without whirling, giving:

$$\dot{\bar{x}}_0=\dot{\bar{y}}_0=0 \tag{5}$$

By assuming the amplitudes ($\Delta\bar{x}$, $\Delta\bar{y}$, $\Delta\dot{\bar{x}}$, $\Delta\dot{\bar{y}}$) to be small, a first order expansion of the pressure can be written as:

$$\bar{p}=\bar{p}_0+\Delta\bar{x}\bar{p}_x+\Delta\bar{y}\bar{p}_y+\Delta\dot{\bar{x}}\bar{p}_{\dot{x}}+\Delta\dot{\bar{y}}\bar{p}_{\dot{y}} \tag{6}$$

Substituting equations (4) and (6) into (1) and (2), retaining only first order terms and separating variables, a total of five equations is obtained:

$$\frac{\partial}{\partial\theta}\left(\frac{\bar{h}_0^3}{\bar{\mu}}\frac{\partial\bar{p}_i}{\partial\theta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{\bar{h}_0^3}{\bar{\mu}}\frac{\partial\bar{p}_i}{\partial\zeta}\right)=$$

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$$
\begin{cases}
\dfrac{\partial \bar{h}_0}{\partial \theta} & \bar{p}_i = \bar{p}_0 \\[2ex]
-\dfrac{\partial}{\partial \theta}\left(\dfrac{3\bar{h}_0^2\cos\theta'}{\bar{\mu}}\dfrac{\partial \bar{p}_0}{\partial \theta}\right) - \dfrac{\partial}{\partial \zeta}\left(\dfrac{3\bar{h}_0^2\cos\theta'}{\bar{\mu}}\dfrac{\partial \bar{p}_0}{\partial \zeta}\right) - \sin\theta' & \bar{p}_i = \bar{p}_x \\[2ex]
-\dfrac{\partial}{\partial \theta}\left(\dfrac{3\bar{h}_0^2\sin\theta'}{\bar{\mu}}\dfrac{\partial \bar{p}_0}{\partial \theta}\right) - \dfrac{\partial}{\partial \zeta}\left(\dfrac{3\bar{h}_0^2\sin\theta'}{\bar{\mu}}\dfrac{\partial \bar{p}_0}{\partial \zeta}\right) + \cos\theta' & \bar{p}_i = \bar{p}_y \\[2ex]
2\cos\theta' & \bar{p}_i = \bar{p}_{\dot{x}} \\[2ex]
2\sin\theta' & \bar{p}_i = \bar{p}_{\dot{y}}
\end{cases}
\tag{7}
$$

In [1] a detailed deduction of the equations is given. The boundary conditions to equation (7) for the dynamic coefficients (p_i, $i=x, y, \dot{x}, \dot{y}$) can be shown to be the same as for the static equilibrium position (p_i, $i=0$) [1]. The boundary conditions at the bearing edges are:

$$\zeta = \pm\frac{L}{D}: \ \bar{p} = \bar{p}_0 = \bar{p}_x = \bar{p}_y = \bar{p}_{\dot{x}} = \bar{p}_{\dot{y}} = 0 \tag{8}$$

Symmetry around the bearing centerplane implies that:

$$\zeta = 0: \frac{\partial \bar{p}}{\partial \zeta} = \frac{\partial \bar{p}_0}{\partial \zeta} = \frac{\partial \bar{p}_x}{\partial \zeta} = \frac{\partial \bar{p}_{\dot{x}}}{\partial \zeta} = \frac{\partial \bar{p}_y}{\partial \zeta} = \frac{\partial \bar{p}_{\dot{x}}}{\partial \zeta} = \frac{\partial \bar{p}_{\dot{y}}}{\partial \zeta} = 0 \tag{9}$$

In the lubricant supply grooves the boundary conditions are:

$$\theta = \theta_1: \ \bar{p} = \bar{p}_0 = \bar{p}_x = \bar{p}_y = \bar{p}_{\dot{x}} = \bar{p}_{\dot{y}} = 0 \tag{10}$$

At the film rupture zone the boundary conditions are given as:

$$\theta = \theta_c: \ \bar{p}_0 = \bar{p}_c, \ \frac{\partial \bar{p}_0}{\partial \theta} = \frac{\partial \bar{p}_0}{\partial \zeta} = 0$$

and

$$\bar{p}_x = \bar{p}_y = \bar{p}_{\dot{x}} = \bar{p}_{\dot{y}} = 0 \tag{11}$$

Variational Principles

To use the Finite Element Method, a variational principle corresponding to equation (7) has to be found:

$$I(\bar{p}_i) = \int_\Omega F_i(\theta, \zeta, \bar{p}_i, \bar{p}'_{i\theta}, \bar{p}'_{i\zeta})\, d\Omega, \quad i = 0, x, y, \dot{x}, \dot{y} \tag{12}$$

If F_i in equation (12) is the functional equivalent to equation (7), the well known Euler-Lagrange equation must be satisfied:

$$F'_{\bar{p}_i} - \frac{\partial}{\partial \theta}F'_{\bar{p}'_{i\theta}} - \frac{\partial}{\partial \zeta}F'_{\bar{p}'_{i\zeta}} = 0, \ i = 0, x, y, \dot{x}, \dot{y} \tag{13}$$

The functionals are:

$$F_i = -\left(-\frac{\bar{h}_0^3}{2\bar{\mu}}\left(\left(\frac{\partial \bar{p}_i}{\partial \theta}\right)^2 + \left(\frac{\partial \bar{p}_i}{\partial \zeta}\right)^2\right) + \right.$$

Nomenclature

A = finite element area

$\bar{B}_{xx}, \bar{B}_{xy}, \bar{B}_{yx}, \bar{B}_{yy}$ = $\dfrac{C\omega B_{xx}}{W}, \dfrac{C\omega B_{xy}}{W}, \dfrac{C\omega B_{yx}}{W}, \dfrac{C\omega B_{yy}}{W}$ = damping coefficients, dimensionless

C = radial clearance, m

D = journal diameter, m

e = journal eccentricity, m

f_{x0}, f_{y0} = bearing reaction force components, dimensionless

F = functional

h = film thickness, m

$\bar{h}$ = h/C = film thickness, dimensionless

$\bar{K}_{xx}, \bar{K}_{xy}, \bar{K}_{yx}, \bar{K}_{yy}$ = $\dfrac{CK_{xx}}{W}, \dfrac{CK_{xy}}{W}, \dfrac{CK_{yx}}{W}, \dfrac{CK_{yy}}{W}$ = stiffness coefficients, dimensionless

L = bearing length, m

N = rotational speed, rps

N = interpolation polynomial

p = film pressure N/m^2

$\bar{p}$ = $p/6\mu_0\omega\left(\dfrac{R}{C}\right)^2$ = film pressure, dimensionless

R = journal radius, m

S = $\dfrac{\mu_0 NLD}{W}\left(\dfrac{R}{C}\right)^2$ = Sommerfeld number, dimensionless

t = time, s

W = static load, N

$\bar{x}, \bar{y}$ = $x/C, y/C$ = journal center coordinates, dimensionless

ϵ = e/C = eccentricity ratio, dimensionless

θ = angular coordinate for bearing film, rad

θ' = angular coordinate for bearing film, rad

μ_0 = reference viscosity of lubricant, Pas

$\bar{\mu}$ = μ/μ_0 = lubricant viscosity, dimensionless

ν = instability whirl frequency, s^{-1}

φ = attitude angle, rad

ω = $2\pi N$ = angular speed of rotation, rad/s

ζ = z/R = axial co-ordinate for bearing film, dimensionless

τ = $t\nu$ = time, dimensionless

c = cavitation

e = finite element

0 = static equilibrium position

Bracketed terms

$[\mathbf{K}_p]$ = "stiffness" matrix

$\{\bar{\mathbf{p}}_i\}$ = unknown nodal pressures

$\{\mathbf{K}_{ui}\}$ = "load" vector

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$$
\left[\begin{array}{l}
\bar{h}_0 \dfrac{\partial \bar{p}_0}{\partial \theta} \\
-\left(\dfrac{3\bar{h}_0^2\cos\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \theta} - \cos\theta'\right) \dfrac{\partial \bar{p}_x}{\partial \theta} - \left(\dfrac{3\bar{h}_0^2\cos\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \zeta}\right) \dfrac{\partial \bar{p}_x}{\partial \zeta} \\
-\left(\dfrac{3\bar{h}_0^2\sin\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \theta} - \sin\theta'\right) \dfrac{\partial \bar{p}_y}{\partial \theta} - \left(\dfrac{3\bar{h}_0^2\sin\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \zeta}\right) \dfrac{\partial \bar{p}_y}{\partial \zeta} \\
2\sin\theta' \\
-2\cos\theta'
\end{array}\right]
\begin{array}{l}
\bar{p}_i = \bar{p}_0 \\
\bar{p}_i = \bar{p}_x \\
\bar{p}_i = \bar{p}_y \\
\bar{p}_i = \bar{p}_{\dot{x}} \\
\bar{p}_i = \bar{p}_{\dot{y}}
\end{array}
\quad (14)
$$

Finite Element Formulation of the Problem

Equation (14) has to be solved with appropriate boundary conditions, equation (8)–(11). This is done by use of the Finite Element Method [2, 3, 4]. Stationary values are found if:

$$\frac{\partial I(\bar{p}_i)}{\partial \bar{p}_i} = 0, \quad i = 0, x, y, \dot{x}, \dot{y} \tag{15}$$

In the interior of an element the pressure distribution $\bar{p}_i(\theta, \zeta)$ is expressed by known polynomials and unknown nodal parametric values:

$$\bar{p}_{ei} = \sum_{j=1}^{n} N_{ij}(\theta, \zeta) \bar{p}_{ij} \tag{16}$$

where n is the number of element nodes and index j indicates node numbers. The equivalent to (15) for stationary nodal values are:

$$\frac{\partial I(\bar{p}_i)}{\partial \bar{p}_{ij}} = 0, \quad i = 0, x, y, \dot{x}, \dot{y} \text{ and } j = 1, \ldots, n \tag{17}$$

Introducing equation (16) in (14) and equation (14) in (12) and performing the differentiation gives:

$$
\frac{\partial I(\bar{p}_i)_j}{\partial \bar{p}_{ij}} =
$$

$$
-\int_A \left(-\frac{\bar{h}_0^3}{\bar{\mu}} \left(\frac{\partial N_{ij}}{\partial \theta} \sum_{k=1}^{n} \frac{\partial N_{ik}}{\partial \theta} \bar{p}_{ik} + \frac{\partial N_{ij}}{\partial \zeta} \sum_{k=1}^{n} \frac{\partial N_{ik}}{\partial \zeta} \bar{p}_{ik}\right) + \right.
$$

$$
\left[\begin{array}{l}
\bar{h}_0 \dfrac{\partial N_j}{\partial \theta} \\
-\left(\dfrac{3\bar{h}_0^2\cos\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \theta} - \cos\theta'\right) \dfrac{\partial N_j}{\partial \theta} - \dfrac{3\bar{h}_0^2\cos\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \zeta} \dfrac{\partial N_j}{\partial \zeta} \\
-\left(\dfrac{3\bar{h}_0^2\sin\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \theta} - \sin\theta'\right) \dfrac{\partial N_j}{\partial \theta} - \dfrac{3\bar{h}_0^2\sin\theta'}{\bar{\mu}} \dfrac{\partial \bar{p}_0}{\partial \zeta} \dfrac{\partial N_j}{\partial \zeta} \\
2\sin\theta' \\
-2\cos\theta'
\end{array}\right] dA = 0
\begin{array}{l}
\bar{p}_i = \bar{p}_0 \\
\bar{p}_i = \bar{p}_x \\
\bar{p}_i = \bar{p}_y \\
\bar{p}_i = \bar{p}_{\dot{x}} \\
\bar{p}_i = \bar{p}_{\dot{y}}
\end{array}
$$

$$, i = 0, x, y, \dot{x}, \dot{y} \tag{18}$$

The integration area A in equation (18) represents the area of one finite element. Combination of the contributions from each element in the solution domain, [3], gives rise to a system of symmetric linear equations. As can be seen from equation (18), is the left hand side of the equation system, represented by the first bracketed term in the integral, equal for the five right hand sides, represented by the terms in the "{ }" bracket:

$$[\mathbf{K}_p]\{\bar{\mathbf{p}}_i\} = \{\mathbf{K}_{ui}\}, \quad i = 0, x, y, \dot{x}, \dot{y} \tag{19}$$

Since the pressure distribution $\bar{p}_0$ enters the expressions for the dynamic coefficients, the solution procedure is as follows. First, the static pressure solution is found and the active film domain established. Next, the right hand sides for the remaining four equations are generated, and equation (19) is solved for these with the same boundary conditions as for the static solution. In the static equilibrium position, the forces acting on the shaft can be expressed as:

$$\left.\begin{array}{l} f_{x0} \\ f_{y0} \end{array}\right\} = \left\{\begin{array}{l} \bar{W} \\ 0 \end{array}\right\} = \frac{6\pi}{L/D} \int_0^{L/D} \int_0^{2\pi} \bar{p}_0 \left\{\begin{array}{l} -\cos\theta' \\ -\sin\theta \end{array}\right\} d\theta d\zeta \tag{20}$$

Where f_{x0} equals the static load, $\bar{W}$, while f_{y0} must be zero. The dynamic coefficients $\bar{K}_{xx}$ and $\bar{K}_{yx}$ can be expressed as:

$$\left.\begin{array}{l} \bar{K}_{xx} \\ \bar{K}_{yx} \end{array}\right\} = \frac{6\pi}{L/D} \int_0^{L/D} \int_0^{2\pi} \bar{p}_x \left\{\begin{array}{l} -\cos\theta' \\ -\sin\theta' \end{array}\right\} d\theta d\zeta \tag{21}$$

A similar integration of $\bar{p}_y$ gives $\bar{K}_{xy}$ and $\bar{K}_{yy}$, the integration of $\bar{p}_{\dot{x}}$ gives $\bar{B}_{xx}$ and $\bar{B}_{yx}$, and the integration of $\bar{p}_{\dot{x}}$ gives $\bar{B}_{xx}$ and $\bar{B}_{yx}$, and the integration of $\bar{p}_{\dot{y}}$ gives $\bar{B}_{xy}$ and $\bar{B}_{yy}$.

Results

The results obtained are shown on Figs. 2–3. Comparisons with results obtained by the Finite Difference Method, [1], imply that the Finite Element Method is well suited for this purpose. The results presented are based on calculations with six noded triangular elements.

The implementation of the Finite Element Method on the dynamic bearing coefficient problem was performed without major problems, except for one thing. The pressure gradients of the static solution appear in the right hand side of the equations for the bearing stiffness, and solution of these equations are extremely sensitive to the accuracy of the

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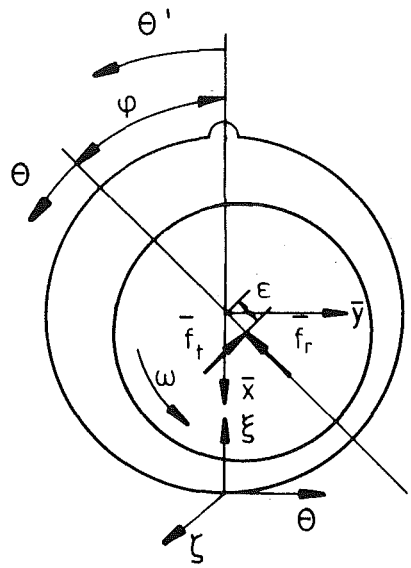

Fig. 1 Geometry and coordinate system of a radial bearing

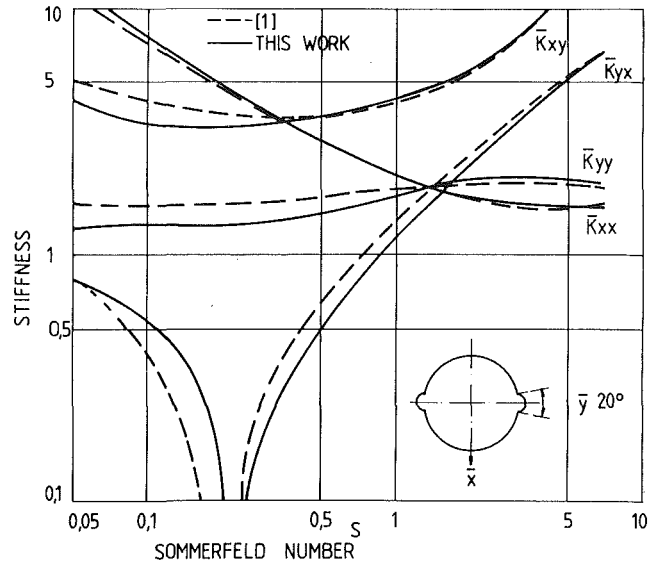

Fig. 2 Stiffness coefficients versus Sommerfeld number for two-axial-groove bearing with $L/D = 0.5$, [1]

gradients. It is thus very important to ensure that the gradient is properly evaluated due to the great variation in the pressure in a hydrodynamic bearing.

The deviation in results obtained here and in [1], is mainly caused by different ways of obtaining the film rupture zone for the bearing. Whereas a rather refined and sophisticated method, based on successive alterations of the mesh during the calculating procedure is used in [1], the results obtained in this paper are achieved by using the original finite element mesh in the lubricant during the calculation.

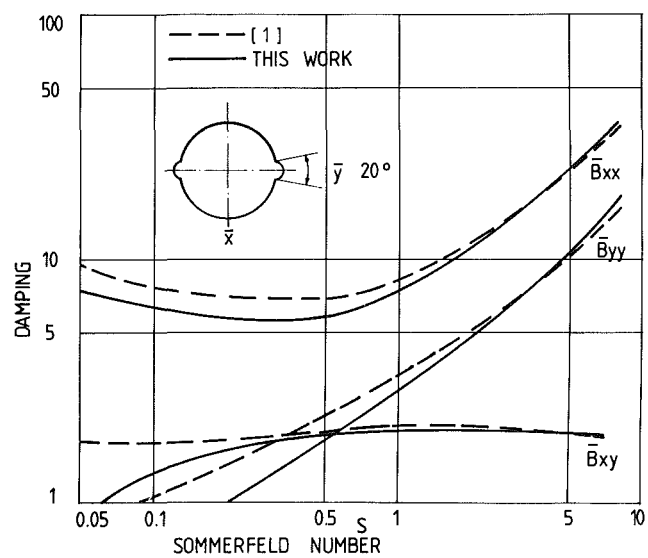

Fig. 3 Damping coefficients versus Sommerfeld number for two-axial-groove bearing with $L/D = 0.5$, [1]

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