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Some Extensions of the Lubrication Theory of Osborne Reynolds

Introduction

The mathematical theory of lubrication began with Osborne Reynolds one century ago, when he set forth to provide scientific explanation for the remarkable results of Beauchamp Tower's experiments (Reynolds, 1886). During the course of his investigation Reynolds linearized the equations of motion for a constant property, Newtonian fluid in laminar flow between rigid surfaces. A crucial step in Reynolds' analysis, when expressed in modern terms, was the realization that owing to the particular geometry of the lubricant film, the problem has two different length scales. Let the length scale across the film be C and let R represent the length scale in the "plane of the film." For conventional lubricant films $(C/R)=O(10^{-3})$; recognition of this fact was essential for the development of lubrication theory.

We find it expedient to normalize, before further discussion, the equations of motion with respect to characteristic length, velocity and time. Let U_* represent the velocity scale in the plane of the film, then the equation of continuity div $\mathbf{v}=0$, where $\mathbf{v}=(u, v, w)$ is the velocity of the lubricant, dictates that the characteristic velocity across the film be $O(U_* C/R)$. Denoting normalized (nondimensional) variables by an overscore bar, we have

$$\{\bar{x}, \bar{y}, \bar{z}\}=\frac{1}{R}\left\{x, \left(\frac{R}{C}\right)y, z\right\}; \{\bar{u}, \bar{v}, \bar{w}\}$$

$$=\frac{1}{U_*}\left\{u, \left(\frac{R}{C}\right)v, w\right\} \tag{1}$$

$$\bar{p}=\mathrm{Re}\left(\frac{C}{R}\right)\frac{p}{\rho U_*^2}\ ;\ \bar{t}=\Omega t$$

and the equations of motion take the form:

$$\frac{C^2\Omega}{\nu}\frac{\partial \bar{u}}{\partial \bar{t}}+\mathrm{Re}\left(\frac{C}{R}\right)\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right)$$

$$=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+\left(\frac{C}{R}\right)^2\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right) \tag{2a}$$

$$\left(\frac{C}{R}\right)^2\left[\frac{C^2\Omega}{\nu}\frac{\partial \bar{v}}{\partial \bar{t}}+\mathrm{Re}\left(\frac{C}{R}\right)\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{v}}{\partial \bar{z}}\right)\right.$$

$$\left.-\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}-\left(\frac{C}{R}\right)^2\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right)\right]=-\frac{\partial \bar{p}}{\partial \bar{y}} \tag{2b}$$

$$\frac{C^2\Omega}{\nu}\frac{\partial \bar{w}}{\partial \bar{t}}+\mathrm{Re}\left(\frac{C}{R}\right)\left(\bar{u}\frac{\partial \bar{w}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{w}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right.$$

$$=-\frac{\partial p}{\partial \bar{z}}+\frac{\partial^2 \bar{w}}{\partial \bar{y}^2}+\left(\frac{C}{R}\right)^2\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right) \tag{2c}$$

Here p is the pressure of the lubricant, $\mathrm{Re}=CU_*/\nu$ is the Reynolds number, and Ω is some characteristic frequency of the flow, say the orbital frequency of a journal. The orientation of the Cartesian coordinate axes (x, y, z) is shown in Fig. 1.

The normalized form of the incompressibility constraint is simply

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{z}}=0 \tag{3}$$

The theory of lubrication proposed by Reynolds is based on the assumption that $(C/R)\rightarrow 0$. Two additional assumptions have also to be made, viz., that $\Omega=O(U_*/R)$ and that Re is bounded; it is only in this sense that the flow of a lubricant is a "slow flow."

In consequence of the above assumptions, the equations of motion reduce to

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2 u}{\partial y^2}\ ;\ \frac{\partial p}{\partial y}=0;\ \frac{\partial p}{\partial z}=\mu\frac{\partial^2 w}{\partial y^2} \tag{4}$$

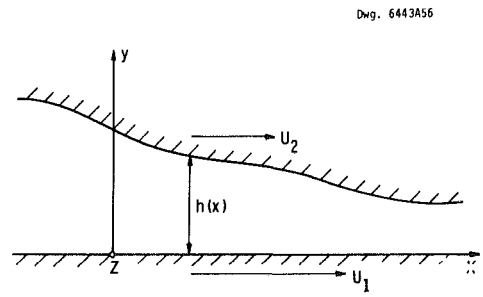

Fig. 1 Bearing surfaces and coordinate axes. (From A. Z. Szeri, ed., *Tribology*, Hemisphere Publishing Corp. With permission)

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when written in terms of the primitive variables. The equation of mass conservation (3) remains unchanged, except for removal of the overscore bar.

It is worthy of note that equation (4) and the order of magnitude analysis that lead to this equation were published 18 years earlier than Prandtl's boundary layer theory, the basis of which is also a recognition of the existence of multiple length scales.

Classical lubrication theory takes its departure from equation (4). The second of these equations assures us that to first approximation the pressure remains constant across the film, thus the first and the third of equation (4) can be formally integrated across the film. The velocity components that are obtained in this fashion were substituted by Reynolds into the averaged (across the film) equation of continuity. In this manner, Reynolds succeeded in deriving an equation in lubricant pressure, now bearing his name:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial z}\right)=6U_0\frac{\partial h}{\partial x} \tag{5}$$

Here h is the film thickness and U_0 is the effective sliding velocity. Let U_1 and U_2 represent the velocity, parallel to the "plane" of the film, of the two bearing surfaces; then $U_0=U_1+U_2$ for a journal bearing and $U_0=U_1-U_2$ for a thrust bearing.

For a journal bearing the pressure equation of Reynolds assumes the form:

$$\frac{\partial}{\partial\theta}\left(H^3\frac{\partial P}{\partial\theta}\right)+\left(\frac{D}{L}\right)^2\frac{\partial}{\partial Z}\left(H^3\frac{\partial P}{\partial Z}\right)=12\pi\frac{dH}{d\theta} \tag{6}$$

Here $\theta=x/R$, $Z=2z/L$, $H=h/C$ and $P=p/\mu N(R/C)^2$

The work of Reynolds received immediate acceptance by the engineering community, but there were great difficulties encountered when attempting to solve the pressure equation (5) in its two dimensional form. There are two simplifications to equation (5), however, which are not only tractable analytically, but which posses definite physical meaning. These two approximations are associated with the limits $(L/D)\rightarrow 0$ and $(D/L)\rightarrow 0$, respectively; the former leading to the "short-bearing" theory (Dubois and Ockvirk, 1953) and the latter to the "long-bearing" theory (Sommerfeld, 1904). Predictions of the short-bearing theory give acceptable accuracy for $L\leq D/4$ while long-bearing solutions become progressively more realistic for increasing aspect ratio, once $L>2D$. Perhaps the first two-dimensional solution of (5) was by Christopherson (1941), using Southwell's relaxation method. Christopherson's work was soon followed by others (e.g., Cameron and Wood, 1949), but it was not until after the advent of high speed computing, that two-dimensional solutions of the Reynolds equation (5) became available (Pinkus, 1958; Raimondi and Boyd, 1958; Pinkus and Sternlicht, 1961).

As numerical solutions of the Reynolds equation became a commonplace, more and more investigators began to ask the question: does the Reynolds pressure equation account for the physics of the problem adequately? In fact classical theory was found to be wanting much earlier, as early as 1916, when Martin found that it cannot satisfactorily explain the existence of continuous hydrodynamic films in highly stressed counterformal contacts, such as found in gears.

There were at least six principal areas in which extension of classical theory seemed desirable. Each of these areas is concerned with the relaxation of one or more constraints of the Reynolds theory. Deleting one or more of the basic assumptions of the classical theory leads to studies in:

(1) *Elastohydrodynamic* (EHD) effects – when the viscosity is dependent on pressure and the surfaces are no longer constrained to be rigid.

(2) *Turbulence* effects – when the lubricant flow Reynolds number is outside its range for laminar flow.

(3) *Inertia* effects – when $(C/R)\,\mathrm{Re}>1$ or when $\Omega>U_*/R$ and bearing performance is influenced by the mass effects of the lubricant.

(4) *Thermohydrodynamic* (THD) effects – when the lubricant viscosity varies strongly with temperature, $\mu=\mu(T)$.

(5) *Non-Newtonian* effects – when lubricant behavior deviates significantly from linearly viscous behavior.

(6) *Compressibility* effects – when the lubricant is a gas.

With heavy reliance on high speed computing and on the large body of literature available, it is now possible to provide at least partial answers to questions raised by researchers, concerning validity of classical theory in modern lubrication practice. I will attempt to sketch some of the developments/extensions of lubrication theory. The paper is organized in six sections, corresponding to the six areas of studies outlined above. The reader is asked to bear in mind that space is at a premium when attempting to cram 100 years of development into a single journal article; in consequence, many significant contributions and contributors are left unlisted, but this is done without malice.

Elastohydrodynamic Theory

It may be shown that concentrated machine contacts, such as found in gears, can be conveniently represented by cylinders and that locally such representation may be reduced to a cylinder-plane geometry (Dowson and Higginson, 1977.) Thus we take the cylinder rolling on a plane as the prototype of EHD contact. If the cylinder is "long," we are permitted to apply long-bearing theory. Accordingly, we take the limit $(D/L)\rightarrow 0$ in equation (6). The result, when written in terms of primitive variables is

$$\frac{d}{dx}\left(\frac{h^3}{\mu}\frac{dp}{dx}\right)=6U_0\frac{dh}{dx} \tag{7}$$

The pressure gradient in the lubricant is given by the first integral of equation (7), and when nondimensionalized, it can be written as

$$\frac{d\bar{p}}{d\theta}=\frac{6h_0n^2}{R}\left[\frac{\bar{h}(\theta)-\bar{h}(\theta_2)}{h^3(\theta)}\cos\theta\right] \tag{8}$$

Here we make reference to Fig. 2. The film thickness is

$$h=-\frac{R}{n}(1+n\cos\theta) \tag{9}$$

and the remaining variables have the definition

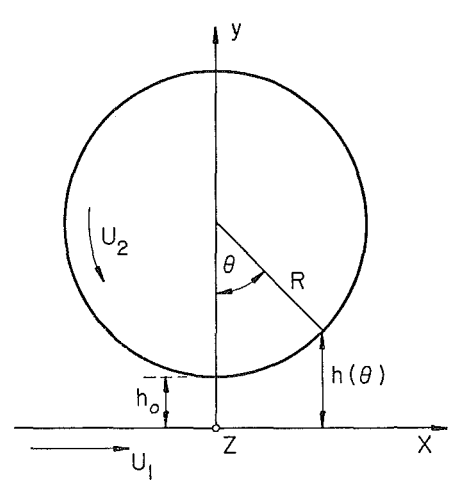

Fig. 2 Cylinder-plane geometry and nomenclature. (From A. Z. Szeri, ed., *Tribology*, Hemisphere Publishing Corp. With permission)

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$$\theta=\sin^{-1}\left(\frac{x}{R}\right), n=\frac{-R}{h_0+R}\ ;\ \bar{p}=\frac{ph_0}{\mu U_0}\ ,\ \bar{h}=\frac{nh}{R}\ ,\ U_0=U_1+U_2$$

Implicit in (8) is the condition that $d\bar{p}/d\theta=0$ at $\theta=\theta_2$, where θ_2 is the position of the cavitation boundary. The pressure distribution is obtained from (8) by integration:

$$\bar{p}=\frac{6h_0n^2}{R}\int_{\theta_1}^{\theta}\cos\theta\ \frac{\bar{h}(\theta)+\bar{h}(\theta_2)}{\bar{h}^3(\theta)}\ d\theta \tag{10}$$

Here $p(\theta_1)=0$ and θ_2 is evaluated from the constraint $p(\theta_2)=0$, in accordance with the Swift-Stieber boundary condition. An excellent discussion on boundary conditions and film cavitation can be found in the review article by Dowson and Taylor (1979).

Let P_y represent the lubricant force per unit length of the cylinder, then

$$P_y=R\int_{\theta_1}^{\theta_2}p\cos\theta\ d\theta=3n^2\int_{\theta_1}^{\theta_2}\frac{\bar{h}(\theta)-\bar{h}(\theta_2)}{\bar{h}^3(\theta)}\sin2\theta d\theta \tag{11}$$

Equation (11) is easily evaluated in closed form with the aid of the Sommerfeld substitution (c.f., Cameron, 1966) to yield

$$\frac{h_0}{R}=2.45\ \frac{\mu U_0}{P_y},\quad n\rightarrow-1 \tag{12}$$

Following Dowson and Higginson (1977) we now estimate the film thickness for a gear tooth contact. Typical values for lubricant viscosity, surface speed and load are:

$\mu=0.075$ Pa.s, $U_1=U_2=0.5$ m/s, $U_0=$ m/s, $P_y=2.65$ MN/m

Substituting these values into (12) and assuming that in the neighborhood of contact the gear geometry is satisfactorily represented by an "equivalent cylinder" of $R=0.025$ m, we obtain

$$h_0=0.0172\ \mu\text{m}$$

This value for film thickness is small in comparison with even the best surface finishes encountered in gears (Cheng, 1984). It is almost two orders of magnitude smaller than the mean film thickness between two steel disks of $R=0.081$ m under a load of $P_y=0.196$ MN/m (Crook, 1958).

The cylinder-plane geometry was first analyzed by Martin in 1916. His negative results on the existence of continuous lubricant film in conformal contacts discouraged theoretical research on lubricated contacts for two decades. Then Peppler (1938) made an important contribution by considering elastic deformation of the surfaces. Though he came to the incorrect conclusion that the maximum oil film pressure can never exceed the corresponding maximum Hertzian contact pressure, he pointed the way to further development. The assumption of a uniform, constant viscosity lubricant was removed by Gatcombe (1945). A useful representation of pressure dependence of viscosity for mineral oils is given by Weibull's equation

$$\mu=\mu_0\exp(\alpha p)$$

where μ_0 and α are material constants for the lubricant (cf. Sargent, 1980). Introducing this expression for μ into the long-bearing Reynolds equation yields

$$\frac{d\bar{q}}{d\theta}=\frac{6h_0n^2}{R}\left(\frac{\bar{h}(\theta)-\bar{h}(\theta_2)}{\bar{h}^3(\theta)}\cos\theta\right);\bar{q}\equiv\frac{h_0}{\mu_0U_0\alpha}\ (1-e^{-\alpha p}) \tag{13}$$

for the gradient of the (dimensionless) reduced pressure, $\bar{q}$. Comparison of equation (8) and (13) tells that any pressure distribution found for the $\mu=$const case can readily be used for pressure dependent viscosity, through the formula

$$p=-\frac{1}{\alpha}\ln(1-\alpha q), q\equiv\frac{\mu_0U_0\alpha}{h_0}\ \bar{q} \tag{14}$$

It was first noted by Blok (1950) that $p\rightarrow\infty$ as $q\rightarrow1/\alpha$. Solid surfaces, of course, cannot sustain unlimited pressure. They will deform and our simple analysis breaks down; nevertheless, it points to the necessity of having to consider both the pressure dependence of the lubricant and the deformation of the surfaces when discussing continuous film lubrication of highly loaded contacts. Figure 3 dramatizes the development of EHD theory. The individual contributions of elastic deformation of the surfaces and of pressure dependence of lubricant viscosity, respectively, are small. But they combine nonlinearly to yield a much increased load capacity and a new film shape.

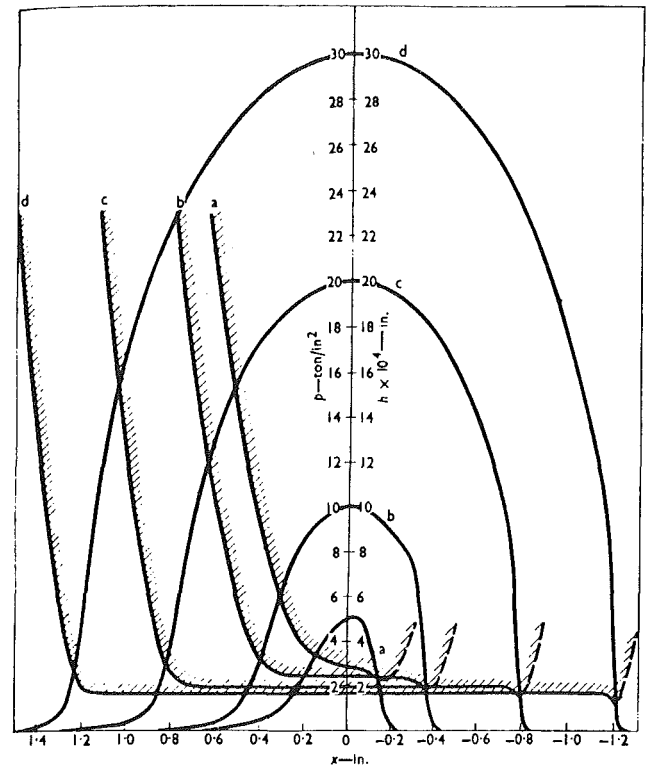

Fig. 3 Pressure distributions and film shapes for the same center-line film thickness
(*a*) Constant viscosity, rigid cylinders;
(*b*) Pressure-dependent viscosity, rigid cylinders;
(*c*) Constant viscosity, elastic cylinders;
(*d*) Pressure-dependent viscosity, elastic cylinders;
(From D. Dowson and G. R. Higginson, *J. Mech. Eng. Sci.*, Vol. 1, No. 1, 1959. With permission)

The first satisfactory solution to account for the effects of both elastic deformation and pressure dependence of viscosity was reported by Grubin (1949). His work, and that of Petrushevich (1951), establishes most of the essential properties of EHD solutions in concentrated contacts.

(1) The film is of nearly uniform thickness in the contact zone, except for a sudden decrease just upstream of the cavitation boundary.
(2) There is sudden peaking of the pressure in the region of the film thickness change. Other than that, the pressure curve follows the shape of the Hertzian ellipse within the contact zone.

For a complete solution of the isothermal EHD problem we need to satisfy simultaneously

(*a*) The Reynolds equation for the lubricant
(*b*) The viscosity-pressure relationship
(*c*) The equation of elasticity for the solid members.

It was discovered very early that the conventional iterative scheme represented by

$$\text{film shape}\ \nearrow\ \text{pressure}\ \nwarrow\ \text{deformation}$$

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does not always converge, particularly when there is large elastic deformation of the surfaces. To remedy this, Dowson and Higginson (1959) proposed the so called "inverse hydrodynamic method."

The inverse hydrodynamic method is essentially a one-dimensional method, but here it works well. As a starting point we perform the indicated differentiation in (7) and write

$$h^3 \frac{d}{dx}\left(\frac{1}{\mu}\frac{dp}{dx}\right) - \frac{dh}{dx}\left(6U_0 - \frac{3h^2}{\mu}\frac{dp}{dx}\right) = 0 \quad (15)$$

Then postulate an arbitrary pressure curve $p = p(x)$. This should have a long inlet sweep, but otherwise should follow closely the pressure for Hertzian contact in a zone of width $2a$

$$p = p_0\sqrt{1 - \frac{x^2}{a^2}}, \quad (16)$$

at least for thin film, i.e., for near Hertzian, conditions. The next step is to locate the point $x = x_a$ on the inlet side of the pressure curve $p(x)$; this is defined by the condition

$$h^3 \frac{d}{dx}\left(\frac{1}{\mu}\frac{dp}{dx}\right) = 0 \quad (17)$$

For an isoviscous lubricant x_a is simply the x-location of the point of inflection on $p(x)$, as $(dp/dx)_a \neq 0$. At this location the film thickness is given by equation (15) as

$$h_a = \left[\frac{2\mu_a U_a}{(dp/dx)_a}\right]^{1/2} \quad (18)$$

The other condition, $dh/dx = 0$, under which (17) is satisfied, is of no interest presently as it cannot occur in the inlet region.

Thus for the arbitrary pressure curve $p(x)$ and given viscosity and speed, we can calculate the film thickness h_a at the particular location x_a. The film thickness at the maximum point of $p(x)$ is available immediately from integration of (7), $h_m = 2h_a/3$, where h_m is the film thickness at $dp(x)/dx = 0$.

Once the film thickness h_a has been determined, the entire film shape corresponding to the given pressure distribution $p(x)$ can be found from the cubic equation

$$K\bar{h}^3 - \bar{h} + 1 = 0;\ K \equiv \frac{h_m^2}{6\mu U_0}\frac{dp}{dx},\ \bar{h} \equiv \frac{h}{h_m} \quad (19)$$

which represents the once integrated Reynolds equation (7).

It is possible to obtain a general picture of film shape without actually solving the equations. Under high loads the pressure curve is expected to be close to the Hertzian curve in the neighborhood of the peak pressure (Hertzian condition). In this region equation (15) and equation (16) yield, on the assumption that $\mu \simeq \mu_{\max}$,

$$\frac{dh}{dx} \simeq \frac{h^3 P_0}{6\mu U_0 a^2} \quad (20)$$

Under near Hertzian conditions in the contact zone h^3 is very small and μ is large, thus we except $h \sim$ constant here. At the trailing edge of the film the pressure will fall rapidly, again following (16); in consequence there must be a sudden decrease in film thickness here so that flow continuity is satisfied.

The surface displacement which is caused by the arbitrary pressure $p(x)$ can, on the other hand, be calculated by the assumption of plane stress. Let $s_2 - s_1 > 0$ be the length of the line load, then the deflection normal to the surface and in the direction of $p(x)$ is given by the formula

$$v = -\frac{1}{\pi}\int_{s_1}^{s_2} p(s)\left\{\alpha\left[\ln((x-s)^2 + y^2) - \frac{y^2}{(x-s)^2 + y^2}\right] - \beta\frac{(x-s)^2}{(x-s)^2 + y^2}\right\}ds \quad (21)$$

where $\alpha = (1 - \sigma^2)/E$ and $\beta = -\sigma(1 + \sigma)/E$.

The inverse hydrodynamic method can now be described as consisting of the following procedures: (i) selection of the design load, (ii) selection of an arbitrary $p^{(1)}(x)$ that is to yield the specified design load, (iii) calculation of $h^{(1)}(x)$ corresponding to $p^{(1)}(x)$, from (19), (iv) calculation of $v^{(1)}(x)$ corresponding to $p^{(1)}(x)$, from (21), (v) comparison of $h^{(1)}(x)$ and $v^{(1)}(x)$. If $|h^{(1)}(x) - h - v^{(1)}(x)|$ is large, modify pressure curve to $p^{(2)}(x), \ldots$, etc.

Assumptions implicit in the inverse hydrodynamic solution are: (*a*) side leakage is neglected, (*b*) the lubricant is incompressible, (*c*) lubricant viscosity depends on pressure only, i.e., thermal effects are neglected and (*d*) the lubricant is Newtonian.

To facilitate representation of their calculated results, Dowson and Higginson group the independent variables of the isothermal EHD problem into four convenient dimensionless parameters that define the problem:

$$\bar{H} = \frac{h_0}{R},\ \bar{W} = \frac{W}{\tilde{E}R},\ \bar{U} = \frac{\mu_0 U_0}{2\tilde{E}R},\ \bar{G} = \alpha\tilde{E}$$

where R is the effective radius of the geometry and $\tilde{E} = E/(1 - \sigma^2)$ is the effective modulus of elasticity. The numerical solutions can then be represented to good accuracy by the formula

$$\bar{H} = 1.6\frac{\bar{G}^{0.6}\bar{U}^{0.7}}{\bar{W}^{0.13}} \quad (22)$$

The practical range of variation of the material parameter, $\bar{G}$, is small, thus the speed parameter, $\bar{U}$, emerges as the dominant variable of the EHD problem. As the speed decreases or as the load increases, the pressure spike moves upstream, and eventually disappears for heavily loaded contacts.

Assumption (*a*) is quite accurate for line contact. Lubricant compressibility, on the other hand, can be quite important, especially at high speeds (Dowson, Higginson and Whittaker, 1963). It will particularly affect the pressure peak, both in magnitude and in location.

Besides the line contact of the cylinder-plane geometry, point contacts can also be experienced at zero load. After the load is applied, the line expands into a rectangle and the point into an ellipse. Cheng (1970) wrote the Reynolds pressure equation in terms of a reduced pressure and applied a Grubin type inlet analysis to contacts whose contacting area is an arbitrary ellipse. The surface deformation was calculated on the basis of classical Hertzian theory. Cheng's solution loses accuracy at high rolling speeds. Full computer solutions for elliptical contacts were made for flooded as well as for starved contacts in a series of papers by Hamrock and Dowson (1981). This work employs a more accurate viscosity-pressure relationship, due to Roelands, than previous works. In order to avoid high gradients in the field of computation, the Reynolds pressure equation is written for the parameter $\phi \equiv PH^{3/2}$. In the elasticity analysis the load area is divided into rectangular areas. The pressure is then assumed to be piecewise constant, so as to avoid the problems associated with the singularity in the formula for deformation. Film thickness formulas for both flooded and starved contacts are given by Hamrock and Dowson (1981). The procedure calls for the simultaneous solution of the elasticity and the Reynolds equation. Recently Rohde and Oh (1975) presented a unified mathematical treatment of both thick and thin film elastohydrodynamic lubrication problems. The equation of elastic deformation and of hydrodynamic lubrication together represent a nonlinear integrodifferential equation for the determination of the pressure P. This system is written by Rohde and Oh in the form of an operator equation, $\Psi(P) = 0$. Newton's method is used to construct approximate solutions to this equation, according to the scheme

$$\Psi(P_n) + \Psi'(P_n)\eta_n = 0,\ \eta_n|_{\partial\Sigma^*} = 0$$

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and

$$P_{n+1} = P_n + \eta_n$$

Here Σ^* denotes that portion of the region Σ where $P > 0$ and Ψ' is the Frechèt derivative of Ψ, given by

$$\Psi'(P)\eta = \frac{d}{d\epsilon}\,\Psi(P + \epsilon\eta)\,|_{\epsilon=0}$$

Analysis of thermal effects were performed by Cheng and Sternlicht (1965) for combined rolling and sliding motion. The results indicated that the basic features of isothermal EHD contacts are preserved when temperature dependence of viscosity is accounted for. In fact the film thickness formula (22) remains applicable, provided that the viscosity in the immediate vicinity of the contact is employed. However, friction is controlled by viscosity, thus thermal effects cannot be neglected when calculating frictional force. Except for nearly pure rolling conditions, rolling friction is always much less than sliding friction. As interfacial slip increases, stress in the lubricant no longer follows from a linear constitutive equation and thermal effects must also be taken into account. At low-slip, sliding friction can be predicted from the Maxwell model for viscoelastic fluids. As sliding velocity increases the rate of increase of friction becomes zero, or even slightly negative. A nonlinear viscous-plastic lubricant model introduced recently is the Bair-Winer model (Bair and Winer, 1979). This model uses material constants determined from tests independent of EHD application.

Turbulence

In modern applications oftentimes bearings of rotating machinery are lubricated by the process-fluid itself, which may have significantly lower kinematic viscosity than conventional lubricants. Another reason for bearings operating in a high Reynolds number range is the present day tendency to design for larger size and for larger surface speed. For a journal bearing the Reynolds number is defined by $\mathrm{Re} = R\omega C/\nu$.

It is well known that the basic flow between eccentric rotating cylinders is stable only for slow rotation of the inner cylinder, the flow losing its stability to a secondary laminar flow, the Taylor vortex flow (Taylor, 1923), upon increasing the rotation. The appearance of this cellular, toroidal vortex flow has a strong influence on the torque required to rotate the inner cylinder, hence the great practical significance of the problem. As the Reynolds number is increased beyond its critical value, the axisymmetric Taylor vortices become unstable to nonaxisymmetric disturbances (Coles, 1965); the bifurcation is now from steady flow to a periodic flow. This periodic flow remains stable for some range of the Reynolds number, eventually giving up its stability to a quasiperiodic flow with two incommensurate frequencies. Upon further increase of rotational speed of the inner cylinder other, more complicated, processes are observed but eventually turbulence occurs (DiPrima and Swinney, 1983). Compounding the complexities here is the fact that the flow is not unique, in the sense that a number of different states are observable under ostensibly the same steady state conditions (Coles, 1965).

Laminar-turbulent transition between concentric cylinders is being pursued very vigorously. That flow transition takes place also between eccentric cylinders was first recognized by Wilcock (1950). Smith and Fuller (1956) investigated the dependence of a coefficient of friction in the Reynolds number, for a lightly loaded journal bearing.

Analytically the problem was attacked by DiPrima (1963). DiPrima's local theory shows the flow to be least stable at the position of maximum film thickness. The critical Reynolds number calculated at this position first decreases, as the eccentricity ratio $0 \le \epsilon \le 1$ increases from zero, and remains below its concentric value in the range $0 < \epsilon < 0.6$. For $\epsilon > 0.6$ it increases rapidly with increasing ϵ. This finding is, however, not in agreement with experimental results. Vohr's (1968) data indicate that the eccentricity has a stabilizing effect over its whole range. Vohr also finds the maximum intensity of vortex motion not at the point of maximum film thickness, as predicted by local theory, but at the 50 deg position downstream from it.

In a second attempt to explain the dependence of Re_c on ϵ, DiPrima and Stuart (1972) examined the whole flow field (nonlocal theory), considering the tangential velocity to be the function of both r and θ. The most dramatic result of the new theory is that it no longer places maximum vortex intensity at the position of maximum instability but at 90 deg downstream from it. This theory also shows $\mathrm{Re}_c(\epsilon)$ to be a monotonically increasing function. Agreement with experiment is good, however, only in the range $0 < \epsilon < 0.5$. In a more recent nonlinear, nonlocal theory, DiPrima and Stuart (1975) found that the position of maximum vortex activity is at $\theta_c = 90$ deg only if the supercritical Reynolds number vanishes and that $\theta_c \to 0$ deg as $\mathrm{Re}_{sc} \to \infty$. The supercritical Reynolds number is defined as $\mathrm{Re}_{sc} = (\mathrm{Re} - \mathrm{Re}_c/\epsilon)$.

Bearings of turbines, generators, hydraulic prime movers and other power generating equipment operate in the superlaminar regime. Current theory is unable to predict flow stability and transitions in these bearings. This inability derives from several sources, only one of which is relevant to this discussion: Flow transition in bearings is reported by isothermal theory to be in the local Reynolds number range $1300 < R_h < 2000$ (DiPrima, 1963; Vohr, 1968). But on analyzing experimental data of e.g., Gardner and Ulschmid (1974), we find transition more likely to occur in the range $400 < R_h < 900$. Partial answer to the puzzle may come from Li's (1978) work, which suggests that in the presence of significant viscous dissipation or a strong positive radial temperature gradient, such as encountered in large bearings, the neutral (marginal) stability is oscillatory and the parameter for critical conditions is lowered from the value that would have been obtained from isothermal theory. The decrease is particularly significant in fluids of large molecular Prandtl number, such as lubricating oils.

Once turbulence has set in, however, the importance of the precise mechanism of transition diminishes. Turbulence is an irregular motion in which the various flow properties, such as velocity and pressure, show random variation with time and with position. Because of this randomness, the instantaneous value of a flow property has little practical significance and it is the averaged value of that property that is of interest.

To make ideas precise, we represent a dependent variable in the equations of motion by the sum of its average (denoted by an upper case letter and superscript bar) and its fluctuating component (denoted by a lower case letter and a prime, so that

$$\{u, v, w\} = \{\bar{U} + u', \bar{V} + v', \bar{W} + w'\};\ p = \bar{P} + p' \qquad (23)$$

An average value, say $\bar{U}$ of the velocity component u, is interpreted as a time average and is calculated according to

$$\bar{U} = \frac{1}{T}\int_0^T u(t+\tau)d\tau,\ \ T_1 < T < T_2$$

Here T_1 is the time scale of the largest eddies and T_2 is the time scale of the "slow" variations of the flow.

Our aim now is to extend the Reynolds pressure equation to turbulent flow, i.e., to write equation (5) in terms of the mean value $\bar{P}$ of the turbulent pressure. To this end we substitute resolution (23) into the Navier-Stokes equations and average the results in accordance with the rules first established by Reynolds (cf. Monin and Yaglom, 1971).

In nondimensional form the equations of motion for the mean flow are

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$$\frac{C^2\Omega}{\nu}\frac{\partial \bar{U}}{\partial \bar{t}}+\left(\frac{C}{R}\right)\mathrm{Re}\left\{\bar{U}\frac{\partial \bar{U}}{\partial \bar{x}}+\bar{V}\frac{\partial \bar{U}}{\partial \bar{y}}+\bar{W}\frac{\partial \bar{U}}{\partial \bar{z}}+\frac{\partial(\overline{u'u'})}{\partial \bar{x}}+\frac{\partial(\overline{u'w'})}{\partial \bar{z}}\right\}+\mathrm{Re}\frac{\partial(\overline{u'v'})}{\partial \bar{y}}=-\frac{\partial \bar{P}}{\partial \bar{x}}+\frac{\partial^2 \bar{U}}{\partial \bar{y}^2}+\left(\frac{C}{R}\right)^2\left(\frac{\partial^2 U}{\partial \bar{x}^2}+\frac{\partial^2 V}{\partial \bar{z}^2}\right) \tag{24a}$$

$$\left(\frac{C}{R}\right)^2\left\{\frac{C^2\Omega}{\nu}\frac{\partial \bar{V}}{\partial \bar{t}}+\left(\frac{C}{R}\right)\mathrm{Re}\left[\bar{U}\frac{\partial \bar{V}}{\partial \bar{x}}+\bar{V}\frac{\partial \bar{V}}{\partial \bar{y}}+\bar{W}\frac{\partial \bar{V}}{\partial \bar{z}}\right]+\mathrm{Re}\left[\frac{\partial(\overline{u'v'})}{\partial \bar{x}}+\frac{\partial(\overline{w'v'})}{\partial \bar{z}}\right]-\frac{\partial^2 \bar{V}}{\partial \bar{y}^2}-\left(\frac{C}{R}\right)^2\left(\frac{\partial^2 \bar{V}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{W}}{\partial \bar{z}^2}\right)\right\}+\left(\frac{C}{R}\right)\mathrm{Re}\frac{\partial(\overline{v'v'})}{\partial \bar{y}}=-\frac{\partial \bar{P}}{\partial \bar{y}} \tag{24b}$$

$$\frac{C^2\Omega}{\nu}\frac{\partial \bar{W}}{\partial \bar{t}}+\left(\frac{C}{R}\right)\mathrm{Re}\left\{\bar{U}\frac{\partial \bar{W}}{\partial \bar{x}}+\bar{V}\frac{\partial \bar{W}}{\partial \bar{y}}+\bar{W}\frac{\partial \bar{W}}{\partial \bar{z}}+\frac{\partial(\overline{u'w'})}{\partial \bar{x}}+\frac{\partial(\overline{w'w'})}{\partial \bar{z}}\right\}+Re\frac{\partial(\overline{v'w'})}{\partial \bar{y}}=-\frac{\partial \bar{P}}{\partial \bar{z}}+\frac{\partial^2 \bar{W}}{\partial \bar{y}^2}+\left(\frac{C}{R}\right)^2\left(\frac{\partial^2 \bar{W}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{W}}{\partial \bar{z}^2}\right) \tag{24c}$$

Implicit in (24) is that the turbulent velocity correlations $\overline{u'u'}, \overline{u'v'}, \ldots$, are all of the same magnitude with characteristic velocity u_*, and that $(u_*/U_*)^2 = O(1)$. The last assumption seems to hold only in free turbulence (Hinze, 1979). Nevertheless, if made, then together with the limit $(C/R) \to 0$ it will yield the equations considered by Constantinescu and others:

$$\frac{\partial \bar{P}}{\partial x}=\frac{\partial \bar{U}}{\partial y}\left(\mu\frac{\partial \bar{U}}{\partial y}-\rho\overline{u'v'}\right);\quad \frac{\partial \bar{P}}{\partial y}=0;$$
$$\frac{\partial \bar{P}}{\partial z}=\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{W}}{\partial y}-\rho\overline{v'w'}\right) \tag{25}$$

For the wall turbulence, in the boundary layer Hinze recommends $(u_*/U_*)^2 = O(C/R)$. This assumption together with the limit $(C/R)^2 \to 0$ leads to the more acceptable equations

$$\frac{\partial \bar{P}}{\partial x}=\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{U}}{\partial y}-\rho\overline{u'v'}\right)+\rho\frac{d\bar{U}}{dt}$$
$$\frac{\partial \bar{P}}{\partial y}=0 \tag{26}$$
$$\frac{\partial \bar{P}}{\partial z}=\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{W}}{\partial y}-\rho\overline{v'w'}\right)+\rho\frac{d\bar{W}}{dt}$$

Here d/dt represents material differentiation relative to the mean flow. According to this second analysis, mean-flow inertia is of the same order of magnitude as the surviving Reynolds stresses; both quantities should be taken into account when $\mathrm{Re}(C/R) = O(1)$. In other words, there can be no turbulence without inertia. To calculate bearing performance under turbulent conditions, Constantinescu and others employ (25), but when taking into account lubricant inertia, they resort to the more acceptable model (26). It will be shown later that in bearings inertia becomes significant at large values of the reduced Reynolds number, $\mathrm{Re}^* = \mathrm{Re}(C/R)$, even under laminar condition, and certainly under turbulent condition.

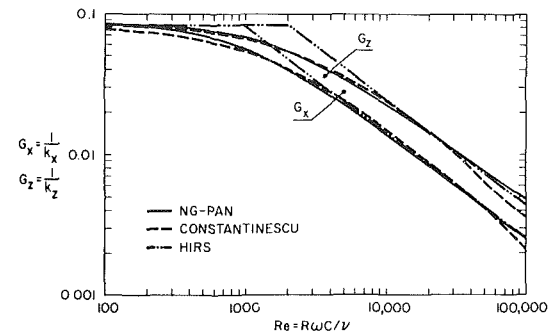

Fig. 4 Variation of the turbulence coefficients k_x and k_z with Reynolds numbers. Comparison of various theories.
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The problem of dealing with turbulence becomes obvious on inspection of equation (25); the Reynolds stress components $\rho\overline{u'v'}, \rho\overline{v'w'}$ appear as additional unknowns, yet we have no additional equations at our disposal. The various theories of turbulence presently available to us aim at closing the turbulence problem by postulating relationship between the mean motion and the Reynolds stress components.

The earliest turbulent lubrication theories employed equation (25) and the ideas of (i) mixing length (Constantinescu, 1959), (ii) eddy viscosity (Ng and Pan, 1965; Elrod and Ng, 1967) and (iii) empirical drag laws (Hirs, 1973; Black and Walton, 1974). In these theories the Reynolds stress is represented in terms of the mean flow. Such representation, when used in conjunction with equation (25), can lead to a relationship between the mean velocity and the pressure gradient. This is a nonlinear relationship and must be used in some iterative process (Elrod and Ng, 1967). Another alternative is to linearize the flow-pressure relationship (Constantinescu, 1959; Ng and Pan, 1967). In either case the mean velocity, now in terms of the mean pressure, is substituted into the equation of continuity in a development that parallels the laminar flow case. It is comforting that several of the available theories yield a formally identical Reynolds equation in turbulent pressure

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu k_x}\frac{\partial \bar{P}}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h^3}{\mu k_z}\frac{\partial \bar{P}}{\partial z}\right)=\frac{1}{2}U_0\frac{\partial h}{\partial x} \tag{27}$$

Here $k_x = k_x(R_h)$ and $k_z = k_z(R_h)$, where $R_h = U_* h/\nu$ is the local Reynolds number, thus in these formulations turbulence is accounted for by weighting the film with some function of R_h. Since $k_x \geq 12$, $k_z \geq 12$, equality sign holding for laminar flow, it is seen that lubrication under turbulent conditions takes place at smaller film thickness than under laminar conditions and in consequence, will yield higher pressures, and higher load capacity. It is in the actual form of the turbulence functions $k_x(x, z; \mathrm{Re})$, $k_z(x, z; \mathrm{Re})$ where the various theories differ, Fig. 4. But the difference between predictions of the various theories is often insignificant, except perhaps at extreme conditions such as high eccentricity in a journal bearing.

One of the simplest means by which to provide closure for the turbulence problem is the eddy viscosity hypothesis of Boussinesq:

$$-\overline{u'v'}=\epsilon_m\frac{\partial \bar{U}}{\partial y},\quad -\overline{v'w'}=\epsilon_m\frac{\partial \bar{W}}{\partial y} \tag{28}$$

Ng and Pan (1965), whose analysis will be sketched for illustration purposes, employed Reichardt's representation of the eddy viscosity for momentum transport, thus

$$\frac{\epsilon_m}{\nu}=\kappa\left(y^+-\delta_l^+\tanh\frac{y^+}{\delta_l^+}\right) \tag{29}$$

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Here κ, δ_l^+ are constants and $y^+ = yV_*/\nu$ is a normalized coordinate, with $V_* = (|\tau_0|/\rho)^{1/2}$ and τ_0 representing wall shear. Substitution of (29) into (28) and (25) and formal integration yields the velocity components

$$\bar{U} = \frac{1}{\mu}\left[\tau_{xy}\left(\frac{h}{2}\right)\int_0^y \frac{dy'}{1+\epsilon_m/\nu} + \frac{\partial \bar{P}}{\partial x}\int_0^y \frac{y'-h/2}{1+\epsilon_m/\nu}\,dy'\right]$$
$$\bar{W} = \frac{1}{\mu}\left[\tau_{zy}\left(\frac{h}{2}\right)\int_0^y \frac{dy'}{1+\epsilon_m/\nu} + \frac{\partial \bar{P}}{\partial z}\int_0^y \frac{y'-h/2}{1+\epsilon_m/\nu}\,dy'\right] \tag{30}$$

These expressions are linearized now by assuming that the flow in the clearance space is a small perturbation of Couette flow (otherwise the solution must be iterative as in the work of Elrod and Ng, 1967)

$$\tau_{xy} = \tau_c + \delta\tau_x;\ \tau_{zy} = \delta\tau_z,\ \frac{\partial \tau_x}{\tau_c} \ll 1,\ \frac{\partial \tau_z}{\tau_c} \ll 1$$

so that $|\tau| = \tau_c + \delta\tau_x + O(\delta\tau_x^2)$. Here τ_c is the turbulent Couette stress. We are then permitted to linearize equation (29) also about its Couette value, to give

$$\frac{\epsilon_m}{\nu}(y;\ |\tau|) = f_c(y) - 1 + g_c(y)\frac{\delta\tau_x}{\tau_c} + O(\delta\tau_x^2) \tag{31}$$

where

$$f_c = 1 + \frac{\epsilon_m}{\nu}(y;\ \tau_c);\ g_c = \tau_c \left.\frac{\partial(\epsilon_m/\nu)}{\partial|\tau|}\right|_{\tau=\tau_c}$$

Neglecting higher order terms in (31) and substituting into (30) we obtain

$$\bar{u} \equiv \frac{\bar{U}}{U_*} = \frac{h\tau_c}{\mu U_*}\int_0^{\bar{y}} \frac{d\eta}{f_c(\eta)} + \left(\tau_{xy}\left(\frac{1}{2}\right) - \tau_c\right)\frac{h}{\mu}\int_0^y \frac{1}{f_c(\eta)}$$
$$\left(1 - \frac{g_c(\eta)}{f_c(\eta)}\right)d\eta + B_x\int_0^{\bar{y}} \frac{\frac{1}{2}-\eta}{f_c(\eta)}\left(1 - \frac{g_c(\eta)}{f_c(\eta)}\right)d\eta \tag{32}$$

and $$\bar{w} \equiv \frac{\bar{W}}{U_*} = \frac{h\tau_{zy}\left(\frac{1}{2}\right)}{\mu U_*}\int_0^{\bar{y}} \frac{d\eta}{f_c(\eta)} + B_z\int_0^{\bar{y}} \frac{\frac{1}{2}-\eta}{f_c(\eta)}\,d\eta$$

with $B_x \equiv -\frac{h^2}{\mu U_*}\frac{\partial \bar{P}}{\partial x}, B_z \equiv -\frac{h^2}{\mu U_*}\frac{\partial \bar{P}}{\partial z}$ and $\bar{y} = y/h$

Observing that in pure shear flow $B_x = 0$ and $\tau_{xy}(1/2) - \tau_c = 0$, we have

$$(h_c^+)^2\int_0^1 \frac{d\eta}{f_c(\eta)} = R_h;\ h_c^+ \equiv \frac{h}{\nu}\frac{\bar{\tau}_c}{\rho}$$

Both $f_c(\bar{y})$ and $g_c(\bar{y})$ are symmetrical with respect to $\bar{y} = 1/2$. Thus satisfaction of the remaining boundary conditions $\bar{U} = U$, $\bar{W} = 0$ at $y = 0$ leads to

$$\tau_{xy}\left(\frac{1}{2}\right) = \tau_c,\ \tau_{zy}\left(\frac{1}{2}\right) = 0$$

Equation (32) can be simplified to

$$\bar{u} = \frac{1}{2} + \frac{(h_c^+)^2}{R_h}\int_{1/2}^{\bar{y}} \frac{d\eta}{h_c(\eta)} + B_x\int^{\bar{y}} \frac{\frac{1}{2}-\eta}{f_c(\eta)}$$
$$\left(1 - \frac{g_c(\eta)}{f_c(\eta)}\right)d\eta,\ \bar{w} = B_z\int_0^{\bar{y}} \frac{\frac{1}{2}-\eta}{f_c(\eta)}\,d\eta \tag{33}$$

Substituting for $\bar{U}$ and $\bar{W}$ in the continuity equation yields the linearized turbulent lubrication equation (27), where the coefficient k_x and k_z are defined by

$$\frac{1}{k_x} = \int_0^1 d\bar{y}\int_0^{\bar{y}} \frac{\frac{1}{2}-\eta}{f_c(\eta)}\left(1 - \frac{g_c(\eta)}{f_c(\eta)}\right)d\eta,\ \frac{1}{k_z}$$
$$= \int_0^1 d\bar{y}\int_0^{\bar{y}} \frac{\frac{1}{2}-\eta}{f_c(\eta)}\,d\eta \tag{34}$$

At high Reynolds numbers the prediction of Ng, Pan, and Elrod and those of Hirs are almost identical and it is in the near transition regime that any significant discrepancy occurs. The turbulent functions k_x, k_z of Ng and Pan and of Constantinescu are smooth functions of the Reynolds number as Re → 0 and they achieve their laminar value, $k_x = k_z = 12$, only at Re = 0. This, of course, models a continuously developing turbulence as Re↑, in complete contrast to experimental evidence (DiPrima and Swinney, 1983).

None of the above theories agree exactly with the calculations of Ho and Vohr (1974) who use the Kolmogoroff–Prandtl model, estimating the eddy-viscosity length scale from the van Driest formula, but results of Elrod and Ng come closest.

There have been various other turbulence models used for predicting bearing performance, such as the low Reynolds number energy dissipation model of Launder and Leschziner (1978b). But these models are cumbersome to use and, at the same time, fail to offer substantially different results. It seems to this writer that the road to further development in modeling turbulent bearing performance is temporarily obstructed, and will reopen again when fluid mechanics provides tribologists with better theories of turbulent shear flow. This is particularly true for the transition-weak turbulence regime, where most bearings tend to operate. It is here that fluid mechanics is presently undergoing significant developments.

Of the theories discussed here, the Ng-Pan theory and the bulk flow theory of Hirs are to be preferred over Constantinescu's model. The bulk flow theory may be extended to flow between rough or grooved surfaces as experimental data become available. Extension of the Ho-Vohr model to include fluid inertia is also possible.

Effects of Lubricant Inertia

According to equation (2), lubricant inertia effects assume importance when $\Omega^* > 1$ and/or $\mathrm{Re}^* > 1$. Here $\Omega^* = C^2\Omega/\nu$ and $\mathrm{Re}^* = (C/R)\mathrm{Re}$ are the reduced frequency and the reduced Reynolds number, respectively. The three limits that are instructive to consider here are (i) $\mathrm{Re}^*/\Omega^* \to 0$, $\mathrm{Re}^* > 1$, (ii) $\mathrm{Re}^*/\Omega^* \to 1$, $\mathrm{Re}^* > 1$ and (iii) $\Omega^*/\mathrm{Re}^* \to 0$, $\Omega^* > 1$.

When one of the bearing surfaces undergoes rapid, small amplitude vibration, the condition $\Omega^* \gg \mathrm{Re}^*$ is approximately satisfied. In this case, we retain the temporal inertia terms in (2) but drop the terms representing convective inertia (Schlichting, 1968). Expressed in primitive variables, the equations of motion have now the reduced form

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2},\ \frac{\partial p}{\partial y} = 0,$$
$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\frac{\partial^2 w}{\partial y^2} \tag{35}$$

We note that these equations are linear.

When a solid body accelerates through a fluid, a force must be applied to accelerate the mass of the body and an added force must be applied to accelerate the mass of fluid set in motion by the body. The added mass coefficient, C_M, is defined

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as the factor which multiplies the mass of the displaced fluid to give the added mass. Studies of added mass can be traced back to Stokes (1843). Recently Chen et al. (1976) expressed equation (35) in terms of the stream function and found an exact solution to the resulting linear fourth order partial differential equation

$$\frac{1}{\mathrm{Re}}\nabla^4\psi-\frac{\partial}{\partial t}\nabla^2\psi=0$$

in terms of Bessel functions. They were able to show that for a rod vibrating in a "large" cylinder filled with viscous fluid, both the added mass coefficient, C_M, and the damping coefficient, C_v, have their maximum at small Ω^* and decrease with $\Omega^*\uparrow$. This was also demonstrated by the analysis of Brennen (1976), who obtained a large Reynolds number limit for the added mass coefficient. Brennen found that for the ratio of C_M obtained at $\Omega^* \to 0$ to its value obtained at $\Omega^* \to \infty$ is $\simeq$ 1.2. Tichy and Modest (1978) solve equation (35) for arbitrary two dimensional surfaces executing normal oscillation and find the classical lubrication solution for pressure to be in error, due to neglect of inertia forces. The analysis was extended by Modest and Tichy (1979) to combined oscillation and sliding. Mulcahy (1980) was able to show that for $\Omega^* < 25$ the added mass coefficient, C_M, is independent of the reduced frequency and that in the range $0 < \Omega^* < 25$ the ratio C_M/C_v is linear in the reduced frequency, according to the formula $C_M/C_v \simeq \Omega^*/10$. For $\Omega^* > 25$ Mulcahy finds that $C_M = C_M(\Omega^*)$ and that it is a decreasing function for $\Omega^*\uparrow$, as indicated by earlier investigators.

In journal or thrust bearings in near steady state but at high Reynolds number we have $\mathrm{Re}^* \gg \Omega^*$. This condition approximates to the limit $\Omega^*/\mathrm{Re}^* \to 0$ and we are permitted to neglect temporal inertia in favor of convective inertia. When written in terms of primitive variables, this limit results in the equations

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \\ 0 &= -\frac{\partial p}{\partial y} \\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{\partial P}{\partial z}+\mu\frac{\partial^2 w}{\partial y^2} \end{aligned} \tag{36}$$

Equation (36) was solved by Reinhardt and Lund (1974) to first order in a perturbation series in Re^*, the zero order solution representing the classical theory of Reynolds. Sestieri and Piva (1982) applied (36) to a plane slider in a numerical analysis of the problem at hand. They show significant inertia effects, up to 40 percent in load, for $\mathrm{Re}^* = 8$. Care must be taken, however, when interpreting the results of Sestieri and Piva. As was pointed out by Constantinescu and Galetuse (1982), a reduced Reynolds number of $\mathrm{Re}^* = 8$ for the slider geometry of Sestieri and Piva translates to $\mathrm{Re} = 25{,}000$ in a conventional journal bearing. For long cylinders at $\mathrm{Re}^* = 30$ San Andres and Szeri (1985) obtain from numerical solutions of the exact equations a 47 percent change in pressure due to inertia, when $C/R = 1.0$. But as $(C/R)\downarrow$ the results from the exact equations decrease monotonically to the results of the classical theory. For $(C/R) = 2 \times 10^{-3}$, the difference between classical and inertial solutions is only in the third significant digit (Kacou and Szeri, 1986) for both bearing force and linearized force coefficients, in the range $0 < \mathrm{Re}^* < 4$. Above $\mathrm{Re} = 2000$ the flow becomes turbulent, but even there inertial effects can be neglected according to Constantinescu and Galetuse (1982).

Solution of the equations of motion is very difficult in the intermediate range, $\Omega^* \simeq \mathrm{Re}^*$. There are two main approaches here to deal with the problem: (i) the method of "averaged inertia" and (ii) the method of regular perturbation in Re^*.

The method of averaged inertia was employed for the steady problem by Osterle, Chou, and Saibel (1957) for a long bearing, and by Constantinescu (Constantinescu, 1970; Constantinescu and Galetuse, 1982). The more recent of these papers solves equation (26) for steady flow and finds that in conventional journal bearings convective inertia effects can be neglected in the range $0 < \mathrm{Re}^* < 20$. Although at the upper end of the range inertia effects do appear, they are still small in comparison with other effects such as thermal distortion. This is in agreement with the conclusion drawn by Kacou and Szeri (1986).

The method of "averaged inertia" was employed by Burton and Hsu (1974) to find that at small downward vertical load the journal center might rise above the bearing center, as was shown experimentally by Black and Walton (1974). Others to use the method of averaged inertia were King and Taylor (1977) and Launder and Leschziner (1978a, b).

The method of averaged inertia leads to a Reynolds type partial differential equation in pressure. When deleting terms of order $(C/R)^2$ in equation (2) (or when employing equation (26) for turbulent flow) we may write

$$\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial y} \end{aligned} \tag{37}$$

here τ_{xy}, τ_{zy} are stress components. When these equations are integrated across the film and use is made of the equation of continuity, we obtain after differentiation

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{h}{\rho}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h}{\rho}\frac{\partial p}{\partial z}\right)=\frac{\partial}{\partial x}\left(\frac{1}{\rho}\tau_{xy}\Big|_0^h\right) \\ &+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\tau_{zy}\Big|_0^h\right)+U_s\frac{\partial^2 h}{\partial x\partial t}-\frac{\partial^2}{\partial x\partial t}\int_0^h u\,dy \\ &-\frac{\partial}{\partial x^2}\int_0^h u^2dy-2\frac{\partial^2}{\partial x\partial y}\int_0^h uw\,dy-\frac{\partial^2}{\partial z\partial t}\int_0^h w\,dy-\frac{\partial^2}{\partial z^2}\int_0^h w^2dy \\ &+U_s\left(\frac{\partial^2}{\partial x^2}\int_0^h u\,dy+\frac{\partial^2}{\partial x\partial z}\int_0^h w\,dy\right) \end{aligned} \tag{38}$$

Here U_s is the surface velocity in the (x, z) plane. We also define average velocity components

$$U=\frac{1}{h}\int_0^{h(x,t)} u\,dy,\quad W=\frac{1}{h}\int_0^{h(x,t)} w\,dy \tag{39}$$

and, assuming that the velocity profiles near the wall are not influenced by fluid inertia, approximate the viscous terms under laminar conditions as

$$\tau_{xy}\Big|_0^h=-\frac{12\mu}{h}\left(U-\frac{1}{2}U_s\right);\ \tau_{zy}\Big|_0^h=-\frac{12\mu}{h}W \tag{40}$$

When, in addition, we use the approximate relationships

$$\int_0^{h(x,t)} u^2dy=hU^2;\ \int_0^{h(x,t)} uw\,dy=hUW;\ \int_0^{h(x,t)} w^2dy=hW^2$$

a pressure equation in the following form is obtained

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$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial P}{\partial z}\right)=12\left\{v(h)-\frac{1}{2}U_s\frac{\partial h}{\partial x}\right\}$$
$$+\frac{h^2}{\nu}\left\{U_s\frac{\partial^2 h}{\partial x\partial t}-\frac{\partial^2(hU)}{\partial x\partial t}\right. \tag{41}$$
$$-\frac{\partial^2(hU^2)}{\partial x^2}-2\frac{\partial^2(hUW)}{\partial x\partial z}-\frac{\partial^2(hW)}{\partial z\partial t}-\frac{\partial^2(hW)^2}{\partial z^2}$$
$$+U_s\frac{\partial^2(hU)}{\partial x^2}+U_s\frac{\partial^2(hW)}{\partial x\partial z}$$

If equation (38) is applied to a squeeze film damper (Szeri, Raimondi, and Giron, 1982), we put

$$h=CH,\ t=\bar{t}/\Omega,\ x=R\theta,\ Z=\frac{L}{2}z,\ p=12\mu\Omega\left(\frac{R}{C}\right)^2\bar{p},$$
$$\mathrm{Re}=\frac{\delta C\Omega}{\nu}$$

and define the characteristic velocity by $U*=\delta\Omega$, where δ is the amplitude and Ω is the frequency of the orbit of the inner cylinder. This leads to the pressure equation:

$$\frac{\partial}{\partial\theta}\left(H^3\frac{\partial\bar{P}}{\partial\theta}\right)+\left(\frac{D}{L}\right)^2\frac{\partial}{\partial Z}\left(H^3\frac{\partial\bar{P}}{\partial Z}\right)=\frac{\partial H}{\partial\bar{t}}$$
$$+\frac{\mathrm{Re}}{12}H^2\left[\frac{\partial^2 H}{\partial\bar{t}^2}-\frac{2}{H}\left(\frac{\partial H}{\partial t}\right)^2\right] \tag{42}$$

Equation (42) suggests that when $\mathrm{Re}=O(12)$, the inertial contribution will be of the same order of magnitude as the viscous contribution.

Equation (42), constructed for bearing aspect ratio $(L/D)<1/2$, shows damping to be independent and the added mass coefficients, i.e., the inertia coefficients, to be linear in the Reynolds number. The most serious assumption in the "method of averaged inertia" is that the shape of the velocity is not influenced by inertia.

When the Reynolds number is small, we may simplify equation (37) for parallel plates. Using the von Karman (1923) similarity transformations, the equation of motion can be reduced to a partial differential equation in two variables. Kuzma (1967) used an iteration scheme to obtain solutions of (37) to order Re. Tichy and Winer (1970) found solution to the next order and calculated the ratio of inertial to non-inertial solution for force to be given by $W/W_{\mathrm{L.T}}=1+0.18$ Re. Jones and Willson (1974) put the perturbation analysis on firmer mathematical foundation and find a large Reynolds number asymptote for constant approach velocity. Grimm (1976) presents a numerical treatment of the problem of flat plates approaching under constant load, and shows convincingly that numerical solutions must, in general, be used for these kind of problems as the perturbation series diverges.

The perturbation solution is valid for $\mathrm{Re}<1$, but was found to be good for $\mathrm{Re}\simeq 10$ (Tichy and Winer, 1970). For arbitrary value of Re, Tichy (1981) introduces an Oseen type linearization

$$u\frac{\partial u}{\partial x}\sim U\frac{\partial u}{\partial x},\ v\frac{\partial u}{\partial y}\sim V\frac{\partial u}{\partial y}$$

for some "slug-flow" U, V. For $\mathrm{Re}\to 0$ this approximate solution very nearly gives the previously obtained low Reynolds number limit (Tichy and Winer, 1970). There is also a high Reynolds number asymptote, linear in the Reynolds number. This suggests, perhaps, that the method of averaged inertia yields the high Reynolds number limit (San Andres, 1986).

In way of a summary one may conclude that two of the limits, viz., $\mathrm{Re}^*/\Omega^*\to 0$ and $\Omega^*/\mathrm{Re}^*\to 0$ are better researched. In the former case $C_M\sim$const in the range $0<\Omega^*<25$ and both C_M and C_V decrease with $\Omega^*\uparrow$. In the second case the bearing stiffness and damping is independent of the Reynolds number, at least in the $0<\mathrm{Re}^*<4$ range. In the intermediate limit, $\Omega^*/\mathrm{Re}^*\to 1$, the various solutions are in contradiction.

Thermal Effects

Classical lubrication theory predicts bearing performance on the assumption that the viscosity of the lubricant is constant and uniform over the film; bearing performance is found then to be heavily dependent on the value of the lubricant viscosity. If now the viscosity becomes a strong function of temperature, results of the classical theory can no longer be expected to apply, especially not in cases where the rise of lubricant temperature across the bearing is substantial.

In many applications, temperature rise across the bearing, although not negligible, remains small. The classical isothermal theory in this case might still be applied, provided that the calculations are based on an "effective" or operating viscosity. The value of the effective viscosity can be arrived at by iteration between solutions of the classical theory and a simple energy balance (Raimondi and Boyd, 1958). The iteration process can be characterized by the formula

$$T_e^{(n+1)}=T_i+\kappa g(T_e^{(n)}) \tag{43}$$

where $T_e^{(n)}$ is the nth iterated value of the effective temperature, T_i is the temperature of the lubricant at inlet and the function $g(T_e^{(n)})$ represent the temperature rise across the bearing due to viscous dissipation, as calculated from classical theory with $\mu=\mu(T_e^{(n)})$. The value of the parameter κ is inferred then from curve fitting to existing data. It has been recommended that $1/2\le\kappa\le 1$. This scheme will work in some situations (Kaufman, Szeri, and Raimondi, 1978).

In large bearings and/or under severe running conditions thermal effects may be significant to the extent, that prediction of bearing performance is no longer possible when based on the assumption of a uniform effective viscosity. Furthermore, since in large bearings the usual limit condition on design is the maximum permissible bearing temperature, the actual temperature distribution itself is of considerable importance. The theory that calculates pointwise variation of temperature and viscosity in the lubricant film, and takes these into account when predicting bearing performance, is the thermohydrodynamic (THD) theory.

Let T_* be the characteristic temperature of the lubricant film and write $\bar{t}=t/T_*$, $\bar{\mu}=\mu/\mu_*$, $\mu_*=\mu(T_*)$, so that $\bar{t}$ and $\bar{\mu}$ are nondimensionalized. Then assuming the lubricant to be incompressible and thus neglecting the work of compression, the normalized energy equation for the lubricant can be put in the form:

$$\bar{u}\frac{\partial\bar{t}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{t}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{t}}{\partial\bar{z}}=\frac{1}{\mathrm{Pe}}\left[\frac{\partial^2\bar{t}}{\partial\bar{y}^2}\right.$$
$$\left.+\left(\frac{C}{R}\right)^2\left(\frac{\partial^2\bar{t}}{\partial\bar{x}^2}+\frac{\partial^2\bar{t}}{\partial\bar{z}^2}\right)\right]+\bar{\mu}\bar{\Lambda}\bar{\phi} \tag{44}$$

The nondimensionalized dissipation function $\bar{\phi}$ has the definition

$$\bar{\Phi}\equiv\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2+\left(\frac{C}{R}\right)^2\left\{\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2\right.$$
$$+\left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2+2\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2+\left(\frac{\partial\bar{v}}{\partial\bar{y}}\right)^2+\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^2\right.$$
$$\left.\left.+\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{x}}+\frac{\partial\bar{w}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{z}}+\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{z}}\right]\right\}$$
$$+\left(\frac{C}{R}\right)^4\left[\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2+\left(\frac{\partial\bar{v}}{\partial\bar{z}}\right)^2\right] \tag{45}$$

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In equation (44), $\mathrm{Pe}=\rho c_p RU_*/k$ is the Peclet number, $\mathrm{Pr}=\mu c_p/k$ is the Prandtl number and $\bar{\Lambda}=(\mu\omega/\rho c_p T_*)(R/C)^2$ is the dissipation number.

If, in accordance with classical theory we take the limit $(C/R)\rightarrow 0$, equation (44) reduces to

$$\bar{u}\frac{\partial \bar{t}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{t}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{t}}{\partial \bar{z}}=\frac{1}{\mathrm{Pe}}\frac{\partial^2 \bar{t}}{\partial \bar{y}^2}+\bar{\mu}\bar{\Lambda}\left[\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2+\left(\frac{\partial \bar{w}}{\partial \bar{y}}\right)^2\right] \tag{46}$$

The first applications of an energy equation to lubrication were by Christopherson (1941) and by Cameron and Wood (1946). Cope (1949) presented a detailed discussion of (44) and arrived at equation (46) via an order of magnitude analysis. But Cope did not stop here; he argued that all the generated heat is spent on increasing the temperature of, and is carried away by, the lubricant. In our analysis this tantamounts to taking the limit $(1/\mathrm{Pe})\rightarrow 0$ in equation (46). Now in a conventional journal bearing $(C/R)\simeq 2\times 10^{-3}$ and at $T_*=40°\mathrm{C}$ an SAE10 oil has $\mathrm{Pr}\simeq 500$, thus $\mathrm{Pe}=\mathrm{O(Re)}$ under these conditions. In the language of equation (46), Cope's assertion is that heat conduction across the film is negligible when $(1/\mathrm{Pe})\rightarrow 0$, or as in our example, when $(1/\mathrm{Re})\rightarrow 0$. But it is exactly at high Reynolds numbers that the thermal boundary layer will be thin and vigorous heat transfer to the solid members will take place. Thus on the one hand, the condition for the "validity" of Cope's adiabatic theory is $(1/\mathrm{Re})\rightarrow 0$, but on the other, it cannot apply there. Nevertheless, there have been a number of publications using this theory (Raimondi, 1966). It was generally accepted that the classical isothermal theory and Cope's adiabatic theory would bracket actual bearing performance (Dowson and March, 1967). This idea was later discarded (Seireg and Ezzat, 1972; McCallion, Yousif, and Lloyd, 1970; Ezzat and Rohde, 1972).

The problem with the adiabatic theory is that when neglecting the conduction term in equation (46), one lowers the order of that equation. Parallel may be drawn here between the present situation and the process whereby discarding the viscous terms in the Navier-Stokes equation under the assumption $(1/\mathrm{Re})\rightarrow 0$ leads to Euler's equation for an ideal fluid.

Among the first to appreciate the importance of the surviving conduction term in equation (46) was Dowson and Hudson (1963). Notable subsequent papers were those by Dowson and March (1967), McCallion et al. (1970), and Ezzat and Rohde (1972), just to name a few. A common theme of the work reported in these papers is to solve equation (46) as an initial boundary value problem, i.e., to specify thermal inlet and boundary conditions and to march the solution in the direction of increasing x. This scheme will work as long as $\bar{u}(\bar{x},\bar{y},\bar{z})\geq 0$ in the whole solution domain. Backflow, however, is encountered in bearings under certain conditions at inlet, near the stationary surface. To deal with this situation one must retain the heat conduction term $(\partial^2\bar{t}/\partial\bar{x}^2)$ in equation (44) even though this term is $\mathrm{O}(10^{-6})$; its retention keeps the equation elliptic in nature and thus applicable to recirculating flow (Suganami and Szeri, 1979a). In journal bearings at large eccentricity this can lead to lubricant temperature which shows a minimum somewhere within the bearing, a distance downstream from the inlet (Suganami and Szeri, 1979b). Boncompain, Fillon, and Frene (1985) first establish the position $u(x,y,z)=0$ in the bearing and then using (46) march into two opposite directions. The temperature effect due to recirculation (operative near the stationary surface) and the "hot oil carry over" (operative near the moving surface) may on occasion combine to become significant.

If $\mu=\mu(T)$ and $T=T(x,y,z)$, equation (4) is no longer applicable and one must start with more general equations which allow for pointwise variation of the viscosity, $\mu=\mu(x,y,z)$. But thermal effects are most significant under superlaminar flow of the lubricant. With this in mind, we write equation (25) for a variable viscosity lubricant in turbulent flow, under the eddy viscosity hypothesis

$$\frac{\partial \bar{P}}{\partial x}=\frac{\partial}{\partial y}\left[\mu\left(1+\frac{\epsilon_m}{\nu}\right)\frac{\partial \bar{U}}{\partial y}\right];\quad \frac{\partial \bar{P}}{\partial y}=0;$$

$$\frac{\partial \bar{P}}{\partial z}=\frac{\partial}{\partial y}\left[\mu\left(1+\frac{\epsilon_m}{\nu}\right)\frac{\partial \bar{W}}{\partial y}\right] \tag{47}$$

In analogy with the isothermal case we may formally integrate (47) and using the continuity equation derive a generalized (variable viscosity, turbulent) Reynolds equation for lubricant pressure

$$\frac{\partial}{\partial x}\left(\frac{\Gamma}{\mu_*}\frac{\partial \bar{P}}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\Gamma}{\mu_*}\frac{\partial \bar{P}}{\partial z}\right)=U_0\frac{\partial F}{\partial x} \tag{48}$$

For journal bearings and for thrust bearings F has the values

$$F=h-\frac{\xi_2(h)}{\xi_1(h)}\ \text{and}\ F=\frac{\xi_2(h)}{\xi_1(h)},$$

respectively. If we put $\mu_* f(x,y,z)\equiv\bar{\mu}(1+\epsilon_m/\nu)$ and

$$\xi_1(x,\phi,z)\equiv\int_0^{\phi}\frac{dy}{f(x,y,z)};\quad \xi_2(x,\phi,z)\equiv\int_0^{\phi}\frac{y\,dy}{f(x,y,z)}$$

then $\Gamma(x,z)$ in (48) has the definition

$$\Gamma(x,z)\equiv-\int_0^{h(x)}\left[\xi_2(x,y,z)-\frac{\xi_2(x,h,z)}{\xi_1(x,h,z)}\xi_1(x,y,z)\right]dy$$

When employing (48), the turbulent flow counterpart of the energy equation (44) should be used (Safar and Szeri, 1973).

Although the distribution of temperature, and therefore of viscosity, is assumed to be known at the start of the solution of equation (48), this equation has nonlinear coefficients as the eddy viscosity is dependent on the shear stress (Safar and Szeri, 1973). For isothermal flow, equation (48) is identical to the pressure equation of Elrod and Ng (1967) and for laminar flow, $\epsilon_m=0$, it reduces to the thermohydrodynamic pressure equation of Dowson and co-workers (Dowson and March, 1967; Dowson and Hudson, 1963). When the coefficients in (48) are linearized, the equation becomes a generalization of the Ng-Pan turbulent equation (Szeri, 1980). In two recent papers the linearized form of (48) was applied to thrust pads (Jeng, Zhou, and Szeri, 1986a, 1986b).

The most significant results of thermohydrodynamic theory may be listed as:

(1) There is an often very significant decrease of lubricant pressure and, consequently, of load carrying capacity.
(2) The isothermal and the adiabatic theories do not bracket THD bearing performance.
(3) Two dimensional viscosity distributions are inadequate to reproduce THD solutions (Boncompain, Fillon, and Frene, 1985).
(4) Due to the change in the laws governing generation and transport of energy in the transition regime, turbulent bearings can run cooler than laminar bearings, Fig. 5 (Suganami and Szeri, 1986b).

Non-Newtonian Effects

In bearings of various types non-Newtonian effects might assume importance due mainly to two circumstances: process fluid lubrication and treatment of the lubricant with polymeric additives. A third circumstance that calls for extension of classical theory to non-Newtonian fluids is relevant to lubrication of EHD contacts.

In EHD contacts the principal features, such as the sudden contraction of the film or the existence of the second pressure

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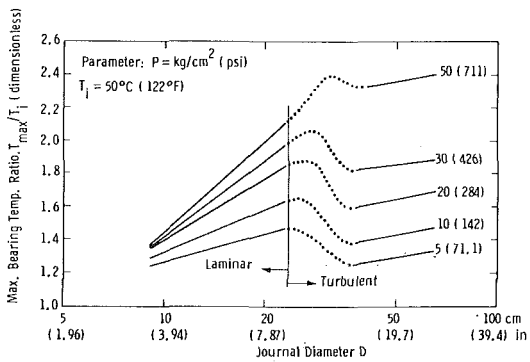

Fig. 5 Predicted maximum bearing temperature versus shaft diameter in a journal bearing at N = 3600 r/min.
(From A. Z. Szeri, ed., *Tribology*, Hemisphere Publishing Corp. With permission)

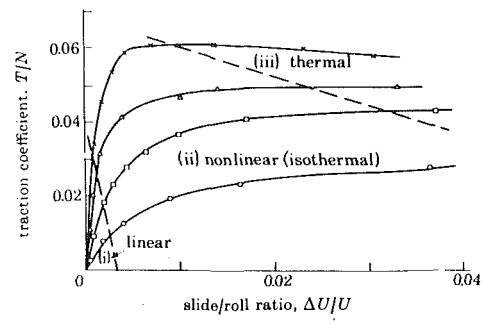

Fig. 6 Typical traction curves measured on a two-disk machine in EHL line contact at varying mean contact pressure, p : x, 1.03; △, 0.68; □, 0.51; ○, 0.40 GPa. (From K. L. Johnson and R. Cameron, R., *Proc. Inst. Mech. Eng.*, 182, 307, 1967. With permission).

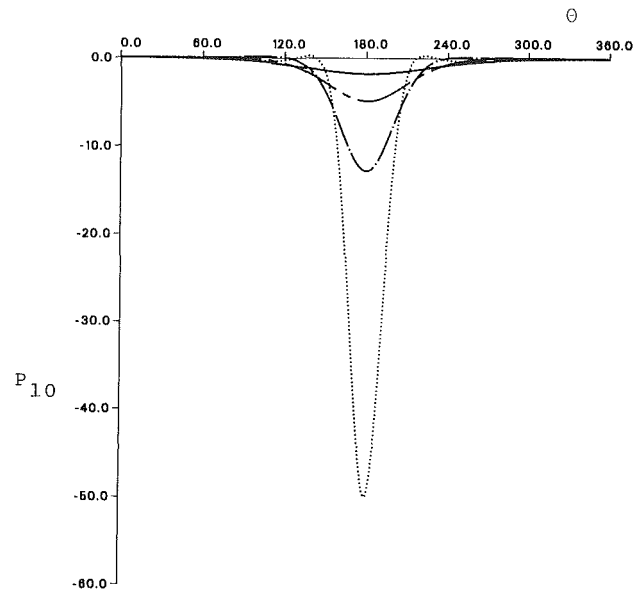

Fig. 7 Pressure correction to order ϵ in a journal bearing lubricated by a fluid of grade three

peak, remain largely unaffected by thermal effects if the maximum Hertzian load $p<0.5$ GPa. If, on the other hand, as it is in practice $p>0.5$ GPa, then thermal effects are so severe that lubricant behavior departs significantly from the linear. In these cases traction forces calculated on the assumption of a Newtonian lubricant are an order of magnitude larger than found experimentally.

In discussing the various regimes of lubricant behavior in EHD contacts we refer to Fig. 6. On this figure we see coefficient of friction plotted against the ratio of sliding to rolling speed for a two disk machine (Johnson and Cameron, 1967). Under conditions of the experiments, the ordinate in proportional to shear stress, τ, and the abscissa to shear-rate, $\dot{\gamma}$. The various curves are drawn for various constant Hertzian pressure. Each of these curves is seen to have a linear portion, $\tau \alpha \dot{\gamma}$, where the fluid is essentially Newtonian. For large p, the region of this type of behavior is limited to very small values of $\dot{\gamma}$. At larger strain rate the lubricant exhibits nonlinear behavior, though it may still be treated by isothermal theories. At still larger strain rate the lubricant enters the zone of predominant thermal effects. Here the shear stress shows a slight decrease with increasing strain rate (Johnson and Tevaarwerk, 1977).

There have been several theories put forth to explain, and to model, traction in the nonlinear region. Fein (1967) and Harrison and Trachman (1972) suggested that the fluid, during its brief stay in the contact zone, would not have time to assume equilibrium values of density, viscosity, and shear modulus. Dyson (1970) suggested that the apparent decrease of viscosity in steady, continuous shear of a linearly viscoelastic fluid can be used to model the nonlinear part of the traction curve. An alternative explanation, that the shear stress-shear rate is nonlinear for the lubricant, was offered by Trachman and Cheng (1972). Smith (1962) proposed that in the high pressure contact zone the lubricant was shearing like a plastic solid.

Johnson and Tevaarwerk (1977) formulated a new rheological equation for the lubricant by adding a linear elastic element to the nonlinear viscous element. No particular specification was made about the type of nonlinear viscous element that should be used. Nevertheless, Johnson and Tevaarwerk used the Eyring hyperbolic sine as the viscous element, as this gave best fit to experimental data; at the same time it had its roots in molecular theory. The rheological equation proposed by Johnson et al. is

$$\dot{\gamma}=\dot{\gamma}_e+\dot{\gamma}_v=\frac{1}{G}\frac{d\tau}{dt}+\frac{\tau_0}{\mu}\sinh\left(\frac{\tau}{\tau_0}\right) \tag{49}$$

Here τ is an "equivalent" or average stress and τ_0 is a representative stress. This rheological model was extended to include thermal effects by Houpert (1985) recently. Based on analytical calculations assuming isothermal EHD conditions and on curve fitting to obtain a thermal correction factor, Houpert constructs a fast numerical algorithm to calculate traction forces.

Recently Bair and Winer (1979) offered a Maxwell model modified with a limiting shear stress, that yields good comparison with experiment. In earlier rheological models, measurements of contact behavior had to be made in order to predict contact behavior. In the Bair and Winer model the material constants, viz., the low shear stress viscosity, μ_0, the limiting elastic shear modulus, G_∞, and the limiting yield shear stress, τ_L, are all from sources other than EHD contact measurements, and are functions of pressure and temperature. The Bair-Winer equation relating stress to motion in the lubricant is

$$\dot{\gamma}=\dot{\gamma}_e+\dot{\gamma}_v=\frac{1}{G_\infty}\frac{d\tau}{dt}-\frac{\tau_L}{\mu_0}\ln\left(1-\frac{\tau}{\tau_L}\right) \tag{50}$$

If the limiting shear stress is very large, then (50) reduces to the classical Maxwell model. If, on the other hand $(1/G_\infty)\rightarrow 0$, the limiting case of viscoplastic behavior, the model yields the nonlinear viscous form

$$\dot{\gamma}=-\frac{\tau_L}{\mu_0}\ln\left(1-\frac{\tau}{\tau_L}\right) \tag{51}$$

The third possibility, elastic-plastic behavior, is obtained by letting the viscosity take very large values, in which case all the

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strain at low stress occurs in the elastic term and at large stress the limiting stress controls (Bair and Winer, 1979).

Turning now to hydrodynamic lubrication applications, we find that one of the early treatments of the problem of lubrication by a non-Newtonian lubricant was by Ng and Saibel (1962). In their regular perturbation treatment of the plane slider they assumed a stress constitutive equation of the form

$$\tau = \mu \frac{\partial u}{\partial y} + \epsilon k \left(\frac{\partial u}{\partial y} \right)^3 \tag{52}$$

Equation (52) arises when one restricts the flow of a thermodynamically compatible third grade fluid of the Rivlin-Ericksen type (Fosdick and Rajagopal, 1980) to the geometry of Ng and Saibel. In another paper Hsu and Saibel (1965) use the constitutive equation

$$\tau + k\tau^3 = \mu \frac{\partial u}{\partial y} \tag{53}$$

which is applicable to both pseudoplastic, $k > 0$, and dilatant, $k < 0$, fluids. They find for the pseudoplastic case that the load capacity is reduced when compared to the performance of a Newtonian lubricant. Fix and Paslay (1967) investigated the lubricant elasticity effects of a Maxwell type material in a long journal bearing. Their most significant finding is a reduction of the attitude angle, which equals $\pi/2$ under Sommerfeld or full film conditions, due to elasticity of the lubricant. In an experimental paper Tao and Philippoff (1967) show a pressure distribution for viscoelastic liquids that is flatter then obtained for a Newtonian fluid. They also indicate a shift into the downstream direction of the positive pressure peak. Considering the in-plane oscillation of a flat plate in a linearly viscoelastic fluid, Tanner (1968) argues that the normal stress effect is unimportant in the usual lubrication application. For an elasto-viscous liquid, i.e., a nonlinear Reiner-Rivlin fluid generalized to include unequal cross stresses, Reiner, Hanin and Harnoy (1969) show a load capacity higher than given by a Newtonian fluid.

It is well known (cf. Giesekus, 1963; Tanner, 1966) that the Stokesian velocity field is a solution to the equations of motion of a plane, steady, noninertial flow of a second grade fluid. This fact has been used to extend the results of lubrication theory to flows involving fluids of second grade (Davis and Walters, 1973). Since the velocity field in both problems is identical, the only quantity that is to be calculated is a "modified" pressure field. But since the equations of motion satisfied by a second grade fluid are of higher order than the Navier-Stokes equations, there remains a question of uniqueness of solution (Fosdick and Rajagopal, 1978; Rajagopal, 1984). For a long journal bearing Davies and Walters find qualitative agreement with the earlier results of Fix and Paslay (1967), concerning the magnitude of the attitude angle. The result of Giesekus and Tanner has been extended to dynamically loaded bearings by Harnoy and Hanin (1974). Tichy (1978) also looks at small amplitude rapid oscillations and includes temporal inertia for a Lodge rubber-like liquid. Tichy finds perturbation solution to order δ, where δ is the amplitude of plate motion. At low Deborah number the viscoelastic effect is found to be negligible. However, Tichy warns that bearing operation at high Deborah number could give rise to various resonance effects. He then goes on to state that the often noted discrepancy between theory and experiment (eg., Harnoy and Philippoff, 1976) may be in part explained by resonance phenomena.

In a series of papers Bourgin (1979) discusses flow of a fluid of a differential type of arbitrary grade. At low values of the Deborah number, $\mathrm{De} = \tau/t_0$ where τ is the relaxation time of the fluid and t_0 is the time of observation, fluids of the differential type of grade 3 can be looked upon as a third grade approximation to a simple fluid in slow motion (Fosdick and Rajagopal, 1980). Three cases are defined by Bourgin: (i) when $\mathrm{De} = O((C/R)^2)$, any differential fluid behaves as a Newtonian fluid, (ii) for typical journal bearing applications $\mathrm{De} = O(C/R)$, thus it is not possible to distinguish between two Rivlin-Ericksen fluids of grade >3 in these cases, and (iii) for typical roller bearing applications $\mathrm{De} = O(1)$ and fluids of the differential type cannot be used. The stress relaxation effect on damper performance is studied by Tichy and Modest (1980) using the constitutive equation for a second grade fluid less the normal stress term and demonstrating strong effects on phase angle even at small values of De. Beris, Armstrong, and Brown (1983) look at slightly eccentric rotating cylinders; in fact their perturbation solution is in terms of the eccentricity. The extra "boundary condition," required for a second grade fluid, is the requirement for single valuedness of the solution in the multiply connected domain.

For differential fluids of complexity 1 Bourgin and Gay (1983) investigate the influence of non-Newtonian effects on load capacity in a finite journal bearing and in a Rayleigh bearing (Bourgin and Gay, 1985). Buckholtz looks at the performance of power law lubricants in a plane slider (Buckholtz, 1985) and in a short bearing (Buckholtz and Hwang, 1985).

Recently Verma, Sharman, and Ariel (1983) investigated flow of a second grade, thermodynamically compatible fluid between corotating porous, parallel disks.

For a thermodynamically compatible third grade fluid the Cauchy stress $\underset{\sim}{T}$ is related to the motion of the fluid as (Fosdick and Rajagopal, 1980).

$$\underset{\sim}{T} = -p\underset{\sim}{I} + \mu \underset{\sim}{A}_1 + \alpha_1 \underset{\sim}{A}_2 + \alpha_2 \underset{\sim}{A}_1^2 + \beta_3 (\mathrm{tr}\underset{\sim}{A}_1^2) \underset{\sim}{A}_1 \tag{54}$$

Here we assume that the fluid is thermodynamically compatible, i.e., it meets the Clausius-Duhem inequality and its specific Helmholtz free energy is a minimum when the fluid is at rest (Fosdick and Rajagopal, 1980). If we set the material constants $\alpha_1 = \alpha_2 = 0$ in (54), we obtain the fluid of complexity 1 (Bourgin, 1979) which can be brought into the form (52) as used by Ng and Saibel (1962) and by Buckholtz (1985), if $N = 3$ in the latter case.

Our aim presently is to substitute equation (54) into the equations of motion:

$$\mathrm{div}\underset{\sim}{T} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \tag{55}$$

and, after normalization, to apply the limit $(C/R) \to 0$. The resulting equation, too lengthy to repeat here (Kacou, Rajagopal, and Szeri, 1986), contains two parameters,

$$\epsilon \equiv \frac{\alpha_1 U}{\mu R} \text{ and } \eta \equiv \frac{\beta_3 U^2}{\mu R^2}$$

The parameter ϵ controls diffusion of vorticity from the boundary and $\eta(R/C)^2$ is proportional to the ratio of the non-Newtonian shear stress to the Newtonian shear stress.

To seek solutions for the equation of motion in a long journal bearing, we may expand the stream-function as

$$\psi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon^n \delta^m \psi_{nm}; \; u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x} \tag{56}$$

The first three perturbation equations are:

$$\frac{\partial^4 \psi_{00}}{\partial y^4} = 0 \tag{57a}$$

$$\frac{\partial^4 \psi_{10}}{\partial y^4} = \frac{\partial \psi_{00}}{\partial x} \frac{\partial^5 \psi_{00}}{\partial y^5} - \frac{\partial \psi_{00}}{\partial y} \frac{\partial^5 \psi_{00}}{\partial x \partial y^4} \tag{57b}$$

$$\frac{\partial^4 \psi_{01}}{\partial y^4} = -2 \frac{\partial^2 \psi_{00}}{\partial y^2} \left(\frac{\partial^3 \psi_{00}}{\partial y^3} \right)^2 - \frac{\partial^2 \psi_{00}}{\partial y^2} \frac{\partial^4 \psi_{00}}{\partial y^4} \tag{57c}$$

These equations are linear and can be solved for ψ_{00}, ψ_{10}, $\psi_{01}, \ldots,$. Proceeding now in the same manner as when

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deriving the classical Reynolds equations, we obtain, at each level of perturbation, a differential equation in P_{00}, P_{10}, P_{01}, . . . , where

$$P=\sum_{n=0}\sum_{m=0}\epsilon^n\delta^m P_{nm} \tag{58}$$

Here P_{00} is the long bearing pressure distribution of classical theory. The first order correction P_{10} is shown plotted in Fig. 7; this profile is very close to the one calculated by Harnoy and Philippoff (1976) for a second grade fluid, using results due to Giesekus and Tanner.

It is fairly clear that viscoelastic effects are not significant at small values of the Deborah number in steady flow. If, however, $\Omega^*>1$ anomalous resonance effects migh occur. There is virtually no information on viscoelastic effects in the flow regime $\Omega^*\sim \mathrm{Re}^*>1$. Here inertia effects might interact nonlinearly with viscoelastic effects, leading to changed bearing performance.

Among the many models which have been proposed in the literature to describe non-Newtonian fluid behavior, one that has gained considerable attention is the BKZ fluid model (Berstein, Kearsley, and Zapas, 1963). The model has been shown to be consistent with statistical modeling of polymeric fluids. It might, thus, be applicable when investigating process fluid lubrication. BKZ fluids have memory and are represented by integral constitutive equations. Because of their memory, BKZ fluids might show significant departure from Newtonian fluids in squeeze film applications. Little information is available presently on numerical simulation of fluids with finite memory flowing at high shear rates, but initial studies on special simple integral constitutive models have already provided very interesting information regarding the structure of the boundary layer. There is much work to be done here by both the rheologist and the numerical analyst.

Hydrodynamic Gas Film Lubrication

The qualitative difference in performance between liquids and gases, in general, vanishes as $\mathrm{M}\to 0$, where the Mach number, M, is the ratio of the fluid velocity to the local velocity of sound. This conclusion also holds for bearings, and at low speeds the behavior of gas film lubricated bearings is similar to that of liquid lubricated bearings. The similarity no longer holds at high surface speeds, however, the main additional phenomena being the compressibility of the lubricant.

Perhaps the earliest mention of air as a lubricant was made by Hirn in 1854. Kingsbury (1897) was the first to construct an air lubricated journal bearing. But the scientific theory of gas lubrication can be considered as an extension of the Reynolds lubrication theory. This extension was made soon after Reynolds' pioneering work: Harrison in 1913 published solutions for "long" slider and journal bearings lubricated with a gas. Nevertheless, the study of gas lubrication remain dormant until the late fifties, when the impetus for development in gas bearings came mainly from the precision instruments and the aerospace industries.

The equations of motion, when written for a compressible fluid are more complicated than their incompressible counterpart (2). Written in normalized form, they contain the following terms in addition to the terms already found in equation (2):

Equation (2*a*):

$$\left(\frac{C}{R}\right)^2\left[\frac{1}{3}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right] \tag{59a}$$

Equation (2*b*):

$$\left(\frac{C}{R}\right)^2\left[\frac{1}{3}\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right] \tag{59b}$$

Equation (2*c*):

$$\left(\frac{C}{R}\right)^2\left[\frac{1}{3}\frac{\partial}{\partial\bar{z}}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right] \tag{59c}$$

The normalized equation of mass conservation now assumes the form

$$\frac{\partial\bar{\rho}}{\partial\bar{t}}+\frac{\partial\bar{\rho}\,\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{\rho}\,\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{\rho}\,\bar{w}}{\partial\bar{z}}=0 \tag{60}$$

in place of (3), where $\bar{\rho}=\rho/\rho_0$ and ρ_0 is some normalizing density.

In writing the Navier-Stokes equations for a compressible Newtonian fluid, equation (2) and (29), we neglected the effect of dilation on pressure or, equivalently, assumed that the fluid pressure is given by the average of the normal stresses. The relevant formula is

$$p-\hat{p}=-\kappa\frac{1}{\rho}\frac{d\rho}{dt} \tag{61}$$

where $\hat{p}=tr(\underset{\sim}{T})/3$, $\underset{\sim}{T}$ is the stress tensor and κ is the bulk viscosity of the fluid. The Stokes assumption (Stokes, 1845) $\kappa=0$ holds only for monatomic gas and is not true in general (Truesdell, 1952) and $(1/\rho)d\rho/dt\neq 0$ in compressible flow. But usually only second order effects are involved in making the assumption $p-\hat{p}=0$ and in much of gas bearing literature the assumption $\kappa=0$ is made.

The density in the equations of motion and continuity is no longer constant and for a gas, density depends on both pressure and temperature, according to the equation of state, $\rho=\rho(p,T)$. But there is simplification as for most lubrication problems the perfect gas law is suitable, so that $\rho=p/RT$, where R is the gas content. The gas viscosity is, again, dependent on both temperature (Kestin and Pilarczyk, 1954) and pressure (cf. Gross, 1962).

The relationships $\rho=\rho(p,T)$, $\mu=\mu(p,T)$ introduce the temperature T as a dependent variable into the problem, in addition to the three velocity components and the pressure. This forces as to having to consider the energy equation for the gas.

The equation for the conservation of energy for a constant property gas is similar to (44) but in addition it contains the compression work term

$$p\,\nabla\cdot\mathbf{v}=\Lambda\,\bar{p}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right) \tag{62}$$

Here $\bar{\Lambda}=\mu\omega R^2/\rho C_p T\,C^2$, as defined earlier.

When taking into account the geometry of the film, the equation of energy takes the form

$$\bar{u}\frac{\partial\bar{t}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{t}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{t}}{\partial\bar{z}}=\frac{1}{\mathrm{Pe}}\frac{\partial^2\bar{t}}{\partial\bar{y}^2}$$
$$+\bar{\Lambda}\left\{\bar{\mu}\left[\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2+\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2\right]+\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)p\right\} \tag{63}$$

Elrod and Burgdorfer (1959) simplified equation (63) for a two-dimensional film of negligible internal energy, flowing between walls held at constant temperature and found that

$$\frac{T_{\max}}{T_0}-1=\mathrm{O}(\mathrm{M}^2)$$

where T_0 is the temperature of the walls.

Consequently, if the bearing Mach number is small, the temperature variation across the lubricant film thickness can be neglected. Accordingly, one commonly accepts the isothermal approximation when analyzing gas bearings.

Another way to estimate the appropriateness of the isothermal approximation is by comparing the transit time of the gas in the bearing to the time scale of conduction (Ausman, 1966).

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Let the temperature of gas, contained between parallel surfaces, exceed the temperature of the walls by ΔT. This temperature difference will, at the centerline, decrease to $\Delta T/3$ in a time $\tau_c = O(c_p \rho h^2/k)$, where c_p, ρ, and k are the specific heat, density and heat conductivity, respectively, for the gas. The transit time for a gas particle through the bearing is τ_t. The criteria for the validity of the isothermal assumption is $\tau_t > \tau_c$, or Pr Re* < O(10). For gas bearings Pr = O(1) and Re* $\ll$ 1; e.g., typically $U_* = O(10^3)$, $C = O(10^{-4})$, R = O(1) and ν = O(1) relative to the kg-cm-sec system, giving Re* = $O(10^{-5})$. Thus the condition $\tau_t > \tau_c$ is, in general, satisfied.

As in usual circumstances Re* = $O(C/R)$, when we take formally the limit $(C/R)^2 \to 0$ in the equations of motion, equaitons (2) and (59), while treating the viscosity as a constant, we again arrive at equation (4). The pressure remains constant across the film to first order, just as in the incompressible film. We can integrate the first and the last of equations (4) and obtain formally the velocity distribution in the film once the kinematic boundary conditions on the solid surfaces are established. The simplest velocity boundary condition is no-slip and this condition will be satisfied on the average provided that $K_n < 0.01$, where the Knudsen number K_n represents the ratio of the mean free path to the film thickness h. When $0.01 < K_n < 15$, slip flow becomes significant (Gross, 1962). In this discussion we assume that the continuum assumption holds.

Assuming no-slip at the walls, the velocity may now be substituted into the equation of continuity (60). Integration across the film yields the Reynolds equation in pressure for a gas lubricated bearing in steady flow:

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial z}\right)=6U_0\frac{\partial(\rho h)}{\partial x} \tag{64}$$

The only difference between this equation and the incompressible equation of Reynolds is that equation (64) accounts for variation of density. We note, however, that in spite of this similarity in appearance, equation (64) is nonlinear. In incompressible lubrication the Reynolds equation is linear and nonlinearity is manifested by the boundary conditions $p = dp/dx = 0$, $p \geq 0$. In the gas lubrication, on the other hand, the boundary conditions are linear, as there can be no cavitation, but the differential equation for pressure is nonlinear.

By expanding the velocity in a Taylor series about a wall position, Burgdorfer (1959) obtained a slip boundary condition to account for molecular effects. The slip flow correction for a gaseous lubricant amounts to substituting

$$\frac{h^3}{\mu}\left(1+6\frac{p_a\lambda_a}{ph}\right)$$

for h^3/μ in (64). Significant load reduction can result from slip flow (Burgdorfer, 1959). That equation (64) is valid for ultra low clearance gas bearings was only recently shown by Gans (1986).

For journal bearings the pressure in equation (64) is normalized with the ambient pressure, p_a. Defining the Compressibility number, Λ, by $\Lambda = 6\mu\omega(R/C)^2/p_a$, where p_a is the ambient pressure, the normalized form of (64) is

$$\frac{\partial}{\partial\theta}\left(H^3P\frac{\partial P}{\partial\theta}\right)+\frac{\partial}{\partial Z}\left(H^3P\frac{\partial P}{\partial Z}\right)=\Lambda\frac{\partial(PH)}{\partial\theta} \tag{65}$$

For a long bearing, $D/L \to 0$, equation (65) is easily integrated to yield

$$\frac{\partial P}{\partial\theta}=\frac{\Lambda}{H^3P}(PH-K) \tag{66}$$

where K is the integration constant. At very high speeds, observing on physical grounds that $\partial P/\partial\theta$ is bounded, equation (66) yields

$$PH=K=P_iH_i, \qquad \Lambda\to\infty \tag{67}$$

where the subscript refers to the inlet.

On the other hand, $\partial P/\partial\theta = O(\Lambda)$ as Λ) as $\Lambda \to 0$ and for the long bearing, equation (65) reduces to

$$\frac{\partial}{\partial\theta}\left(H^3\frac{\partial P}{\partial\theta}\right)-\Lambda H\frac{\partial P}{\partial\theta}=O(\Lambda^2), \qquad \Lambda\to 0 \tag{68}$$

From the last two equations we draw the important conclusions (Ausman, 1966) that

(1) At low speeds gas bearing behavior is approximated by incompressible bearing behavior
(2) At high speeds ph = const and the load capacity becomes independent of speed and dependent on the inlet (ambient) pressure
(3) At high speeds side leakage becomes negligible as both long bearing and finite bearing solutions approach the asymptote ph = const (Scheinberg, 1953).

Numerical solutions of equation (65) are well documented and are available in the literature (Raimondi, 1961; Elrod and Malanoski, 1960). These solutions are obtained under the boundary conditions that $P = 1$ at the edge of the bearing.

Although closed form solutions of (65) are not available, we are able to obtain analytical solutions in asymptotic cases. The pressure equation (64) may be linearized at small values of the eccentricity ϵ by assuming that

$$P=1+\epsilon P_\epsilon \tag{69}$$

Substituting (69) into (65) one obtains the first order perturbation equation

$$\frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}-\Lambda\right)P_\epsilon\right]+\frac{\partial^2P_\epsilon}{\partial Z^2}=\Lambda\sin\theta \tag{70}$$

This formulation, known as the "linearized P" solution (Ausman, 1959) yields a load that is linearly dependent on the eccentricity ratio. This, of course is a consequence of linearization and seems to be valid for $\epsilon < 0.3$ (Ausman, 1961). In an effort to increase the range of validity of the perturbation solution, in his second attempt Ausman (1961) employed the series representation

$$\psi=1+\epsilon\psi_\epsilon; \quad \psi\equiv PH \tag{71}$$

This latter formulation is known as the "linearized PH" solution.

For large Λ, equation (65) may be written as (Pan, 1980)

$$\frac{\partial\psi}{\partial\theta}=\frac{1}{\Lambda}\left[\frac{\partial}{\partial\theta}\left(H^3P\frac{\partial P}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(H^3P\frac{\partial P}{\partial z}\right)\right] \tag{72}$$

Neglecting the right hand side upon taking the limit $\Lambda \to \infty$ would lead to

$$\psi=\text{const}; \quad \Lambda\to\infty$$

and satisfaction of the boundary conditions, viz.,

$$\frac{\partial\psi}{\partial\theta}=\epsilon\sin\theta \quad \text{at} \quad Z=\pm\frac{L}{D}$$

would not be possible. Clearly the small parameter $1/\Lambda$ multiplies the highest derivative in (72) and one is faced with a singular perturbation problem (Nayfeh, 1973). To overcome the difficulty of obtaining meaningful linear differential equations for P at $1/\Lambda \to 0$, Pan (1980) uses matched asymptotic expansion. The key for matching the inner (boundary layer) and outer ($|Z| < L/D$) solutions is the mass content rule (Elrod and Burgdorfer, 1959).

$$\frac{1}{2\pi}\int_0^{2\pi}H^3P^2d\theta=1+\frac{3}{2}\epsilon^2 \tag{73}$$

which is obtained by integrating (65) across the film.

Gas bearings find a multitude of applications in industry.

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The density of information storage on magnetic tape or disk can be increased by reducing the spacing between the read-write head and the recording medium. This provides an excellent application for the theory of gas lubricated sliders. In machine tools gas lubrication allows attainment of high speeds with minimal heat generation. Precise linear and rotational indexing without oil contamination or excess vibration can be achieved when using gas bearings.

Summary

This is an attempt at presenting some of the more important extensions of the lubrication theory of Osborne Reynolds. I do not claim that the material is complete, nor that it is particularly up-to-date. In fact, I hope that the reader realizes the difficulties associated with an undertaking such as this and treats it with indulgence. Again, I must apologize for having had to leave out a number of significant developments, e.g., externally pressurized gas bearings, and for failing to mention some of the most important names in Tribology.

Nevertheless, I offer this work as a tribute to the genius of Osborne Reynolds, whom we are celebrating at the 1986 Tribology Conference.

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