The figure shows a pendulum-like arrangement for measuring the radius of gyration \((k)\) of a rotor (i.e. polar mass moment of inertia \(I_p=m k^2\)). The rotor is suspended from massless and inextensible wires. Each wire of length \((a)\) is fixed at radius \((b)\) from the rotor center of mass. From small amplitude motions \(\theta(t)\), the procedure requires recording the natural period of motion and then extracts the radius of gyration from a simple engineering formula.

a) **Clearly outlining** physical considerations and assumptions DERIVE the equation of motion for the arrangement shown, i.e. 
\[ I_p \ddot{\theta} + K \theta = 0, \]
where \(K = W (b^2/a)\) is a torsional “stiffness” depending on the wires’ length and disposition and the rotor weight.

Show a rigorous procedure, define variables, establish principles used, note assumptions, and highlight steps in modeling and analysis.

b) Measurements were conducted with a single stage compressor rotor. The rotor weighs 150 lbf, and \(a=100 \text{ inch}, \ b=10 \text{ inch}\). The rotor performed 10 oscillations in 30 seconds. Determine, the system natural frequency \([\text{rad/s}]\), the rotor polar mass moment of inertia \(I_p[\text{lbf-inch-s}^2]\), and the radius of gyration \(k [\text{inch}]\)

**NOTE:** Neatness, organization, explanations in complete sentences, and clarity of procedure are a must for full grade.
Assumptions:
no friction, small angular motions, rotations about cg, cables are massless and inextensible.

Let: \( \theta'' = \frac{d^2}{dt^2} \theta \)

For small angular motions \( \theta(t) \) about the center of mass, the equation of motion is

\[
I_p \cdot \theta'' = -2TH \cdot b \tag{2}
\]

where \( TH \) is the component of the cable tension in the horizontal plane

from the EOM for vertical motions of rotor cg:

\[
M \cdot y'' = 2 \cdot TV - W \tag{3}
\]

where \( TV \) is the component of the cable tension in the vertical plane

For small amplitude angular motions. From [1], \( \phi = \frac{b}{a} \cdot \theta \) [4a] and: \( y'' := 0 \)

**Constraint: arc length** \( s = \phi \cdot a = \theta \cdot b \)
Thus,
\[ T_v := \frac{W}{2} \quad \text{and} \quad T_H = T_v \cdot \frac{\sin \phi}{\cos \phi} \quad \text{with} \quad \frac{W \cdot \phi}{2} = \frac{W \cdot \theta}{a} \quad [4b] \]

Substitution of [4b] into [2] renders the desired EOM:
\[ I_p \cdot \dddot{\theta} + k_\theta \cdot \dot{\theta} = 0 \quad [6a] \quad \text{where} \quad k_\theta = \frac{W \cdot b^2}{a} \]

since:
\[ W := m \cdot g \quad \text{and} \quad I_p = m \cdot r_k^2 \]

Then [6a] reduces to:
\[ \dddot{\theta} + \frac{b^2}{a \cdot r_k^2} \cdot \dot{\theta} = 0 \quad [6b] \]

The natural frequency of the oscillatory system is defined as:
\[ \omega_n = \left( \frac{k_\theta}{I_p} \right)^{\frac{1}{2}} \quad [7] \]

and the natural period of motion is:
\[ T_n = \frac{2 \cdot \pi}{\omega_n} \]

(b) Engineering calculations

\[ T_n := \frac{30}{10} \cdot s \quad \omega_n := \frac{2 \cdot \pi}{T_n} \quad \omega_n = 2.094 \ \text{rad/s} \]

\[ a := 100 \cdot \text{in} \quad W := 150 \cdot \text{lb} \]

\[ b := 10 \cdot \text{in} \quad k_\theta := \frac{W \cdot b^2}{a} \quad k_\theta = 150 \text{lb-in} \quad \text{stiffness} \]

from [7], the polar moment of inertia is
\[ I_p := \frac{k_\theta}{\omega_n^2} \quad I_p = 34.196 \text{lb-in-s}^2 \]

and the radius of gyration is
\[ r_k := \left( \frac{I_p \cdot g}{W} \right)^{\frac{1}{2}} \quad r_k = 9.382 \text{in} \]
The equation of motion can be easily derived using conservation of mechanical energy:

\[
T = \frac{1}{2} I_p \left( \frac{d}{dt} \theta \right)^2 \quad \text{kinetic energy}
\]

\[
V = m \cdot g \cdot h \quad \text{change in potential energy}
\]

\[
h = a \cdot (1 - \cos(\phi)) = a \cdot \frac{\phi^2}{2}
\]

for small angles, and \( \phi = \frac{b \cdot \theta}{a} \)

Thus,

\[
V = m \cdot g \cdot \frac{b^2}{a} \cdot \frac{\theta^2}{2} = \frac{1}{2} k_0 \cdot \theta^2
\]

where \( k_0 = \frac{W}{a} \cdot \frac{b^2}{a} \)

Hence from \( \frac{d}{dt} (T + V) = 0 \),

\[
I_p \cdot \frac{d^2 \theta}{dt^2} + k_0 \cdot \theta = 0
\]