The figure shows the dynamic free response (amplitude [ft] versus time [sec]) of a simple mechanical structure. Static load measurements determined the structure stiffness $K=1000$ lbf/in. From the measurements, determine

- **a)** damped period of motion $T_d$ (sec)
- **b)** damped natural frequency $\omega_d$ [rad/s],
- **c)** Using the concept of log-dec ($\delta$), if applicable, determine the system damping ratio $\xi$. Explain your method
- **d)** Undamped natural frequency $\omega_u$ [rad/s],
- **e)** Estimate the system equivalent mass, $M_e$ [lb]
- **f)** Estimate the system damping coefficient, $C_e$ [lb s/in]

### Identification of parameters

**DISPLACEMENT (ft) vs time (sec)**

(a) Determine damped period of motion: $T_d := \frac{0.6 \cdot \text{sec}}{4}$ from 4 periods of damped motion

(b) Determine damped natural frequency: $\omega_d := \frac{2 \cdot \pi}{T_d}$

(c) Determine damping ratio from log-dec:

Select two amplitudes of motion (well spaced) and count number of periods in between

$X_o := 0.23 \cdot \text{ft}$ after $n := 5$ periods $X_n := 0.05 \cdot \text{ft}$

**Log-dec** is derived from ratio: $\delta := \frac{1}{n} \cdot \ln \left( \frac{X_o}{X_n} \right)$

$\delta = 0.305$

from log-dec formula $\delta = \frac{2 \cdot \pi \cdot \xi}{\left( 1 - \xi^2 \right)^{0.5}}$ \[\xi := \frac{\delta}{\left( 4 \cdot \pi^2 + \delta^2 \right)^{0.5}}\]

$\xi = 0.049$

$$
\xi = 0.049
$$

**T_d = 0.15 \text{ sec}$$

**$\omega_d = 41.888 \text{ rad/sec}$**
Note that approximate formula: \( \frac{\delta}{2 \cdot \pi} = 0.049 \) is a very good estimation of damping ratio.

(d) Determine damped natural frequency:

\[
\omega_n := \frac{\omega_d}{\left(1 - \xi^2\right)^{0.5}}
\]

\( \omega_n = 41.937 \text{ rad/sec} \)

a little higher than the damped frequency (recall damping ratio is small)

(e) Determine system mass:

Static tests conducted on the structure show its stiffness to be \( K = 1 \times 10^3 \text{ lbf/in} \)

and from the equation for natural frequency, the equivalent system mass is

\[
M_e := \frac{K}{\omega_n^2}
\]

\( M_e = 219.526 \text{ lb} \)

(f) Determine system damping coefficient:

\[
C_e := \xi \cdot 2 \cdot (K \cdot M_e)^{0.5}
\]

\( C_e = 2.314 \text{ lbf \cdot sec/in} \)

**Note:**

**Actual values of parameters are**

\( M = 220 \text{ lb} \quad C = 2.4 \text{ lbf \cdot sec/in} \quad \zeta = 0.05 \)
The car of mass $M=500$ kg is traveling at constant speed $V_0=50$ kilometer/hour when it hits a rigid wall. A spring ($K$) and a viscous dashpot ($C$) represent the car front bumper system. The system natural frequency $f_n=3$ Hz, and the dashpot provides critical damping. Disregard friction on the car wheels and ground.

a) Derive the EOM for the car after the collision using the coordinate $Y(t)$ that has its origin at the car location when the bumper first touches the wall. Provide initial conditions in speed and displacement. [10]

b) State the solution to the EOM, i.e. give the system response $Y(t)$ as a function of the system parameters [natural frequency, damping ratio, etc] and initial conditions. [10]

c) Find the maximum bumper deflection and the time, after collision, when this event occurs. Compare the calculated deflection with the maximum deflection for a system without damping. Comment on your findings. [5]

d) Sketch the motion $Y(t)$ vs. time (to 0.5 sec) labeling the axes with physical units and noting important parameters. Will the car bounce from the wall? Explain your answer. [5]

---

**a) EQN of motion for car after collision, use Y coordinate: (Y=0, no bumper deflection)**

$$M \frac{d^2 Y}{dt^2} = -F_{\text{damper}} - F_{\text{spring}}$$

Thus the EOM is:

$$M \frac{d^2 Y}{dt^2} + C \frac{d Y}{dt} + K Y = 0 \quad (1)$$

with I.C's at $t=0$

$$Y(0) = Y_0 = 0 \text{ m} \quad \text{no bumper deflection, and}$$

$$\frac{d Y}{dt} = V_0 \quad V_0 := 50 \text{ kph} = \frac{13.889}{3600} \text{ m/s} \quad \text{car speed at instant of collision with wall.}$$

Known: natural frequency and mass of car

$$f_n := 3 \text{ Hz} \quad \omega_n := f_n \cdot 2 \pi \quad \text{natural period of motion}$$

$$M := 500 \text{ kg} \quad K := M \cdot \omega_n^2$$

$$K = 1.777 \times 10^5 \text{ N/m} \quad T_n := \frac{1}{f_n}$$

**Transient response of critically damped system, step force $F_0=0$**

$$C \frac{d Y}{dt} + K Y = F_0$$

**at** $t=0$

$$Y_0 := 0 \text{ m} \quad V_0 := \frac{13.889}{3600} \text{ m/s} \quad \text{and} \quad F_0 := 0 \text{ N} \quad Y_{ss} := \frac{F_0}{K}$$

From cheat sheet, soln is:

$$Y(t) = e^{ss \cdot t} \left( A_1 + t \cdot A_2 \right) + Y_{ss} \quad (2)$$

where

$$ss := -\omega_n \quad \text{root of characteristic eqn.}$$

$$A_1 := Y_0 - Y_{ss}$$

$$A_2 := V_0 - A_1 \cdot ss$$

The dynamic response of the car $Y(t)$ is given by:

$$Y(t) := e^{ss \cdot t} \cdot A_2$$

The graph below displays the car motion for a time equal to 3 periods of natural motion (undamped). Note the overshoot and largely damped response w/o oscillations. The car does not bounce from the wall.

**without damping, max bumper deflection is**

Based on PCME

$$\frac{1}{2} M \cdot V_0^2 = \frac{1}{2} K \cdot X_{\text{max}}^2$$

$$X_{\text{max}} := \frac{V_0}{\omega_n}$$

$$X_{\text{max}} = 0.737 \text{ m}$$
A significant reduction in deflection.

Note how quickly the dashpot dissipates the initial kinetic energy.

Comparison not for exam for completeness, compare the response with that of a lightly UNDERDAMPED system

\[ \zeta = 0.05 \]

\[ \omega_d := \omega_n \sqrt{1 - \zeta^2} \]

\[ x(t) := \frac{V_0}{\omega_d} e^{-\zeta \omega_n t} \cdot \sin(\omega_d t) \]

Bumper velocity is null at time \( t_a \)

\[ t_a := \frac{-1}{ss} \]

\[ t_a = 0.053 \text{ s} \]

Maximum deflection:

\[ X_{\text{max}} = 0.737 \text{ m} \]

\[ \frac{X_{\text{max}}}{Y(t_a)} = 2.718 \]

A significant reduction in deflection. Note how quickly the dashpot dissipates the initial kinetic energy.

Note that model with little damping predicts the car will bounce from wall.

Graph not for exam

Comparison not for exam for completeness, compare the response with that of a lightly UNDERDAMPED system

\[ f_n = 3 \text{ Hz} \]

\[ T_n = 0.333 \text{ s} \]

**Calculation of maximum bumper deflection**

It occurs when velocity is 0 m/s

(Take time derivative of soln, Eq. (3), to obtain)

\[ V(t) := A(1 + ss \cdot t) \cdot e^{ss \cdot t} \]

Note how large is the force in the bumper. No wonder why a car crash is always a disastrous event!

"Felt" by car

\[ F_w(t) := K \cdot Y(t) + C \cdot V(t) \]

\[ \text{Kelas\_max} := K \cdot Y(t_a) \]

\[ \text{Kelas\_no\_damping} := K \cdot X_{\text{max}} \]

\[ F_{\text{elas\_max}} = 4.816 \times 10^4 \text{ N} \]

\[ F_{\text{elas\_no\_damping}} = 1.309 \times 10^5 \text{ N} \]

\[ K = 1.777 \times 10^5 \frac{\text{N}}{\text{m}} \]
The sketch shows a test stand for automobile bumpers. Each of the two shock absorbers consists of a spring having a stiffness $K=3000\ \text{N/m}$ that acts concentrically with a dashpot ($C$). The bumper mass ($M$) is 40 kg. The system is at rest in the static equilibrium position when a force $F$ having an impulse of 2000 N-s acting over a **short interval** is applied at the centerline of the bumper.

(a) determine the value $C$ that will bring the bumper to rest in the shortest possible time without rebound

(b) If $C=1500\ \text{Ns/m}$ determine the displacement $x(t)$ of the bumper.

(c) Find the bumper’s maximum displacement from the equilibrium position and its time of occurrence. Is the result reasonable? Explain.

Assume:
bumper is rigid - undeformable
X=0 denotes SEP.

For motions from the static equilibrium position, the equation of motion is:

$$M \frac{d^2 X}{dt^2} + 2C \frac{d X}{dt} + 2KX = F$$

where

- $M := 40\ \text{kg}$ bumper mass, regarded as rigid - non deformable
- $K := 3000\ \text{N/m}$ absorber stiffness and damping coefficients
- $C$ (damping coefficient to be determined)

The natural frequency equals

$$\omega_n := \left( \frac{2 \cdot K}{M} \right)^{\frac{1}{2}}$$

$$\omega_n = 12.247\ \text{rad/s}$$

$$f_n := \frac{\omega_n}{2 \cdot \pi}$$

$$f_n = 1.949\ \text{Hz}$$

(a) The system must be critically damped to bring the damper to rest in the shortest possible time. Thus, the amount of physical damping $C$ must equal (for each shock absorber)

$$C_{\text{crit}} := 2 \cdot (2 \cdot K \cdot M)^{\frac{1}{2}}$$

$$C_{\text{crit}} = 979.796\ \text{N} \cdot \text{s/m}$$

$$C := \frac{1}{2} C_{\text{crit}}$$

$$C = 489.898\ \text{N} \cdot \text{s/m}$$

Amount of damping on each absorber

(b) If $C := 1500\ \text{Ns/m}$, the damping ratio is

$$\xi := \frac{2 \cdot C}{C_{\text{crit}}}$$

$$\xi = 3.062$$

i.e. an overdamped system
The solution of the EOM, with RHS = 0, is:

\[
x(t) = e^{s_1 t} + e^{s_2 t}
\]
\[
\frac{dx}{dt} = A \cdot e^{s_1 t} + B \cdot e^{s_2 t}
\]

The roots of the characteristic equation are

\[
s_1 := -\xi \cdot \omega_n + \omega_n \left(\xi^2 - 1\right)^{\frac{1}{2}}
\]
\[
s_2 := -\xi \cdot \omega_n - \omega_n \left(\xi^2 - 1\right)^{\frac{1}{2}}
\]

satisfying the initial conditions. At t=0 s

\[
x(0) = 0 = A + B \quad \text{Thus,} \quad B = -A \quad \text{no initial displacement}
\]

The impulse produces an initial velocity equal to

\[
v(0) = v_0 = \frac{\text{Imp}}{M} = A \cdot (s_1 - s_2)
\]

as explained in class

\[
v_0 := \frac{\text{Imp}}{M} \quad v_0 = 50 \frac{\text{m}}{\text{s}}
\]

initial velocity is quite large!

(180 km/hour!)

\[
A := \frac{v_0}{(s_1 - s_2)} \quad A = 0.705 \text{ m}
\]

for graphing

\[
T_n := \frac{1}{f_n}
\]
\[
T_{\text{max}} := 3 \cdot T_n
\]

Note large overshoot (max deflection) and return to equilibrium with no oscillations.

\[
\text{VELOCITY}
\]
\[
v(t) := A \cdot e^{s_1 t} - e^{s_2 t}
\]
\[
v(0 \cdot s) = 50 \frac{\text{m}}{\text{s}}
\]

\[
\text{DISPLACEMENT}
\]
\[
x(t) := A \cdot \left(e^{s_1 t} - e^{s_2 t}\right)
\]
Max bumper displacement (deflection) occurs when velocity equals 0, i.e
\[ v = 0 = A \left( e^{s_1 t_M} - s_1 e^{s_2 t_M} \right) \]
\[ \frac{s_1 t_M}{s_2} = \frac{s_2}{s_1} \]

Take natural log of both sides to find
\[ (s_1 - s_2) t_M = \ln \left( \frac{s_2}{s_1} \right) \]
\[ t_M := \frac{\ln \left( \frac{s_2}{s_1} \right)}{s_1 - s_2} \]
\[ t_M = 0.05 \text{ s} \]

And the maximum bumper deflection is
\[ x(t_M) = 0.618 \text{ m} \]

This is a large deflection! probably bumper will deform plastically before this occurs.
Example Problem: Seismic instrument

The figure shows a seismic instrument with its case rigidly attached to a vibrating surface. The instrument displays the motion $Y(t)$ relative to the case motion $Z(t)$.

- Derive the equation of motion for the instrument using the relative motion $Y(t)$ as the output variable. Show all assumptions and modeling steps for full credit.
- Determine the instrument natural frequency $\omega_n$ [rad/s] and critical damping $D_c$ [lb.s/in] for $M=0.02$ lb and $K=13$ lb/in.
- When will the instrument show $Y=0$, i.e. no motion?
- If the vibrating surface moves with a constant acceleration of 3 g, what will the instrument display at s-s? Give value in inches.

Note: the sensor works under all attachment configurations, i.e. vertical, horizontal, top, bottom, etc.

a) EQN of motion using $Y$ (relative) coordinate:

The relationship between the absolute motion $X(t)$ of the sensor mass and the case motion is

$X(t) = Z(t) + Y(t)$  \hspace{1cm} (1)

PCLM (Newton's Law) refers to inertial (absolute) references:

$M \frac{d^2}{dt^2} X = -F_{\text{damper}} - F_{\text{spring}}$  \hspace{1cm} (2)

with

$F_{\text{damper}} = D \frac{d}{dt} Y$ \hspace{1cm} $F_{\text{spring}} = K \cdot Y$ \hspace{1cm} (3)

Substitution of (3) and (1) into (2) gives the EOM as:

$M \frac{d^2}{dt^2} Y + D \frac{d}{dt} Y + K \cdot Y = -M \frac{d^2}{dt^2} Z$  \hspace{1cm} (4)

b) Determine system parameters:

$W := 0.02 \cdot \text{lbf}$ \hspace{1cm} $g = 386.089 \frac{\text{in}}{s^2}$

$M := \frac{W}{g}$ \hspace{1cm} $K := 13 \frac{\text{lbf}}{\text{in}}$ \hspace{1cm} $D := \frac{\text{lbf} \cdot \text{s}}{\text{in}}$

Damping coeffic.not specified

natural frequency $\omega_n := \left( \frac{K}{M} \right)^{\frac{5}{2}}$

Critical damping $D_c := 2 \left( K \cdot M \right)^{\frac{5}{2}}$

$\omega_n = 500.957 \frac{\text{rad}}{s}$

$D_c = 0.052 \frac{\text{lbf} \cdot \text{s}}{\text{in}}$
Damping ratio.

\[ \zeta := \frac{D}{2(K \cdot M)^{\frac{5}{2}}} \]

c) Instrument will record NO motion \( Y=0 \), if \( Z(t)=\text{cte} \) or \( dZ/dt=\text{cte} \), i.e. the "vibrating surface" remains stationary or moves at constant speed.

d) If the vibrating surface moves with a constant acceleration equal to 3 g's. Then, the instrument will display, at steady state (after transients have disappeared due to damping):

\[ Y_s = -3 \cdot g \frac{M}{K} \]

Let's calculate the instrument dynamic response:

For example, let \( \zeta = 0.10 \)

Then the sensor response for a sudden (3g) acceleration becomes with zero initial conditions:

\[ Y_0 := 0 \cdot m \quad V_0 := 0 \cdot \frac{m}{s} \quad \omega_d := \omega_n \left(1 - \zeta^2\right)^{\frac{5}{2}} \]

\[ C_1 := (Y_0 - Y_s) \quad C_2 := \left[ \frac{V_0 + \zeta \cdot \omega_n (Y_0 - Y_s)}{\omega_d} \right] \]

\[ Y(t) := \left( e^{-\zeta \cdot \omega_n \cdot t} \right) \cdot \left( C_1 \cdot \cos(\omega_d \cdot t) + C_2 \cdot \sin(\omega_d \cdot t) \right) + Y_s \]

\[ T_d := \left( \frac{2\pi}{\omega_d} \right) \]

\[ Y_s = -0.117 \text{ mm} \]

Response of seismic instrument to sudden acceleration

SEE Video_seismic_instrument
The spring-mass-damper system represents a package cushioning an electronic component. The package rests on a hard floor. A pulse force \( F(t) \) of very, very short duration is exerted on the package as shown. The component vibrates without rebounding. Let \( K=60 \) lb/in, \( M=5.9 \) lb, and \( C=0.04 \) lb-sec/in,

a) Calculate the system natural frequency (Hz) and damping ratio [8].
b) Provide an engineering estimation (value) for the maximum system velocity (ft/sec).
c) The maximum deflection (ft) of the spring (displacement of system) and the time when it occurs.
d) Draw a graph of the system displacement \( x \) vs. time. Label all physical coordinates.

**KEY:** KNOWLEDGE base: DOES APPLIED FORCE ACT OVER A LONG TIME OR NOT? IS time \( T_o \) very long?

**Pulse:** force magnitude \( F := 2000 \text{ lbf} \) applied during \( T_o := 0.001 \text{ sec} \)

(a) **Find the natural frequency, natural period of motion and viscous damping ratio:**

\[ K := 60 \text{ lb/in} \quad M := 5.9 \text{ lb} \quad C := 0.04 \text{ lb-sec/in} \]

**NATURAL FREQUENCY:**

\[ \omega_n := \sqrt{\frac{K}{M}} \quad f_n := \frac{\omega_n}{2\pi} \quad \omega_n = 62.66 \text{ rad/sec} \quad f_n = 9.973 \text{ Hz} \]

**DAMPING RATIO:**

\[ \zeta := \frac{C}{2M\omega_n} \quad \zeta = 0.021 \quad \text{a small value} \]

**DAMPED NATURAL FREQUENCY:**

\[ \omega_d := \omega_n \left(1 - \zeta^2\right)^{\frac{1}{2}} \quad \omega_d = 62.647 \text{ rad/sec} \]

\[ T_d := \frac{2\pi}{\omega_d} \quad T_d = 0.1 \text{ sec} \]

The damping ratio is very small, thus the system is very **lightly damped.** \( \frac{T_n}{T_o} = 100.274 \)

(b) **engineering judgment:** The (damped) natural period of motion is very large compared to the time the load acts. Hence, the pulse can be treated as an **impulse,** which changes "instantaneously" the **initial velocity** of the system. This initial velocity equals

\[ \text{Impulse} := F \cdot T_o \quad \text{Impulse} := \frac{F \cdot T_o}{M} \]

\[ V_o := \frac{\text{Impulse}}{M} \quad (1) \]

\[ V_o = 130.877 \text{ in/sec} \]
This **initial velocity is (of course) the maximum** ever, since damping will remove the initial kinetic energy due to the impact until the system returns to rest.

The motion starts from rest, \( X(0)=0 \). Thus, the response of the system becomes (cheat sheet):

\[
X(t) := \frac{V_0}{\omega_d} \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_d \cdot t) \tag{2}
\]

However, since damping is so small \( \zeta \approx 0 \), the **engineering approximation** leads to:

\[
X_a(t) := \frac{V_0}{\omega_n} \cdot \sin(\omega_n \cdot t) \quad \text{with velocity} \quad V_a(t) := V_0 \cdot \cos(\omega_n \cdot t) \tag{3}
\]

The **maximum displacement** is:

\[
X_{\text{max}} := \frac{V_0}{\omega_n}
\]

which occurs at a time \( \sim 1/4 \) natural period of motion, i.e

Let's graph the responses (displacement and velocity):

Note how close the engineering approximation is with respect to the exact solution, in particular during the first period of motion.

\[
X_{\text{max}} = 2.089 \text{ in}
\]

\[
\frac{T_n}{4} = 0.025 \text{ sec}
\]

\[
\frac{1}{2} \cdot M \cdot V_0^2 = \frac{1}{2} \cdot K \cdot X_{\text{max}}^2
\]

\[
V_0 = 10.906 \text{ ft/sec}
\]
Maximum displacement happens when speed = 0, i.e.
\[
\frac{d}{dt} \left( \frac{V_0}{\omega_d} \cdot e^{-\zeta \omega_n t} \cdot \sin(\omega_d t) \right) = 0
\]
Take time derivative of \( X(t) \) and obtain time \( t_\_ \)
\[
-\zeta \omega_n \sin(\omega_d t_\_) + \omega_d \cos(\omega_d t_\_) = 0
\]
\[
\tan(\omega_d t_\_) = \frac{\omega_d}{\omega_n \zeta} = \frac{\left(1 - \zeta^2\right)^{0.5}}{\zeta}
\]
Let:
\[
\theta_\_ := \arctan \left( \frac{\left(1 - \zeta^2\right)^{0.5}}{\zeta} \right)
\]
\[
\frac{\theta_\_ \cdot 180}{\pi} = 88.803
\]
and time for maximum displacement is
\[
t_\_ := \frac{\theta_\_}{\omega_d}
\]
\[
\frac{t_\_}{T_n} = 0.247 \quad t_\_ = 0.025 \text{ sec}
\]
and the maximum deflection is
\[
X(t_\_) = 2.022 \text{ in}
\]
\[
X_{\text{max}} = 2.089 \text{ in}
\]
\[
\frac{X(t_\_)}{X_{\text{max}}} = 0.968
\]
compare it to engineering approx:
K excellent agreement!
Q3: Description of motion in a moving reference frame

An instrument package installed in the nose of a rocket is cushioned against vibration with a soft spring-damper. The rocket, fired vertically from rest, has a constant acceleration \( a_0 \). The instrument mass is \( M \), and the support stiffness is \( K \) with damping \( C \). The instrument-support system is underdamped. The relative motion of the instrument with respect to the rocket is of importance.

a) Derive the equation of relative motion for the instrument
b) Give or find an analytical expression for the relative displacement of the instrument vs. time. Express your answer with well defined parameters and variables.

c) Given \( M=1 \) kg, \( K=1 \) N/mm, and damping ratio \( \zeta=0.10 \). Find the natural frequency & damping coefficient of the system
d) For \( a_0=3g \), find the steady-state displacement of the instrument, relative to rocket and absolute.

Definitions: coordinate systems

X(t)  Absolute displacement of instrument recorded from ground

Z(t)  Absolute displacement of rocket from ground.

\[ Y = X - Z \]  displacement of instrument relative to rocket

\[ a_Z = \frac{d^2}{dt^2} Z = a_0 \]  acceleration of rocket fired from REST

sensor parameters:  \( M := 1 \cdot \text{kg} \)  \( K := 1000 \cdot \frac{\text{N}}{\text{m}} \)

\( a_0 := 3 \cdot \text{g} \)  \( \zeta := 0.10 \)

Equation of motion for instrument

from free body diagram, let \( Y=(X-Z)>0 \)

\[ M \cdot \frac{d^2}{dt^2} X = -W - F_s - F_d \quad (1) \]

where

\[ F_s = -W + K \cdot (X - Z) = -W + K \cdot Y \quad (2a) \]

is the spring force supporting instrument. The dashpot force is

\[ F_d = C \cdot \frac{d}{dt} (X - Z) \quad (2b) \]

Substitute Eqs. (2) into Eq.(1):
\[
M \frac{d^2}{dt^2} X = -W + W - K \cdot Y - C \frac{d}{dt} Y
\]

But interest is in the relative motion \( Y \);

\[
M \frac{d^2}{dt^2} (Y + Z) + K \cdot Y + C \frac{d}{dt} Y = 0
\]

hence, substitute \( X = Y + Z \)

\[
M \left( \frac{d^2}{dt^2} Y + C \frac{d}{dt} Y \right) + K \cdot Y = -M \cdot a_Z = -M \cdot a_o
\]

(3) is the desired EOM.

**Find natural frequency and damping coefficient**

natural frequency of sensor is:

\[
\omega_n := \left( \frac{K}{M} \right)^{0.5}
\]

\[
\omega_n = 31.623 \text{ rad/s}
\]

Natural period:

\[
T_n := \frac{2 \cdot \pi}{\omega_n}
\]

\[
T_n = 0.199 \text{ s}
\]

damping coefficient.

\[
\zeta := 2 \cdot (K \cdot M)^{0.5}
\]

\[
C = 6.325 \text{ N.s/m}
\]

damped natural frequency

\[
\omega_d := \omega_n \cdot (1 - \zeta^2)^{0.5}
\]

\[
\omega_d = 31.464 \text{ rad/s}
\]

**Solution of ODE - prediction of relative motion**

The solution of ODE Eq. (3) with null initial conditions since motion starts from rest is (Use cheat sheet)

\[
Y = Y_S + e^{-\zeta \cdot \omega_n \cdot t} \cdot (C_1 \cdot \cos(\omega_d\cdot t) + C_2 \cdot \sin(\omega_d\cdot t))
\]

\[
C_1 := Y_o - Y_S \quad \quad C_2 := \frac{(V_o + \zeta \cdot \omega_n \cdot C_1)}{\omega_d}
\]

\[
C_1 = 0.029 \text{ m} \quad \quad C_2 = 2.957 \times 10^{-3} \text{ m}
\]

and, after long time \( Y \) approaches:

\[
Y_S = -0.029 \text{ m}
\]

To find the instrument absolute displacement, first determine the absolute motion of the rocket, i.e.

\[
V_Z(t) := a_o \cdot t \quad \quad Z(t) := a_o \cdot \frac{t^2}{2}
\]

The **absolute displacement of the sensor** is \( X = Y + Z \)

after very-long times, times removes the homogenous (transient) response; and the sensor reaches its steady state motion

\[
X_{SS}(t) := Z(t) + Y_S
\]
not for Quiz  Lets graph the relative and absolute displacements of the sensor

\[
Y(t) := Y_s + e^{-\zeta \omega_n t} \cdot (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))
\]

without damping

\[
Y_\text{t}(t) := Y_s \cdot (1 - \cos(\omega_n t))
\]

for plots, set \( T_{\text{max}} := 7 \cdot T_n \)

Relative displacement of sensor w/r to rocket

\[
\begin{align*}
Y &= Y_s \\
Y_{\text{rel}} &= Y_t - Y_s \quad \text{(Underdamped)} \\
Y_{\text{und}} &= Y_\text{t} - Y_s \quad \text{(UNDAMPED)}
\end{align*}
\]

Displacements - Sensor (X) and Rocket (Z)

\[
X = Y + Z
\]

\[
kmh := \frac{1}{3.6} \cdot \frac{\text{m}}{\text{s}}
\]
A device designed to determine the moment of inertia of a **wheel-tire assembly** consists of a 2 mm diameter steel suspension wire, 2 m long, and a mounting plate, to which it is attached the wheel-tire assembly. The suspension wire is fixed at its upper end and hangs vertically. When the system oscillates as a **torsional pendulum**, the period of oscillation without the wheel tire assembly is 4 seconds. With the wheel tire assembly mounted to the support plate, the period of oscillation is 25 seconds. Determine the mass moment of inertia of the wheel tire assembly. Recall that the shear modulus for steel is \( G = 82.7 \cdot 10^9 \text{ N/m}^2 \) and the torsional stiffness of the cable is \( K_\theta = \frac{\pi d^4}{32}G/L \). 

Consider the system does not have any damping - and there is no external moment acting.

The general EOM for rotations \( \theta(t) \) about a fixed axis (o-o) is

\[
I_0 \cdot \frac{d^2 \theta}{dt^2} + k_\theta \cdot \theta = 0 \quad (1)
\]

where \( I_0 \) is a mass moment of inertia and \( K_\theta \) is a torsional stiffness.

The natural frequency of the system is

\[
\omega_n = \left( \frac{K_\theta}{I_0} \right)^{1/2} = \left( \frac{2 \cdot \pi}{2 \cdot T_n} \right) \quad (2)
\]

where \( T_n \) is the natural period of motion.

Let:

\[
G := 82.7 \cdot 10^9 \frac{\text{N}}{\text{m}^2} \quad \text{Shear modulus for steel}
\]

\[
L := 2 \cdot \text{m} \quad \text{cable length}
\]

\[
d := 2 \cdot \text{mm} \quad \text{cable diameter}
\]
\[ K_0 := \frac{G}{L} \left( \frac{\pi \cdot d^4}{32} \right) \]

\[ K_0 = 0.065 \text{ N} \cdot \text{m rad} \]

From (2)

\[ I_o = \left( \frac{T_n}{2 \cdot \pi} \right)^2 \cdot K_0 \]

**first case:** period of oscillation of assembly alone (w/o tire)

Find moment of inertia of support plate

\[ I_{\text{plate}} := \left( \frac{T_{n1}}{2 \cdot \pi} \right)^2 \cdot K_0 \]

\[ I_{\text{plate}} = 0.026 \text{ kg m}^2 \]

**second case:** period of oscillation of assembly with wheel included

Find moment of inertia of system (plate+wheel)

\[ I_{\text{system}} := \left( \frac{T_{n2}}{2 \cdot \pi} \right)^2 \cdot K_0 \]

\[ I_{\text{system}} = 1.028 \text{ kg m}^2 \]

**Solve: find mass moment of inertia of wheel-tire**

\[ I_{\text{wheel\_tire}} := I_{\text{system}} - I_{\text{plate}} \]

\[ I_{\text{wheel\_tire}} = 1.002 \text{ kg m}^2 \]
Application example

A two-blade composite-aluminum of a wind turbine is supported on a shaft & bearing system so that it is free to rotate about its centroidal axis. A weight $W = 100 \, \text{kg}$ is taped to one of the blades at a distance $R = 20 \, \text{m}$ from the axis of rotation as shown. When a blade is pushed a small angle from the vertical position and released, the wind turbine is found to oscillate 2 periods of natural motion in 1 minute. Assume there is no friction at the support bearings.

* Determine the wind turbine centroidal mass moment of inertia ($I_T$) in (kg.m$^2$).
* The turbine weighs 2500 kg. Determine the radius of gyration.
Make a free body diagram of turbine swinging (oscillating) about pivot O with angle \( \theta(t) \)

![FREE BODY DIAGRAM](image)

**Assume:**
- no drag (frictionless bearings & no air drag)
- center of mass of turbine = center of rotation O

Let \( I_T \): mass moment of inertia of turbine; hence EOM (moments) about O is

\[
(I_T + M R^2) \ddot{\theta} = I_O \ddot{\theta} = -W R \sin \theta
\]

For small \( \theta \) angles, Eq. (1) reduces to the linear form

\[
I_O \ddot{\theta} + M g R \theta = 0
\]

i.e. the typical equation of a pendulum with natural period given as

\[
T_n = \frac{2 \pi}{\omega_n} = 2 \pi \left( \frac{I_o}{M g R} \right)^{1/2}
\]

Hence, the wind turbine mass moment of inertia can be easily determined from the measured period of oscillation

\[
I_T = - M R^2 + \left( \frac{T_n}{2 \pi} \right)^2 M g R
\]

Given

\[WT\]

Wind turbine weight

\[W\]

Added weight

\[Rx, Ry\]

Reaction forces

\[\theta\]

Angle
$W = 100 \text{ kgf}$ taped to one of the blades at a distance $R = 20 \text{ m}$ from the axis of rotation. Natural period $T_n = 60 \text{ seconds} / 2 \text{ periods of natural motion} = 30 \text{ sec}$. Note $M = W / g = 100 \text{ kg}$

$\begin{align*}
I_T &= -MR^2 + \left(\frac{T_n}{2\pi}\right)^2 M g R \\
I_T &= 4.071 \times 10^5 \text{ kg m}^2 
\end{align*}$

Since the turbine weighs $W_T = 2500 \text{ kgf}$, and from $I_T = M_T r_k^2$, the radius of gyration is

$\begin{align*}
r_k &= \left[\frac{I_T}{M_T}\right]^{1/2} \\
r_k &= 12.761 \text{ m}
\end{align*}$
Example - Cushioning of package

The EOM after the package first contacts hard ground is:

\[
M \cdot \frac{d^2 z}{dt^2} + c \cdot \frac{dz}{dt} + k \cdot z = M \cdot g
\]

\[1\]
\[M := 2 \cdot \text{kg}\]
\[k := 50 \cdot \text{N/mm}\]

k represents the stiffness of the packaging material. M is the component mass.

**Assume no damping, c=0**

with I.C's: \(z(0) = 0\) \(\frac{dz}{dt} = v_o\)

Equation [1] is valid for \(z>0\), i.e. as long as packaging material is being compressed.

Note that if package rebounds, then \(F_{spring} = 0\)

\[\omega_n := \left(\frac{k}{M}\right)^{0.5}\]
\[\omega_n = 158.114 \text{ rad/s}\]
\[f_n := \frac{\omega_n}{2 \cdot \pi}\]
\[f_n = 25.165 \text{ Hz}\] Natural frequency of system

Natural period: \(T_n := \frac{\pi}{\omega_n}\)
\[T_n = 0.04 \text{ s}\]

\[z_{ss} := \frac{(M \cdot g)}{k}\] [2]
\[z_{ss} = 0.392 \text{ mm}\] s-s response - after transients die out

from free fall, assume no air drag. The package velocity when touching ground is \(v_o := (2 \cdot g \cdot h)^{0.5}\)

The EOM of motion w/o damping is:

\[
M \cdot \frac{d^2 z}{dt^2} + k \cdot z = W
\]

[3] with I.C. \(z_0 := 0 \cdot \text{m}\)
\[v_0 := 2.214 \frac{\text{m}}{\text{s}}\]

**Dynamic response for undamped system**, \(\xi := 0\)

The solution of this ODE is very simple, i.e. the superposition of the particular solution (\(z_{ss}\)) and the homogeneous solution (periodic with natural frequency).

\[z(t) = z_{ss} + A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t)\] [4a] displacement of package

\[v(t) = \frac{dz}{dt} = \omega_n \cdot \left(-A_c \cdot \sin(\omega_n \cdot t) + A_s \cdot \cos(\omega_n \cdot t)\right)\] [4b] velocity of package

\[a(t) = \frac{dv}{dt} = -\omega_n^2 \cdot \left(-A_c \cdot \cos(\omega_n \cdot t) + A_s \cdot \sin(\omega_n \cdot t)\right)\] [4c] acceleration of package
The constants $A_c$ and $A_s$ are determined from the initial conditions. At time $t=0$ s, the package material is not (yet) deflected $z(0)=0$ and the package initial velocity is that of the free fall, $v(0)=v_0$

From Eq [4a]: \[ z(0) = z_0 = 0 = z_{ss} + A_c \quad A_c := -z_{ss} \quad [5a] \]

From Eq [4b] \[ v(0) = v_o = A_s \cdot \omega_n \quad A_s := \frac{v_o}{\omega_n} \quad [5b] \]

\[ z(t) := z_{ss} + A_c \cdot \cos\left(\omega_n t\right) + A_s \cdot \sin\left(\omega_n t\right) \quad [4a] \]

**NOTE** that package REBOUNDS when it crosses $z=0$ on its upward motion (remember when you jump on a trampolin.)

**What is the maximum dynamic displacement (deflection)?**

Prior to obtaining this deflection, let's recall some trigonometry:

Let: \[ A = \left( A_c^2 + A_s^2 \right)^{0.5} \]

and in the formula \[ A_c \cdot \cos\left(\omega_n t\right) + A_s \cdot \sin\left(\omega_n t\right) \]

\[ A \cdot \left( \frac{A_c}{A} \cdot \cos\left(\omega_n t\right) + \frac{A_s}{A} \cdot \sin\left(\omega_n t\right) \right) \quad [\text{**]} \]

Now, define a phase angle $\Phi$ such that

\[ \cos(\Phi) = \frac{A_c}{A} \quad \sin(\Phi) = \frac{A_s}{A} \]

and write expression [\text{**} above as

\[ A \cdot \left( \cos(\Phi) \cdot \cos(\omega_n t) + \sin(\Phi) \cdot \sin(\omega_n t) \right) \]

or

\[ A \cdot \cos(\omega_n t - \Phi) \]

where \[ \Phi := \arctan\left( \frac{v_o}{\omega_n \cdot (-z_{ss})} \right) + \pi \]

\[ \Phi = 1.599 \text{ radians} \]

Hence, with the aid of the simple trigonometry "trick" we write the displacement $z(t)$, Eqn [4a] as

\[ z(t) := z_{ss} + A_c \cdot \cos(\omega_n t) + A_s \cdot \sin(\omega_n t) \quad [4a] \]

\[ z(t) := z_{ss} + A \cdot \cos(\omega_n t - \Phi) \quad [6a] \]

displacement of package

$T_{nn} := 0.04 \cdot s$
Let's return to the question, **what is the maximum displacement?**

Clearly, the trig function \( \cos(x) \) can be at most +1 or -1, hence the

\[
z(0 \cdot s) = 0 \text{m}
\]

**Max dynamic displacement is**

\[
z_{\text{max}} := z_{\text{ss}} + A
\]

\[
z_{\text{max}} := z_{\text{ss}} + \left[ \frac{v_0}{\omega_n} \right]^2 \cdot 2^{0.5}
\]

\[
z_{\text{max}} = 14.403 \text{ mm}
\]

\[
z_{\text{ss}} = 0.392 \text{ mm}
\]

\( z > 0 \) means travel downwards, i.e. compression of packaging material.

**Motion does not die since there is no damping.**

The response found is strictly valid for \( z > 0 \), i.e. when packaging material "spring" is compressed. "Spring" of package can not be stretched.

**Velocity \([\text{m/s}], v(t) = dz/dt\)**

\[
v(t) := -A \cdot \omega_n \cdot \sin(\omega_n \cdot t - \Phi)
\]

\[
\text{what is maximum speed? } v_{\text{max}} := A \cdot \omega_n = 1 + \left( \frac{z_{\text{ss}} \cdot \omega_n}{v_0} \right)^2 \cdot 2^{0.5} \cdot v_0 = 2.215 \frac{\text{m}}{\text{s}}
\]

\[
v_{\text{max}} = 2.215 \frac{\text{m}}{\text{s}}
\]

\[
\frac{v_{\text{max}}}{v_0} = 1
\]

\[
v(0 \cdot s) = 2.214 \frac{\text{m}}{\text{s}}
\]

\[
v_0 = 2.214 \frac{\text{m}}{\text{s}}
\]
acceleration \([\text{m/s}^2]\), \(a(t) = \frac{dv}{dt}\),

\[
a(t) := -A \cdot \omega_n^2 \cdot \left(\cos(\omega_n t - \Phi)\right)
\]  \[6c\]

what is maximum accel? \(a_{\text{max}} := A \cdot \omega_n^2\)

\[
a_{\text{max}} = 350.256 \frac{\text{m}}{\text{s}^2}
\]

since

\[
A := \left[ \frac{z_{ss}^2}{\omega_n^2} + \left(\frac{v_0}{\omega_n}\right)^2 \right]^5
\]

then

\[
a_{\text{max}} = A \cdot \omega_n^2 = \left[ z_{ss}^2 \omega_n^4 + \left(\frac{v_0}{\omega_n}\right)^2 \right]^5
\]

but

\[
z_{ss} \cdot \omega_n^2 = \left(\frac{M \cdot g}{k}\right) \cdot \frac{k}{M} = g
\]

\[
\left(\frac{v_0 \cdot \omega_n}{g}\right)^2 = 2 \cdot g \cdot h \cdot \frac{k}{M \cdot g} \cdot \frac{1}{g} = \frac{2 \cdot h}{z_{ss}}
\]

Hence:

\[
a_{\text{max}} := g \cdot \left[1 + \frac{2 \cdot h}{z_{ss}}\right]^{1/5}
\]  \[7c\]

where \(z_{ss} = \frac{W}{k}\)

Note that if packaging stiffness (k) is very large then \(z_{ss}\) is very small, and hence, the maximum (peak) acceleration can be quite large!!!

Max acceleration if no damping: \(a_{\text{max}} = 350.256 \frac{\text{m}}{\text{s}^2}\)

\[\frac{a_{\text{max}}}{g} = 35.716\]

peak decelerations can be much larger than 1g!

\[
a(0\cdot s) = 9.807 \frac{\text{m}}{\text{s}^2}
\]

negative (peak) acceleration denotes upward force
The force from the cushioning into the package is \( F = -kz - cv = -W + M \text{ acc} \)

\[
F(t) := (-g + a(t)) \cdot M
\]

\( F_{\text{max}} := M \cdot (a_{\text{max}} + g) \)

\( F_{\text{max}} = 720.125 \text{ N} \)

\( M \cdot g = 19.613 \text{ N} \)

\( F < 0 \) means upwards force
- spring being compressed

\( \frac{F_{\text{max}}}{M \cdot g} = 36.716 \)

Quite large!! Much larger than component weight \( (W) \)

The package will REBOUND when \( z = 0 \) \((z < 0)\) and \( \frac{dz}{dt} = V < 0 \).

Approximately at

\[
t_{\text{rebound}} := \frac{T_n}{2}
\]

\( t_{\text{rebound}} = 0.02 \text{ s} \)

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