LECTURE 14: DEVELOPING THE EQUATIONS OF MOTION FOR TWO-MASS VIBRATION EXAMPLES

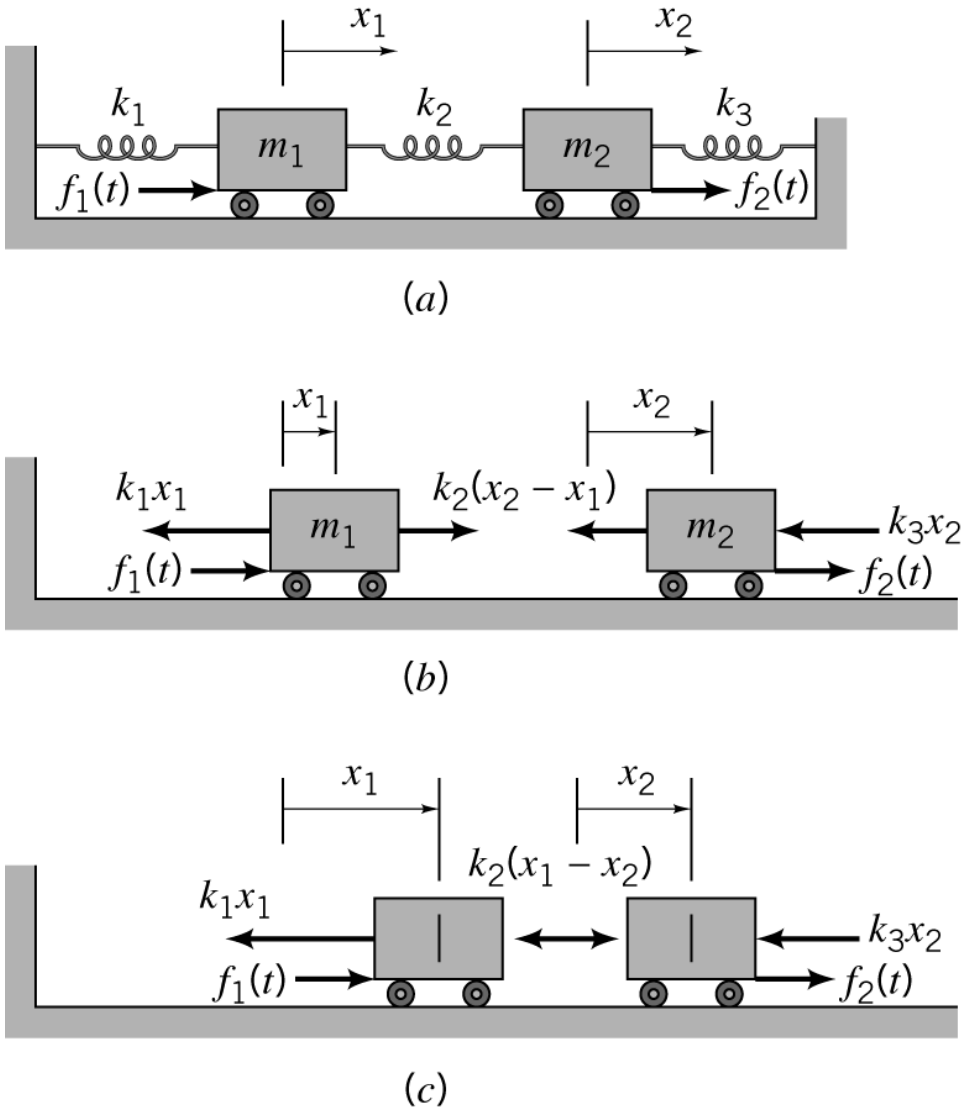


Figure 3.47 a. Two-mass, linear vibration system with spring connections. b. Free-body diagrams. c. Alternative free-body diagram.

***Equations of Motion Assuming*:** $x_1, x_2 > 0\,;\, x_2 > x_1$

The connecting spring is in tension, and the connecting spring-force magnitude is $f_{12} = k_2(x_2 - x_1)$. From figure 3.47B:

$$\begin{aligned} &mass\ m_1 : \Sigma f_{x1} = f_1(t) - k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1 \\ &mass\ m_2 : \Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2\,, \end{aligned} \tag{3.122}$$

with the resultant differential equations:

$$\begin{aligned} &m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = f_1(t) \\ &m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = f_2(t)\ . \end{aligned} \tag{3.123}$$

***Equations of Motion Assuming*:** $x_1, x_2 > 0\,;\, x_2 < x_1$

The spring is in compression, and the connecting-spring force magnitude is $f_{12} = k_2(x_1 - x_2)$. From figure 3.47C:

$$mass\ m_1 : \Sigma f_{x1} = f_1(t) - k_1 x_1 - k_2(x_1 - x_2) = m_1 \ddot{x}_1$$

$$mass\ m_2 : \Sigma f_{x2} = f_2(t) + k_2(x_1 - x_2) - k_3 x_2 = m_2 \ddot{x}_2\ .$$

Rearranging these differential equations gives Eqs.(3.122).

Steps for obtaining the correct differential equations of motion:

a. Assume displaced positions for the bodies and decide whether the connecting spring forces are in tension or compression.

b. Draw free-body diagrams that conform to the assumed displacement positions and their resultant reaction forces (i.e., tension or compression).

c. Apply $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the free body diagrams to obtain the governing equations of motion.

The matrix statement of Eqs.(3.123) is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} . \tag{3.124}$$

The mass matrix is diagonal, and the stiffness matrix is symmetric. A stiffness matrix that is not symmetric and cannot be made symmetric by multiplying one or more of its rows by constants indicates a system that is or can be dynamically unstable. You have made a mistake, if in working through the example problems, you arrive at a nonsymmetric stiffness matrix. Also, for a neutrally-stable system, the diagonal entries for the mass and stiffness matrices must be greater than zero.

The center spring "**couples**" the two coordinates. If $k_2 = 0$, the following "**uncoupled**" equations result

$$m_1 \ddot{x}_1 + k_1 x_1 = f_1(t)$$

$$m_2 \ddot{x}_2 + k_3 x_2 = f_2(t) \ .$$

These uncoupled equations of motion can be solved separately using the same procedures of the preceding section.

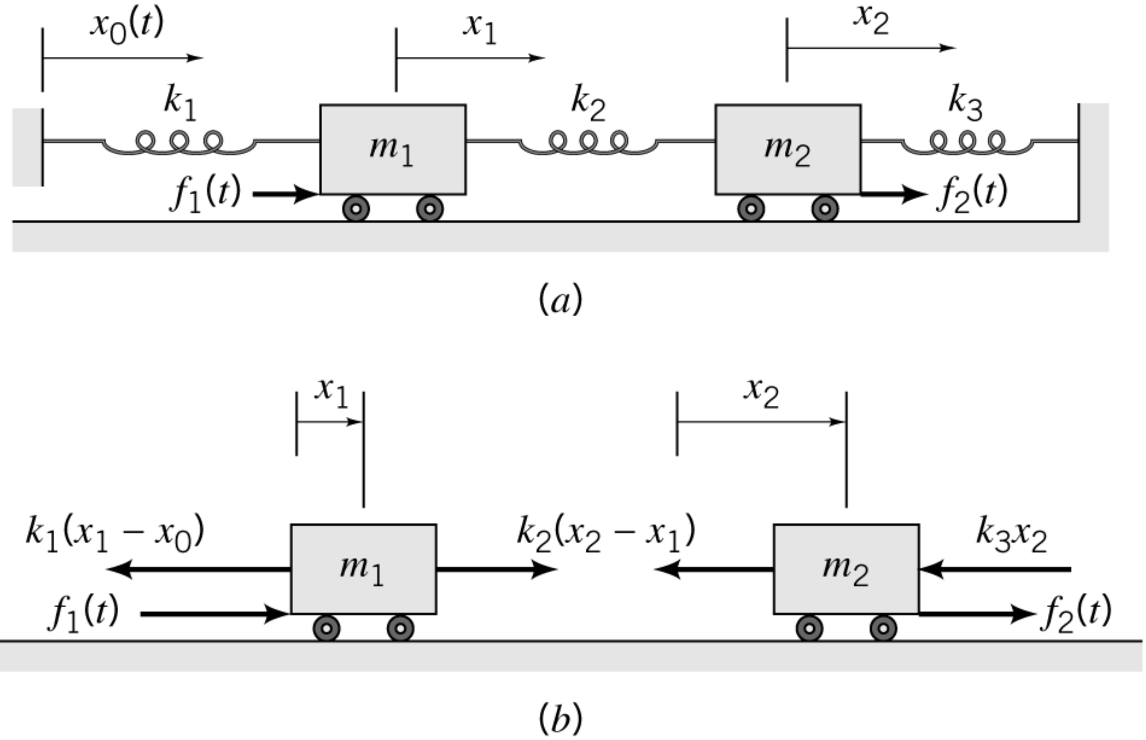


Figure 3.48 a. Two-mass, linear vibration system with motion of the left-hand support. b. Free-body diagram for assumed motion $x_2 > x_1 > x_0 > 0$.

Base Excitation from the Left-Hand Wall

Assume that the left-hand wall is moving creating base excitation via $x_0(t)$. From the free-body diagram for assumed motion $x_2 > x_1 > x_0 > 0$,

$$\begin{aligned} &mass\ m_1 : \Sigma f_{x1} = f_1(t) - k_1(x_1 - x_0) + k_2(x_2 - x_1) = m_1 \ddot{x}_1 \\ &mass\ m_2 : \Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2 , \end{aligned} \qquad (3.125)$$

In matrix notation

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) + k_1 x_0(t) \\ f_2(t) \end{Bmatrix} . \quad (3.126)$$

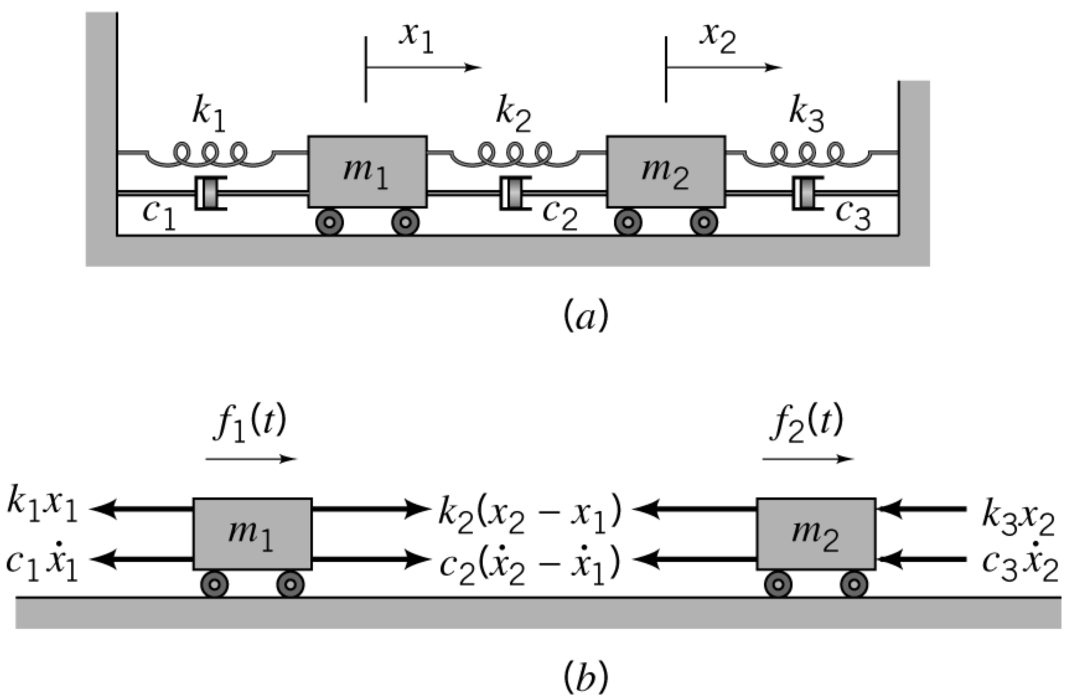


Figure 3.49 a. Two-mass, linear vibration system with spring and damper connections. b. Free-body diagram for $x_2 > x_1 > 0$, $\dot{x}_2 > \dot{x}_1 > 0$

Connection with Dampers

Assumed motion conditions:

a. Both m_1 and m_2 are moving to the right $(\dot{x}_1 > 0; \dot{x}_2 > 0)$, and

b. The velocity of m_2 is greater than the velocity of m_1 $(\dot{x}_2 > \dot{x}_1)$.

Based on this assumed motion, tension is developed in left and center dampers, but compression is developed in the right damper. The tension in damper 1 is $c_1\dot{x}_1$, the tension in damper 2 is $c_2(\dot{x}_2 - \dot{x}_1)$, and the compression in damper 3 is $c_3\dot{x}_2$.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the free body diagrams of figure 3.52B gives:

$$mass\ m_1 : \Sigma f_{x1} = f_1(t) - k_1 x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) = m_1\ddot{x}_1$$

$$mass\ m_2 : \Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 - c_2(\dot{x}_2 - \dot{x}_1) - c_3\dot{x}_2 = m_2\ddot{x}_2$$

These equations can be rearranged as

$$\begin{aligned} &m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t) \\ &m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = f_2(t)\,. \end{aligned} \tag{3.128}$$

In Matrix format

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & (c_2+c_3) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} . \quad (3.129)$$

Similar steps are involved in the development of equations of motion for systems connected by dampers that held for spring connections; namely,

a. Assume relative magnitudes for the bodies' velocities and decide whether the connecting damper forces are in tension or compression.

b. Draw free-body diagrams that conform to the assumed velocity conditions and their resultant damper forces (i.e., tension or compression).

c. Apply $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{r}}$ to the free-body diagrams to obtain the governing equations of motion.

The spring and damper forces can be developed sequentially.

Base Excitation via Right-hand wall motion

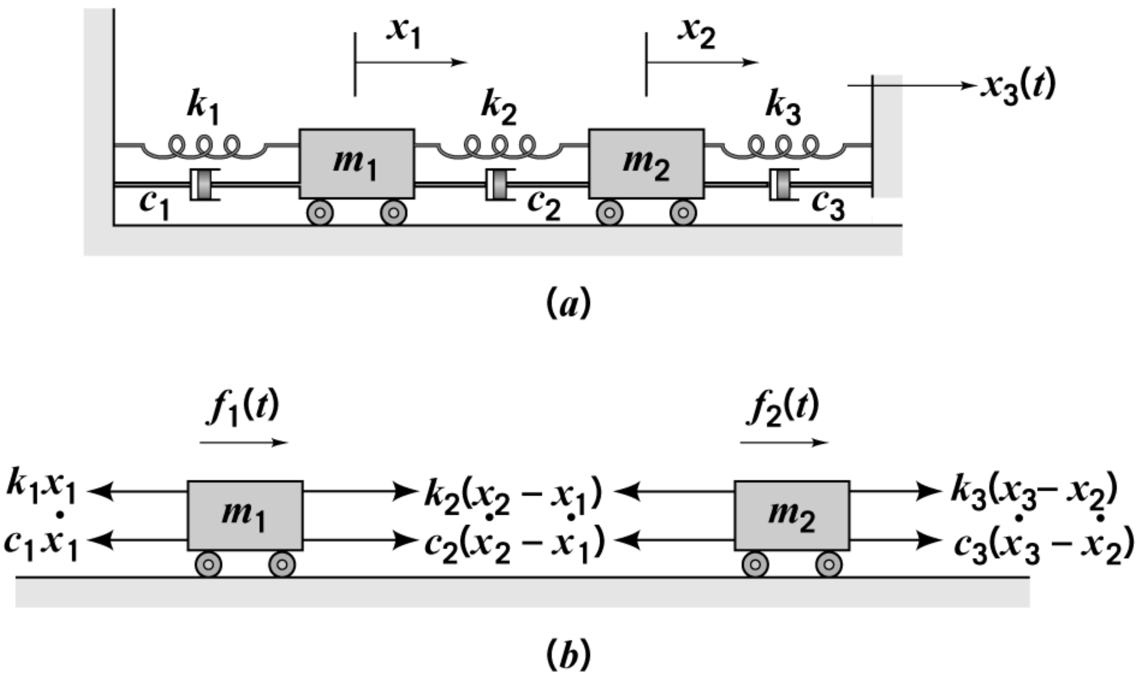


Figure 3.50 a. Coupled two-mass system with motion of the right-hand support defined by $x_3(t)$. b. Free-body diagram corresponding to assumed motion defined by $x_3 > x_2 > x_1 > 0$ and $\dot{x}_3 > \dot{x}_2 > \dot{x}_1 > 0$.

mass m_1 ,

$$\Sigma f_{x1} = f_1(t) - k_1 x_1 + k_2(x_2 - x_1) - c_1 \dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1)$$
$$= m_1 \ddot{x}_1$$

mass m_2 ,

$$\Sigma f_{x2} = f_2(t) - k_2(x_2 - x_1) + k_3(x_3 - x_2) - c_2(\dot{x}_2 - \dot{x}_1) + c_3(\dot{x}_3 - \dot{x}_2)$$
$$= m_2 \ddot{x}_2 \, .$$

Matrix Statement

Base excitation causes the additional forcing functions on the right. The stiffness and damping matrices should always be symmetric. If they are not, you have made a mistake.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & (c_2 + c_3) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} +$$

$$\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) + k_3 x_3(t) + c_3 \dot{x}_3(t) \end{Bmatrix} . \tag{3.131}$$

Developing the Equations of Motion for a Double Pendulum

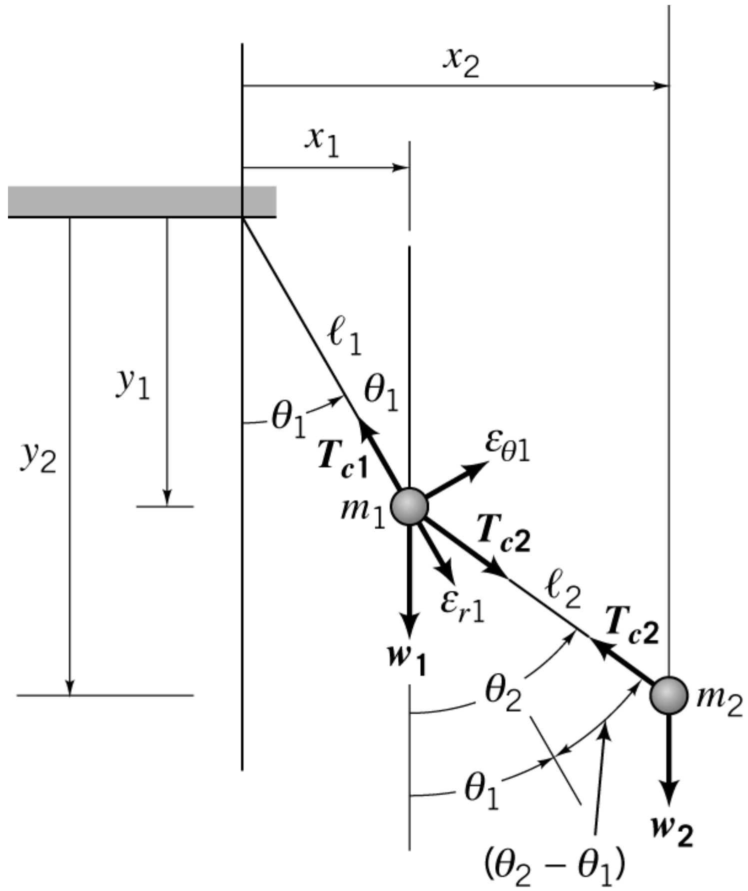


Figure 3.51 Free-body diagram for the double pendulum of figure 3.25.

Equations of motion for mass m_1 :

$$\begin{aligned}\Sigma f_{r1} &= w_1 \cos\theta_1 + T_{c2}\cos(\theta_2 - \theta_1) - T_{c1} \\ &= m_1(\ddot{r} - r\dot{\theta}_1^2) = -m_1 l_1 \dot{\theta}_1^2 \\ \Sigma f_{\theta 1} &= T_{c2}\sin(\theta_2 - \theta_1) - w_1 \sin\theta_1 \\ &= m_1(r\ddot{\theta} + 2\dot{r}\dot{\theta}_1) = m_1 l_1 \ddot{\theta}_1 \ .\end{aligned} \tag{3.131}$$

The second equation provides one equation in the two unknowns $(T_{c2}, \ddot{\theta}_1)$.

Simple -pendulum equations of motion,

$$\Sigma f_r = w\cos\theta - T_c = -ml\dot{\theta}^2$$
$$\Sigma f_\theta = -w\sin\theta = ml\ddot{\theta} \quad (3.82)$$

Equations of motion for mass m_2:

$$\Sigma f_{x2} = -T_{c2}\sin\theta_2 = m_2\ddot{x}_2$$
$$\Sigma f_{y2} = w_2 - T_{c2}\cos\theta_2 = m_2\ddot{y}_2 \ . \quad (3.132)$$

We now have two additional unknowns $(\ddot{y}_2, \ddot{x}_2)$.

Kinematics from figure 3.54:

$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \ , \quad y_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \ . \quad (3.133a)$$

Differentiating with respect to time gives:

$$\dot{x}_2 = l_1\cos\theta_1\dot{\theta}_1 + l_2\cos\theta_2\dot{\theta}_2$$
$$\dot{y}_2 = -l_1\sin\theta_1\dot{\theta}_1 - l_2\sin\theta_2\dot{\theta}_2 \ . \quad (3.133b)$$

Differentiating again gives:

$$\ddot{x}_2 = l_1\cos\theta_1\ddot{\theta}_1 - l_1\sin\theta_1\dot{\theta}_1^2 + l_2\cos\theta_2\ddot{\theta}_2 - l_2\sin\theta_2\dot{\theta}_2^2$$
$$\ddot{y}_2 = -l_1\sin\theta_1\ddot{\theta}_1 - l_1\cos\theta_1\dot{\theta}_1^2 - l_2\sin\theta_2\ddot{\theta}_2 - l_2\cos\theta_2\dot{\theta}_2^2 \quad . \qquad (3.133c)$$

Substituting these results into Eq.(3.132) gives:

$$-T_{c2}\sin\theta_2 = m_2(l_1\cos\theta_1\ddot{\theta}_1 - l_1\sin\theta_1\dot{\theta}_1^2 + l_2\cos\theta_2\ddot{\theta}_2 - l_2\sin\theta_2\dot{\theta}_2^2)$$
$$w_2 - T_{c2}\cos\theta_2 = m_2(-l_1\sin\theta_1\ddot{\theta}_1 - l_1\cos\theta_1\dot{\theta}_1^2 - l_2\sin\theta_2\ddot{\theta}_2 - l_2\cos\theta_2\dot{\theta}_2^2) \quad . \qquad (3.134)$$

The second of Eq.(3.131) and Eqs.(134) provide three equations for the three unknowns $(T_{c2}, \ddot{\theta}_1, \ddot{\theta}_2)$.

Eqs.(3.134-a) $\times$ $\cos\theta_2$ - Eqs.(3.134-b) $\times$ $\sin\theta_2$ gives

$$-w_2\sin\theta_2 = m_2[l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + l_1\dot{\theta}_1^2\sin(\theta_2-\theta_1)] \quad , \qquad (3.135)$$

Eqs.(3.134-a) $\times$ $\cos\theta_1$ - Eqs.(3.134-b) $\times$ $\sin\theta_1$ gives $T_{c2}\ \sin(\theta_2-\theta_1)+..$ Substituting for $T_{c2}\ \sin(\theta_2-\theta_1)$ into the second of Eq.(3.131) gives (with a lots of algebra) :

$$-w_2 \sin\theta_1 + m_2[-l_1\ddot{\theta}_1 - l_2\ddot{\theta}_2\cos(\theta_2 - \theta_1)$$
$$+ l_2\dot{\theta}_2^2\sin(\theta_2 - \theta_1)] - w_1\sin\theta_1 = m_1 l_1 \ddot{\theta} \ .$$

This the second of the two required differential equations. In matrix format the model is

$$\begin{bmatrix} l_1(m_1 + m_2) & m_2 l_2 \cos(\theta_2 - \theta_1) \\ m_2 l_1 \cos(\theta_2 - \theta_1) & m_2 l_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$
$$= \begin{Bmatrix} -(w_1 + w_2)\sin\theta_1 + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \\ -w_2 \sin\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \end{Bmatrix} . \tag{3.137}$$

Note that this inertia matrix is neither diagonal nor symmetric, but it can be made symmetric; e.g., multiply the first equation by l_1 and the second equation by l_2. As with the stiffness matrix, the inertia matrix should be either symmetric, or capable of being made symmetric. Also, correct diagonal entries are positive.

The linearized version of this equation is obtained by assuming that both θ_1 and θ_2 are small (i.e., $\sin\theta_1 \cong \theta_1$; $\cos\theta_1 \cong 1$, etc.) and can be stated

$$
\begin{bmatrix} (m_1+m_2)l_1^2 & m_2 l_2 l_1 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}
+ \begin{bmatrix} (w_1+w_2)l_1 & 0 \\ 0 & w_2 l_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \ .
$$

We have “symmetricized” the inertia matrix and now have a diagonal stiffness matrix. The inertia matrix couples these two degrees of freedom.

Lecture 15. EIGENANALYSIS FOR 2DOF VIBRATION EXAMPLES

Thinking about solving coupled linear differential equations by considering the problem of developing a solution to the following homogeneous version of Eq.(3.124)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 . \quad (3.139)$$

To find a solution to the one-degree-of-freedom problem $m\ddot{x} + kx = 0$, we "guessed" a solution of the form $x = A\cos\omega t \Rightarrow \ddot{x} = -\omega^2 A\cos\omega t$. Substituting this guess netted

$$(-\omega^2 + k/m)A\cos\omega t = 0 \ .$$

A nontrivial solution ($A \neq 0$) requires that $\omega = \omega_n = \sqrt{k/m}$.

For Eq.(3.139) we will guess

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos\omega t \Rightarrow \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos\omega t \ . \quad (3.140)$$

Substituting this guessed solution gives

$$\left[-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \right] \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos\omega t = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} ,$$

or

$$\begin{bmatrix} [-m_1\omega^2 + (k_1 + k_2)] & -k_2 \\ -k_2 & [-m_2\omega^2 + (k_2+k_3)] \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.141)$$

Solving for a_1, a_2 via Cramer's rule gives:

$$a_1 = \frac{1}{\Delta} \begin{vmatrix} 0 & -k_2 \\ 0 & -m_2\omega^2 + (k_2+k_3) \end{vmatrix} = \frac{0}{\Delta} ,$$

$$a_2 = \frac{1}{\Delta} \begin{vmatrix} -m_1\omega^2 + (k_1+k_2) & 0 \\ -k_2 & 0 \end{vmatrix} = \frac{0}{\Delta} ,$$

where Δ is the determinant of the coefficient matrix. For a nontrivial solution $(a_1, a_2 \neq 0)$, $\Delta = 0$; i.e., the coefficient matrix must be singular.

$$\Delta = [(k_1 + k_2) - m_1\omega^2]\,[(k_2 + k_3) - m_2\omega^2] - k_2^2$$
$$= m_1 m_2 \omega^4 - [m_1(k_2+k_3) + m_2(k_1+k_2)]\,\omega^2 + (k_1+k_2)(k_2+k_3) - k_2^2 = 0. \quad (3.142)$$

<u>This is the characteristic equation.</u> It is quadratic and defines two natural frequencies $\omega_{n1}^2, \omega_{n2}^2$ versus the single natural frequency for the one-degree-of-freedom vibration examples.

Numerical Example:

$$m_1 = 1\,kg,\ m_2 = 2\,kg\ ,\ k_1 = k_2 = k_3 = 1 N/m. \quad (3.143)$$

For these data, the differential Eq.(3.139) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0, \quad (3.144)$$

and the frequency Eq.(3.142) becomes

$$\omega^4 - 3\,\omega^2 + \frac{3}{2} = 0 \Rightarrow \omega^2 = \frac{3}{2} \pm \frac{\sqrt{3}}{2}\ ;$$

with the solutions:

$$\omega_{n1}^2 = .633975\,\sec^{-2} \Rightarrow \omega_{n1} = .7962\,\sec^{-1}$$
$$\omega_{n2}^2 = 2.36603\,\sec^{-2} \Rightarrow \omega_{n2} = 1.538\,\sec^{-1}\ . \quad (3.145)$$

The first (lowest) root $\omega_{n1}^2 = .634\ \sec^{-2}$ is the first eigenvalue and defines the first natural frequency $\omega_{n1} = .796\,rad/\sec$. The next root $\omega_{n2}^2 = 2.366\ \sec^{-2}$ is the second eigenvalue and defines the second natural frequency $\omega_{n2} = 1.510\,rad/\sec$.

Solving for the a_1, a_2 coefficients. Substituting the data of Eq.(3.145) into Eq.(3.141) gives

$$\begin{bmatrix} -\omega^2 + 2 & -1 \\ -1 & -2\omega^2 + 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

Now substituting $\omega^2 = \omega_{n1}^2 = .634\,\sec^{-2}$ gives

$$\begin{bmatrix} -.633975 + 2 & -1 \\ -1 & -1.26796 + 2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = 0$$

The coefficient-matrix determinant is zero, which implies that there is only one independent equation for the two unknowns. Hence, we can use either equation to solve for the ratios of the two unknowns. Setting $a_{11} = 1$ gives:

$$1.36603(1) - a_{21} = 0 \Rightarrow a_{21} = 1.36603$$

$$-1(1) + .732050\ a_{21} = 0 \Rightarrow a_{21} = 1.36603\ .$$

Hence, the first "eigenvector" is

$$(a_{i1}) = \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.36603 \end{Bmatrix} . \qquad (3.146)$$

Multiplying this vector by any finite constant (positive or negative) will yield an equally valid first eigenvector, since the vector is defined only in terms of the ratio of its components. In vibration problems, an eigenvector is also called a "mode shape."

Substituting $\omega^2 = \omega_{n2}^2 = 2.3360\,\sec^{-2}$ into Eq.(3.141), nets

$$(a_{i2}) = \begin{Bmatrix} a_{21} \\ a_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -.36603 \end{Bmatrix} .$$

The matrix of eigenvectors is

$$[A] = \begin{bmatrix} 1.0 & 1.0 \\ 1.36603 & -.36603 \end{bmatrix} . \qquad (3.147)$$

Figure 3.52 illustrates the two eigenvectors.

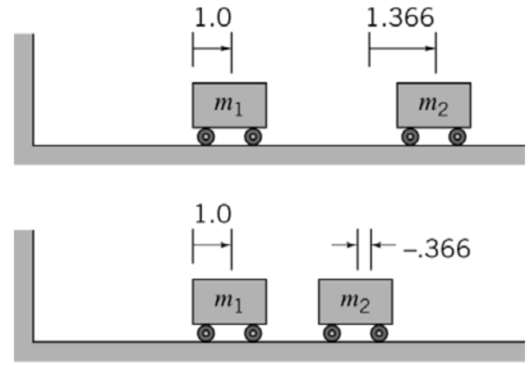


Figure 3.52 Eigenvectors for the two-mass system of figure 3.47, with the numerical values of Eq.(3.143).

Consider the following coordinate transformation for Eq.(3.139)

$$(x_i) = [A](q_i) \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0 \\ 1.36603 & -.36603 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} ,(3.148a)$$

$$(\ddot{x}_i) = [A](\ddot{q}_i) \ . \qquad (3.148b)$$

where the right vector (q_i) is the vector of *modal* coordinates.

Substituting from Eqs.(3.148) into Eq.(3.124) gives

$$[M][A](\ddot{q}_i) + [K][A](q_i) = (f_i) \ . \qquad (3.149)$$

Premultiplying Eq.(3.149) by the transpose of $[A]$ gives

$$[A]^T[M][A](\ddot{q}_i) + [A]^T[K][A](q_i) = [A]^T(f_i) \ . \qquad (3.150)$$

We can now show by substitution (for this example problem) that

$$[A]^T[M][A] = [M_q]$$
$$[A]^T[K][A] = [K_q] \ , \qquad (3.148)$$

where the "modal mass matrix" $[M_q]$ and "modal stiffness matrix $[K_q]$ are diagonal. Note

$$[A]^T[M][A] = \begin{bmatrix} 1.0 & 1.36603 \\ 1.0 & -.36603 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1.0 & 1.0 \\ 1.36603 & -.36603 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 1.36603 \\ 1.0 & -.36603 \end{bmatrix} \begin{bmatrix} 1.0 & 1.0 \\ 2.73206 & -.73206 \end{bmatrix} \qquad (3.149)$$

$$= \begin{bmatrix} 4.732 & .0000 \\ .0000 & 1.268 \end{bmatrix} = [M_q] \ .$$

The modal mass matrix $[M_q]$ is diagonal, with the first and second "modal masses" defined by $m_{q1} = 4.732 \ ; \ m_{q2} = 1.268$.

We want to "normalize" the eigenvectors *with respect to the mass matrix* such that the modal mass matrix $[M_q]$ reduces to the identity matrix [I]. The modal-mass coefficient for the jth mode is defined by $(a_{ji})^T[M](a_{ji}) = m_{qj}$. Dividing the jth eigenvector (a_{ji}) by $m_{qj}^{1/2}$ will yield an eigenvector with a modal mass equal to 1, yielding

$$[A^*]^T[M][A^*] = [I] \ . \qquad (3.150)$$

Normalizing the current eigenvector set means dividing the first and second eigenvectors by $(M_{q1})^{1/2} = \sqrt{4.732} = 2.175$ and $(M_{q2})^{1/2} = \sqrt{1.268} = 1.126$, respectively, obtaining

$$[A^*] = \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix} . \qquad (3.151)$$

You may want to repeat calculations for this set of eigenvectors to confirm that the modal mass matrix is now the identity matrix. Proceeding with this normalized version of the eigenvector matrix to verify that the modal stiffness matrix is diagonal yields

$$[A^*]^T[K][A^*] = [A^*]^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix}$$

$$= \begin{bmatrix} .45970 & .62796 \\ .88807 & -.32506 \end{bmatrix} \begin{bmatrix} .29144 & 2.10120 \\ .79622 & -1.53819 \end{bmatrix}$$

$$= \begin{bmatrix} .63345 & -.00000 \\ -.00000 & 2.3660 \end{bmatrix} = [K_q].$$

The normalized matrix of eigenvectors yields a diagonalized modal stiffness matrix [K_q]; moreover, the diagonal entries are the eigenvalues defined in Eq.(3.142); i.e.,

$$[A^*]^T[K][A^*] = [\Lambda] = \begin{bmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix}, \qquad (3.152)$$

where $[\Lambda]$ is the diagonal matrix of eigenvalues.
The resultant modal equations are:

$$\ddot{q}_1 + .63345\, q_1 = (A_1)^T (f_1) = Q_1$$

$$\ddot{q}_2 + 2.3660\, q_1 = (A_2)^T (f_2) = Q_2\ .$$

The transformation from modal to physical coordinates is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}.$$

Modal Units

Given that $[A^*]^T[M][A^*] = [I]$, and $[A^*]^T[K][A^*] = [\Lambda]$, the units for an entry in normalized eigenvector matrix $[A^*]$ is $mass^{-1/2}$. Hence, for the SI system, the eigenvector units are $kg^{-1/2}$; for the USA standard unit system, the units are $slug^{-1/2}$. From the coordinate transformation $(x)_i = [A^*](q)_i$, the units for a modal coordinate is $mass^{1/2} \times length$. For the SI and USA standard systems, the appropriate units are, respectively, $meter\ kg^{1/2}$ and $ft\ slug^{1/2}$. Looking at the first of Eq.(3,154), a dimensional analysis yields

$$\ddot{q}_1 (L m^{1/2} T^{-2}) + \omega_{n1}^2 (T^{-2})\, q_1 (L m^{1/2}) = a_{11} (m^{-1/2}) f_1(F)$$
$$+ a_{12} (m^{-1/2}) f_2(F)$$

$$\therefore\ m L T^{-2} = F\ .$$

confirming the correctness of these dimensions.

Lessons:

a. Vibration problems can have multiple degrees of freedom.

b. Multiple-degree-of-freedom (MDOF) vibration problems can be coupled by either the stiffness (linear spring-mass system) or inertia (double pendulum) matrices.

c. For a neutrally stable system, the inertia and stiffness matrices should be symmetric and the diagonal elements should be positive.

d. Free vibrations of a MDOF vibration problem leads to an eigenvalue problem. The solution to the eigenvalue problem yields eigenvalues, ω_{ni}^2, which define the natural frequencies ω_{ni}, and eigenvectors that define the system mode shapes.

e. The matrix of eigenvectors $[A]$ can be normalized such that it diagonalizes the original inertia and stiffness matrices as

$$[A^*]^T[M][A^*] = [I] \ , \quad [A^*]^T[K][A^*] = [\Lambda] \ ,$$

where $[I]$ is the identity matrix, and $[\Lambda]$ is the diagonal matrix of eigenvalues.

Lecture 16. SOLVING FOR TRANSIENT MOTION USING MODAL COORDINATES

Free Motion

With normalized eigenvectors, the matrix version of the modal Eq.(3.125) is

$$(\ddot{q})_i + [\Lambda](q_i) = (Q_i) = [A^*]^T (f_i) = \begin{bmatrix} A^*_{11} & A^*_{21} \\ A^*_{12} & A^*_{22} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} A^*_{11} f_1 + A^*_{21} f_2 \\ A^*_{12} f_1 + A^*_{22} f_2 \end{Bmatrix} . \quad (3.153)$$

(Q_i)= the modal force vector. Uncoupled, modal differential equations:

$$\ddot{q}_1 + \omega_{n1}^2 q_1 = Q_1 = A^*_{11} f_1 + A^*_{21} f_2$$
$$\ddot{q}_2 + \omega_{n2}^2 q_2 = Q_2 = A^*_{12} f_1 + A^*_{22} f_2 \ . \quad (3.154)$$

The homogeneous version of Eq.(3.154)

$$\ddot{q}_{1h} + \omega_{n1}^2 q_{1h} = 0 \ , \quad \ddot{q}_{2h} + \omega_{n2}^2 q_{2h} = 0 \ ,$$

have solutions:

$$q_{1h}(t) = A_1 \cos\omega_{n1} t + B_1 \sin\omega_{n1} t$$
$$q_{2h}(t) = A_2 \cos\omega_{n2} t + B_2 \sin\omega_{n2} t \ .$$

Adding particular solutions $q_{1p}(t), q_{2p}(t)$ corresponding to specific right-hand forcing functions, yield the complete solution

$$q_1(t) = q_{1p}(t) + A_1 \cos\omega_{n1} t + B_1 \sin\omega_{n1} t$$
$$q_2(t) = q_{2p}(t) + A_2 \cos\omega_{n2} t + B_2 \sin\omega_{n2} t \ .$$

The constants A_i, B_i must be determined from the modal-coordinate initial conditions.

To obtain modal-coordinate initial conditions:

$$(x) = [A^*](q) \Rightarrow [A^*]^T [M](x) = [A^*]^T [M][A^*](q) = [I](q)$$
$$[A^*]^T [M](x) = [I](q) = (q) \Rightarrow [A^*]^{-1} = [A^*]^T [M] \ .$$

Hence, the modal-coordinate initial conditions are defined by

$$(q_0) = [A^*]^T [M](x_0) \ .$$

Similarly, the modal-velocity initial conditions are defined by

$$(\dot{q}_0) = [A^*]^T[M](\dot{x}_0) \ .$$

For the prior example

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}, \quad (3.144)$$

$$[A^*] = \begin{bmatrix} .45970 & .88807 \\ .62796 & -.32506 \end{bmatrix}, \quad (3.151)$$

with

$$\begin{aligned} \omega_{n1}^2 &= .633975 \sec^{-2} \Rightarrow \omega_{n1} = .7962 \sec^{-1} \\ \omega_{n2}^2 &= 2.36603 \sec^{-2} \Rightarrow \omega_{n2} = 1.538 \sec^{-1} \ . \end{aligned} \quad (3.145)$$

Modal coordinate initial conditions:

$$\begin{Bmatrix} q_{10} \\ q_{20} \end{Bmatrix} = [A^*]^T[M] \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix}$$

$$= \begin{bmatrix} .45970 & .62796 \\ .88807 & -.32506 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix}$$

$$= \begin{bmatrix} .45970 & 1.2559 \\ .88807 & -.65012 \end{bmatrix} \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = \begin{Bmatrix} .45970 x_{10} + 1.2559 x_{20} \\ .88807 x_{10} - .65012 x_{20} \end{Bmatrix} .$$

Modal velocity initial conditions:

$$\begin{Bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{Bmatrix} = [A^*]^T[M] \begin{Bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{Bmatrix} = \begin{Bmatrix} .45970 \dot{x}_{10} + 1.2559 \dot{x}_{20} \\ .88807 \dot{x}_{10} - .65012 \dot{x}_{20} \end{Bmatrix} . \quad (3.155b)$$

With these initial conditions, we can solve Eqs.(3.132) for any specified modal force terms $Q_1(t), Q_2(t)$, obtaining a complete modal-coordinate solution defined by $q_1(t), q_2(t)$. From Eq.(3.123a) the solution for the physical coordinates can then be stated

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = q_1(t) \begin{Bmatrix} A^*_{11} \\ A^*_{21} \end{Bmatrix} + q_2(t) \begin{Bmatrix} A^*_{12} \\ A^*_{22} \end{Bmatrix} .$$

Hence, the physical coordinate response vector is a linear sum of the modal solutions $q_1(t), q_2(t)$ *times their respective mode* *shapes* (*or eigenvectors*).

Example. With $x_{10} = 1\,cm$, $x_{20} = \dot{x}_{10} = \dot{x}_{20} = 0$, Eqs. (3.155) gives $\dot{q}_{10} = \dot{q}_{20} = 0$; $q_{10} = .4597\,cm$, $q_{20} = .88807\,cm$, and:

$$\ddot{q}_1 + \omega_{n1}^2\, q_1 = 0 \Rightarrow q_1 = q_{1h} = .4597\cos(.796\,t)$$

$$\ddot{q}_2 + \omega_{n2}^2\, q_2 = 0 \Rightarrow q_2 = q_{2h} = .8881\cos(1.538\,t) \ .$$

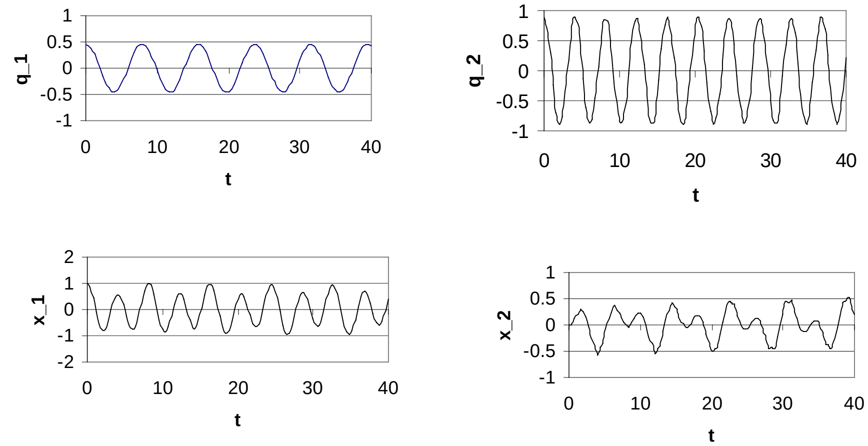
The corresponding physical-coordinate solution is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = .4597\cos(.796\,t)\begin{Bmatrix} .4597 \\ .6280 \end{Bmatrix} + .8881\cos(1.538\,t)\begin{Bmatrix} .8881 \\ -.3251 \end{Bmatrix} \ . \qquad (3.157)$$

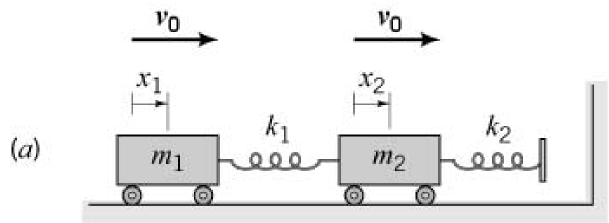
Any disturbance to the system from initial conditions (or external forces) will result in combined motion at the two natural frequencies. Note that the physical initial conditions are correctly represented,

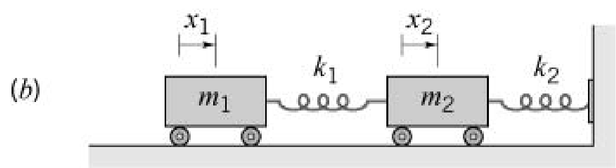
$$\begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = .4597\begin{Bmatrix} .4597 \\ .6280 \end{Bmatrix} + .8881\begin{Bmatrix} .8881 \\ -.3251 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.0 \end{Bmatrix} cm \ .$$

Figure 3.53 Solution from Eq.(3.157) for q_1, q_2, x_1 and x_2 for $x_{10} = 1\,cm,\ x_{20} = 0\ cm \Rightarrow q_{10} = -.527\,cm;\ q_{20} = .527\,cm$.



Modal Transient Example Problem 1. Free Undamped Motion





The 2-mass model illustrated in *a* is just about to collide with a wall. The right-hand spring will cushion the shock of the collision. Before collision, the system's model is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \ . \qquad (\text{i})$$

Once contact is established, the system looks like the model of frame *b* and is governed by the matrix differential equations of motion

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1+k_2) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \ . \ (\textbf{ii})$$

The physical parameters are:

$$m_1 = 150\, kg, m_2 = 100\, kg \ , \ k_1 = k_2 = 1.5\times 10^4 N/m,$$

yielding

$$\begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1.5\times 10^4 & -1.5\times 10^4 \\ -1.5\times 10^4 & 3.0\times 10^4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \ . (\textbf{ii})$$

Starting from the instant of contact, the physical initial conditions are:

$$x_1(0) = x_2(0) = 0 \ , \ \dot{x}_1(0) = \dot{x}_2(0) = v_0 = 4\, m/\text{sec} \ .$$

Solve for $x_1(t), x_2(t)$ *and the reaction forces* $k_1[x_1(t) - x_2(t)],\ k_2 x_2(t)$.

Solution. Following the procedures of the preceding examples, the eigenvalues and natural frequencies are:

$$\omega_{n1}^2 = 41.886\,\text{sec}^{-2} \Rightarrow \omega_{n1} = 6.4720\,\text{sec}^{-1}$$
$$\omega_{n2}^2 = 358.11\,\text{sec}^{-2} \Rightarrow \omega_{n2} = 18.924\,\text{sec}^{-1} \ . \qquad \textbf{(iv)}$$

The matrix of unnormalized eigenvectors is

$$[A] = [(a)_1, (a)_2] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ .5811 & -2.5811 \end{bmatrix} .$$

The matrix of normalized eigenvectors follows from $[A^*]^T[M][A^*] = [I]$ as

$$[A^*] = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix} .$$

Hence, the model is

$$\ddot{q}_1 + 41.886 q_1 = 0 \ , \quad \ddot{q}_2 + 358.11 q_2 = 0$$
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} .$$

From $(q_0) = [A^*]^T[M](x_0)$, the modal-coordinate initial

conditions are zero. From $(\dot{q}_0) = [A^*]^T[M](\dot{x}_0)$, the modal-velocity initial conditions are

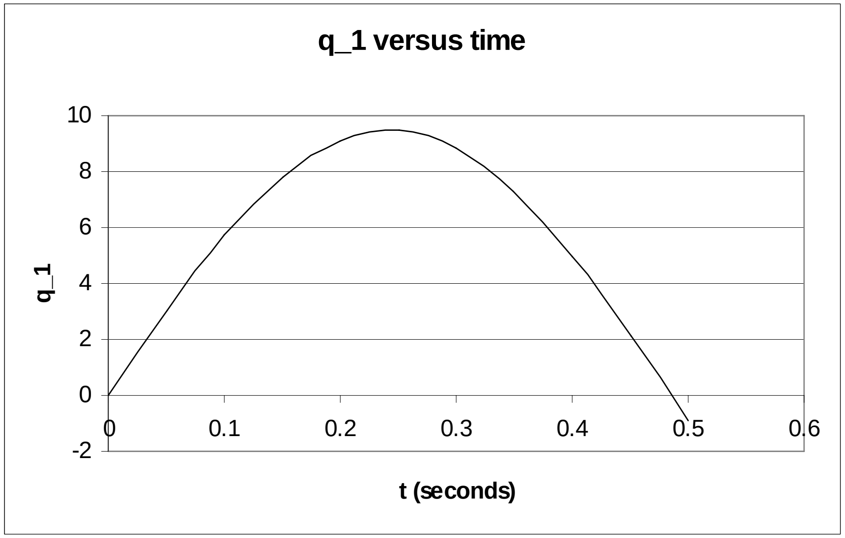
$$\begin{Bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{Bmatrix} = \begin{bmatrix} .073767 & .042866 \\ .035002 & -.090344 \end{bmatrix} \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$$
$$= \begin{bmatrix} 11.065 & 4.2866 \\ 5.2503 & -9.0344 \end{bmatrix} \begin{Bmatrix} 4 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 61.4 \\ -15.136 \end{Bmatrix} \text{sec}^{-1} \ . \qquad \textbf{(v)}$$

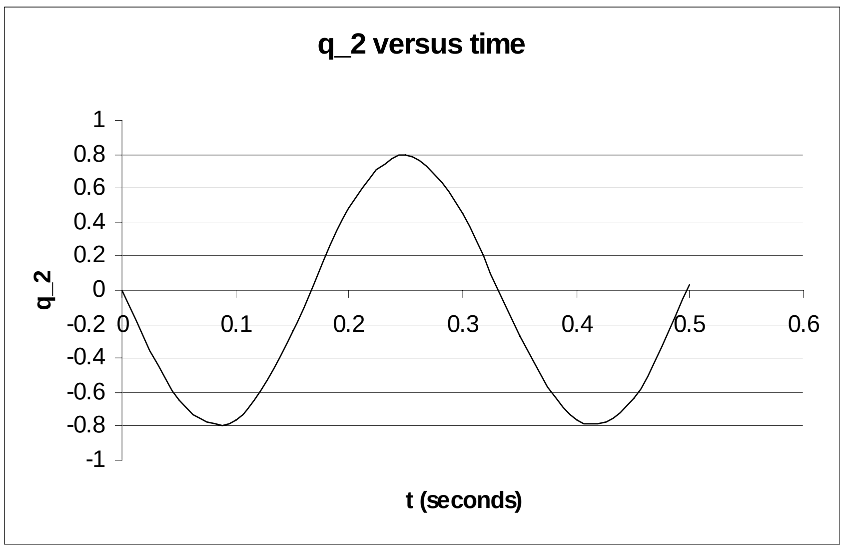
In terms of initial conditions, the solution to $\ddot{q}_i + \omega_{ni}^2 q_i = 0$ is $q_i = q_{i0} \cos\omega_{ni} t + (\dot{q}_{i0}/\omega_{ni}) \sin\omega_{ni} t$. Hence, the modal solutions are:

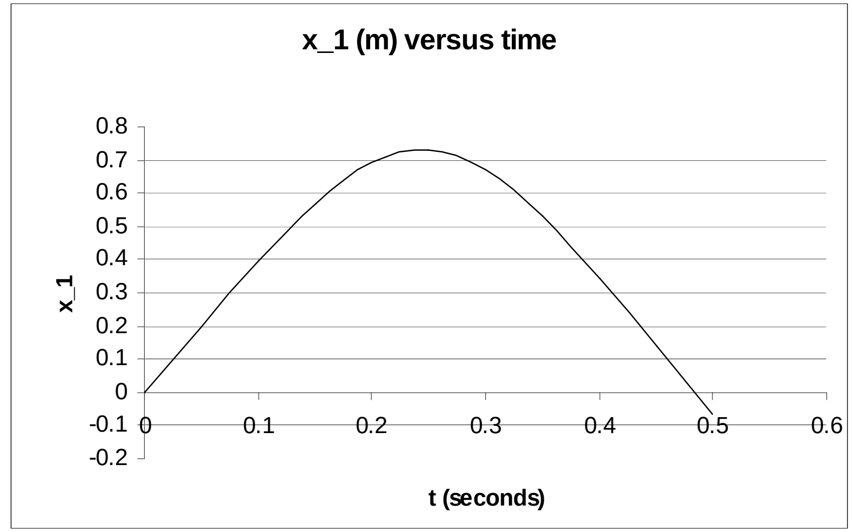
$$q_1 = q_{1h} + 0 = \frac{61.406}{6.472} \sin 6.472 t = 9.488 \sin 6.472 t$$
$$q_2 = q_{2h} + 0 = \frac{-15.136}{18.924} \sin 18.92 t = -.800 \sin 18.924 t \ .$$

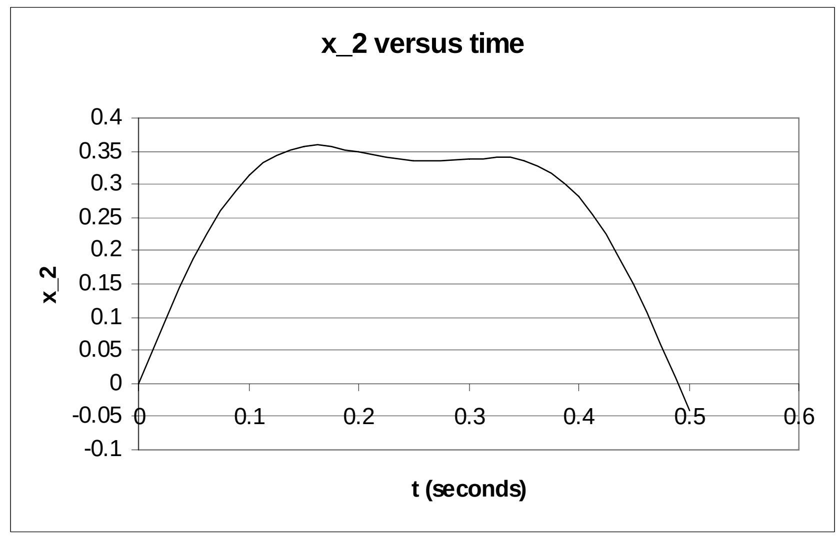
From $(x) = [A^*](q)$, the solution for the physical coordinates is

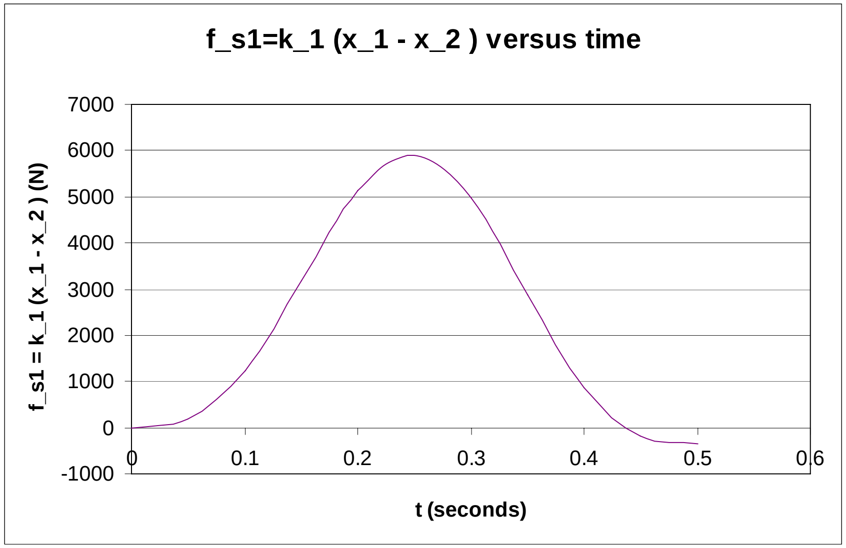
$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = q_1(t) \begin{Bmatrix} .0738 \\ .0429 \end{Bmatrix} + q_2(t) \begin{Bmatrix} .0350 \\ -.0903 \end{Bmatrix} .$$

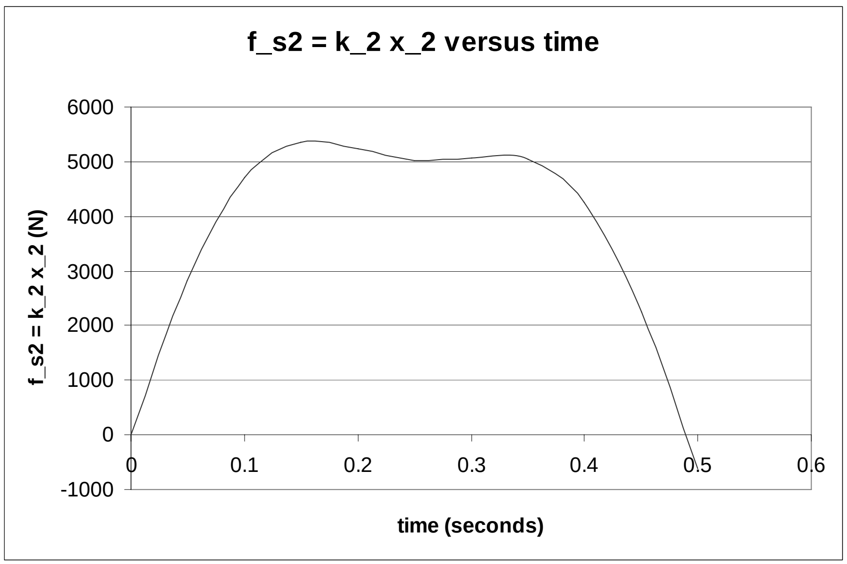

q_1 versus time
q_1
t (seconds)
10
8
6
4
2
0
-2
0
0.1
0.2
0.3
0.4
0.5
0.6


q_2 versus time
q_2
t (seconds)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6


x_1 (m) versus time
x_1
t (seconds)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6


x_2 versus time
x_2
t (seconds)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6





Do the numbers seem right? We can do a quick calculation to see if the peak force and deflection seem to be reasonable. Suppose both bodies are combined, so that one body has a mass of $m = m_1 + m_2 = 100 + 150 = 250\,kg$. Then a conservation of energy equation to find the peak deflection is

$$T_0 + V_0 = T_f + V_f$$

$$\therefore \frac{m}{2} v_0^2 = \frac{k}{2}\Delta^2 \Rightarrow \frac{250}{2} 4^2 = \frac{1.5 \times 10^4}{2}\Delta^2 \Rightarrow \Delta = .516\,m$$

The predicted peak deflection for $x_2(t)$ is 0.35 m, which is on the order of magnitude for the estimate, but lower. We would expect the correct number to be lower, because two masses with a spring between them will produce a lower collision force than a single rigid body with an equivalent mass.

Note that the spring-mass system loses contact with the wall when $f_{s2} = k_2 x_2$ changes sign at about $t = 0.048\,\text{sec}$. After this time, the right spring is disengaged, and the model becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \, .$$

Damping and Modal Damping Factors

The SDOF harmonic oscillator equation of motion is

$$m\ddot{Y} + c\dot{Y} + kY = f(t) \ .$$

yields the homogeneous equation

$$\ddot{Y}_h + 2\zeta\omega_n \dot{Y}_h + \omega_n^2 Y_h = 0 \ ,$$

Assumed solution: $Y_h = Ae^{st}$ yields:

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) A e^{st} = 0 \ .$$

Since $A \neq 0$, and e^{st},

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad \textbf{(3.23)}$$

For $\zeta<1$, the roots are

$$\begin{aligned} s &= -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n \pm j\,\omega_d \ . \end{aligned} \qquad \textbf{(3.24)}$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called the *damped natural frequency*. The homogeneous solution looks like

$$Y_h = \boldsymbol{A_1}\, e^{(-\zeta\omega_n + j\omega_d)t} + \boldsymbol{A_2}\, e^{(-\zeta\omega_n - j\omega_d)t} \ , \qquad \textbf{(3.25)}$$

where $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are *complex* coefficients. The solution can be stated

$$Y_h = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \ ,$$

where A and B are real constants.

MDOF vibration problems with general damping matrices modeled by,

$$[M](\ddot{x}) + [C](\dot{x}) + [K](x) = 0 \ ,$$

also have complex roots (eigenvalues) and eigenvectors.

The matrix of real eigenvectors, based on a symmetric stiffness $[K]$ and mass $[M]$ matrices, will only diagonalize a damping matrix that can be stated as a linear summation of $[K]$ and $[M]$; i.e.,

$$[C] = \alpha[M] + \beta[K]$$

For a damping matrix of this particular (and very unlikely) form, the modal damping matrix is defined by

$$\begin{aligned} [C_q] &= [A^*]^T[C][A^*] = \alpha[A^*]^T[M][A^*] + \beta[A^*]^T[K][A^*] \\ &= \alpha[I] + \beta[\Lambda] \ , \end{aligned}$$

where $[I]$ and $[\Lambda]$ are the identity matrix and the diagonal matrix of eigenvectors, respectively. With this damping-matrix format, an n-degree-of-freedom vibration problem will have modal differential equations of the form:

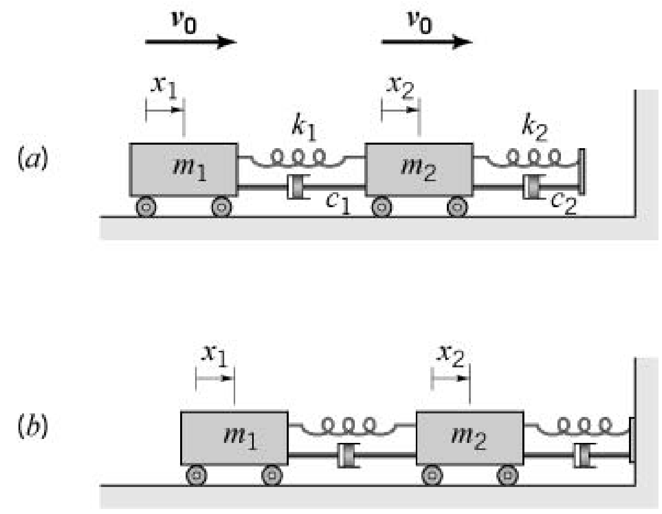
$$\ddot{q}_1 + (\alpha + \beta\omega_{n1}^2)\dot{q}_1 + \omega_{n1}^2 q_1 = Q_1$$
$$\ddot{q}_2 + (\alpha + \beta\omega_{n2}^2)\dot{q}_2 + \omega_{n2}^2 q_2 = Q_2$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\ddot{q}_n + (\alpha + \beta\omega_{nn}^2)\dot{q}_n + \omega_{nn}^2 q_n = Q_n \ .$$

This is not a very useful or generally applicable result. For lightly damped systems, damping is more often introduced directly in the undamped modal equations via :

$$\ddot{q}_1 + 2\zeta_1\omega_{n1}\dot{q}_1 + \omega_{n1}^2 q_1 = Q_1$$
$$\ddot{q}_2 + 2\zeta_2\omega_{n2}\dot{q}_2 + \omega_{n2}^2 q_2 = Q_2$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\ddot{q}_n + 2\zeta_n\omega_{nn}\dot{q}_n + \omega_{nn}^2 q_n = Q_n \ .$$

The damping factors $\zeta_1, \zeta_2, \ldots, \zeta_n$ are specified for each modal differential equation, based on measurements or experience.

Transient Modal Example Problem 2, Free-Motion with Modal Damping



For the preceding collision problem, with viscous dampers and during contact the vibration model is now

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & (c_1 + c_2) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \ . \qquad (3.129)$$

We will rework the problem using assumed modal damping of

10% for each mode; i.e., $\zeta_1 = \zeta_2 = 0.10$. Hence

$$2\zeta_1 \omega_{n1} = 2 \times 0.1 \times 6.472 = 1.294$$

$$2\zeta_2 \omega_{n2} = 2 \times 0.1 \times 18.92 = 3.785$$

The modal differential equation model now looks like:

$$\ddot{q}_1 + 1.294\dot{q}_1 + 41.886 q_1 = 0 \ , \quad \ddot{q}_2 + 3.785\dot{q}_2 + 358.11 q_2 = 0$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} .$$

The solution to the differential equation $\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = 0$ is

$$Y = e^{-\zeta\omega_n t} (A \cos\omega_d t + B \sin\omega_d t) \ . \qquad \textbf{(3.27)}$$

For the initial conditions $Y(0) = 0$, $\dot{Y}(0) = \dot{Y}_0$, the constants A, B are solved from $Y(0) = 0 = A$, and from

$$\dot{Y} = -\zeta\omega_n e^{-\zeta\omega_n t} B \sin\omega_d t + e^{-\zeta\omega_n t} \omega_d B \cos\omega_d t \ ,$$

$$\dot{Y}(0) = \dot{Y}_0 = B\omega_d \Rightarrow B = \dot{Y}_0 / \omega_d .$$

The solution is

$$Y = \frac{\dot{Y}_0}{\omega_d} e^{-\zeta \omega_n t} \sin\omega_d t \ , \ Y = \frac{\dot{Y}_0}{\omega_n \sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin\omega_d t.$$

Applying this result to the present example gives,

$$\omega_{d1} = 6.472\sqrt{1 - .1^2} = 6.440 \ , \ \omega_{d2} = 18.92\sqrt{1 - .1^2} = 18.82 \ ,$$

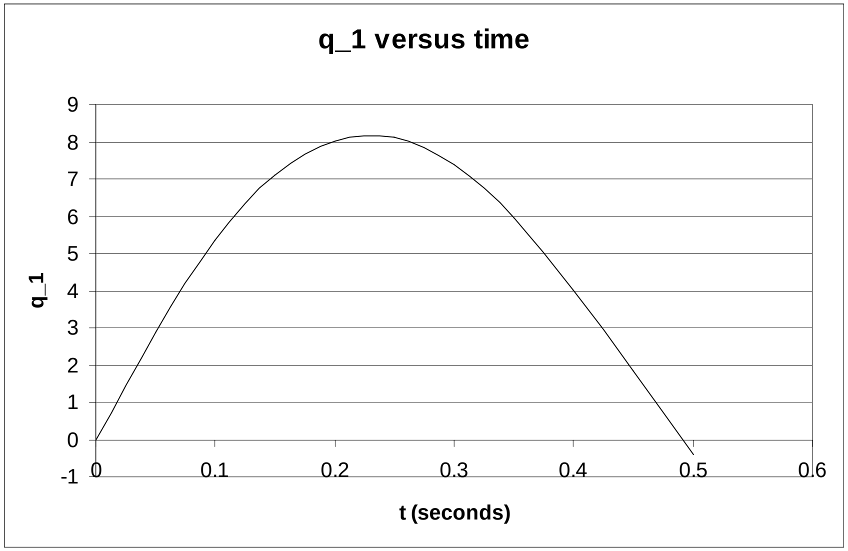
and the modal coordinate solutions are:

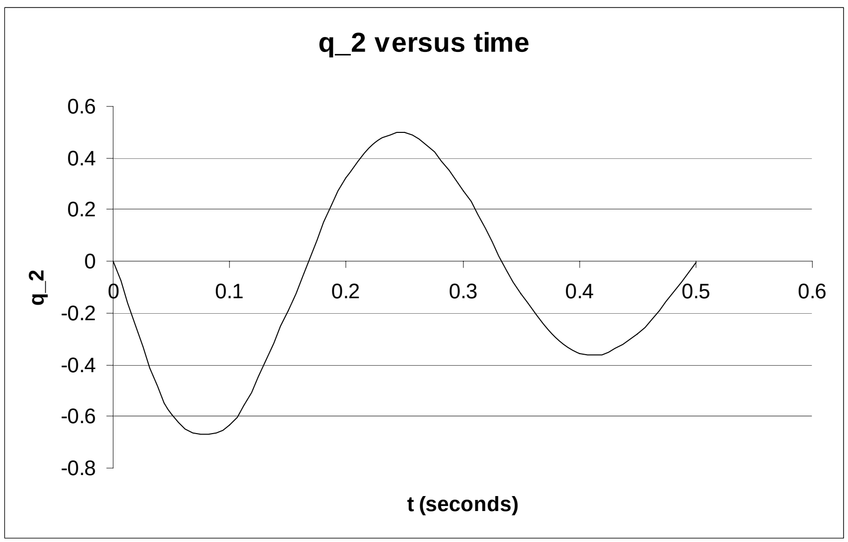
$$q_1 = \frac{61.406}{6.440} e^{-.6472t} \sin 6.440t = 9.535 e^{-.6472t} \sin 6.404t$$

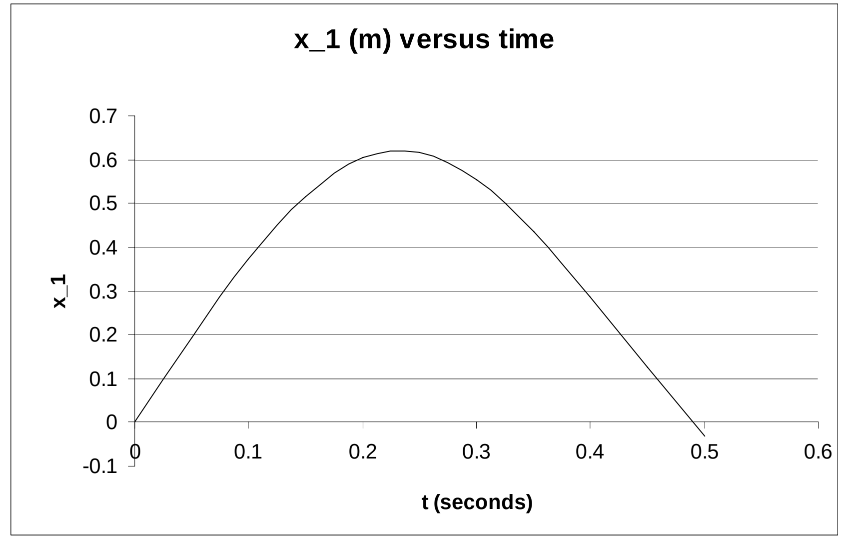
$$q_2 = \frac{-15.136}{18.82} e^{-1.892t} \sin 18.82t = -.804 e^{-1.892t} \sin 18.82t \ ,$$

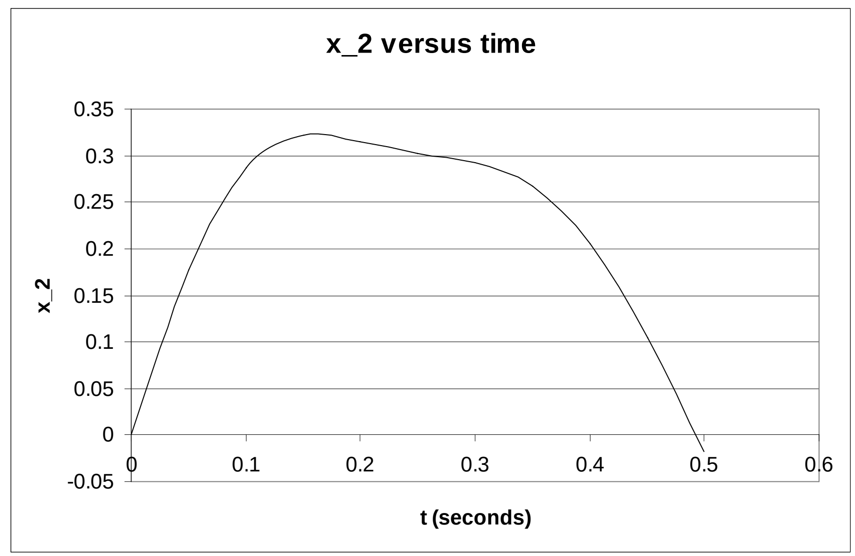
The transformation to obtain the physical coordinates remains unchanged as

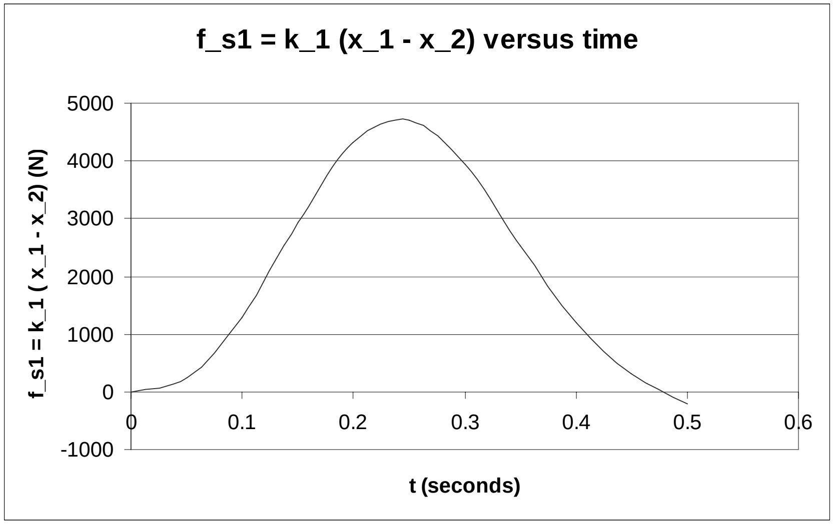
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = q_1(t) \begin{Bmatrix} .0738 \\ .0429 \end{Bmatrix} + q_2(t) \begin{Bmatrix} .0350 \\ -.0903 \end{Bmatrix} .$$

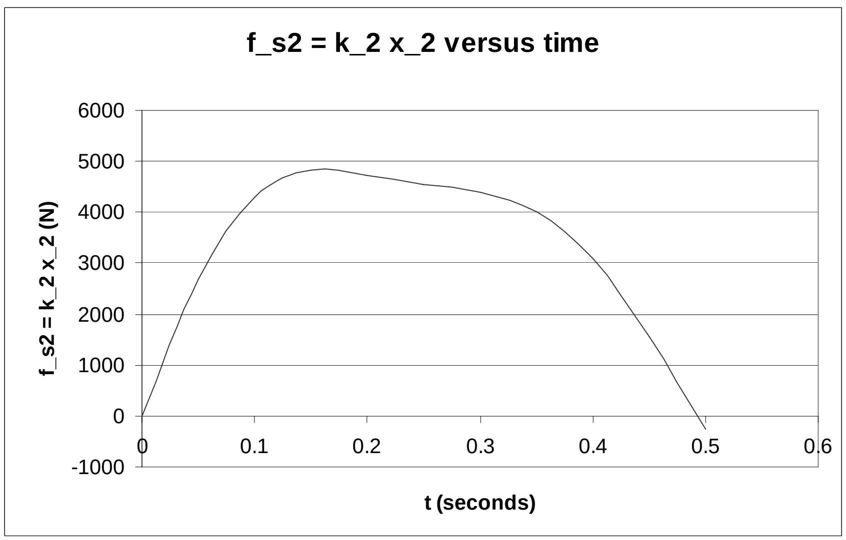

q_1 versus time
q_1
t (seconds)
9
8
7
6
5
4
3
2
1
0
-1
0
0.1
0.2
0.3
0.4
0.5
0.6


q_2 versus time
q_2
t (seconds)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
0.1
0.2
0.3
0.4
0.5
0.6


x_1 (m) versus time
x_1
t (seconds)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6


x_2 versus time
x_2
t (seconds)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6



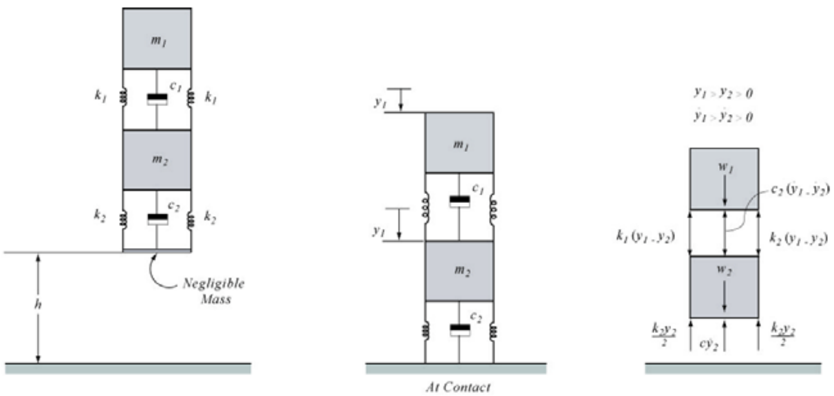


Ad
ding damping reduces the peak force in spring 2 from 5363 N to

4820 N.

Lecture 17. MORE TRANSIENT MOTION USING MODAL COORDINATES

Example



At the left is an assembly that is released from a height $h = 2ft$ above the ground. In the middle, the assembly has just contacted the ground. The subsequent positions of m_1 and m_2 are defined, respectively, by y_1 and y_2. At the time of contact,

$y_1(0) = y_2(0) = 0$, and

$\dot{y}_1(0) = \dot{y}_2(0) = v_0 = \sqrt{2gh} = \sqrt{2 \times 32.2 \times 2} = 11.35 ft/\sec$ ..

Engineering-analysis tasks:

a. Draw free-body diagrams and derive the equations of motion.

b. State the matrix equations of motion.

c. Solve for two cycles of motion for the lowest natural frequency.

Equation of Motion from Free-body diagrams:

$$\begin{aligned} m_1 \ddot{y}_1 &= \sum f_{y_1} = w_1 - k_1 (y_1 - y_2) - c_1 (\dot{y}_1 - \dot{y}_2) \\ m_2 \ddot{y}_2 &= \sum f_{y_2} = w_2 - k_2 y_2 - c_2 \dot{y}_2 + k_1 (y_1 - y_2) + c_1 (\dot{y}_1 - \dot{y}_2) \ . \end{aligned} \quad (1)$$

Matrix Format:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & (c_1 + c_2) \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} . \quad (2)$$

Equations for Modal Coordinates using Modal damping to account for internal damping

$$(\ddot{q})_i + [2\zeta\omega_n](\dot{q}_i) + [\Lambda](q_i) = (Q_i) = [A^*]^T (f_i)$$

$$= \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} A_{11}^* w_1 + A_{21}^* w_2 \\ A_{12}^* w_1 + A_{22}^* w_2 \end{Bmatrix} .$$

Component modal differential equations:

$$\ddot{q}_1 + 2\zeta_1\omega_{n1}\dot{q}_1 + \omega_{n1}^2 q_1 = Q_1 = A_{11}^* w_1 + A_{21}^* w_2$$
$$\ddot{q}_2 + 2\zeta_2\omega_{n2}\dot{q}_2 + \omega_{n2}^2 q_2 = Q_2 = A_{12}^* w_1 + A_{22}^* w_2 \ . \qquad (3)$$

The homogeneous version of Eq.(3) is

$$\ddot{q}_{1h} + 2\zeta_1\omega_{n1}\dot{q}_{1h} + \omega_{n1}^2 q_{1h} = 0 \ , \quad \ddot{q}_{2h} + 2\zeta_2\omega_{n2}\dot{q}_{2h} + \omega_{n2}^2 q_{2h} = 0 \ ,$$

with solutions:

$$q_{1h}(t) = e^{-\zeta_1\omega_{n1}t}(A_1 \cos\omega_{d1}t + B_1 \sin\omega_{d1}t)$$
$$q_{2h}(t) = e^{-\zeta_2\omega_{n2}t}(A_2 \cos\omega_{d2}t + B_2 \sin\omega_{d2}t) \ .$$

The particular solutions $q_{1p}(t), q_{2p}(t)$ corresponding to Eq.(3) are

$$q_{1p} = (A_{11}^* w_1 + A_{21}^* w_2)/\omega_{n1}^2 \ , \ q_{2p} = (A_{12}^* w_1 + A_{22}^* w_2)/\omega_{n2}^2 \ ,$$

yielding the complete modal-coordinate solutions

$$q_1(t) = \frac{(A_{11}^* w_1 + A_{21}^* w_2)}{\omega_{n1}^2} + e^{-\zeta_1\omega_{n1}t}(A_1 \cos\omega_{d1}t + B_1 \sin\omega_{d1}t)$$

$$q_2(t) = \frac{(A_{12}^* w_1 + A_{22}^* w_2)}{\omega_{n2}^2} + e^{-\zeta_2\omega_{n2}t}(A_2 \cos\omega_{d2}t + B_2 \sin\omega_{d2}t) \ .$$

The constants A_i, B_i must be determined from the modal-coordinate initial conditions.

The modal-coordinate initial conditions are defined by

$$(q_0) = [A^*]^T[M](y_0) \ .$$

Similarly, the modal-velocity initial conditions are defined by

$$(\dot{q}_0) = [A^*]^T[M](\dot{y}_0) \ .$$

A previous undamped model had the physical parameters:

$$m_1 = 150\,kg, m_2 = 100\,kg\ ,\ k_1 = k_2 = 1.5\times 10^4 N/m,$$

yielding

$$[M] = \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix},\quad [K] = \begin{bmatrix} 1.5\times 10^4 & -1.5\times 10^4 \\ -1.5\times 10^4 & 3.0\times 10^4 \end{bmatrix}$$

$$\{w_i\} = \begin{Bmatrix} 1457 \\ 981 \end{Bmatrix}.$$

The eigenvalues and natural frequencies are:

$$\omega_{n1}^2 = 41.886\,\sec^{-2} \Rightarrow \omega_{n1} = 6.4720\,\sec^{-1}$$

$$\omega_{n2}^2 = 358.11\,\sec^{-2} \Rightarrow \omega_{n2} = 18.924\,\sec^{-1}\ .$$

The matrix of normalized eigenvectors is

$$[A^*] = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix}.$$

From,

$$\ddot{q}_1 + 2\zeta_1\omega_{n1}\dot{q}_1 + \omega_{n1}^2 q_1 = Q_1 = A_{11}^* w_1 + A_{21}^* w_2$$

$$\ddot{q}_2 + 2\zeta_2\omega_{n2}\dot{q}_2 + \omega_{n2}^2 q_2 = Q_2 = A_{12}^* w_1 + A_{22}^* w_2\ , \qquad (3)$$

the model (with 5% modal damping) is

$$\ddot{q}_1 + 6.47\dot{q}_1 + 41.886\,q_1 = 149.5$$

$$\ddot{q}_2 + 18.92\,\dot{q}_2 + 358.11\,q_2 = -37.63$$

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} .073767 & .035002 \\ .042866 & -.090344 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}.$$

From $(q_0) = [A^*]^T[M](y_0)$, the modal-coordinate initial conditions are zero. From $(\dot{q}_0) = [A^*]^T[M](\dot{y}_0)$, the modal-velocity initial conditions are

$$\begin{Bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{Bmatrix} = \begin{bmatrix} .073767 & .042866 \\ .035002 & -.090344 \end{bmatrix}\begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix}\begin{Bmatrix} 11.35 \\ 11.35 \end{Bmatrix}$$

$$= \begin{bmatrix} 11.065 & 4.2866 \\ 5.2503 & -9.0344 \end{bmatrix}\begin{Bmatrix} 11.35 \\ 11.35 \end{Bmatrix} = \begin{Bmatrix} 174.22 \\ -42.95 \end{Bmatrix}.$$

Substituting into,

$$q_1(t) = \frac{(A_{11}^* w_1 + A_{21}^* w_2)}{\omega_{n1}^2} + e^{-\zeta_1 \omega_{n1} t}(A_1 \cos\omega_{d1} t + B_1 \sin\omega_{d1} t)$$

$$q_2(t) = \frac{(A_{12}^* w_1 + A_{22}^* w_2)}{\omega_{n2}^2} + e^{-\zeta_2 \omega_{n2} t}(A_2 \cos\omega_{d2} t + B_2 \sin\omega_{d2} t) \ ,$$

nets

$$q_1(t) = \frac{149.5}{41.866} + e^{-0.3235t}(A_1 \cos 6.462t + B_1 \sin 6.462t)$$

$$q_2(t) = \frac{-37.63}{358.11} + e^{-0.946t}(A_2 \cos 18.90t + B_2 \sin 18.90t) \ ,$$

where

$$\omega_{d1} = \omega_{n1}\sqrt{1-\zeta_1^2} = 6.470\sqrt{1-0.0025} = 6.462$$

$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta_2^2} = 18.92\sqrt{1-0.0025} = 18.90 \ .$$

Imposing initial conditions for $q_1(t)$

$$q_1(0) = 0 = \frac{149.5}{41.866} + A_1 \Rightarrow A_1 = -3.571$$

Further

$$\dot{q}_1(t) = -0.3235\, e^{-0.3235t}(A_1 \cos 6.462t + B_1 \sin 6.462t)$$

$$+ 6.462\, e^{-0.3235t}(-A_1 \sin 6.462t + B_1 \cos 6.462t)$$

$$\therefore \ \dot{q}_1(0) = 174.22 = -0.3253\, A_1 + 6.462 B_1$$

$$B_1 = \frac{174.22 + 0.3253 \times -3.571}{6.462} = 26.78$$

The complete solution satisfying the initial conditions is

$$q_1(t) = 3.571 + e^{-0.3235t}(-3.571 \cos 6.462t + 26.78 \sin 6.462t)$$

Similarly, the complete solution for $q_2(t)$ is

$$q_2(t) = -0.1051 + e^{-0.946t}(0.1051 \cos 18.90t + B_2 \sin 18.90t)$$

$$\dot{q}_2(t) = -0.946\, e^{-0.946t}(0.1051 \cos 18.90t + B_2 \sin 18.90t)$$
$$+ e^{-0.946t}\, 18.90(0.1051 \sin 18.90t - B_2 \cos 18.90t)$$

$$\therefore \ \dot{q}_2(0) = -42.95 = -0.946 \times 0.1051 - 18.90 B_2 \Rightarrow B_2 = 2.267$$

The complete solution for $q_2(t)$ satisfying the initial conditions is

$$q_2(t) = -0.1051 + e^{-0.946t}(0.1051\cos 18.90t + 2.267\sin 18.90t)$$

The physical solution is

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = q_1(t)\begin{Bmatrix} .073767 \\ .042866 \end{Bmatrix} + q_2(t)\begin{Bmatrix} 0.035002 \\ -0.090344 \end{Bmatrix} .$$

Lecture 18. FORCED HARMONIC MOTION FOR 2DOF EXAMPLES.

Steady-State Solutions due to Harmonic Excitation
2DOF model— undamped case.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} f_{1o} \\ f_{2o} \end{Bmatrix} \sin\omega t \ , \qquad (3.158)$$

with a harmonic-excitation force vector on the right hand side.

Continuing with the model defined by Eqs.(3.124), (3.145), and (3.151) the modal differential Eqs.(3.153) are

$$\begin{Bmatrix} \ddot{q}_1 + .6340\, q_2 \\ \ddot{q}_1 + 2.336\, q_2 \end{Bmatrix} = \begin{bmatrix} .4597 & .6280 \\ .8881 & -.3251 \end{bmatrix} \begin{Bmatrix} f_{10} \\ f_{20} \end{Bmatrix} \sin\omega t$$

$$= \begin{Bmatrix} .4597 f_{10} + .6280 f_{20} \\ .8881 f_{20} - .3251 f_{20} \end{Bmatrix} \sin\omega t \ .$$

For $f_{10} = f_{20} = 1\,N$, the component modal differential equations are:

$$\ddot{q}_1 + .6340\, q_1 = 1.087 \sin\omega t$$

$$\ddot{q}_2 + 2.366\, q_1 = .563 \sin\omega t \ .$$

Assuming steady-state solutions of the form $q_{1ss} = b_1 \sin\omega t$; $q_{2ss} = b_2 \sin\omega t$, and solving for the unknowns yields

$$q_{1ss}(\omega,t) = \left(\frac{1.087}{\omega_{n1}^2}\right) \frac{\sin\omega t}{1 - (\omega/\omega_{n1})^2} \ ,$$

$$\omega_{n1} = \sqrt{.6340} = .7962 \ rad/\text{sec}$$

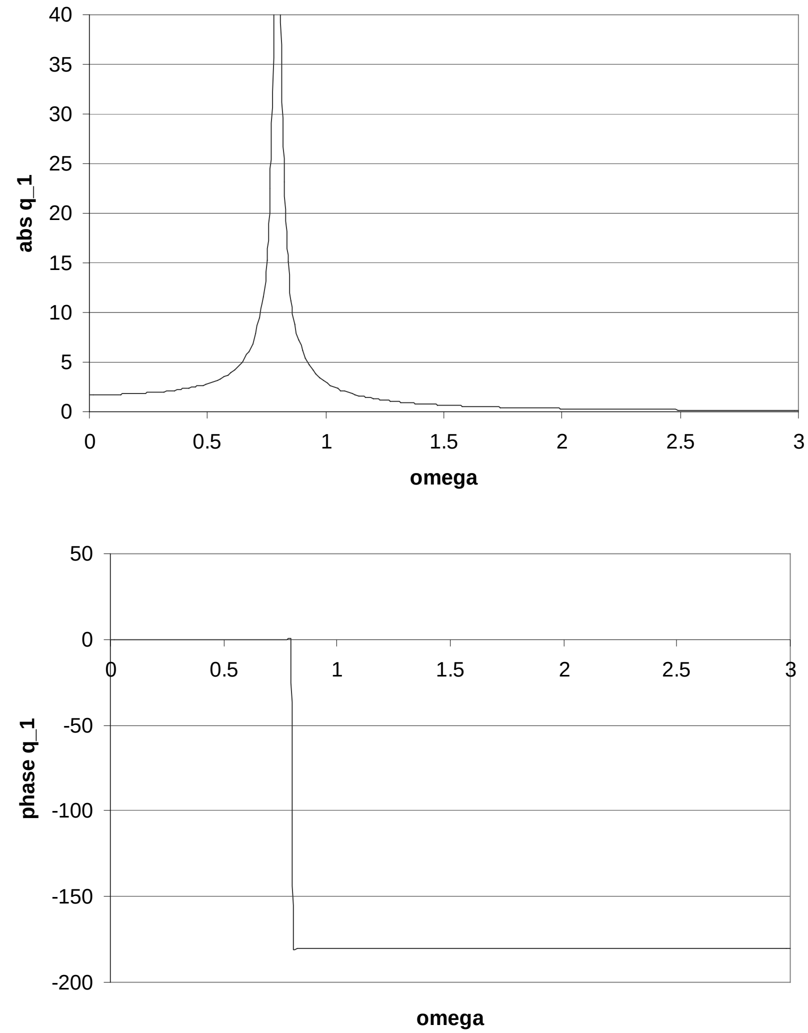
$$q_{2ss}(\omega,t) = \left(\frac{.563}{\omega_{n2}^2}\right) \frac{\sin\omega t}{1 - (\omega/\omega_{n2})^2} \ , \qquad (3.159)$$

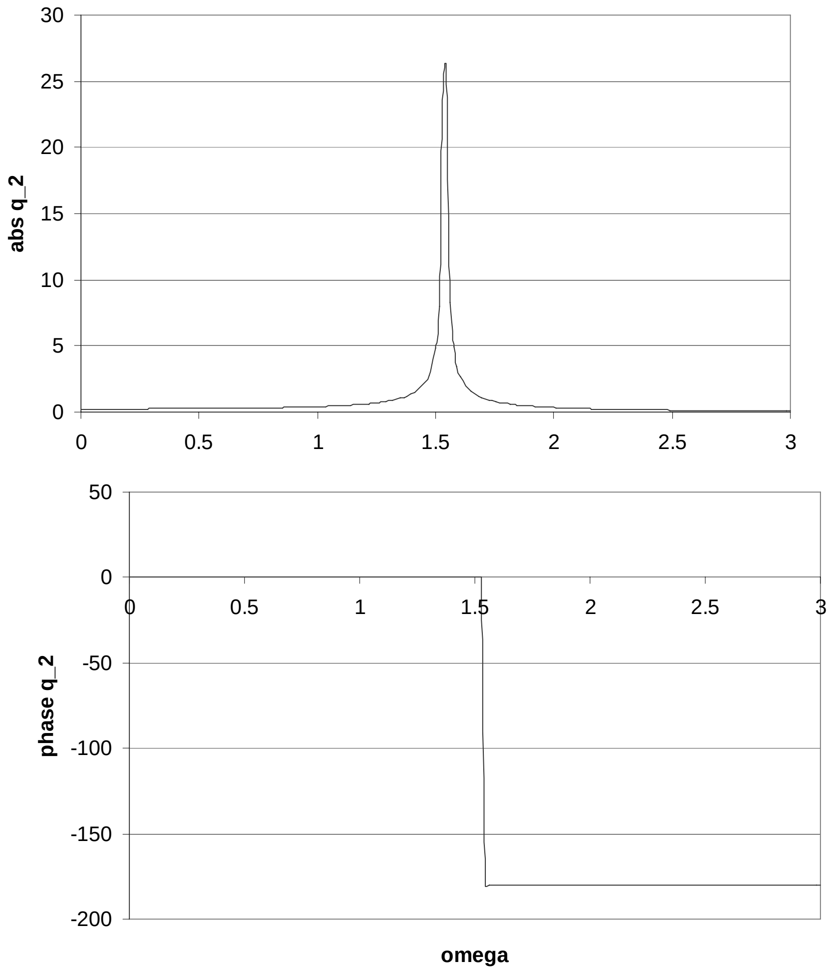
$$\omega_{n2} = \sqrt{2.3660} = 1.538 \ rad/\text{sec} \ .$$

The steady-state, modal-coordinate amplitudes are functions of the ratio of the excitation frequency to the natural frequencies.

The physical coordinates are obtained from the coordinate transformation $(x_i) = [A^*](q_i)$ as

$$\begin{Bmatrix} x_1(\omega,t) \\ x_2(\omega,t) \end{Bmatrix}_{ss} = q_{1ss}(\omega,t) \begin{Bmatrix} .4597 \\ .6280 \end{Bmatrix} + q_{2ss}(\omega,t) \begin{Bmatrix} .8881 \\ -.3251 \end{Bmatrix} . \qquad (3.160)$$





Figur
e 3.57 Steady-state amplitudes and phase for the modal-coordinate solution of Eq.(3.159)

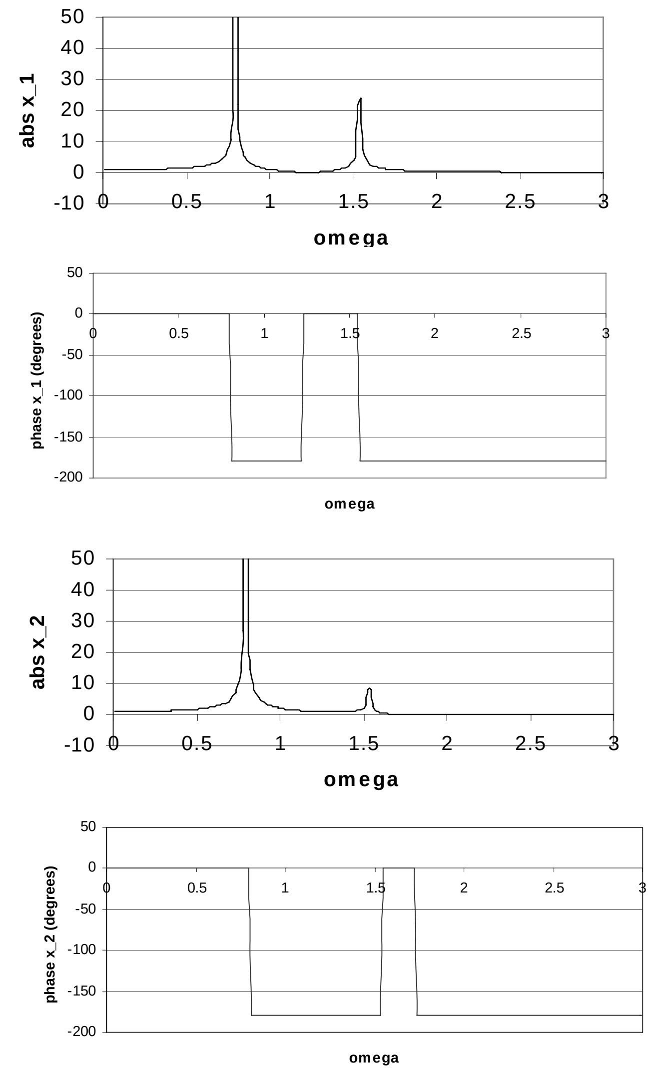
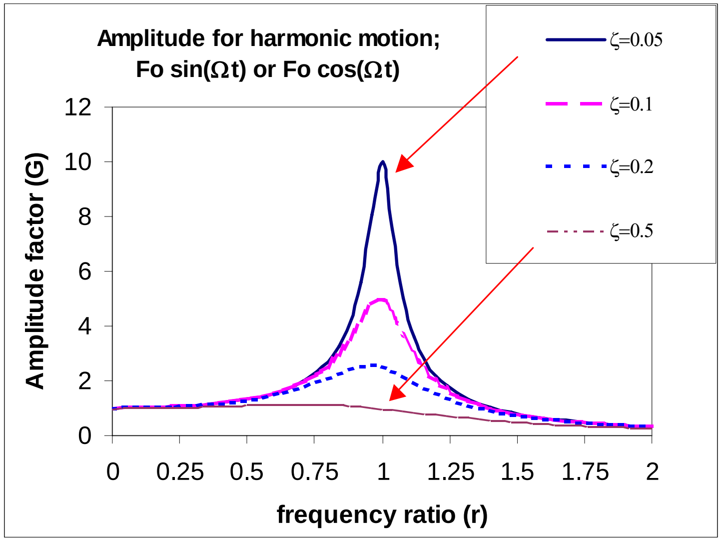


Figure 3.58 Steady-state amplitudes and phase for the physical-coordinate solution of Eq.(3.160).



Modal Truncation. Note from Eq.(3.159) that

$$q_{1ss} \approx 1/\omega_{n1}^2 \;,\; q_{2ss} \approx 1/\omega_{n2}^2$$

Think of a problem with many degrees of freedom with harmonic excitation over a restricted frequency range $[0, \bar{\omega}]$. A mode with a natural frequency ω_i well above the top excitation frequency $\bar{\omega}$ can be "dropped" from the model, because:

(i) its contribution is primarily static; i.e., $\omega_{ni}/\bar{\omega} \ll 1$ and $1/[1-(\omega/\omega_{ni})^2] \cong 1$, and $q_{iss} \cong Q_i/\omega_{ni}^2$.

(ii) the static contribution (being proportional to $1/\omega_{ni}^2$) is itself small.

By using a "modal truncation" a model with several hundred degrees of freedom can frequently be modeled adequately with a greatly-reduced-dimension model by dropping higher-order modes.

2DOF model — Modal Damping

With modal damping, the modal differential equations become

$$\ddot{q}_1 + 2\zeta_1(0.7962)\dot{q}_1 + .6340 q_2 = (.4597 f_{10} \;+\; .6280 f_{20})\sin\omega t$$

$$\ddot{q}_2 + 2\zeta_2(1.5284)\dot{q}_2 + 2.336 q_2 = (.8881 f_{20} \;-\; .3251 f_{20})\sin\omega t \;,$$

where $\omega_{n1} = 0.7962$, $\omega_{n2} = 1.5284$. For $f_{10} = f_{20} = 1N$, and $\zeta_1 = \zeta_2 = 0.1$,the component modal differential equations are:

$$\ddot{q}_1 + .01592\dot{q}_1 + .6340 q_2 = 1.087\sin\omega t$$

$$\ddot{q}_2 + 0.3057\dot{q}_2 + 2.336 q_2 = 0.5630\sin\omega t \;,$$

1DOF ***Forced Excitation***

$$m\ddot{Y} + c\dot{Y} + kY = f_o \sin\omega t \ . \qquad \textbf{(3.32)}$$

Dividing through by the mass m gives

$$\ddot{Y} + 2\zeta\omega_n \dot{Y} + \omega_n^2 Y = (f_o/m)\sin\omega t \ . \qquad \textbf{(3.33)}$$

The steady-solution defined can be restated

$$Y_p(t) = Y_{op} \sin(\omega t + \psi) \qquad \textbf{(3.36)}$$

where Y_{op} is the amplitude of the solution, and ψ is the phase between the solution $Y_p(t)$ and the input excitation force $f(t) = f_o \sin\omega t$. The solution for Y_{op} is

$$\frac{Y_{op}}{f_o/k} = \frac{1}{\{[1 - r^2]^2 + 4\zeta^2 r^2\}^{1/2}} = H(r) \ . \qquad (3.38)$$

where

$$\frac{\omega}{\omega_n} = \mathit{Frequency\ ratio} = r \ , \quad \frac{f_o}{k} = \mathit{static\ deflection}$$

Note that Eq.(3.38) can be stated

$$\begin{aligned} Y_{op} &= \frac{f_o}{k} \times \frac{1}{\{[1 - r^2]^2 + 4\zeta^2 r^2\}^{1/2}} \\ &= \frac{f_o}{k} \times \frac{m}{m} \times \frac{1}{\{[1 - r^2]^2 + 4\zeta^2 r^2\}^{1/2}} \\ &= \frac{f_o}{m} \times \frac{1}{\omega_n^2} \times \frac{1}{\{[1 - r^2]^2 + 4\zeta^2 r^2\}^{1/2}} \end{aligned} \qquad (3.38a)$$

The phase is

$$\begin{aligned} \psi(\omega/\omega_n) &= \tan^{-1}\left(\frac{D}{C}\right) = \tan^{-1}\left[\frac{-2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}\right] \\ &= -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right] . \end{aligned} \qquad (3.39)$$

The steady-state solutions are

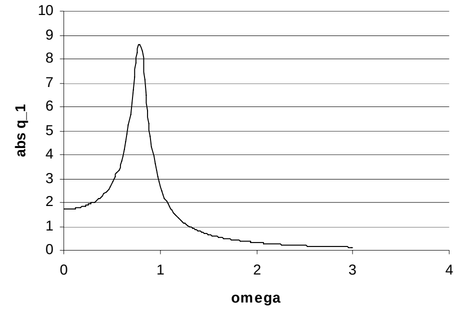
$$q_{1ss} = \frac{1.087}{\omega^2_{n1}} \frac{\sin(\omega t + \psi_1)}{\left[\left(1 - \left(\frac{\omega}{\omega_{n1}}\right)^2\right)^2 + 4\zeta^2_1\left(\frac{\omega}{\omega_{n1}}\right)^2\right]^{1/2}}$$

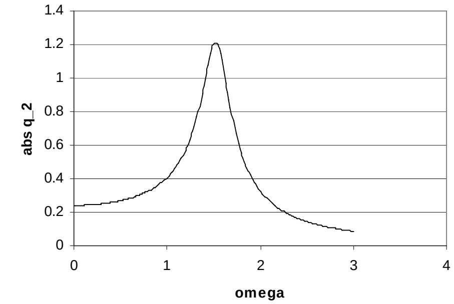
$$q_{2ss} = \frac{0.563}{\omega^2_{n2}} \frac{\sin(\omega t + \psi_2)}{\left[\left(1 - \left(\frac{\omega}{\omega_{n2}}\right)^2\right)^2 + 4\zeta^2_2\left(\frac{\omega}{\omega_{n2}}\right)^2\right]^{1/2}} ,$$

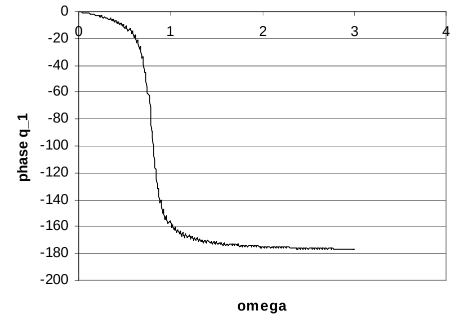
where $\omega_{n1} = 0.7962\, rad/\sec$, $\omega_{n2} = 1.538\, rad/\sec$, and

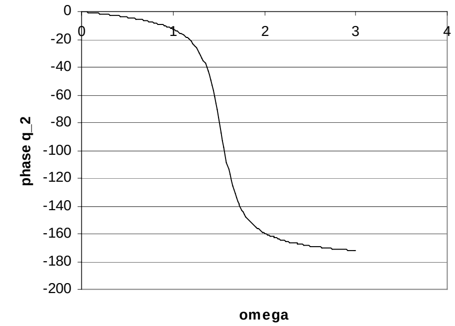
$$\psi_i = -\tan^{-1}\left(\frac{2\zeta_i \frac{\omega}{\omega_{ni}}}{1 - \left(\frac{\omega}{\omega_{ni}}\right)^2}\right) .$$

The modal synchronous response solution is shown below. Modal damping is seen to sharply reduce the peak amplitudes and significantly shift the phase.

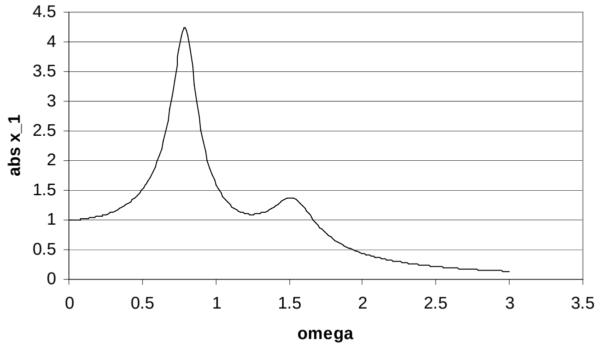


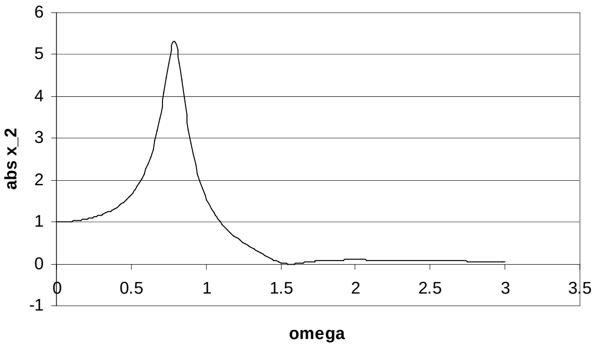






The physical solution is given below. Note the dramatic reduction in amplitudes due to 10% damping.





Direct (non Modal) Harmonic Solution

Substituting the assumed steady-state solution, $x_1 = X_1 \sin\omega t,\ x_2 = X_2 \sin\omega t$, into Eq.(3.158) yields

$$\begin{bmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin\omega t = \begin{Bmatrix} f_{10} \\ f_{20} \end{Bmatrix} \sin\omega t \ . \tag{3.161}$$

Solving for the unknowns using Cramer's rule gives

$$X_1 = \frac{f_{10}\,[\,-m_2\omega^2 + (k_2 + k_3)\,] + f_{20}k_2}{m_1 m_2\omega^4 - [m_1(k_2+k_3) + m_2(k_1+k_2)]\ \omega^2 + (k_1+k_2)(k_2+k_3) - k_2^2}$$

$$X_2 = \frac{f_{10}\ k_2 + f_{20}[(k_1+k_2) - m_1\omega^2]}{m_1 m_2\omega^4 - [m_1(k_2+k_3) + m_2(k_1+k_2)]\ \omega^2 + (k_1+k_2)(k_2+k_3) - k_2^2} \ . \tag{3.162}$$

The denominator equals Δ, the determinant that we set equal to zero to find the eigenvalues and natural frequencies.

Forced Harmonic Response with Damping in Multi-Degree-of-Freedom Systems.

$$[M](\ddot{x}_i)+[C](\dot{x}_i)+[K](x_i)=(f_{i0})\sin\omega t \ , \qquad (3.163)$$

where (x_i) is an n-dimensional vector of physical coordinates, and (f_{i0}) is an n-dimensional vector of physical forces.

Assumed solution

$$(x_{iss}) = (x_{is}) \sin\omega t + (x_{ic}) \cos\omega t$$

$$(\dot{x}_{iss}) = \omega(x_{is}) \cos\omega t - \omega(x_{ic}) \sin\omega t \qquad (3.164)$$

$$(\ddot{x}_{iss}) = -\omega^2(x_{is}) \sin\omega t - \omega^2(x_{ic}) \cos\omega t.$$

Substituting into Eq.(3.163) gives

$$[M](-\omega^2(x_{is})\sin\omega t-\omega^2(x_{ic})\cos\omega t)$$

$$+[C](\omega(x_{is})\cos\omega t-(x_{ic})\sin\omega t)$$

$$+[K]((x_{is})\sin\omega t+(x_{ic})\cos\omega t) = (f_{i0})\sin\omega t \ .$$

Equating coefficients of $\sin\omega t, \cos\omega t$ on both sides of this equation gives

$$\cos\omega t: -\omega^2[M](x_{ic})+\omega[C](x_{is})+[K](x_{ic})=(0)$$

$$\sin\omega t: -\omega^2[M](x_{is})-\omega[C](x_{ic})+[K](x_{is})=(f_{i0}) \ .$$

Combining these equations gives,

$$\begin{bmatrix} -\omega^2[M]+[K] & \omega[C] \\ -\omega[C] & -\omega^2[M]+[K] \end{bmatrix} \begin{Bmatrix} (x_{ic}) \\ (x_{is}) \end{Bmatrix} = \begin{Bmatrix} (0) \\ (f_{i0}) \end{Bmatrix} ,(3.165)$$

a single matrix equation in the 2-n unknowns $(x_{ic}),(x_{is})$.

The solutions for the steady-state response components is normally given in terms of amplitude and phase as defined by

$$x_i=x_{is}\sin\omega t+x_{ic}\cos\omega t=X_i\sin(\omega t+\varphi_i) \ ,$$

where

$$X_i=(x_{ic}^2+x_{is}^2)^{-1/2} \ , \quad \varphi_i=\tan^{-1}(x_{ic}/x_{is}) \ . \qquad (3.166)$$

Note that the phase angle defines the phase of the response with respect to the input excitation vector that is driving the system at $\sin\omega t$. In general, with damping the eigenvalues and eigenvectors are complex, and solutions based on eigen analysis are significantly more complicated.

Lessons

a. For undamped systems with symmetric mass $[M]$ and stiffness $[K]$ matrices, the eigenvalues and eigenvectors are real.

b. The matrix of real eigenvectors $[A]$ can be used to uncouple the symmetric stiffness and mass matrices yielding uncoupled modal differential equations.

c. The matrix of eigenvectors can be normalized such that $[A^*]^T[M][A^*] = [I]$ and $[A^*]^T[K][A^*] = [\Lambda]$.

d. Modal coordinate and velocity initial conditions can be calculated from the transformation $(x_i) = [A^*](q_i)$ as $(q_{i0}) = [A^*]^T[M](x_{i0})$ and $(\dot{q}_{i0}) = [A^*]^T[M](\dot{x}_{i0})$.

e. For free motion, the physical solution consists of a summation of the modal solutions times their eigenvectors.

f. With harmonic excitation, the physical solution consists of a summation of the modal solutions times their eigenvectors.

g. Harmonic solutions can be directly calculated for vibration systems defined by symmetric stiffness and mass matrices and general damping matrices. The solution is obtained in terms of amplitudes and phase values at each

excitation frequency.