# Lecture 21. MORE PLANAR KINEMATIC EXAMPLES

## 4.5c Another Slider-Crank Mechanism

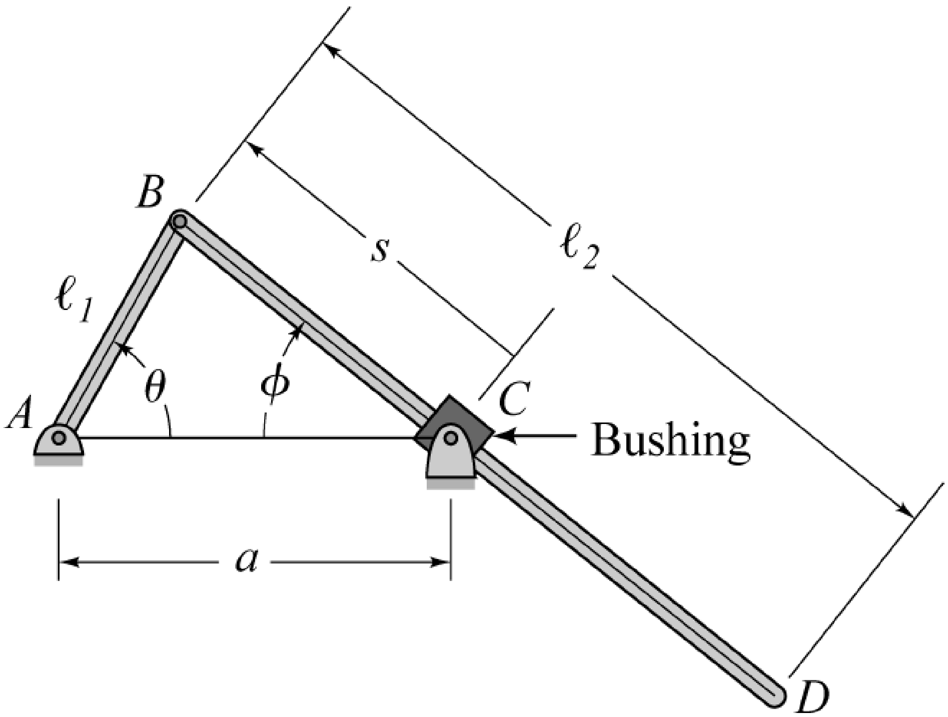



**Figure 4.24** Alternative slider-crank mechanism.

Engineering-analysis task: *For* $\dot{\theta} = \omega$ = *constant, determine* $\varphi$ *and S and their first and second derivatives for one cycle of* $\theta$.

***Geometric Approach***. From figure 4.22:

$$X: \quad l_1 \cos\theta + S\cos\varphi = a$$

$$Y: \quad l_1 \sin\theta = S\sin\varphi \quad . \qquad \textbf{(4.25)}$$

Reordering these equations to

$$S\cos\varphi = a - l_1\cos\theta$$
$$S\sin\varphi = l_1\sin\theta\ , \tag{4.27a}$$

emphasizes that θ is the input coordinate, with φ and $S$ the output coordinates. These equations are nonlinear but can be solved for φ and $S$ in terms of θ, via

$$S^2(\cos^2\varphi + \sin^2\varphi) = (a - l_1\cos\theta)^2 + (l_1\sin\theta^2)$$
$$\therefore\quad S^2 = a^2 - 2al_1\cos\theta + l_1^2\ , \tag{4.27b}$$

and

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{l_1\sin\theta}{a - l_1\cos\theta}\ . \tag{4.27c}$$

Differentiating Eqs.(4.25) nets:

$$\dot{S}\cos\varphi - S\sin\varphi\,\dot{\varphi} = l_1\sin\theta\,\dot{\theta}$$
$$\dot{S}\sin\varphi + S\cos\varphi\,\dot{\varphi} = l_1\cos\theta\,\dot{\theta}\ , \tag{4.28a}$$

or

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} \dot{S} \\ S\dot{\varphi} \end{Bmatrix} = l_1 \ \omega \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix} . \qquad \textbf{(4.28b)}$$

Differentiating Eq.(4.28a) w.r.t. time nets:

$$\begin{aligned} \ddot{S}\cos\varphi - S\sin\varphi\,\ddot{\varphi} - S\cos\varphi\,\dot{\varphi}^2 &= l_1\sin\theta\,\ddot{\theta} + 2\dot{S}\dot{\varphi}\sin\varphi \\ &\quad + l_1\cos\theta\,\dot{\theta}^2 \\ \ddot{S}\sin\varphi + S\cos\varphi\,\ddot{\varphi} - S\sin\varphi\,\dot{\varphi}^2 &= l_1\cos\theta\,\ddot{\theta} - 2\dot{S}\dot{\varphi}\cos\varphi \\ &\quad - l_1\sin\theta\,\dot{\theta}^2 \ . \end{aligned} \qquad \textbf{(4.29)}$$

Substituting $\dot{\theta} = \omega$ , $\ddot{\theta} = 0$ ,and rearranging gives:

$$\begin{aligned} \ddot{S}\cos\varphi - S\ddot{\varphi}\sin\varphi &= l_1\omega^2\cos\theta + 2\dot{S}\dot{\varphi}\sin\varphi + S\cos\varphi\,\dot{\varphi}^2 \\ \ddot{S}\sin\varphi + S\ddot{\varphi}\cos\varphi &= -l_1\omega^2\sin\theta - 2\dot{S}\dot{\varphi}\cos\varphi + S\sin\varphi\,\dot{\varphi}^2 \ . \end{aligned} \qquad \textbf{(4.30a)}$$

The matrix equation for the unknown $\ddot{S}$ and $S\ddot{\varphi}$ is

$$\begin{aligned} &\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} \ddot{S} \\ S\ddot{\varphi} \end{Bmatrix} \\ &= \begin{Bmatrix} l_1\omega^2\cos\theta + 2\dot{S}\,\dot{\varphi}\,\sin\varphi + S\,\dot{\varphi}^2\cos\varphi \\ -l_1\omega^2\sin\theta - 2\dot{S}\,\dot{\varphi}\,\cos\varphi + S\,\dot{\varphi}^2\sin\varphi \end{Bmatrix} . \end{aligned} \qquad \textbf{(4.30b)}$$

The engineering-analysis task is accomplished by executing the following sequential steps:

*1.* Vary $\theta$ over the range $[\ 0, 2\pi\ ]$, yielding discrete values $\theta_i$ .

*2.* For each $\theta_i$ value, solve Eq.(4.25) to determine corresponding values for $\varphi_i$ and $S_i$.

*3.* Enter Eqs.(4.28) with known values for $\theta_i$ , $\varphi_i$ and $S_i$. to determine $\dot{\varphi}_i$ and $\dot{S}_i$.

*4* Enter Eqs.(4.30) with known values for $\theta_i$ , $\varphi_i$ , $S_i$ , $\dot{\varphi}_i$ and $\dot{S}_i$ to determine $\ddot{\varphi}_i$ and $\ddot{S}_i$.

5. Plot $\dot{\varphi}_i$, $\dot{\theta}_i$ , $\ddot{\varphi}_i$ and $\ddot{S}_i$ versus $\theta_i$ .

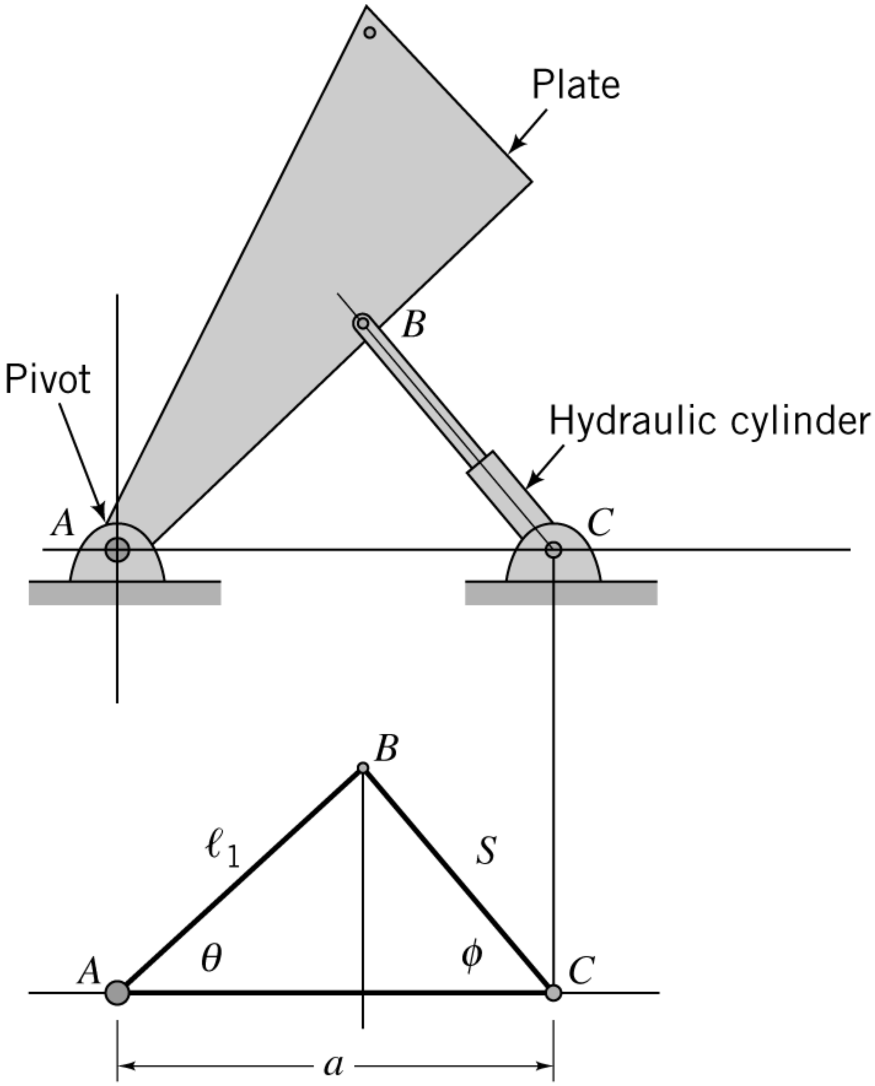



**Figure 4.24** Alternate version of the mechanism in figure 4.22 with *S* as the input.

For S as the input, Eqs.(4.28a) are reordered as:

$$S\dot{\varphi}\sin\varphi + l_1\dot{\theta}\sin\theta = \dot{S}\cos\varphi$$

$$S\dot{\varphi}\cos\varphi - l_1\dot{\theta}\cos\theta = -\dot{S}\sin\varphi \ ,$$

to define $\dot{\varphi}$ and $\dot{\theta}$. Rearranging Eqs.(4.29) defines $\ddot{\varphi}$ and $\ddot{\theta}$ via:

$$-S\sin\varphi\,\ddot{\varphi} - l_1\sin\theta\,\ddot{\theta} = -\ddot{S}\cos\varphi + S\cos\varphi\,\dot{\varphi}^2 + 2\dot{S}\dot{\varphi}\sin\varphi + l_1\cos\theta\,\dot{\theta}^2$$

$$S\cos\varphi\,\ddot{\varphi} - l_1\cos\theta\,\ddot{\theta} = -\ddot{S}\sin\varphi + S\sin\varphi\,\dot{\varphi}^2 - 2\dot{S}\dot{\varphi}\cos\varphi - l_1\sin\theta\,\dot{\theta}^2\ .$$

The basic geometry of figures 4.22 and 4.24 tends to show up regularly in planar mechanisms.

***Vector, Two-Coordinate-System Approach for Velocity and Acceleration Relationships***

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_o + \hat{\dot{\rho}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{4.1}$$

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_o + \hat{\ddot{\boldsymbol{\rho}}} + 2\boldsymbol{\omega} \times \hat{\dot{\boldsymbol{\rho}}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

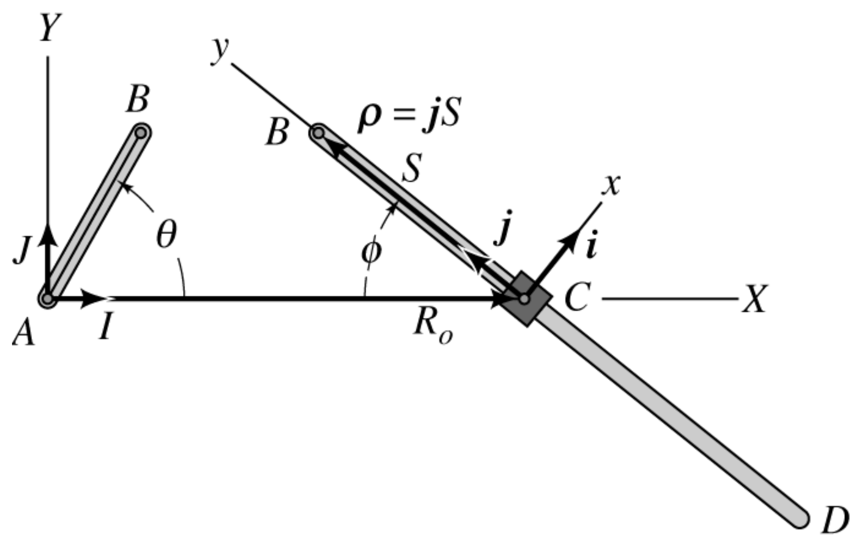



**Figure 4.26**
Two-coordinate arrangement for rod *BD* of the slider-crank mechanism in figure 4.22.

The vector $\boldsymbol{\omega}$ is defined as the angular velocity of the *x*, *y* system relative to the *X*,*Y* system. From figure 4.26 , using the right-hand-screw convention,

$$\boldsymbol{\omega} = -\boldsymbol{K}\dot{\varphi} = -\boldsymbol{k}\dot{\varphi} \ .$$

Given that $\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt}|_{X,Y}$ , we obtain by direct differentiation

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{K}\ddot{\varphi} = -\boldsymbol{k}\ddot{\varphi} \ .$$

From figure 4.26,

$$\boldsymbol{\rho} = \boldsymbol{j}S \ .$$

Differentiating this vector holding $\boldsymbol{j}$ constant gives

$$\dot{\hat{\boldsymbol{\rho}}} = \boldsymbol{j}\dot{S} \ .$$

Differentiating again gives

$$\ddot{\hat{\boldsymbol{\rho}}} = \boldsymbol{j}\ddot{S} \ .$$

Substituting these results into the definitions provided by Eqs.(4.1) gives:

$$\dot{\boldsymbol{r}}_{\boldsymbol{B}} = \boldsymbol{v}_{\boldsymbol{B}} = 0 + \boldsymbol{j}\dot{S} - \boldsymbol{k}\dot{\varphi} \times \boldsymbol{j}S$$

$$\ddot{\boldsymbol{r}}_{\boldsymbol{B}} = \boldsymbol{a}_{\boldsymbol{B}} = 0 + \boldsymbol{j}\ddot{S} + 2(-\boldsymbol{k}\dot{\varphi}) \times \boldsymbol{j}\dot{S} - \boldsymbol{k}\ddot{\varphi} \times \boldsymbol{j}S - \boldsymbol{k}\dot{\varphi} \times (-\boldsymbol{k}\dot{\varphi} \times \boldsymbol{j}S)$$

Carrying through the cross products and completing the algebra nets:

$$\begin{aligned} \boldsymbol{v}_{\boldsymbol{B}} &= \boldsymbol{i}S\dot{\varphi} + \boldsymbol{j}\dot{S} \\ \boldsymbol{a}_{\boldsymbol{B}} &= \boldsymbol{i}(S\ddot{\varphi} + 2\dot{S}\dot{\varphi}) + \boldsymbol{j}(\ddot{S} - S\dot{\varphi}^2) \end{aligned} \qquad \textbf{(4.31)}$$

By comparison to figure 4.25, the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ of the $x,y$ system coincide with the unit vectors $\boldsymbol{\varepsilon}_{\varphi}$ and $\boldsymbol{\varepsilon}_{\mathbf{r2}}$ used in the polar-coordinate solution for $\boldsymbol{v}_{\boldsymbol{B}}$ and $\boldsymbol{a}_{\boldsymbol{B}}$.

Returning to figure 4.25, we can apply Eq.(4.3) to state the velocities and accelerations of points $A$ and $B$ as:

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}$$
$$\boldsymbol{a_B} = \boldsymbol{a_A} + \dot{\boldsymbol{\omega}}_\mathbf{1} \times \boldsymbol{r_{AB}} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega_1} \times \boldsymbol{r_{AB}}) \ .$$

Since point $A$ is fixed, $\boldsymbol{v_A} = \boldsymbol{a_A} = 0$. From the right-hand-rule convention, $\boldsymbol{\omega_1} = \boldsymbol{K}\dot{\theta}$, $\dot{\boldsymbol{\omega}}_\mathbf{1} = \boldsymbol{K}\ddot{\theta}$. From figure 4.25, $\boldsymbol{r_{AB}} = l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$. Substituting, we obtain

$$\boldsymbol{v_B} = 0 + \boldsymbol{K}\dot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)$$

$$\boldsymbol{a_B} = 0 + \boldsymbol{K}\ddot{\theta}_1 \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) + \boldsymbol{K}\dot{\theta} \times [\boldsymbol{K}\dot{\theta} \times l_1(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta)].$$

Carrying out the cross products and algebra gives:

$$\begin{aligned} \boldsymbol{v_B} &= l_1\dot{\theta}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) \\ \boldsymbol{a_B} &= l_1\ddot{\theta}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) - l_1\dot{\theta}^2(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \ . \end{aligned} \tag{4.32}$$

The results in Eqs.(4.32) are given in terms of $\boldsymbol{I}$ and $\boldsymbol{J}$ unit vectors, versus $\boldsymbol{i}$ and $\boldsymbol{j}$ for Eq.(4.31).

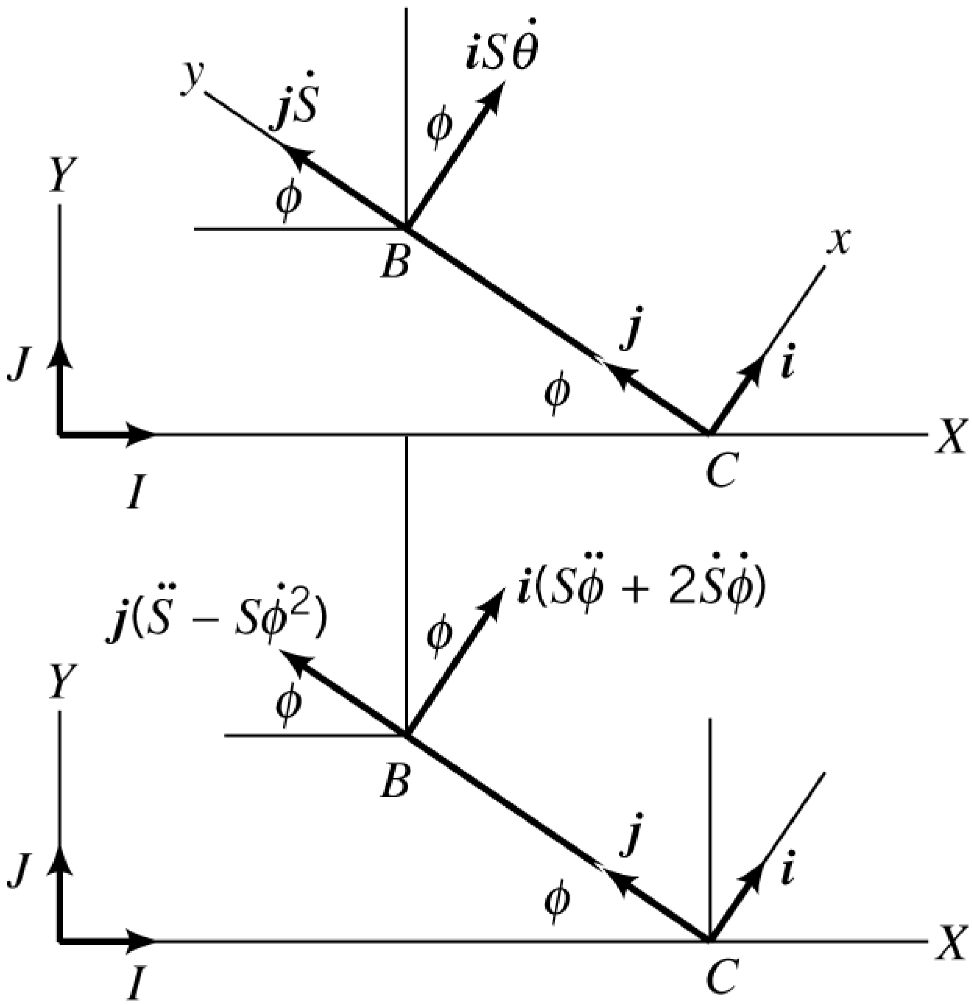



**Figure 4.27** Velocity and acceleration definitions for the velocity point $B$ in the $x$, $y$ system.

From figure 4.27:

$$\boldsymbol{v_B} = S\dot{\varphi}(\boldsymbol{I}\sin\varphi + \boldsymbol{J}\cos\varphi) + \dot{S}(-\boldsymbol{I}\cos\varphi + \boldsymbol{J}\sin\varphi)$$

$$\begin{aligned} \boldsymbol{a_B} &= (S\ddot{\varphi} - S\dot{\varphi}^2)(\boldsymbol{I}\sin\varphi + \boldsymbol{J}\cos\varphi) \\ &+ (\ddot{S} + 2\dot{S}\dot{\varphi})(-\boldsymbol{I}\cos\varphi + \boldsymbol{J}\sin\varphi) \end{aligned} \tag{4.33}$$

Equating these definition with the result from Eqs.(4.32) gives:

$$\boldsymbol{I}: \quad -l_1\dot{\theta}\sin\theta = -\dot{S}\cos\varphi - S\dot{\varphi}\sin\varphi$$

$$\boldsymbol{J}: \quad l_1\dot{\theta}\cos\theta = \dot{S}\sin\varphi + S\dot{\varphi}\cos\varphi \quad ,$$

and

$$\boldsymbol{I}: \; -l_1\ddot{\theta}\sin\theta - l_1\dot{\theta}^2\cos\theta = -(\ddot{S} - S\dot{\varphi}^2)\cos\varphi + (S\ddot{\varphi} + 2\dot{S}\dot{\varphi})\sin\varphi$$

$$\boldsymbol{J}: \quad l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta = (\ddot{S} - S\dot{\varphi}^2)\sin\varphi + (S\ddot{\varphi} + 2\dot{S}\dot{\varphi})\cos\varphi \; ,$$

which repeats our earlier results.

***Solution for the Velocity and Acceleration of Point D***

The simplest approach ( given that we now know $\varphi, \dot{\varphi}$, and $\ddot{\varphi}$ ) is the direct vector formulation. Applying Eqs.(4.3) to points *A* and *B* gives:

$$\boldsymbol{v}_{\boldsymbol{B}} = \boldsymbol{v}_{\boldsymbol{A}} + \boldsymbol{\omega}_{\mathbf{1}} \times \boldsymbol{r}_{\boldsymbol{AB}}$$

$$\boldsymbol{a}_{\boldsymbol{B}} = \boldsymbol{a}_{\boldsymbol{A}} + \dot{\boldsymbol{\omega}}_{\mathbf{1}} \times \boldsymbol{r}_{\boldsymbol{AB}} + \boldsymbol{\omega}_{\mathbf{1}} \times (\boldsymbol{\omega}_{\mathbf{1}} \times \boldsymbol{r}_{\boldsymbol{AB}}) \; .$$

We have already worked through these equations, obtaining solutions for $\boldsymbol{v}_{\boldsymbol{B}}$ and $\boldsymbol{a}_{\boldsymbol{B}}$ in Eqs.(4.32). We can also apply Eqs.(4.3) to points *B* and *D*, since they are points on a rigid body (unlike point $C$ ), obtaining:

$$\boldsymbol{v_D} = \boldsymbol{v_B} + \boldsymbol{\omega_2} \times \boldsymbol{r_{BD}}$$

$$\boldsymbol{a_D} = \boldsymbol{a_B} + \dot{\boldsymbol{\omega}}_{\boldsymbol{2}} \times \boldsymbol{r_{BD}} + \boldsymbol{\omega_2} \times (\boldsymbol{\omega_2} \times \boldsymbol{r_{BD}}) \ .$$

Substituting from Eq.(4.34) for $\boldsymbol{v_B}$ and $\boldsymbol{a_B}$ plus $\boldsymbol{\omega_2} = -\boldsymbol{K}\dot{\varphi}$ , $\dot{\boldsymbol{\omega}}_{\boldsymbol{2}} = -\boldsymbol{K}\ddot{\varphi}$ , and $\boldsymbol{r_{BD}} = l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$ into these equation gives:

$$\boldsymbol{v_D} = l_1\dot{\theta}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) - \boldsymbol{K}\dot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)$$

$$\begin{aligned}\boldsymbol{a_D} &= l_1\ddot{\theta}(-\boldsymbol{I}\sin\theta + \boldsymbol{J}\cos\theta) - l_1\dot{\theta}^2(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \\ &- \boldsymbol{K}\ddot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi) - \boldsymbol{K}\dot{\varphi} \times [-\boldsymbol{K}\dot{\varphi} \times l_2(\boldsymbol{I}\cos\varphi - \boldsymbol{J}\sin\varphi)] \ .\end{aligned}$$

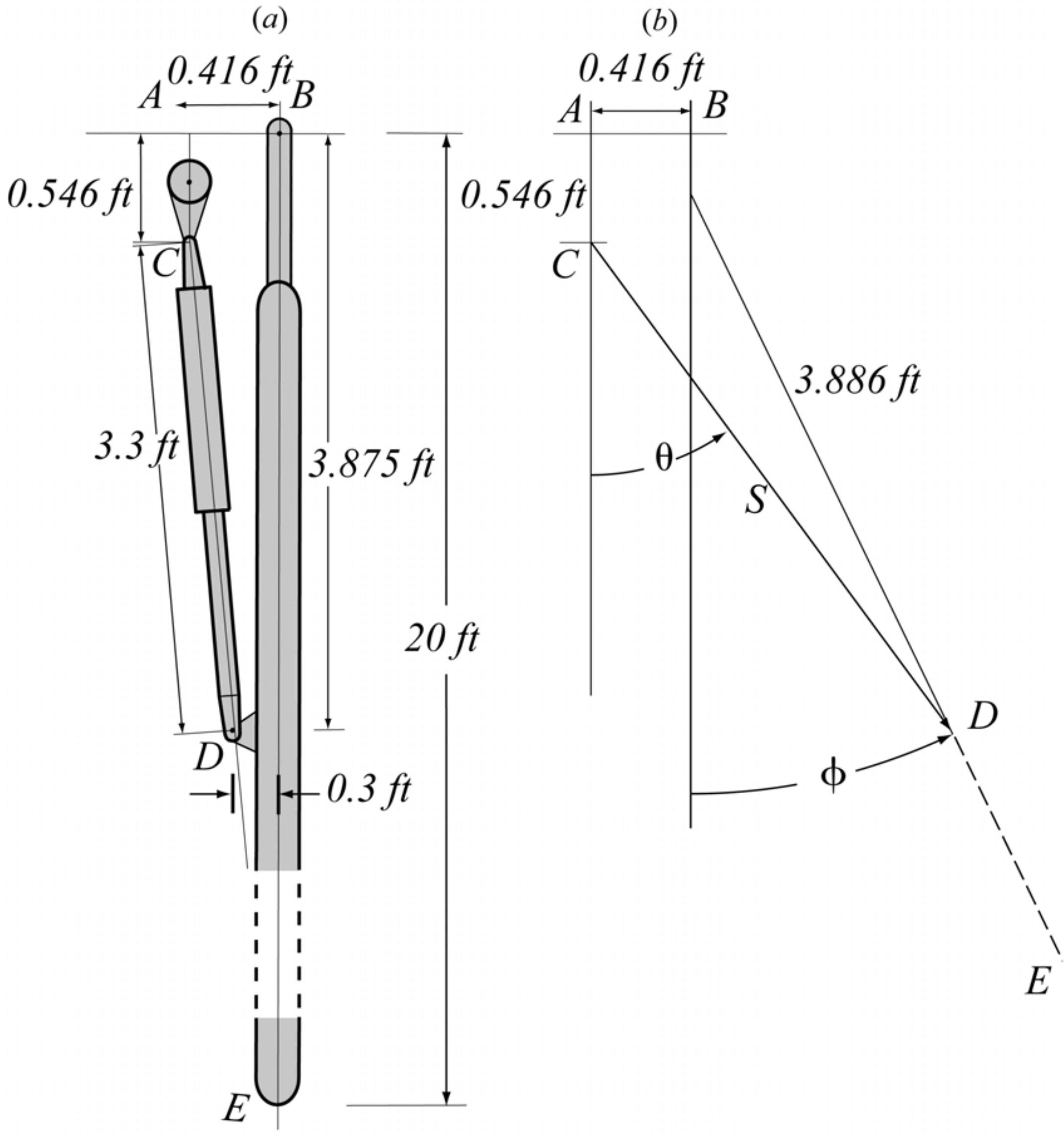
Carrying out the cross products and gathering terms yields:

$$\boldsymbol{v_D} = -\boldsymbol{I}(l_1\dot{\theta}\sin\theta + l_2\dot{\varphi}\sin\varphi) + \boldsymbol{J}(l_1\dot{\theta}\,co\theta - l_2\dot{\varphi}\cos\varphi)$$

$$\begin{aligned}\boldsymbol{a_D} &= \boldsymbol{I}(-l_1\ddot{\theta}\sin\theta - l_1\dot{\theta}^2\cos\theta - l_2\ddot{\varphi}\sin\varphi - l_2\dot{\varphi}^2\cos\varphi) \\ &+ \boldsymbol{J}(l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta - l_2\ddot{\varphi}\cos\varphi + l_2\dot{\varphi}^2\sin\varphi) \ .\end{aligned}$$

These are general equations for $\boldsymbol{v_D}$ and $\boldsymbol{a_D}$ . Substituting $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$ completes the present effort, with $\varphi, \dot{\varphi}$ , and $\ddot{\varphi}$ defined, respectively, by Eqs.(4.28), (4.29), and (4.30).

**Lesson:** The “best” method for finding the velocity and acceleration of a specific point is frequently not the “best” method for finding kinematic relationships.




**Example Problem 4.6** Figure XP4.6a provides a top view of a power-gate actuator. An electric motor drives a lead screw mounted in the arm connecting points *C* and *D*. Lengthening arm *CD* closes the gate; shortening it opens the gate. During closing action, arm *CD* extends from a length of 3.3 ft to 4.3 ft in about 17 seconds to proceed from a fully open to fully closed positions. The gate reaches its steady extension rate quickly at the outset and decelerates rapidly when the gate nears the closed position.

**Tasks:**

*a*. Draw the gate actuator in a general position and derive governing equations that define the orientations of bars *CD* and *BC* as a function of the length of arm CD.

*b*. Assume that arm *CD* extends at a constant rate (gate is closing) and determine a relationship for the angular velocities of arms *CD* and *BE*.

c. Continuing to assume that bar *CD* extends at a constant rate, determine a relationship for the angular accelerations of arms *CD* and *BE*.

**Solution** From figure XP4.6b:

$$\begin{aligned}&\textit{horizontal}\ :\\&S\sin\theta = .416 + 3.886\sin\varphi\ ;\ 3.886 = \sqrt{3.875^2 + .3^2}\\&\textit{vertical}\ :\\&S\cos\theta = 3.886\cos\varphi - .546\end{aligned}$$

**(i)**

*S* is the input and $\theta, \varphi$ are the unknown output variables. Differentiating Eq.(i) with respect to time gives:

$$\dot{S}\sin\theta + S\cos\theta\,\dot{\theta} = 4.92\cos\varphi\,\dot{\varphi}\ ,\ \dot{S}\cos\theta - S\sin\theta\,\dot{\theta} = -4.92\sin\varphi\,\dot{\varphi}$$

**(ii)**

Rearranging Eq.(ii) and putting them in matrix format gives

$$\begin{bmatrix} 4.92\cos\varphi & -S\cos\theta \\ -4.92\sin\varphi & S\sin\theta \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \end{Bmatrix} = \dot{S} \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix}. \qquad \textbf{(iii)}$$

Differentiating Eq.(ii) gives:

$$\ddot{S}\sin\theta + 2\dot{S}\dot{\theta}\cos\theta + S\cos\theta\,\ddot{\theta} - S\sin\theta\,\dot{\theta}^2$$
$$= 3.886\cos\varphi\,\ddot{\varphi} - 3.886\cos\varphi\,\dot{\varphi}^2$$

$$\ddot{S}\cos\theta - 2\dot{S}\dot{\theta}\sin\theta - S\sin\theta\,\ddot{\theta} - S\cos\theta\,\dot{\theta}^2$$
$$= -3.886\sin\varphi\,\ddot{\varphi} - 3.886\cos\varphi\,\dot{\varphi}^2$$

Rearranging the equations gives:

$$3.886\cos\varphi\,\ddot{\varphi} + S\cos\theta\,\ddot{\theta}$$
$$= \ddot{S}\sin\theta + 2\dot{S}\dot{\theta}\cos\theta - S\sin\theta\,\dot{\theta}^2 + 3.886\cos\varphi\,\dot{\varphi}^2 = g_1$$

$$-3.886\sin\varphi\,\ddot{\varphi} + S\sin\theta\,\ddot{\theta}$$
$$= \ddot{S}\cos\theta - 2\dot{S}\dot{\theta}\sin\theta - S\cos\theta\,\dot{\theta}^2 + 3.886\cos\varphi\,\dot{\varphi}^2 = g_2$$

**(iv)**

In matrix format, Eq.(iv) becomes

$$\begin{bmatrix} 3.886\cos\varphi & -S\cos\theta \\ 3.886\sin\varphi & S\sin\theta \end{bmatrix} \begin{Bmatrix} \ddot{\varphi} \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}. \qquad \textbf{(v)}$$

Figure XP4.6c illustrates the solution for $\theta, \varphi$, $\dot{\theta}, \dot{\varphi}$, and $\ddot{\theta}, \ddot{\varphi}$ versus *S*.

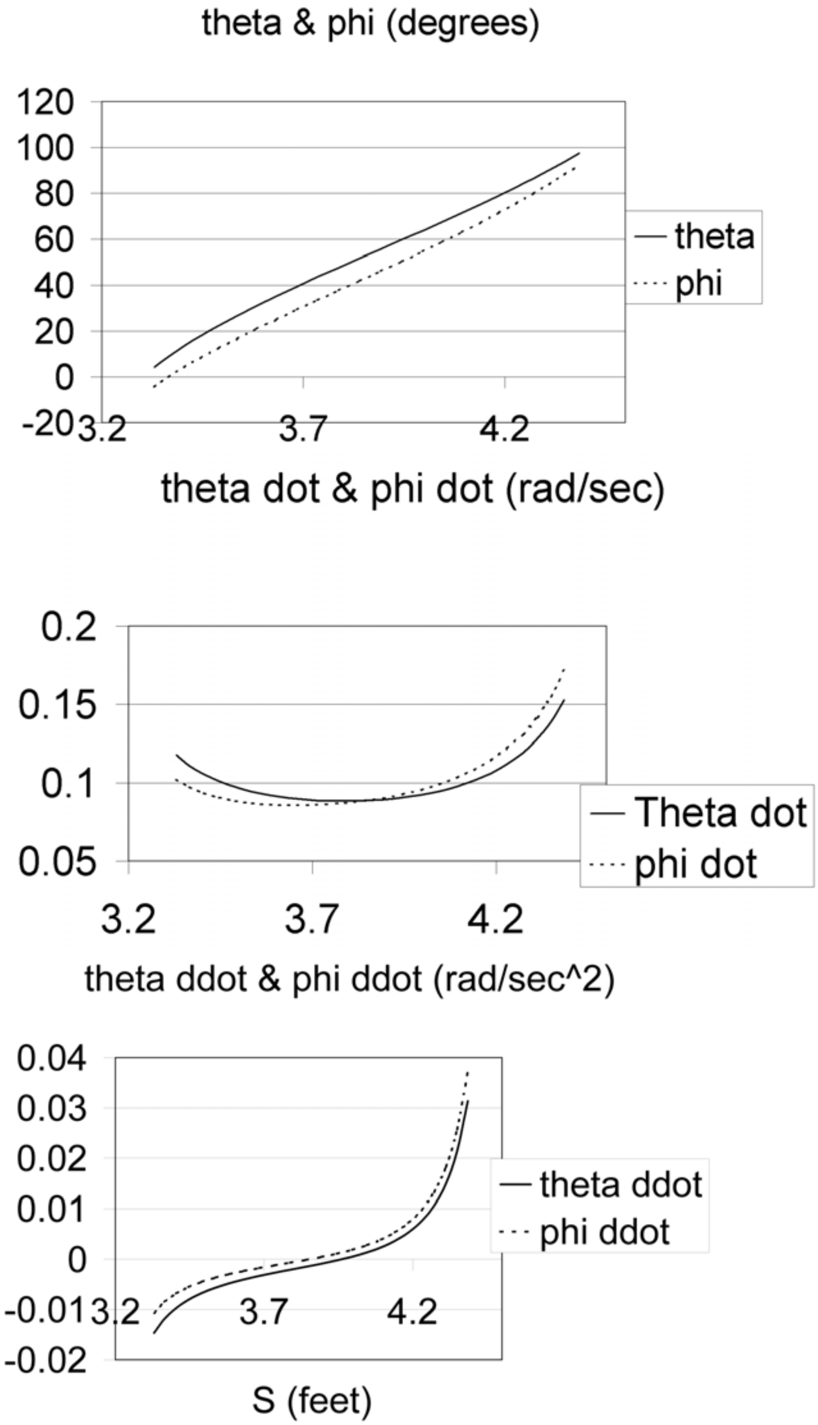



**Figure XP4.6c** Angular positions, velocity, and accelerations versus *S*