

Lecture 28. MORE COMPOUND-PENDULUM EXAMPLES

Spring-Connection Vibration Examples

Nonlinear-Linear Spring relationships

We considered linearization of the pendulum equation earlier in this section. Linearization of connecting spring and damper forces for small motion of a pendulum is the subject of this lecture.

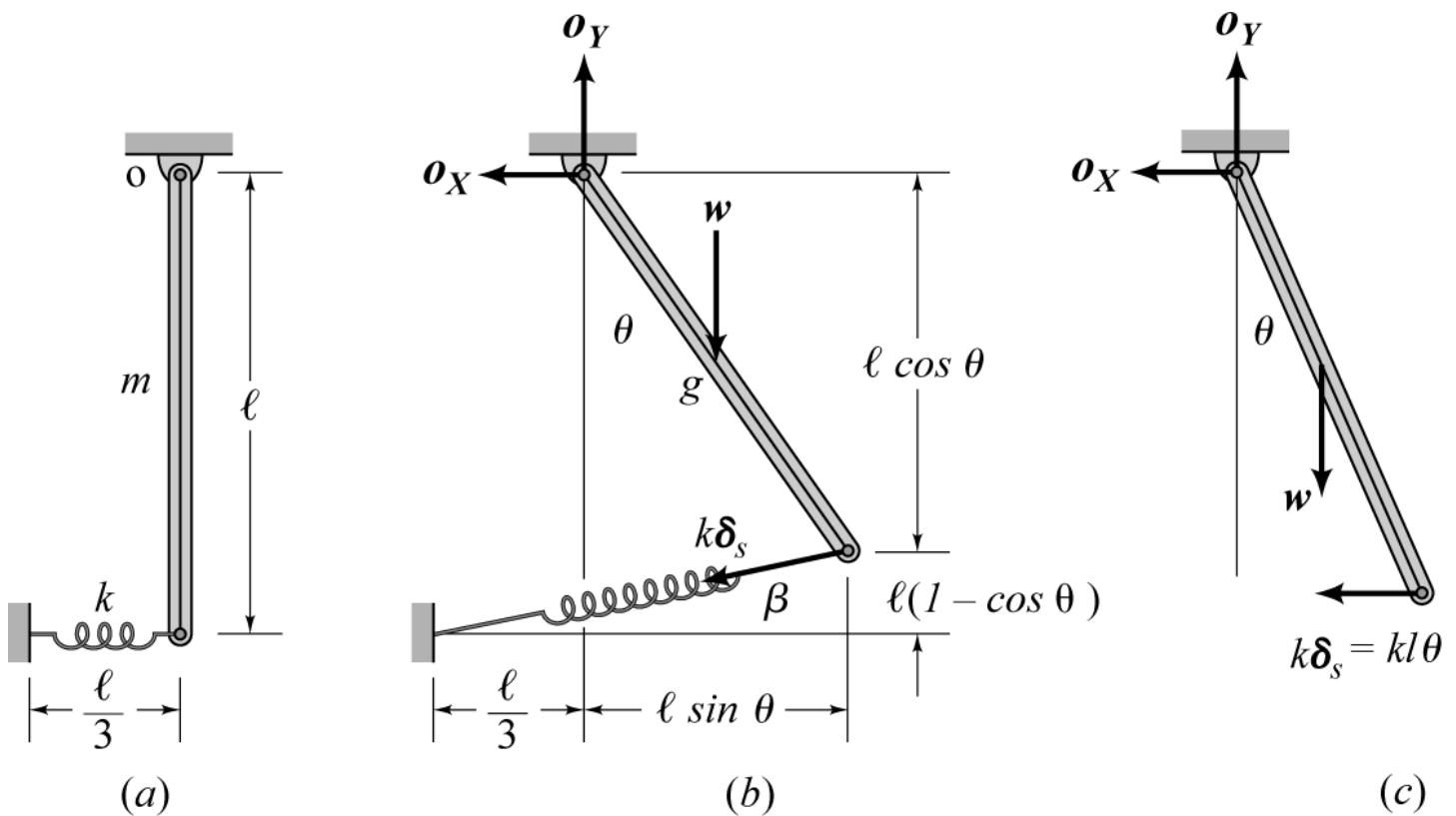


Figure 5.18 Compound pendulum with spring attachment to ground. (a) At rest in equilibrium, (b) General position, (c) Small-angle free-body diagram

The spring has length $l/3$ and is undeflected at $\theta = 0$. The following engineering-analysis tasks apply:

- a. Draw free-body diagrams and derive the EOM*
- b. For small θ develop the linearized EOM.*

Figure 5.18B provides the free-body diagram illustrating the stretched spring. The deflected spring length is

$$\begin{aligned}
 l_s^2 &= \left(\frac{l}{3} + l \sin \theta \right)^2 + [l(1 - \cos \theta)]^2 \\
 &= l^2 \left(\frac{1}{9} + \frac{2 \sin \theta}{3} + \sin^2 \theta + 1 - 2 \cos \theta + \cos^2 \theta \right) \quad (5.56) \\
 &= \frac{l^2}{9} (19 + 6 \sin \theta - 18 \cos \theta) .
 \end{aligned}$$

Hence, the spring force is

$$f_s = k \delta_s = k \left(l_s - \frac{l}{3} \right) ,$$

and it acts at the angle β from the horizontal defined by

$$\sin \beta = l(1 - \cos \theta)/l_s , \quad \cos \beta = \left(\frac{l}{3} + l \sin \theta \right)/l_s . \quad (5.57)$$

The pendulum equation of motion is obtained by a moment equation about the pivot point, yielding

$$\begin{aligned}\Sigma M_o &= I_o \ddot{\theta} = -w \frac{l}{2} \sin \theta - k \delta_s \cos \beta \times l \cos \theta - k \delta_s \sin \beta \times l \sin \theta \\ &= -w \frac{l}{2} \sin \theta - k \delta_s l \cos(\theta - \beta)\end{aligned}$$

Substituting for $\delta_s, \cos \beta, \sin \beta$ (plus a considerable amount of algebra) yields

$$\begin{aligned}\frac{m l^2}{3} \ddot{\theta} + \frac{w l}{2} \sin \theta \\ + \frac{k l^2}{3} (\cos \theta + 3 \sin \theta) \left[1 - \frac{1}{(19 + 6 \sin \theta - 18 \cos \theta)^{1/2}} \right] = 0 .\end{aligned}\tag{5.58}$$

This is a “geometric” nonlinearity. The spring is linear, but the finite θ rotation causes a nonlinearity.

For small θ , expanding $\delta_s = l_s - l/3$ with l_s defined by Eq.(5.56) in a Taylor’s series expansion gives $\delta_s \approx l\theta$. Also, for small θ , a Taylor series expansion gives $\beta \approx \sin \beta \approx -3\theta^2/2 \approx 0$; hence, for small θ , $\cos(\theta - \beta) \approx \cos \theta \approx 1$ the spring force acts perpendicular to the pendulum axis. For small θ , the moment equation reduces to

$$\frac{ml^2}{3}\ddot{\theta} = -w\frac{l}{2}\theta - k(l\theta) \times 1 \times l \quad (5.59)$$

$$\therefore \frac{ml^2}{3}\ddot{\theta} + \left(\frac{wl}{2} + kl^2\right)\theta = 0 .$$

For small θ , the spring deflection is $\delta_s = l\theta$, the spring force, $f_s \cong -k\delta_s = -kl\theta$, acts perpendicular to the pendulum, and the moment of the spring force about o is $kl^2\theta$. Also, note that the spring force is independent of its initial spring length. Figure 5.18c provides the small-angle free body diagram. From Eq.(5.59), the natural frequency is

$$\omega_n = \sqrt{\frac{3g}{2l} + \frac{3k}{m}} ,$$

showing (as expected) an increase in the pendulum natural frequency due to the spring's stiffness.

Nonlinear-Linear Damper forces

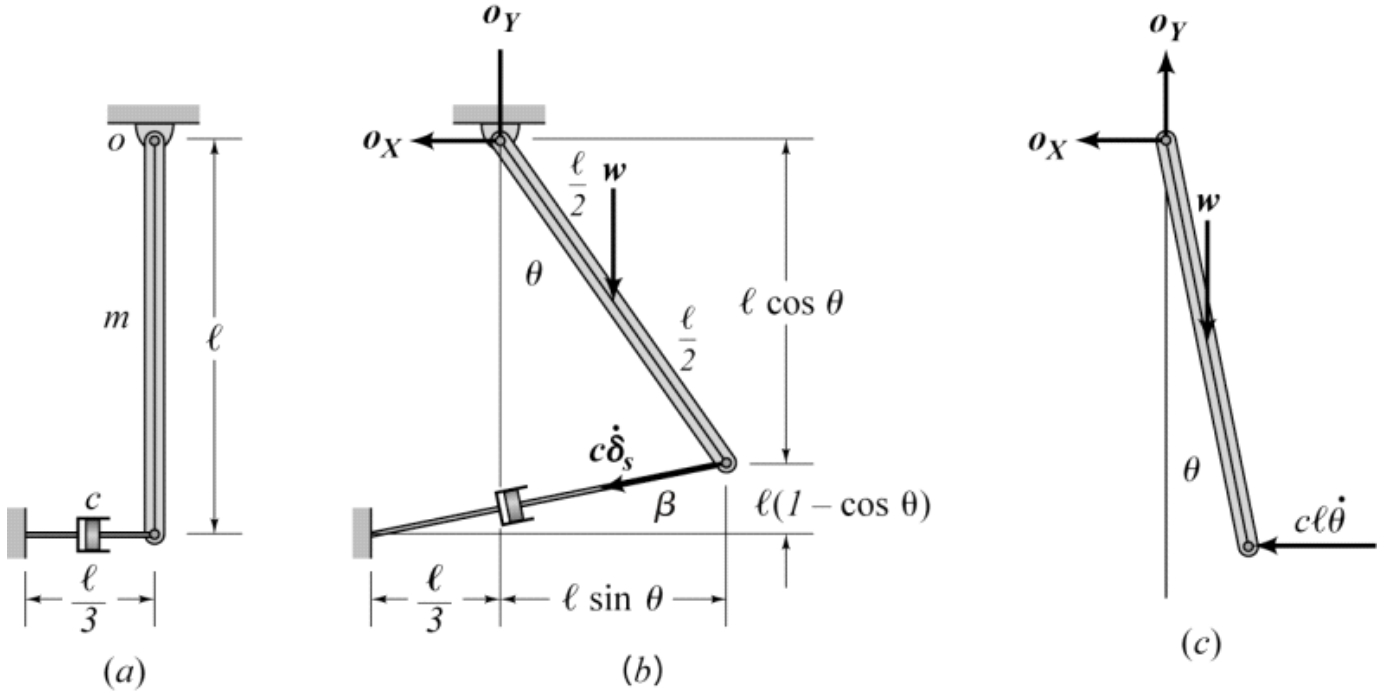


Figure 5.19 Compound pendulum. (a) At rest in equilibrium, (b) General-position free-body diagram, (c) Small rotation free-body diagram

For large θ the damper reaction force, $f_d = -c\dot{\delta}_s$, acts at the angle β from the horizontal. From Eq.(5.56),

$$\dot{\delta}_s = \dot{l}_s = l(\cos \theta/3 + \sin \theta)\dot{\theta} / \sqrt{\frac{19}{9} + \frac{2 \sin \theta}{3} - 2 \cos \theta}$$

For small θ , $\beta \approx 0$, and the damping force acts perpendicular to the pendulum axis and reduces to $f_d = -c l \dot{\theta}$, where $v_\theta = l \dot{\theta}$ is the pendulum's circumferential velocity at the attachment point. Figure 5.19c provides a “small θ ” free-body diagram, yielding the following equation of motion,

$$\Sigma M_o = I_o \ddot{\theta} = -w \frac{l}{2} \theta - l \times c l \dot{\theta} \Rightarrow \frac{m l^2}{3} \ddot{\theta} + c l^2 \dot{\theta} + \frac{w l}{2} \theta = 0 ,$$

with $I_o = m l^2 / 3$ defined in Appendix C. The natural frequency and damping factor are:

$$\omega_n = \sqrt{\frac{3g}{2l}} , \quad 2\zeta\omega_n = c l^2 \times \frac{3}{m l^2} = \frac{3c}{m} .$$

As with the spring, for small θ the damping force $f_d = -c l \dot{\theta}$ is independent of the initial damper length.

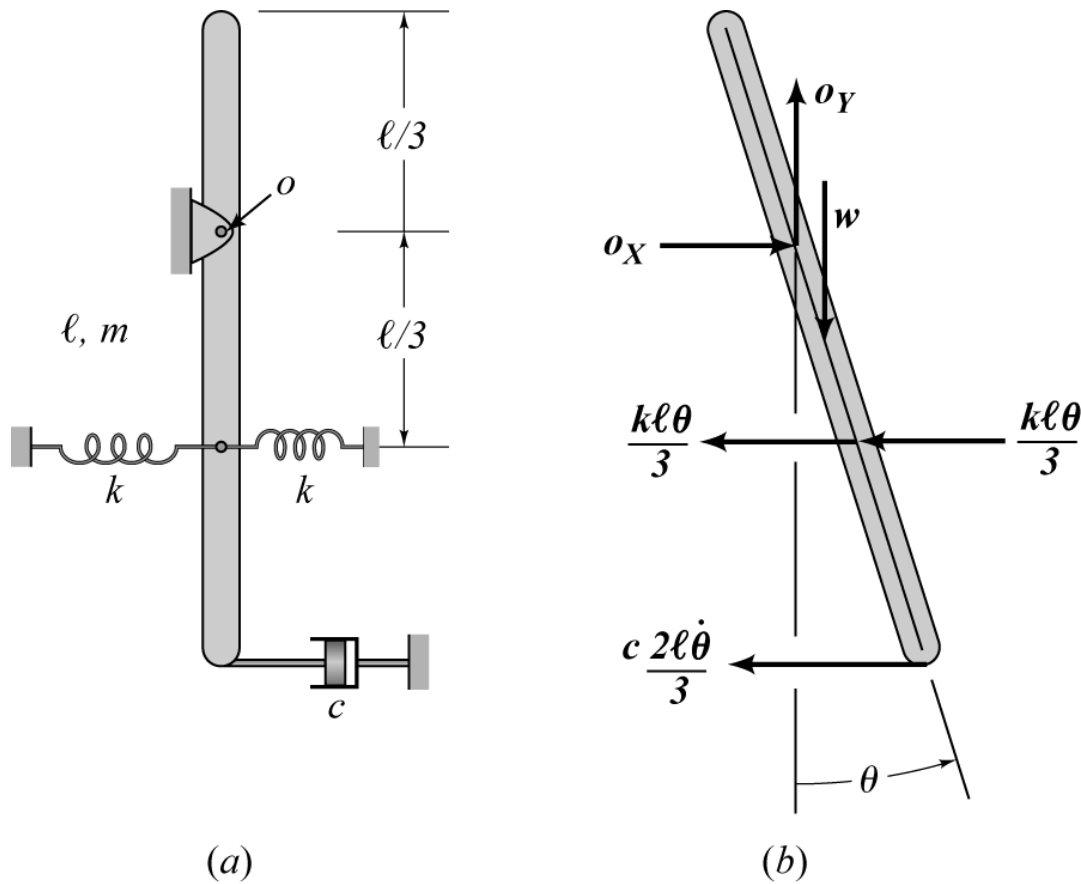


Figure XP5.3 (a) Pendulum attached to ground by two linear springs and a viscous damper, (b) Coordinate and free-body diagram

Figure XP5.3a shows a pendulum with mass $m = .5 \text{ kg}$ and length $\ell = 1 \text{ m}$ supported by a pivot point located $\ell/3$ from the pendulum's end. Two linear spring with stiffness coefficient $k = 15 \text{ N/m}$ are attached to the pendulum a distance $\ell/3$ down from the pivot point, and a linear damper with damping coefficient $c = .5 \text{ Nsec/m}$ is attached to the pendulum's end. The spring is undeflected when the pendulum is vertical. The following engineering analysis tasks apply to this system:

- a. Draw a free-body diagram and derive the differential equation of motion.
- b. Determine the natural frequency and damping factor.

A “small θ ” free-body diagram is given in figure 5.22B.

Taking moments about the pivot point gives

$$I_o \ddot{\theta} = \Sigma M_o = -w \frac{l}{6} \sin \theta - 2 \times k \frac{l\theta}{3} \times \frac{l}{3} - c \frac{2l\dot{\theta}}{3} \times \frac{2l}{3}$$

$$\therefore \frac{ml^2}{9} \ddot{\theta} = -\frac{wl}{6} \sin \theta - \frac{2kl^2}{9} \theta - \frac{4cl^2}{9} \dot{\theta} ,$$

From Appendix C and the parallel-axis formula,

$I_o = ml^2/12 + m(l/6)^2 = ml^2/9$. For $\sin \theta \approx \theta$, the linearized EOM is

$$\ddot{\theta} + \frac{4c}{m} \dot{\theta} + \left(\frac{3g}{2l} + \frac{2k}{m} \right) \theta = 0 .$$

The natural frequency and damping factor are:

$$\omega_n = \left(\frac{3 \times 9.81}{2 \times 1} + \frac{30}{.5} \right)^{1/2} = 8.64 \frac{\text{rad}}{\text{sec}} \Rightarrow f_n = 1.38 \text{ Hz}$$

$$2\zeta\omega_n = \frac{4c}{m} \Rightarrow \zeta = \frac{4c}{2\omega_n m} = \frac{4 \times .5}{2 \times 8.64 \times .5} = .231$$

EOM from Energy, neglecting damping

Select datum through pivot point; hence, $V = -wl/6 \cos \theta$.

$$T + V = T_0 + V_0 \Rightarrow I_o \frac{\dot{\theta}^2}{2} + 2 \times \frac{k}{2} \left(\frac{l\theta}{3} \right)^2 - w \frac{l}{6} \cos \theta = 0 + 0$$

Differentiating w.r.t. θ gives

$$I_o \ddot{\theta} + \frac{2kl^2}{9} \theta + \frac{wl}{6} \sin \theta = 0$$

$I_o = ml^2/9$, and for small θ , $\sin \theta \approx \theta$

$$\frac{ml^2}{9} \ddot{\theta} + \frac{kl^2}{9} \theta + \frac{wl}{6} \theta = 0 \Rightarrow \ddot{\theta} + \left(\frac{3g}{2l} + \frac{k}{m} \right) \theta = 0$$

Spring Supported Bar — Preload and Equilibrium

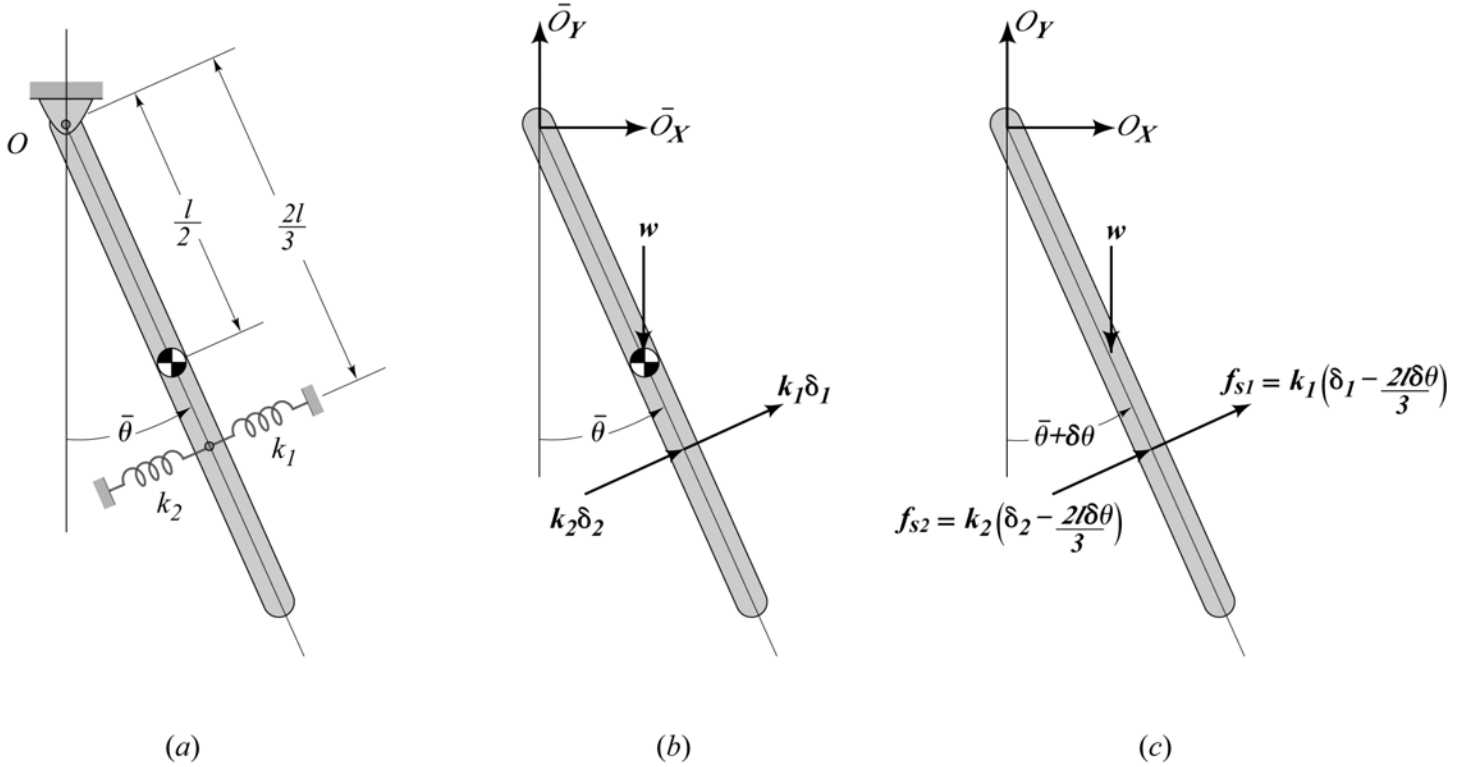


Figure 5.20 Uniform bar, (a) In equilibrium at the angle $\bar{\theta}$, (b) Equilibrium free-body diagram, (c) Displaced position free-body diagram

Bar of mass m and length l in equilibrium at $\theta = \bar{\theta}$ with linear springs having stiffness coefficients k_1, k_2 counteracting the weight w . The springs act at a distance $2l / 3$ from the pivot support point and have been preloaded (stretched or compressed) to maintain the bar in its equilibrium position.

Draw a free-body diagram, derive the EOM, and determine the natural frequency.

Equilibrium Conditions. Taking moments about O in Figure 5.20b gives

$$\Sigma M_O = 0 = w \frac{l}{2} \sin \bar{\theta} - k_1 \delta_1 \frac{2l}{3} - k_2 \delta_2 \frac{2l}{3} . \quad (5.84)$$

Non-equilibrium reaction forces.

Figure 5.20c provides a free-body diagram for a general displaced position defined by the rotation angle $\theta + \delta\theta$. For small $\delta\theta$ the spring-support point moves the perpendicular distance $\delta_s = (2l/3)\theta$. Hence, the stretch of the upper spring decreases from δ_1 to $\delta_1 - (2l/3)\theta$, and the compression of the lower spring decreases from δ_2 to $\delta_2 - (2l/3)\theta$. The spring reaction forces are:

$$f_{s1} = k_1 \left(\delta_1 - \frac{2l}{3} \delta\theta \right) , \quad f_{s2} = k_2 \left(\delta_2 - \frac{2l}{3} \delta\theta \right) .$$

Moment Equation about O

$$\begin{aligned}
 \Sigma M_O = I_O \delta \ddot{\theta} &= -w \frac{l}{2} \sin(\bar{\theta} + \delta\theta) + \frac{2l}{3} k_1 (\delta_1 - \frac{2l}{3} \delta\theta) \\
 &\quad + \frac{2l}{3} k_2 (\delta_2 - \frac{2l}{3} \delta\theta) \\
 &\cong -w \frac{l}{2} (\sin \bar{\theta} + \cos \bar{\theta} \delta\theta) + \frac{2l}{3} k_1 \delta_1 + \frac{2l}{3} k_2 \delta_2 \\
 &\quad - (\frac{2l}{3})^2 (k_1 + k_2) \delta\theta \\
 &= (-w \frac{l}{2} \sin \bar{\theta} + \frac{2l}{3} k_1 \delta_1 + \frac{2l}{3} k_2 \delta_2) - w \frac{l}{2} \cos \bar{\theta} \delta\theta \\
 &\quad - (\frac{2l}{3})^2 (k_1 + k_2) \delta\theta ,
 \end{aligned}$$

after dropping second-order terms in $\delta\theta$. Rearranging provides the EOM ,

$$\begin{aligned}
 \frac{ml^2}{3} \delta \ddot{\theta} + [(\frac{2l}{3})^2 (k_1 + k_2) + w \frac{l}{2} \cos \bar{\theta}] \delta\theta &= 0 \\
 &= -w \frac{l}{2} \sin \bar{\theta} + \frac{2l}{3} k_1 \delta_1 + \frac{2l}{3} k_2 \delta_2 .
 \end{aligned} \tag{5.85}$$

The right-hand side of Eq.(5.85) is zero from the equilibrium

result of Eq.(5.84). If the bar is in equilibrium in a vertical position ($\theta = 0$, $\cos \theta = 1$), the weight contribution to the EOM reverts to the compound pendulum results of Eq.(3.60). For a horizontal equilibrium position, ($\theta = \pi/2$, $\cos \theta = 0$), and the weight term is eliminated. The natural frequency is

$$\omega_n^2 = \frac{3}{ml^2} \left[\left(\frac{2l}{3} \right)^2 (k_1 + k_2) + w \frac{l}{2} \cos \bar{\theta} \right] \quad (5.86)$$

$$\therefore \omega_n = \sqrt{\frac{4(k_1 + k_2)}{3m} + \frac{3g}{2l} \cos \bar{\theta}} .$$

Alternative Equilibrium Condition In figure 5.21a, the lower spring is also assumed to be in tension with a static stretch δ_2 , developing the tension force $k_2 \delta_2$ at equilibrium. Taking moments about O gives the static equilibrium requirement

$$\Sigma M_O = 0 = -w \frac{l}{2} \sin \bar{\theta} + k_1 \delta_1 \frac{2l}{3} - k_2 \delta_2 \frac{2l}{3} . \quad (5.87)$$

The $\delta\theta$ rotation increases the stretch in the lower spring from δ_2 to $\delta_2 + (2l/3)\delta\theta$, decreases the stretch in the upper spring from δ_1 to $\delta_1 - (2l/3)\delta\theta$, and the reaction forces are:

$$f_{s1} = k_1 \left(\delta_1 - \frac{2l}{3} \delta\theta \right) , f_{s2} = k_2 \left(\delta_2 + \frac{2l}{3} \delta\theta \right) .$$

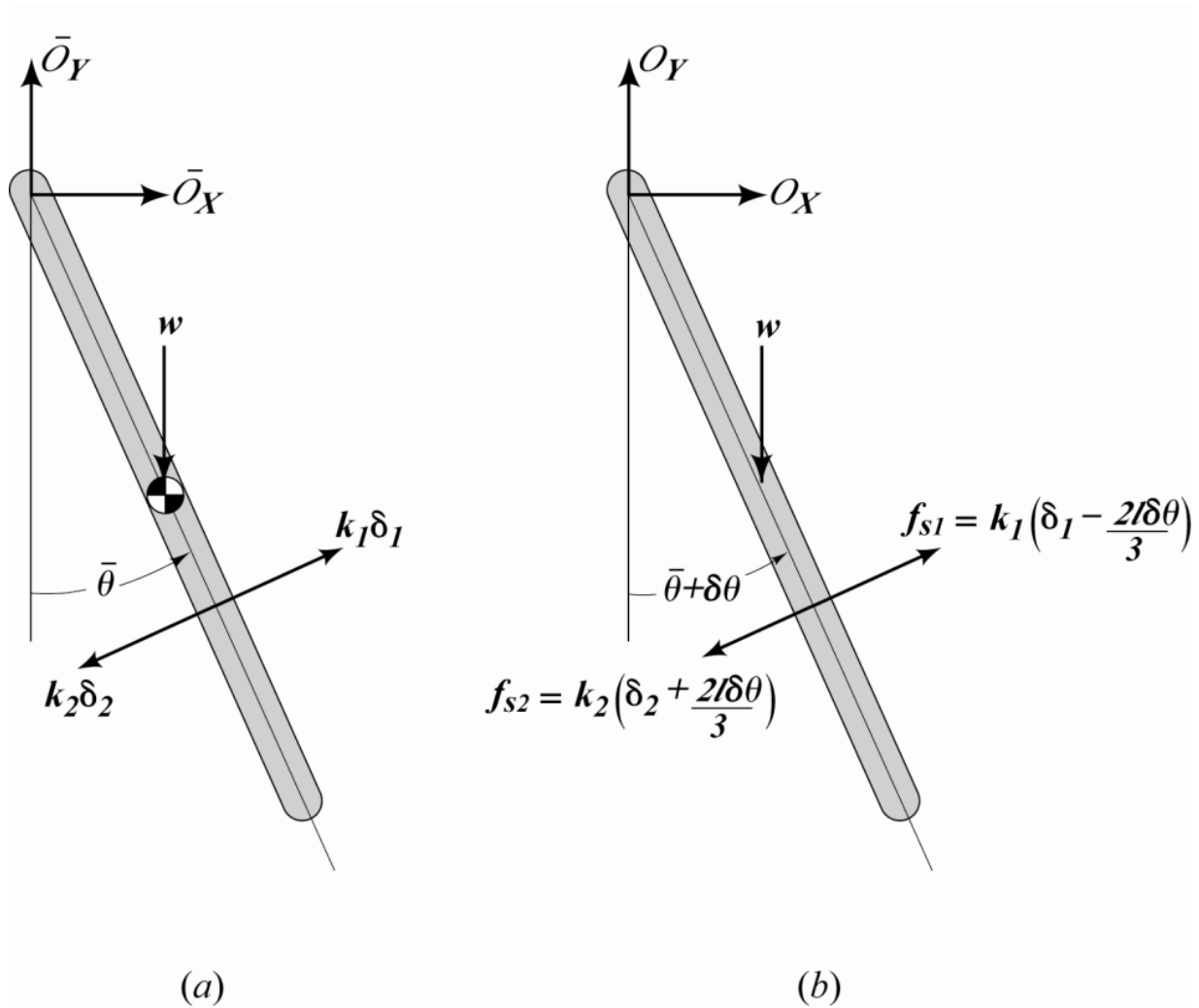


Figure 5.21 Uniform bar: (a) Alternative static equilibrium free-body diagram, (b) Displaced-position free-body diagram

From figure 5.21b,

$$\begin{aligned}
\Sigma M_O = I_O \delta \ddot{\theta} &= -w \frac{l}{2} \sin(\bar{\theta} + \delta\theta) + \frac{2l}{3} k_1 (\delta_1 - \frac{2l}{3} \delta\theta) \\
&\quad - \frac{2l}{3} k_2 (\delta_2 + \frac{2l}{3} \delta\theta) \\
&\cong -w \frac{l}{2} (\sin \bar{\theta} + \cos \bar{\theta} \delta\theta) + \frac{2l}{3} k_1 \delta_1 - \frac{2l}{3} k_2 \delta_2 \\
&\quad - (\frac{2l}{3})^2 (k_1 + k_2) \delta\theta \\
&= (-w \frac{l}{2} \sin \bar{\theta} + \frac{2l}{3} k_1 \delta_1 - \frac{2l}{3} k_2 \delta_2) \\
&\quad + [w \frac{l}{2} \cos \bar{\theta} - (\frac{2l}{3})^2 (k_1 + k_2)] \delta\theta ,
\end{aligned}$$

and the EOM is (again)

$$\begin{aligned}
\frac{ml^2}{3} \delta \ddot{\theta} + [(\frac{2l}{3})^2 (k_1 + k_2) + w \frac{l}{2} \cos \bar{\theta}] \delta\theta &= 0 \\
&= -w \frac{l}{2} \sin \bar{\theta} + \frac{2l}{3} k_1 \delta_1 - \frac{2l}{3} k_2 \delta_2 .
\end{aligned} \tag{5.88}$$

The right-hand side is zero from the equilibrium requirement of Eq.(5.87), and Eq.(5.88) repeats the EOM of Eq.(5.85).

The lesson from this second development is: For small motion about equilibrium, the same EOM is obtained *irrespective* of the initial equilibrium forces in the (linear) springs. The spring-force contributions to the differential equation arise from the *change* in the equilibrium forces due to a *change in position*. This is the same basic outcome that we obtained for a mass m supported by linear springs in figure 3.7. The change in equilibrium angle θ changes w 's contribution to the EOM, because $-w(l/2) \sin \theta$, the moment due to w , is a *nonlinear* function of θ .

Prescribed acceleration of a Pivot Support Point

Moment equations for the fixed-axis rotation problems of the preceding section were taken about a *fixed* pivot point, employing the moment equation

$$M_{oz} = I_o \ddot{\theta} , \quad (5.26)$$

where o identifies the axis of rotation. The problems involved in this short section concerns situations where the pivot point is accelerating, and the general moment equation,

$$M_{oz} = I_o \ddot{\theta} + m(\mathbf{b}_{og} \times \ddot{\mathbf{R}}_o)_z , \quad (5.24)$$

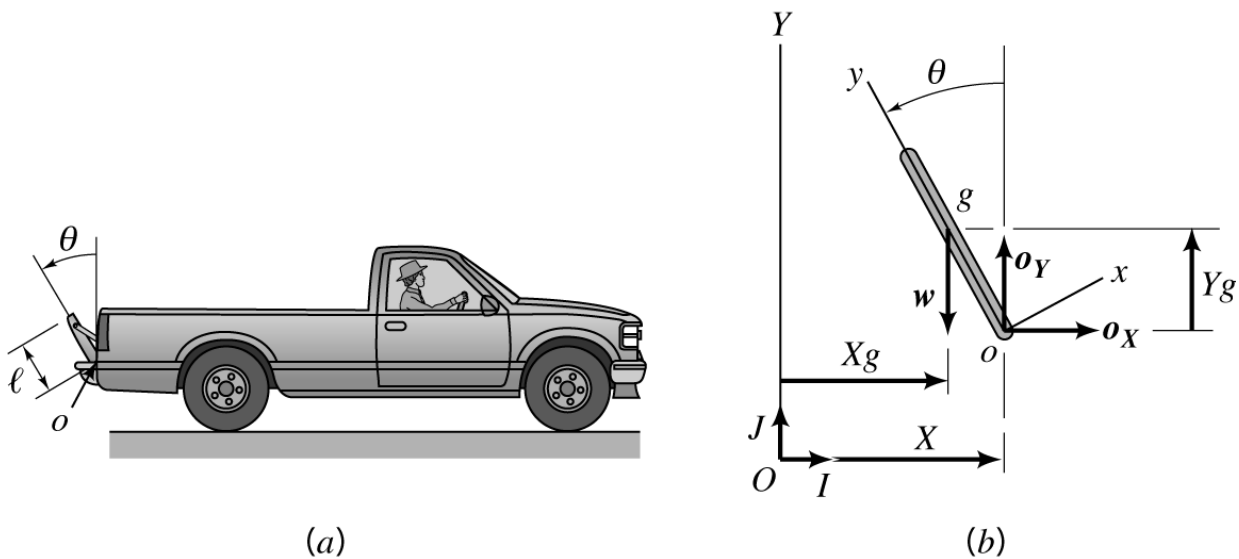
is required. In applying Eq.(5.24), recall the following points:

a. Moments are being taken about the body-fixed axis o , and I_o is the moment of inertia through axis o .

b. The vector \mathbf{b}_{og} goes from a z axis through o to a z axis through the mass center at g .

c. The positive rotation and moment sense in Eq.(5.24) correspond to a counter clockwise rotation for θ .

The last term in the moment equation is positive because the positive right-hand-rule convention for the cross-product in this term coincides with the $+\theta$ sense. For a rigid body with a positive clockwise rotation angle this last term requires a negative sign.



(a)

(b)

Figure 5.24 (a). An accelerating pickup truck with a loose tail gate. (b). Free-body diagram for the tail gate.

The pickup has a constant acceleration of $g/3$. Neglecting friction at the pivot and assuming that the tailgate can be modeled as a uniform plate of mass m , carry out the following engineering tasks:

- Derive the governing equation of motion.
- Assuming that the tailgate starts from rest at $\theta = 0$, what will $\dot{\theta}$ be at $\theta = \pi/2$?
- Determine the reactions at pivot point o as a function of θ (only).

In applying Eq.(5.24) for moments about axis o , we can observe

$$\mathbf{b}_{og} = \frac{l}{2} (-I \sin \theta + J \cos \theta)$$

$$\ddot{\mathbf{R}}_o = I \ddot{X} = I \frac{g}{3}$$

$$\mathbf{b}_{og} \times \ddot{\mathbf{R}}_o = -K \frac{gl}{6} \cos \theta .$$

Hence, Eq.(5.24) gives

$$\begin{aligned} \Sigma M_o &= w \frac{l}{2} \sin \theta = \frac{m l^2}{3} \ddot{\theta} - m \frac{gl}{6} \cos \theta \\ \therefore \quad \frac{m l^2}{3} \ddot{\theta} &= \frac{w l}{2} \sin \theta + \frac{w l}{6} \cos \theta . \end{aligned} \tag{5.61}$$

We have now completed *Task a*. We can use the energy-integral substitution to integrate this nonlinear equation of motion as

$$\ddot{\theta} = \frac{d}{d\theta} \left(\frac{\dot{\theta}^2}{2} \right) = \frac{3g}{2l} \sin \theta + \frac{g}{2l} \cos \theta. \tag{5.62}$$

Multiplying by $d\theta$ reduces both sides of this equation to exact differentials. Integrating both sides with the initial condition $\dot{\theta}(\theta=0)=0$ gives

$$\begin{aligned}\frac{\dot{\theta}^2}{2} &= \frac{g}{l} \Big|_0^\theta \left(-\frac{3}{2} \cos u + \frac{1}{2} \sin u \right) \\ &= \frac{g}{l} \left[\frac{1}{2} \sin \theta + \frac{3}{2} (1 - \cos \theta) \right] .\end{aligned}\tag{5.63}$$

Hence, at $\theta = \pi / 2$, $\dot{\theta}(\pi/2) = 2\sqrt{g/l}$, and we have completed *Task b*. We used the energy-integral substitution, but note that the tail gate's mechanical energy is not conserved. The truck's acceleration is adding energy to the tail gate.

Moving on to *Task c*, stating $\Sigma \mathbf{f} = m \ddot{\mathbf{R}}_g$ for the mass center gives:

$$\begin{aligned}\Sigma f_X &= o_X = m \ddot{X}_g \\ \Sigma f_Y &= o_Y - w = m \ddot{Y}_g\end{aligned}\tag{5.64}$$

We need to determine \ddot{X}_g, \ddot{Y}_g in these equations. From figure 5.19B,

$$X_g = X - \frac{l}{2} \sin \theta \quad ; \quad Y_g = \frac{l}{2} \cos \theta \quad .$$

Differentiating twice with respect to time gives:

$$\ddot{X}_g = \ddot{X} - \frac{l}{2} \cos \theta \ddot{\theta} + \frac{l}{2} \sin \theta \dot{\theta}^2 = \frac{g}{3} - \frac{l}{2} \cos \theta \ddot{\theta} + \frac{l}{2} \sin \theta \dot{\theta}^2$$

$$\ddot{Y}_g = -\frac{l}{2} \sin \theta \ddot{\theta} - \frac{l}{2} \cos \theta \dot{\theta}^2 .$$

Substituting into Eqs.(5.64) gives:

$$\begin{aligned} o_X &= m \left(\frac{g}{3} - \frac{l}{2} \cos \theta \ddot{\theta} + \frac{l}{2} \sin \theta \dot{\theta}^2 \right) \\ o_Y - w &= -m \left(\frac{l}{2} \sin \theta \ddot{\theta} + \frac{l}{2} \cos \theta \dot{\theta}^2 \right) , \end{aligned} \tag{5.65}$$

where \ddot{X} has been replaced with $g/3$, the pick-up truck's acceleration. Substituting from Eqs.(5.62) and (5.63) for $\ddot{\theta}$ and $\dot{\theta}^2$, respectively, (and some algebra) gives:

$$\begin{aligned} o_X &= w \left(\frac{1}{3} + 2 \sin \theta - \frac{3}{2} \sin 2\theta \right) \\ o_Y &= w \left(\frac{11}{8} - \frac{3}{2} \cos \theta + \frac{9}{8} \cos 2\theta - \frac{3}{8} \sin 2\theta \right) , \end{aligned}$$

and completes *Task c*.

The decision to use the general moment Eq.(5.24) and sum moments about the pivot point o instead of the mass center g saves a great deal of effort in arriving at the differential equation of motion. To confirm this statement, consider the following moment equation about g

$$\Sigma M_g = o_X \frac{l}{2} \cos \theta + o_Y \frac{l}{2} \sin \theta = I_g \ddot{\theta}$$

Substituting from Eq.(5.65) for o_X, o_Y gives

$$I_g \ddot{\theta} = \frac{l}{2} \cos \theta m \left(\frac{g}{3} - \frac{l}{2} \cos \theta \ddot{\theta} + \frac{l}{2} \sin \theta \dot{\theta}^2 \right) + \frac{l}{2} \sin \theta \left[w - m \left(\frac{l}{2} \sin \theta \ddot{\theta} + \frac{l}{2} \cos \theta \dot{\theta}^2 \right) \right] .$$

Gathering like terms gives

$$\begin{aligned} \left[I_g + \frac{ml^2}{4} (\sin^2 \theta + \cos^2 \theta) \right] \ddot{\theta} &= \frac{wl}{6} \cos \theta + \frac{wl}{2} \sin \theta \\ &+ \frac{ml}{4} \dot{\theta}^2 (\sin \theta \cos \theta - \sin \theta \cos \theta) . \end{aligned}$$

Simplifying these equations gives Eq.(5.62) ,the original differential equation of motion.

The lesson from this short section is: In problems where a pivot support point has a prescribed acceleration, stating the moment equation (correctly) about the pivot point will lead to the governing equation of motion much more quickly and easily than taking moments about the mass center.

Note: Energy is not conserved with base acceleration!