Lecture 29. GENERAL MOTION/ROLLING-WITHOUT-SLIPPING EXAMPLES
General equations of Motion

Force Equation

$$\sum f_{iX} = m\ddot{R}_{gX} \ , \quad \sum f_{iY} = m\ddot{R}_{gY} \ . \tag{5.15}$$

General Moment Equation

$$\sum M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_{\boldsymbol{z}} \ . \tag{5.24}$$

Moments About the Mass Center

$$M_{gz} = I_g \ddot{\theta} \ . \tag{5.25}$$

Moments About a Fixed axis

$$M_{oz} = I_o \ddot{\theta} \ . \tag{5.26}$$

The examples of this lecture will be analyzed using $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ and $\Sigma M_{gz} = I_g \ddot{\theta}$.

Kinetic Energy of a Rigid Body

$$T = \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{g}}|^2}{2} + \frac{I_g \ \dot{\theta}^2}{2} \ , \tag{5.205}$$

A Cylinder Rolling Down An Inclined Plane

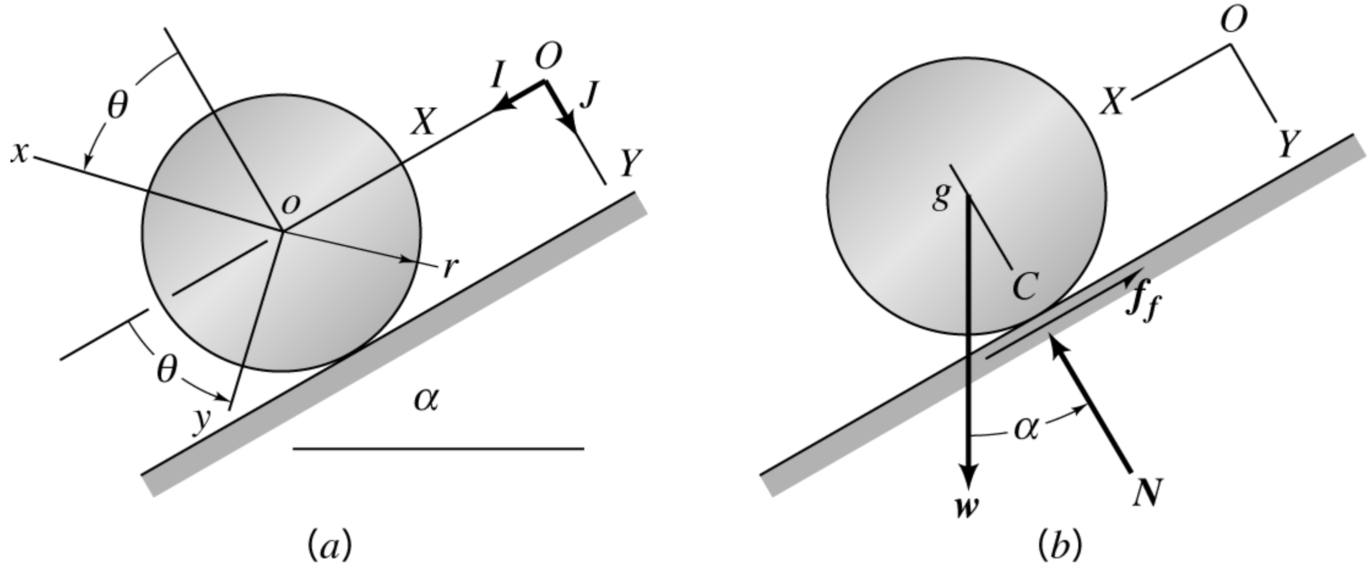


Figure 5.25 (*a*) Uniform disk of radius *r* and mass *m* rolling (without slipping) down an inclined plane, (*b*) Free-body diagram. The static Coulomb coefficient of friction for the plane is μ_s.

Derive the governing differential equation of motion for rolling without slipping.

Force Equation Components

$$\Sigma f_X = w\sin\alpha - f_f = m\ddot{X}$$
$$\Sigma f_Y = w\cos\alpha - N = m\ddot{Y} = 0 \;\Rightarrow\; N = W\cos\alpha. \tag{5.75}$$

Moment Equation

$$\Sigma M_{\theta g} = f_f r = I_g \ddot{\theta} = \frac{mr^2}{2}\ddot{\theta}. \quad (5.76)$$

Eq.(5.76) and the first of Eq.(5.75) constitute two equations in the three unknowns: $\ddot{\theta}, \ddot{X}$, and f_f.

Rolling-without-slipping kinematic constraint

$$\ddot{X} = r\ddot{\theta} \ . \quad (5.77)$$

Solving for f_f from Eq.(5.76), and substituting into the first of Eq.(5.75) gives

$$m\ddot{X} = w\sin\alpha - m\frac{r}{2}\ddot{\theta}.$$

Now, substituting for $\ddot{\theta} = \ddot{X}/r$ from Eq.(5.77) gives

$$m\ddot{X} + m\frac{r}{2}\cdot\frac{\ddot{X}}{r} = \frac{3m}{2}\ddot{X} = w\sin\alpha \ \Rightarrow \ \ddot{X} = \frac{2g}{3}\sin\alpha. \quad (5.78)$$

Using θ as the dependent variable, solve the first of Eq.(5.75) for f_f and then substitute into Eq.(5.76) to obtain

$$I_g\ddot{\theta} = r(w\sin\alpha - m\ddot{X}) \ ,$$

Substituting $\ddot{X} = r\ddot{\theta}$ from Eq.(5.77) gives

$$(I_g + mr^2)\ddot{\theta} = \frac{3mr^2}{2}\ddot{\theta} = wr\sin\alpha. \qquad (5.79)$$

Comparisons of Eqs.(5.78) and (5.79) show basically the same equation.

Friction Force

$$\textit{From,} \quad rf_f = m\frac{r^2}{2}\ddot{\theta}\ , \quad f_f = 0 \ \Rightarrow \quad \ddot{\theta} = 0$$

$$\textit{From,} \quad w\sin\alpha - f_f = m\ddot{X}\ , \quad f_f = 0 \ \Rightarrow \quad \ddot{X} = g\sin\alpha\ .$$

The friction force causes the cylinder to rotate and reduces the acceleration to $\ddot{X} = (2/3)g\sin\alpha$.

How much Coulomb friction is required to prevent slipping?

$$f_f = w\sin\alpha - m\ddot{X} = w\sin\alpha - m(\frac{2g}{3}\sin\alpha) = \frac{w}{3}\sin\alpha\ . \qquad (5.80)$$

Since, $N = w\cos\alpha$, the "required" static Coulomb friction force coefficient to prevent slipping is

$$\mu_s(\textit{required}) = \frac{f_f}{N} = \frac{w\sin\alpha}{3} \times \frac{1}{w\cos\alpha} = \frac{\tan\alpha}{3}. \qquad (5.81)$$

If this calculated value is less than or equal to μ_s, the wheel will roll without slipping. If it greater than μ_s, the wheel will slip, and $f_f = \mu_d N$.

Slipping Motion Equations of Motion. For slipping $f_f = \mu_d N = \mu_d w \cos\alpha$, the cylinder has two degrees of freedom, and the equations of motion are

$$w\sin\alpha - \mu_d w\cos\alpha = m\ddot{X}$$

$$\frac{mr^2}{2}\ddot{\theta} = r\mu_d w\cos\alpha \ .$$

Deriving the equation of Motion for Rolling-Without-Slipping from Conservation of Energy

Without slipping, energy is conserved and $T + V = T_0 + V_0$. Taking the origin of the *X, Y* system as the datum for potential energy,

$$V_g = -wX\sin\alpha \ . \qquad \text{(i)}$$

The kinetic energy is defined to be

$$T = \frac{m}{2}\dot{X}^2 + \frac{mr^2}{4}\dot{\theta}^2 = \frac{m}{2}\dot{X}^2 + \frac{mr^2}{4}(\frac{\dot{X}}{r})^2 = \frac{3m}{4}\dot{X}^2 \ , \qquad \text{(ii)}$$

where the rolling without slipping relationship $\dot{X} = r\dot{\theta}$ has been used. Substituting from (i) and (ii) gives

$$\frac{3m}{2}\left(\frac{\dot{X}^2}{2}\right) - wX\sin\alpha = T_0 + V_0 \ .$$

Differentiating w.r.t. X

$$\frac{3m}{2}\frac{d}{dX}\left(\frac{\dot{X}^2}{2}\right) - w\sin\alpha = 0 \ \Rightarrow \ \frac{3m}{2}\ddot{X} = w\sin\alpha \ .$$

An Imbalanced Cylinder Rolling Down an Inclined Plane

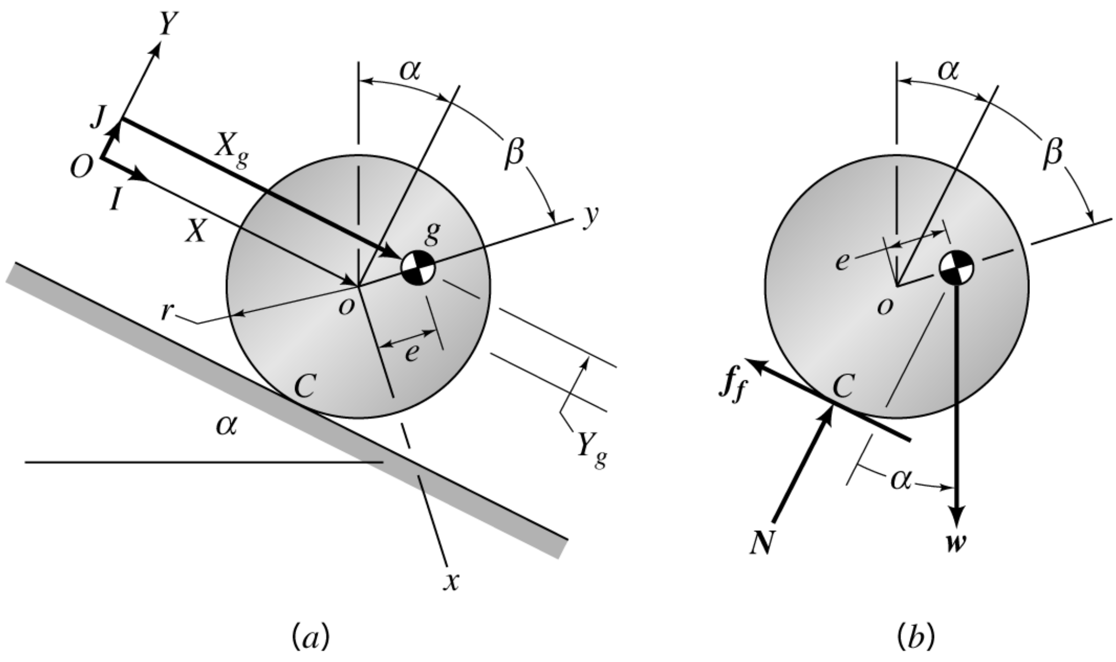


Figure 5.26 (a) Disk with its mass center displaced a distance e from its geometrical center, rolling down an inclined plane, (b) Free-body diagram.

The disk's mass center is located in the *X, Y* system by:

$$X_g = X + e\sin\beta \ , \quad Y_g = e\cos\beta \ . \tag{5.83}$$

The rolling-without-slipping kinematic condition for figure 5.26A is

$$\ddot{X} = r\ddot{\beta} \ . \tag{5.84}$$

Without slipping, the disk has two variables, *X* and β, but only one degree of freedom. The radius of gyration of the disk about point *o* is k_{og} ; hence, $I_o = m k_{og}^2$.

Derive the governing differential equation of motion.

Solution A. Take moments about the mass center.

Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ for the disk's mass center gives:

$$\begin{aligned} \Sigma f_X &= w\sin\alpha - f_f = m\ddot{X}_g \\ \Sigma f_Y &= N - w\cos\alpha = m\ddot{Y}_g \ . \end{aligned} \tag{5.85}$$

In reviewing these equations, note that $\ddot{Y}_g \neq 0$.

Taking moments about the mass center, the moment equation is

$$\Sigma M_g = Ne\sin\beta + f_f(r + e\cos\beta) = I_g\ddot{\beta} \ . \tag{5.86}$$

Note that positive moments are in the $+\beta$, clockwise direction. Also, from the parallel-axis formula, $I_g = I_o - me^2 = m(k_g^2 - e^2)$.

Eqs.(5.84), (5.85) and (5.86) provide four equations in the six unknowns: $\ddot{X}, \ddot{\beta}, \ddot{X}_g, \ddot{Y}_g, N,$ and f_f.

Kinematics. Differentiating Eqs.(5.83) once with respect to time gives

$$\dot{X}_g = \dot{X} + e\cos\beta\dot{\beta} \ , \quad \dot{Y}_g = -e\sin\beta\dot{\beta} \ .$$

Differentiating a second time and substituting for $\ddot{X} = r\ddot{\beta}$ gives

$$\ddot{X}_g = r\ddot{\beta} + e\cos\beta\ddot{\beta} - e\sin\beta\dot{\beta}^2 \ , \quad \ddot{Y}_g = -e\sin\beta\ddot{\beta} - e\cos\beta\dot{\beta}^2, \tag{5.87}$$

which provides our final two equations.

The governing equation of motion is obtained by the following steps:

a. Substitute for $\ddot{X}_g, \ddot{Y}_g$ from Eq.(5.87) into Eqs (5.85), obtaining

$$f_f = w\sin\alpha - m(r\ddot{\beta} + e\cos\beta\,\ddot{\beta} - e\sin\beta\,\dot{\beta}^2)$$
$$N = w\cos\alpha - m(e\sin\beta\,\ddot{\beta} + e\cos\beta\,\dot{\beta}^2) \quad . \tag{5.88}$$

b. Substitute for N and f_f into Eq.(5.86), obtaining

$$I_g\ddot{\beta} = [w\cos\alpha - m(e\sin\beta\,\ddot{\beta} + e\cos\beta\,\dot{\beta})^2]e\sin\beta +$$
$$[w\sin\alpha - m(r\ddot{\beta} + e\cos\beta\,\ddot{\beta} - e\sin\beta\,\dot{\beta}^2)](r + e\cos\beta) \quad .$$

After a fair amount of algebra, the governing equation is

$$m(k_{og}^2 + r^2 + 2re\cos\beta)\ddot{\beta} - mer\sin\beta\,\dot{\beta}^2$$
$$= w[r\sin\alpha + e\sin(\beta + \alpha)]. \tag{5.89}$$

Note: $I_o = mk_{og}^2 \Rightarrow I_g = I_o - me^2 = m(k_{og}^2 - e^2)$.

This equation reduces to Eq.(5.79) if $e = 0$, and $I_o = mr^2/2$.

The energy-integral substitution, $\ddot{\beta} = d(\dot{\beta}^2/2)/d\beta$, converts Eq.(5.89) to

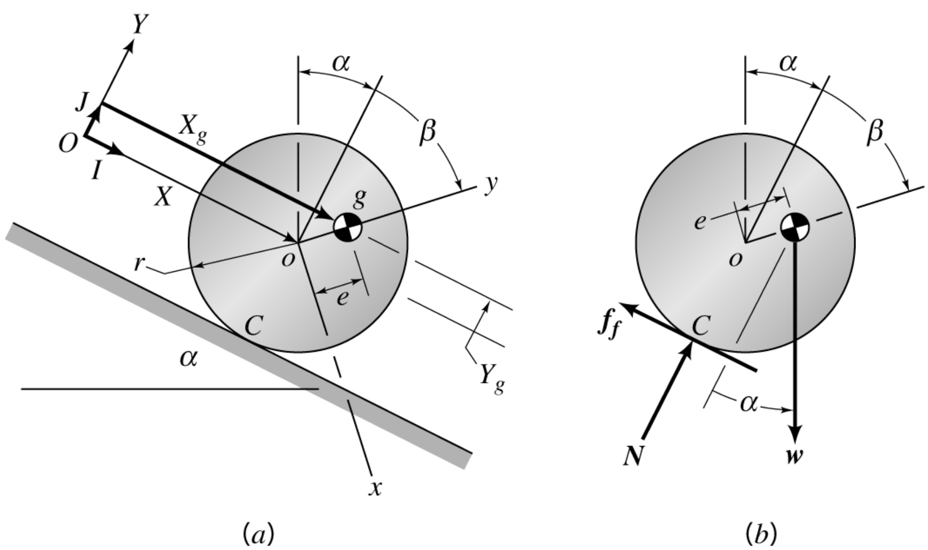
$$\frac{d}{d\beta}[m(k_{og}^2 + r^2 + 2re\cos\beta)(\frac{\dot{\beta}^2}{2})] = w[r\sin\alpha + e\sin(\beta + \alpha)].$$

For the boundary conditions ($\beta(0)=0$; $\dot{\beta}(0)=0$); integration gives

$$m(k_{og}^2 + r^2 + 2re\cos\beta)(\frac{\dot{\beta}^2}{2}) = w[r\sin\alpha\beta - e\cos(\beta+\alpha) + e\cos\alpha] \ . \tag{5.90}$$

Without slipping, there is no energy dissipation.

Solution B: Take moments about C.



For moments about C, the EOM is

$$\sum \boldsymbol{M_C} = I_C\ddot{\beta} - m(\boldsymbol{b_{Cg}} \times \ddot{\boldsymbol{R}}_C) \ ,$$

where clockwise moments are positive (positive β rotation).

Kinematics:

$$\boldsymbol{b_{Cg}} = \boldsymbol{I} e \sin\beta + \boldsymbol{J}(r + e\cos\beta)\ ,\ \ddot{\boldsymbol{R}}_{\boldsymbol{c}} = \boldsymbol{J} r \dot{\beta}^2$$

$$\therefore\ \boldsymbol{b_{Cg}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{C}} = \boldsymbol{K} e r \dot{\beta}^2 \sin\beta$$

Inertia properties:

$$I_g = I_o - me^2 = mk_{og}^2 - me^2$$

$$I_C = I_g + m\,|\,b_{gC}\,|^2 = (mk_{og}^2 - me^2) + m[(r + e\cos\beta)^2 + (e\sin\beta)^2]$$
$$= mk_{og}^2 + m(r^2 + 2re\cos\beta)$$

External moment about C due to weight

$$\begin{aligned}\boldsymbol{M_C} &= -(\boldsymbol{r_{Cg}} \times \boldsymbol{w}) \\ &= -[\boldsymbol{I} e\sin\beta + \boldsymbol{J}(r + e\cos\beta)] \times w(\boldsymbol{I}\sin\alpha + \boldsymbol{J}\cos\alpha)) \\ &= \boldsymbol{K}(we\sin\beta\cos\alpha + rw\sin\alpha + we\sin\alpha\, cos\,\beta \\ &= \boldsymbol{K}[wr\sin\alpha + we\sin(\alpha + \beta)]\end{aligned}$$

Plugging in the results gives (again)

$$m(k_{og}^2 + r^2 + 2re\cos\beta)\ddot{\beta} - mer\sin\beta\,\dot{\beta}^2 = w[r\sin\alpha + e\sin(\beta + \alpha)]. \tag{5.89}$$

This approach is obviously quicker.

Deriving the equation of Motion from Conservation of Energy

Applying $T+V=T_0+V_0$, and using the origin of the X, Y system for the gravity potential-energy function gives

$$\frac{I_g}{2}\dot{\beta}^2+\frac{m}{2}(\dot{X}_g^2+\dot{Y}_g^2)-wX\sin\alpha+we\cos(\alpha+\beta)$$
$$=(T_0+V_0)\ . \tag{5.223}$$

This example has three coordinates X_g, Y_g, β. To eliminate unwanted coordinates, we need the rolling-without-slipping condition $X=r\beta$, plus the kinematic conditions,

$$X_g=X+e\sin\beta=r\beta+e\sin\beta\ \Rightarrow\ \dot{X}_g=r\dot{\beta}+e\cos\beta\,\dot{\beta}$$

$$Y_g=e\cos\beta\ \Rightarrow\ \dot{Y}_g=-e\sin\beta\,\dot{\beta}\ .$$

Substituting into Eq.(5.223) gives

$$m(k_{og}^2+r^2+2er\cos\beta)\frac{\dot{\beta}^2}{2}$$
$$-w[r\beta\sin\alpha-e\cos(\alpha+\beta)]=(T_0+V_0)\ , \tag{5.224}$$

Differentiating with respect to β gives the DEQ. of motion

$$m(k_{og}^2 + r^2 + 2er\cos\beta)\ddot{\beta} - mer\sin\beta\,\dot{\beta}^2 = w[r\sin\alpha + e\sin(\alpha+\beta)] \quad . \tag{5.204}$$