Lecture 32. TORSION EXAMPLES HAVING MORE THAN ONE DEGREE OF FREEDOM

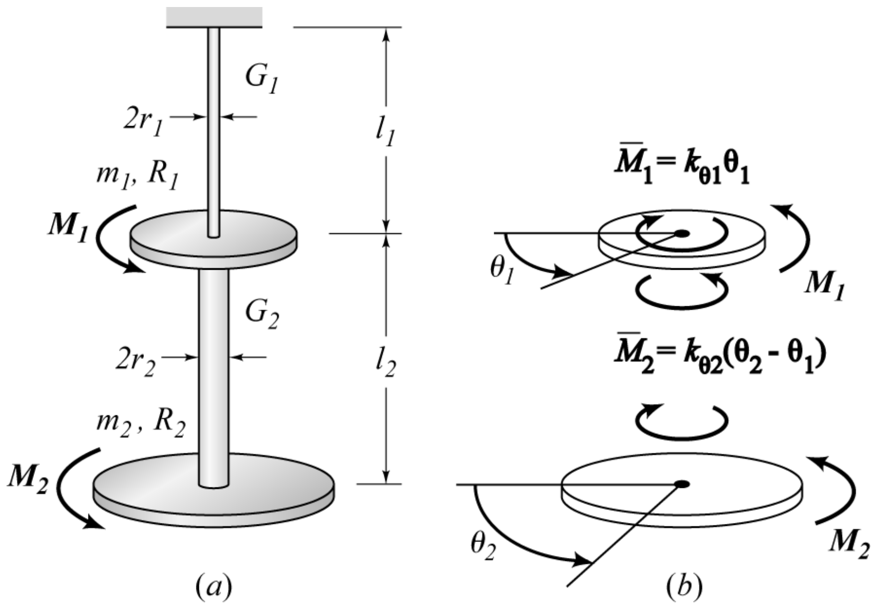


Figure 5.40 (a) Two-disk, torsional vibration example, (b) coordinates and free-body diagram for $\theta_1 > \theta_2 > 0$

Torsional Vibration Examples

We worked through a one-degree-of-freedom, torsional-vibration example in section 5.4, starting with the model of figure 5.10. Figure 5.35 illustrates a two-degree-of-freedom extension to this example. The upper disk has mass m_1, radius R_1, and is connected to "ground" by a circular shaft of radius r_1 length L_1, and shear modulus G_1. The lower disk has mass m_2, radius R_2, and is connected to disk 1 by a circular shaft of radius r_2, length L_2, and shear modulus G_2. The rotation angles θ_1 and θ_2 define the orientations of the two disks. The shafts have zero elastic deflections and moments when these angles are zero.

As with the example of figure 5.10, twisting the upper disk through the angle θ_1 will develop the reaction moment,

$$\bar{M}_1 = -k_{\theta 1}\theta_1 = -\frac{G_1 J_1}{l_1}\theta_1 = -\frac{G_1}{l_1}\frac{\pi r_1^4}{2}\theta_1 \ , \qquad (5.144)$$

acting on top of the upper disk. The negative sign in this equation implies that the moment is acting in a direction opposite to a positive θ_1 rotation direction. The reaction moment acting on the bottom of disk 1 and the top of disk 2 is proportional to the difference between the rotation angles θ_1 and θ_2. Assuming that θ_2 is greater than θ_1, the reaction moment acting *on disk 1* from the lower shaft is

$$\bar{M}_2 = k_{\theta 2}(\theta_2 - \theta_1) = \frac{G_2 J_2}{l_2}(\theta_2 - \theta_1) = \frac{G_2}{l_2}\cdot\frac{\pi r_2^4}{2}(\theta_2 - \theta_1) \ . \qquad (5.145)$$

The positive sign for the moment implies that it is acting in the $+\theta_1$ direction, i.e., acting to rotate disk 1 in a positive $+\theta_1$ direction. The negative of this moment acts on the top of disk 2. In addition, assume that the applied moments $M_1(t)$ and $M_2(t)$ are acting, respectively, on disks 1 and 2. Individually summing moments about the axis of symmetry for the two bodies including these external moments yields:

$$M_1(t) - \bar{M}_1 + \bar{M}_2$$
$$= M_1(t) - k_{\theta 1}\theta_1 + k_{\theta 2}(\theta_2 - \theta_1) = I_{g1}\ddot{\theta}_1 = \frac{m_1 R_1^2}{2}\ddot{\theta}_1 \tag{5.146}$$
$$M_2(t) - \bar{M}_2 = M_2(t) - k_{\theta 2}(\theta_2 - \theta_1) = I_{g2}\ddot{\theta}_2 = \frac{m_2 R_2^2}{2}\ddot{\theta}_2 \ .$$

The matrix statement of these equations is

$$\begin{bmatrix} I_{g1} & 0 \\ 0 & I_{g2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (k_{\theta 1} + k_{\theta 2}) & -k_{\theta 2} \\ -k_{\theta 2} & k_{\theta 2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} M_1(t) \\ M_2(t) \end{Bmatrix} \ . \tag{5.147}$$

The inertia matrix is diagonal, and the stiffness matrices is symmetric.

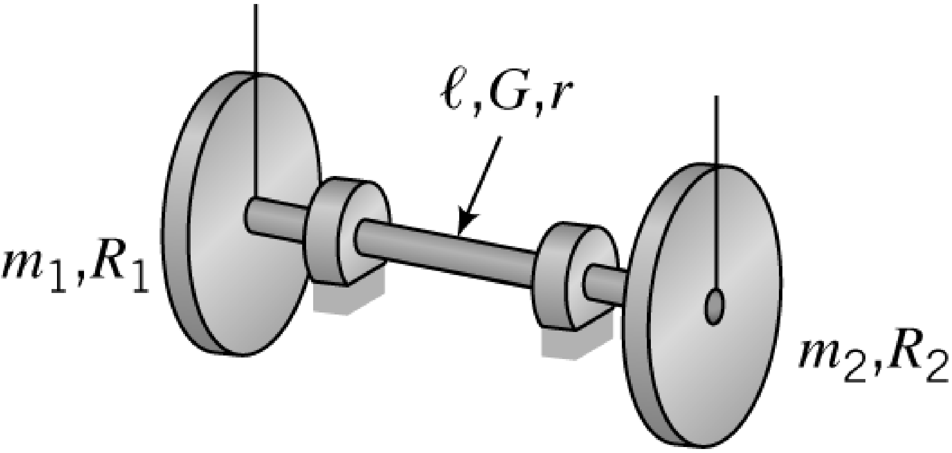


Figure 5.36 Unrestrained, two-disk torsional-vibration example.

The governing differential equations for the model of figure 5.36 are obtained from Eqs.(5.147) by setting the upper stiffness $k_{\theta 1}$ equal to zero. Eq.(5.147) becomes

$$\begin{bmatrix} I_{g1} & 0 \\ 0 & I_{g2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{\theta 2} & -k_{\theta 2} \\ -k_{\theta 2} & k_{\theta 2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} M_1(t) \\ M_2(t) \end{Bmatrix} \qquad (5.151)$$

Substituting the assumed solution, $(\theta_1, \theta_2)^T = (a_1, a_2)^T \cos \omega_n t$ into the homogeneous version of Eq.(5.151) nets

$$\begin{bmatrix} (-\omega_n^2 I_{g1} + k_{\theta 2}) & -k_{\theta 2} \\ -k_{\theta 2} & (-\omega_n^2 I_{g2} + k_{\theta 2}) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \ . \qquad (5.152)$$

The characteristic equation is obtained by setting the determinant equal to zero and is:

$$\omega_n^4 I_{g1} I_{g2} - \omega_n^2 k_{\theta 2}(I_{g1} + I_{g2}) = 0$$

$$\therefore\ \omega_{n1}^2 = 0;\ \omega_{n2}^2 = k_{\theta 2}(I_{g1} + I_{g2})/I_1 I_{g2}.$$

For the numbers in Eq.(5.150),

$$\begin{aligned} &G = 8.27 \times 10^{10} Pa \\ &l_1 = l_2 = .3m,\ r_1 = r_2 = .013m \ \Rightarrow\ k_{\theta 1} = \ k_{\theta 2} = 6184\, Nm/ra \\ &m_1 = 5kg\ ,\ R_1 = .15m \ \Rightarrow\ I_{g1} = .05625\, kg\, m^2 \\ &m_2 = 10kg\ ,\ R_2 = .15m \ \Rightarrow\ I_{g2} = .1125\, kg\, m^2 \end{aligned} \qquad (5.150)$$

the eigenvalues are $\omega_{n1}^2 = 0,\ \omega_{n2}^2 = 1.649 \times 10^5 \sec^{-2}$, and the natural frequencies are $\omega_{n1} = 0\ ,\ \omega_{n2} = 406.\, rad/\sec$. Substituting $\omega_{n1}^2 = 0$ into Eq.(5.152), gives

$$\begin{bmatrix} 6184 & -6184 \\ -6184 & 6184 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = 0\ .$$

The determinant of the coefficient matrix is clearly zero, and the first eigenvector can be defined from either scalar equation by setting $a_{11} = 1$, obtaining $a_{21} = 1$, and the first eigenvector $(\omega_{n1} = 0)$ is

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The second mode is obtained by substituting the second eigenvalue into Eq.(5.152), (plus substituting for I_1, I_2 and $k_{\theta 2}$ from Eq.(5.150)) obtaining

$$\begin{bmatrix} -3092. & -6184. \\ -6184. & -12368. \end{bmatrix} \begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} = 0 \ .$$

Again, the determinant of the coefficient matrix is zero, and the second eigenvector is

$$\begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} = \begin{Bmatrix} 1. \\ -0.5 \end{Bmatrix} .$$

The matrix of eigenvectors is

$$[A] = \begin{bmatrix} 1 & 1 \\ 1 & -.5 \end{bmatrix} .$$

The first step in obtaining the modal differential equations is taken by introducing the modal coordinates, via the coordinate transformation, $(\theta_i) = [A](q_i)$,

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -.5 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} .$$

Substituting $(\theta_i) = [A](q_i)$ into Eq.(5.151) and then premultiplying by the transpose matrix $[A]^T$ gives the uncoupled modal differential equations:

$$(I_{g1}+I_{g2})\ddot{q}_1 = M_1(t)+M_2(t)$$

$$(I_{g1}+.25I_{g2})\ddot{q}_2 + 13900.\ q_2 = M_1(t) - .5M_2(t)\ .$$

Observe that the first modal coordinate q_1 (with the zero eigenvalue and natural frequency) defines rigid-body rotation of the rotor with zero relative rotation between θ_1 and θ_2. The second modal coordinate q_2 defines relative motion with the two disks moving in opposite directions.

The complete model is

$$(I_{g1}+I_{g2})\ddot{q}_1 = M_1(t)+M_2(t)$$

$$(I_{g1}+.25I_{g2})\ddot{q}_2 + 13900\ q_2 = M_1(t) - .5M_2(t)$$

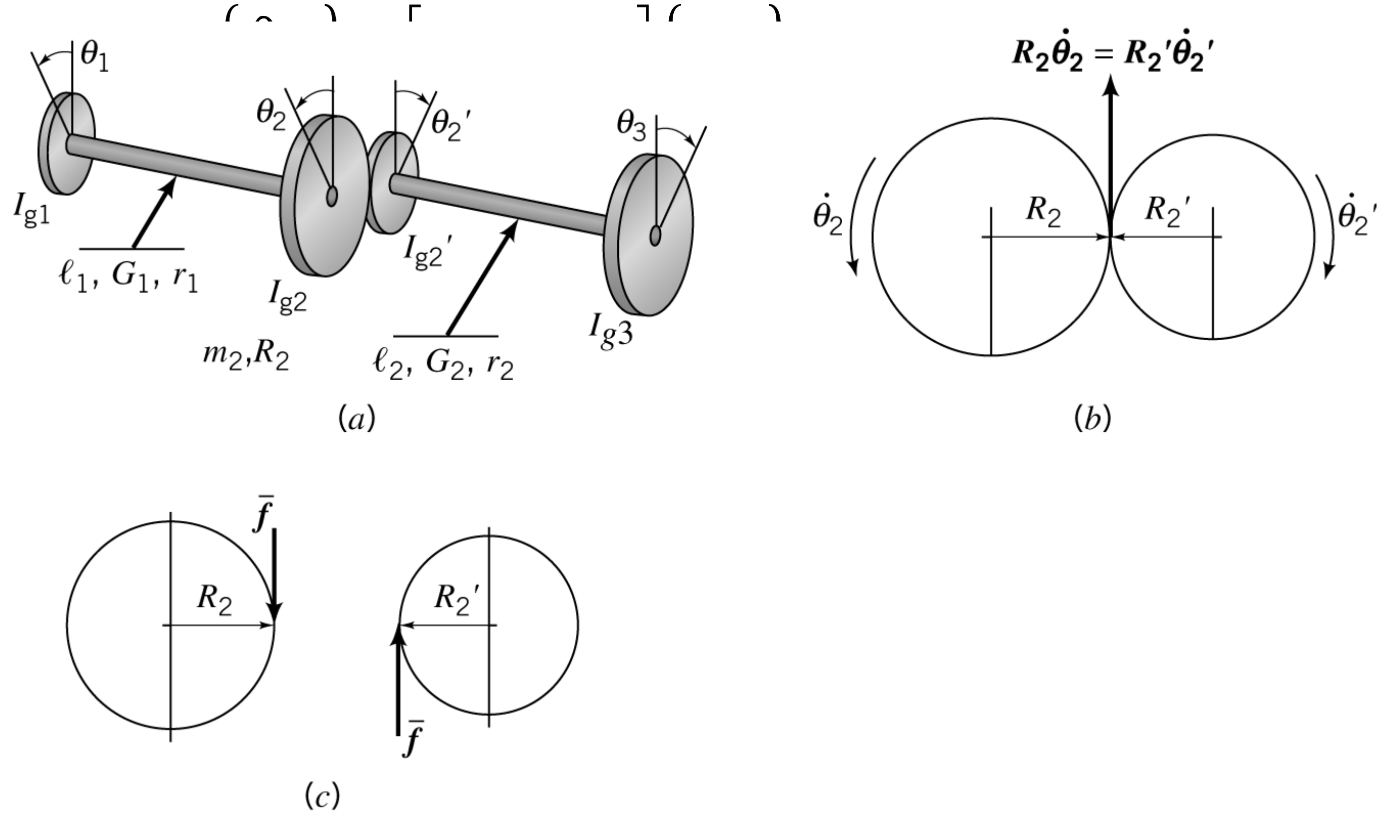


Figure 5.37 (a). Two, two-disk rotors in frictionless bearings with coupled motion. (b). Kinematic constraint between disks 2 and $2'$. (c). Reaction-force components between disks 2 and $2'$.

Figure 5.37 shows two rotors, each with two disks connected by a shaft. The rotations of disks 1 and 2 on rotor 1 are defined by the counter clockwise rotations θ_1, θ_2. The rotations of disks $2'$ and 3 of rotor 2 are defined by the clockwise rotations θ_2', θ_3. The radii of disks 2 and $2'$ are, respectively, R_2, R_2'. The two disks are "geared" together with no relative slipping between their edges; hence, their rotation variables are related by the kinematic constraint equations:

$$R_2\theta_2 = R_2'\theta_2' \Rightarrow R_2\dot{\theta}_2 = R_2'\dot{\theta}_2' \Rightarrow R_2\ddot{\theta}_2 = R_2'\ddot{\theta}_2' \ . \qquad (5.153)$$

Figure 5.37C illustrates the reaction force $\bar{f}$ acting between disks 2 and $2'$. Stating the equations of motion for the disks of the two rotors gives:

Rotor 1 (+ counter clockwise rotations and moments)

$$I_{g1}\ddot{\theta}_1 + k_{\theta 1}(\theta_1 - \theta_2) = 0$$
$$I_{g2}\ddot{\theta}_2 + k_{\theta 1}(\theta_2 - \theta_1) = -\bar{f}R_2 \qquad (5.154a)$$

Rotor 2 (+ clockwise rotations and moments)

$$I'_{g2}\ddot{\theta}'_2 + k_{\theta 2}(\theta'_2 - \theta_3) = \bar{f}R'_2$$
$$I_{g3}\ddot{\theta}_3 + k_{\theta 2}(\theta_3 - \theta'_2) = 0 \ . \qquad (5.154b)$$

Eqs.(5.153) and (5.154) comprise five equations in the five unknowns $(\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}'_2, \ddot{\theta}_3, \bar{f})$. Equating $\bar{f}$ in the second of Eqs.(5.154a) and the first of Eqs.(5.154b) eliminates this variable. Also, Eq.(5.153) can be used to eliminate $\theta'_2, \ddot{\theta}'_2$ yielding the following three coupled differential equations:

$$I_{g1}\ddot{\theta}_1 + k_{\theta 1}(\theta_1 - \theta_2) = 0$$

$$[I_{g2} + (R_2/R'_2)^2 I'_{g2}]\ddot{\theta}_2 - k_{\theta 1}\theta_1 + \theta_2[k_{\theta 1} + k_{\theta 2}(R_2/R'_2)^2]$$
$$-k_{\theta 2}(R_2/R'_2)\theta_3 = 0$$

$$I_{g3}\ddot{\theta}_3 + k_{\theta 2}[\theta_3 - (R_2/R'_2)\theta_2] = 0 \ .$$

These equations define a three-degree-of-freedom problem with the matrix statement

$$\begin{bmatrix} I_{g1} & 0 & 0 \\ 0 & I_{g2}+I_{g2}^{/}(R_2/R_2^{/})^2 & 0 \\ 0 & 0 & I_{g3} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} +$$

$$\begin{bmatrix} k_{\theta 1} & -k_{\theta 1} & 0 \\ -k_{\theta 1} & [k_{\theta 1}+k_{\theta 2}(R_2/R_2^{/})^2] & -k_{\theta 2}(R_2/R_2^{/}) \\ 0 & -k_{\theta 2}(R_2/R_2^{/}) & k_{\theta 2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = 0 \ . \tag{5.155}$$

Because neither of the component rotors are connected to ground, one of the eigenvalues is zero, and the remaining two roots (and their associated eigenvectors) can be determined analytically.

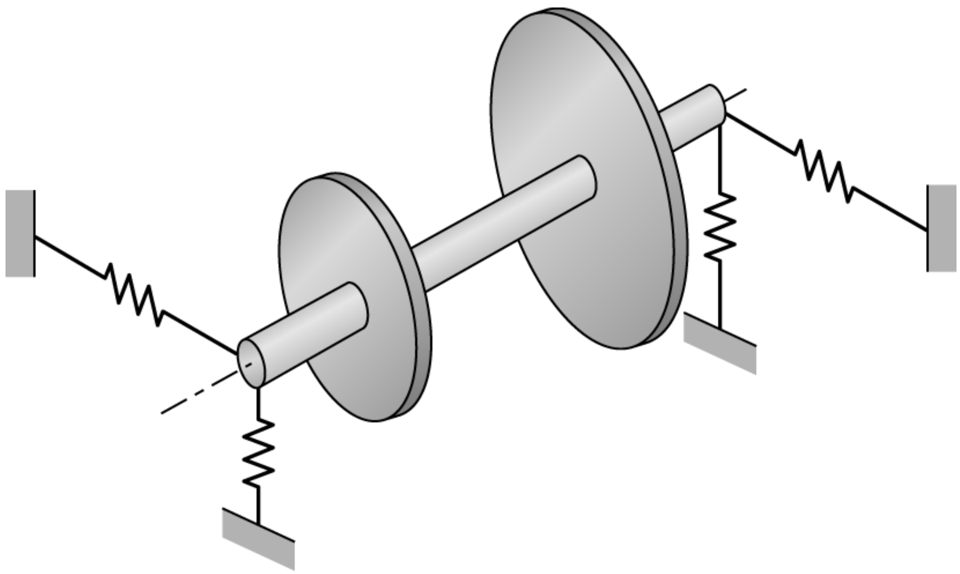


Figure 5.38 Lumped-parameter rotor model including two disks.

Unlike linear extension/compression springs that deliver a reaction force opposing the direction of displacement, or torsional springs that produces a reaction torque to oppose twisting, deflecting or rotating the end of a beam normally produce both a reaction moment and a reaction force. The coupling of displacements and rotations and reaction forces and moments can be confusing, and we will start with a simple example of a cantilevered beam with an attached weight. Displacement of the beam's end creates a reaction force but no reaction moment.

Example A. Cantilever Beam supporting a disk that does not rotate. Zero moments about the right end. The bar has mass *m*. The beam has length *l* and section modulus *EI*.

If it is pulled down and then released (*the cable remains in tension*), *what is the natural frequency of free motion for the disk?*

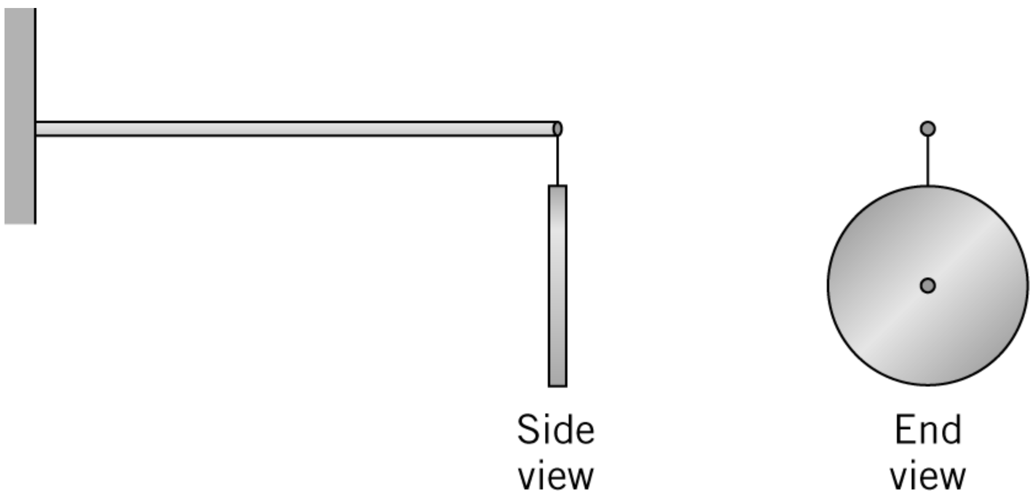


Figure XP 5.2b Cantilever beam with the disk hanging from its end.

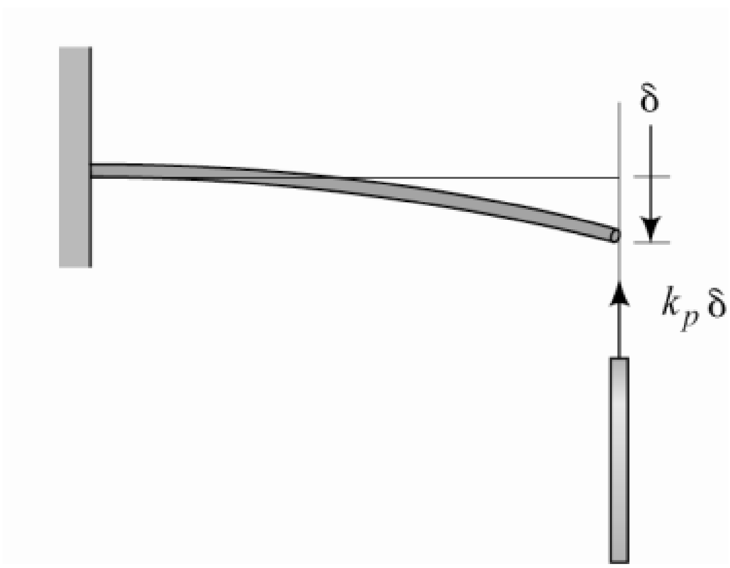
From strength of materials, a beam with a zero moment at its free end and a lateral load f will deflect a distance δ. δ and *f* are related by

$$\delta = \frac{f l^3}{3EI} \quad .$$

In this equation, $F_d = l^3/3EI$ defines the displacement "flexibility coefficient" for the beam's end. We want the displacement stiffness coefficient (for zero moment at the beam's end)

$$k_d(zero\ end\ moment) = \frac{3EI}{l^3} \ .$$

For motion about the equilibrium position, the free-body diagram for the disk (massless beam) is shown below. (δ remains small enough that the cord remains in tension.)



Free-body diagram.

From the free-body diagram, the equation of motion is,

$$m\ddot{\delta} = -k_p\delta \ \Rightarrow \ m\ddot{\delta} + \frac{3EI}{l^3}\delta = w \ ,$$

and the natural frequency is

$$\omega_n = \sqrt{\frac{3EI}{ml^3}}$$

Suppose the beam has length $l = 750\,mm$ and a circular cross section with diameter $d = 25\,mm$. It is made from steel with modulus of elasticity $E = 2.1 \times 10^{11}\,Pa(N/m^2)$. The disk has radius $r = 250\,mm$, thickness $b = 25\,mm$ and is also made from steel (density $= \gamma = 7830\,kg/m^3$). As a first step, the bending section modulus is

$$EI = E \times \frac{\pi r^4}{4} = 2.1 \times 10^{11}\,\frac{N}{m^2} \times \frac{3.1416(.0125)^4}{4}\,m^4$$

$$= 4030.\,Nm^2 \quad .$$

The mass is

$$m = \pi r^2 b\gamma = 3.1416 \times .25^2(m^2) \times .025(m) \times 7830(\frac{kg}{m^3})$$

$$= 38.4\,kg \quad .$$

Substituting, the natural frequency is

$$\omega_n = [\frac{3EI}{ml^3}]^{1/2} = [\frac{3 \times 4030\,Nm^2.}{38.4\,kg \times .75^3(m^3)}]^{1/2} = 27.3\,\sec^{-1} \quad .$$

where the dimensions of a kg are $N\sec^2/m$.

Figure 5.39a represents the more general situation with a disk rigidly attached to the end of a cantilevered beam. If the disk is displaced from its equilibrium position, the disk will rotate in the *Y*-*Z* plane through the angle β_Y Because of the disk's moment of inertia, a moment will now result at the beam's end due to disk rotation. Figure 5.39b provides the free-body diagram for the displaced and rotated disk, including the applied force and moment pair (f_X, M_Y) and reaction force and moment pair $(\bar{f}_X, \bar{M}_Y)$.

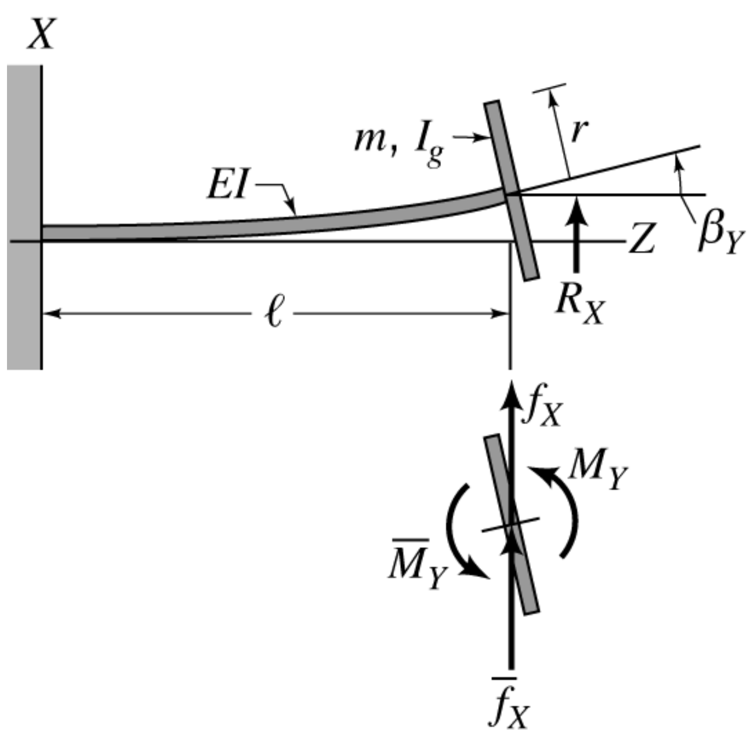


Figure 5.39

(*a*) Cantilevered beam supporting a thin circular disk at its right end.

(*b*) Free-body diagram for the disk.

The equations of motion are easily stated from figure 5.39b . Applying $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ and $\Sigma M_{gy} = I_g \ddot{\beta}_Y$, the disk's equations of motion are:

$$f_X + \bar{f}_X = m\ddot{R}_X \ , \ \ M_Y + \bar{M}_y = I_g \ddot{\beta}_Y \ . \qquad \textbf{(5.156)}$$

Defining the reaction force $\bar{f}_X$ and moment $\bar{M}_Y$ in terms of the displacement and rotation coordinates R_X, β_Y is the principal difficulty in completing these equations.

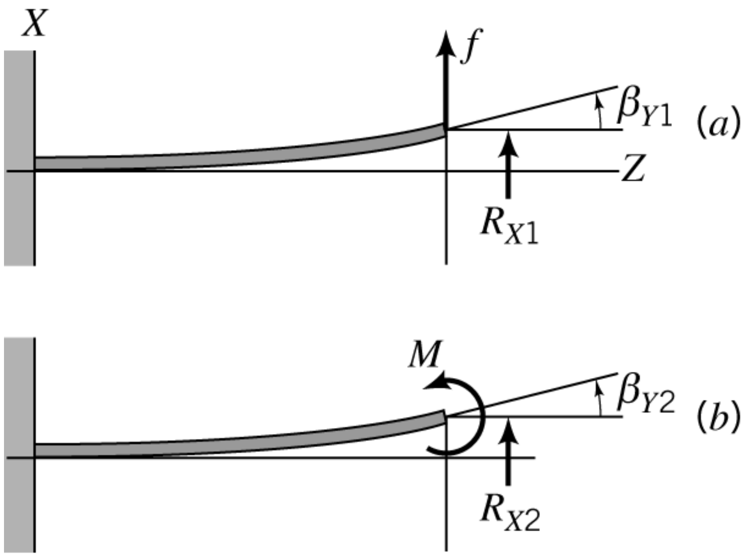


Figure 5.40 (a). Cantilevered beam with an applied end force. (b). Applied moment

Figure 5.40a illustrates the beam with a concentrated load f applied at its end, yielding (from strength of materials) the displacement and rotation.

$$R_{X1} = fl^3/3EI \ , \quad \beta_{Y1} = fl^2/2EI \ . \qquad \text{(5.157a)}$$

We used the displacement result in the first example. Similarly, figure 5.40b illustrates a moment M applied to the beam's end, yielding

$$R_{X2} = \frac{Ml^2}{2EI} \;, \quad \beta_{Y2} = \frac{Ml}{EI} \;.$$

Combining these results as $R_X = R_{X1} + R_{X2}$ and $\beta_Y = \beta_{Y1} + \beta_{Y2}$ gives

$$\frac{fl^3}{3EI} + \frac{Ml^2}{2EI} = R_X$$

$$\frac{fl^2}{2EI} + \frac{Ml}{EI} = \beta_Y \;.$$

In matrix format, these equations are

$$\begin{bmatrix} l^3/3EI & l^2/2EI \\ l^2/2EI & l/EI \end{bmatrix} \begin{Bmatrix} f \\ M \end{Bmatrix} = \begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix} .$$

This coefficient matrix is a “flexibility” matrix $[F]$. An F_{ij} flexibility-matrix entry is the displacement (or rotation) at point i due to a unit load (or moment) at point j . Multiplying through by $[F]^{-1}$ gives

$$\begin{Bmatrix} f \\ M \end{Bmatrix} = \frac{12EI}{l^3}\begin{bmatrix} 1 & -l/2 \\ -l/2 & l^2/3 \end{bmatrix}\begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix}. \qquad \textbf{(5.158)}$$

This coefficient matrix is the stiffness matrix $[k]=[F]^{-1}$. The form of this equation tends to be confusing if we think of it as defining the *applied* loads (f, M) as the output due to input displacements and rotations (R_X, β_Y). However, the following statement makes sense when defining the <u>*reaction*</u> force and moment of figure 5.39b due to displacements and rotations

$$\begin{Bmatrix} \bar{f}_X \\ \bar{M}_Y \end{Bmatrix} = -\begin{bmatrix} 12EI/l^3 & -6EI/l^2 \\ -6EI/l^2 & 4EI/l \end{bmatrix}\begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix}. \qquad \textbf{(5.159)}$$

Substituting this result into Eqs.(5.156) gives the matrix equation of motion,

$$\begin{bmatrix} m & 0 \\ 0 & I_g \end{bmatrix}\begin{Bmatrix} \ddot{R}_X \\ \ddot{\beta}_Y \end{Bmatrix} = \begin{Bmatrix} f_X \\ M_Y \end{Bmatrix} + \begin{Bmatrix} \bar{f}_X \\ \bar{M}_Y \end{Bmatrix}. \qquad \textbf{(5.160)}$$

or

$$\begin{bmatrix} m & 0 \\ 0 & I_g \end{bmatrix} \begin{Bmatrix} \ddot{R}_X \\ \ddot{\beta}_Y \end{Bmatrix} + \begin{bmatrix} 12EI/l^3 & -6EI/l^2 \\ -6EI/l^2 & 4EI/l \end{bmatrix} \begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix} = \begin{Bmatrix} f_X \\ M_Y \end{Bmatrix} . \tag{5.160}$$

An entry k_{ij} for the stiffness matrix is the negative reaction force (or moment) at point i due to a displacement (or rotation) at station j , *with all other displacements and rotations equal to zero*.

Example Problem 5.1 The cantilevered beam of figure 5.39 has length $l = 750\,mm$, a circular cross section with diameter $d = 25\,mm$. It is made from steel with modulus of elasticity $E = 2.1 \times 10^{11}\,Pa(N/m^2)$. The disk has radius $r = 250\,mm$, thickness $b = 25\,mm$ and is also made from steel (density = $\gamma = 7\,830\,kg/m^3$). The following engineering-analysis tasks apply:

a. Determine the inertia and stiffness matrices and state the matrix equation of motion.

b. Determine the eigenvalues, natural frequencies, and eigenvectors.

Solution. As a first step, the bending section modulus is

$$EI = E\times\frac{\pi r^4}{4} = 2.1\times 10^{11}\frac{N}{m^2}\times\frac{3.1416(.0125)^4}{4}m^4$$
$$= 4030.\,Nm^2\ .$$

Continuing, the stiffness coefficients are:

$$k_{11} = \frac{12EI}{l^3} = 12\times 4030.\,Nm^2\times\frac{1}{.75^3 m^3} = 1.15\times 10^5\frac{N}{m}$$

$$k_{12} = k_{21} = \frac{6EI}{l^2} = 6\times 4030.\,Nm^2\times\frac{1}{.75^2 m^2} = 4.30\times 10^4 N/rad$$

$$k_{22} = \frac{4EI}{l} = 4\times 4030\,Nm^2\times\frac{1}{.75\,m} = 2.15\times 10^4 Nm/rad\ .$$

The inertia-matrix entries are:

$$m = \pi r^2 b\gamma = 3.1416\times .25^2(m^2)\times .025(m)\times 7830(\frac{kg}{m^3})$$
$$= 38.4\,kg$$

$$I_g = \frac{mr^2}{4} = \frac{38.4(kg)\times .25^2(m^2)}{4} = 0.60\,kgm^2\ .$$

Substituting these results into Eq.(5.160) defines the model as

$$\begin{bmatrix} 38.4. & 0 \\ 0 & 0.60 \end{bmatrix} \begin{Bmatrix} \ddot{R}_X \\ \ddot{\beta}_Y \end{Bmatrix} + \begin{bmatrix} 1.15\times 10^5 & -4.30\times 10^4 \\ -4.30\times 10^4 & 2.15\times 10^4 \end{bmatrix} \begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix} = \begin{Bmatrix} f_X \\ M_Y \end{Bmatrix} . \qquad \textbf{(i)}$$

Substituting the solution $(r_X, \beta_Y)^T = (A_1, A_2)^T \cos\omega t$ into the homogeneous version of Eq.(i) nets

$$\begin{bmatrix} -38.4\,\omega^2 + 1.15\times 10^5 & -4.30\times 10^4 \\ -4.30\times 10^4 & -0.60\,\omega^2 + 2.15\times 10^4 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad . \textbf{(ii)}$$

The characteristic equation is

$$23.04\,\omega^4 - 8.946\times 10^5\,\omega^2 + 6.235\times 10^8 = 0 \quad .$$

The eigenvalues and natural frequencies defined by the roots of this equation are:

$$\omega_{n1}^2 = 709.9\ \text{sec}^{-2} \Rightarrow \omega_{n1} = 26.6\,\text{sec}^{-1}$$

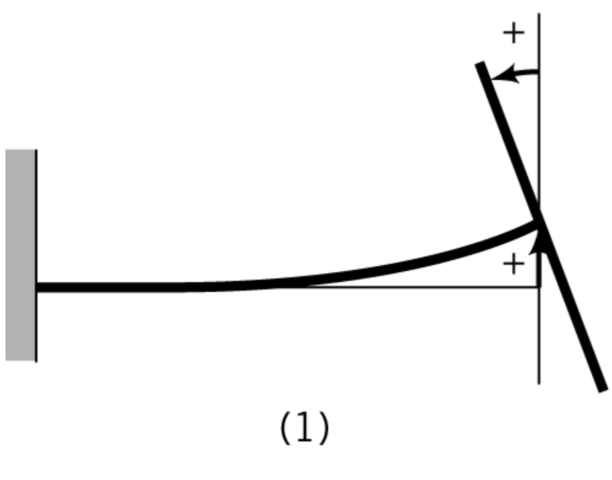
$$\omega_{n2}^2 = 3.812 \times 10^4\ \text{sec}^{-2} \Rightarrow \omega_{n2} = 195.2\,\text{sec}^{-1}\ . \qquad \textbf{(iii)}$$

The first natural frequency $\omega_{n1} = 26.6\,\text{sec}^{-1}$ is slightly lower than the initial example result $\omega_n = 27.3\,\text{sec}^{-1}$

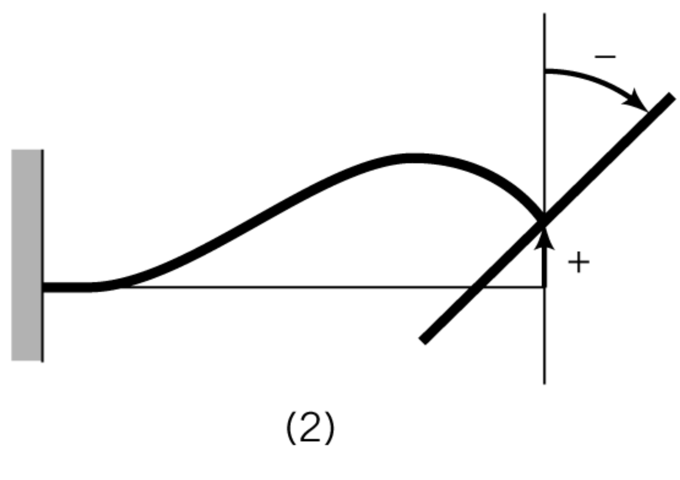
Substituting ω_{n1}^2 and ω_{n2}^2 into Eq.(ii), the corresponding eigenvectors are:

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 2.04 \end{Bmatrix}, \quad \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -31.4 \end{Bmatrix}. \qquad \textbf{(iv)}$$

As illustrated in figure XP 5.1a, the displacement and rotation



(1)



(2)

are in phase for the first mode and out of phase for the second.
Figure XP 5.2a Calculated mode shapes of Eq.(iv); not to scale.

Figure XP 5.2c Cantilever beam with the end disk forced to move up and down but prevented from rotating. ($\beta_Y = 0$).

The disk in Figure XP 5.2c is constrained by rollers that prevent rotation. For $\beta_Y = 0$, the example has only one degree of freedom R_X, and Eq.(5.160)

$$\begin{bmatrix} m & 0 \\ 0 & I_g \end{bmatrix} \begin{Bmatrix} \ddot{R}_X \\ \ddot{\beta}_Y \end{Bmatrix} + \begin{bmatrix} 12EI/l^3 & -6EI/l^2 \\ -6EI/l^2 & 4EI/l \end{bmatrix} \begin{Bmatrix} R_X \\ \beta_Y \end{Bmatrix} = \begin{Bmatrix} f_X \\ M_Y \end{Bmatrix} . \tag{5.160}$$

gives:

$$m\ddot{R}_X + \frac{12EI}{l^3} R_X = f_X \ , \ M_Y = -\frac{6EI}{l^2} R_X \ .$$

The first equation is the equation of motion for R_X. The second equation defines the reaction moment that the constraint rollers must provide to keep the disk from rotating. The natural frequency is defined by

$$\omega_n = [\frac{12EI}{ml^3}]^{1/2} = [\frac{12 \times 4030.}{38.4\,kg \times .75^3 (m^3)}]^{1/2} = 54.6 \sec^{-1} \ .$$

The moment restraint on the disk has doubled the lowest natural frequency.

Note that we have determined the displacement stiffness for a cantilever beam whose end is deflected with zero rotation to be

$$k_d(zero\ end\ rotation) = \frac{12EI}{l^3} \ .$$

Example Problem 5.5. The framed structure has two square floors. The first floor has mass $m_1 = 2000\,kg$ and is supported to the foundation by four solid columns with square cross sections. These columns are cantilevered from the foundation and are welded to the bottom of the first floor. The second floor has mass $m_2 = 2000\,kg$ and is supported from the first floor by four solid columns with square cross sections. These columns are welded to the top of the first floor and are hinged to the second floor. The bottom and top columns have length $l_1 = 6\,m$ and $l_2 = 4\,m$. The top and bottom beams' cross-sectional dimensions are $b_1 = 100\,mm$ and $b_2 = 75\,mm$. They are made from steel with a modulus of elasticity $E = 2.1 \times 10^{11}\,N/m^2$. A model is required to account for motion of the foundation due to earthquake excitation defined by $x(t)$.

Tasks:

a. Select coordinates, draw a free-body diagram, derive the equations of motion.

b. State the equations of motion in matrix format and solve for the eigenvalues and eigenvectors. Draw the eigenvectors.

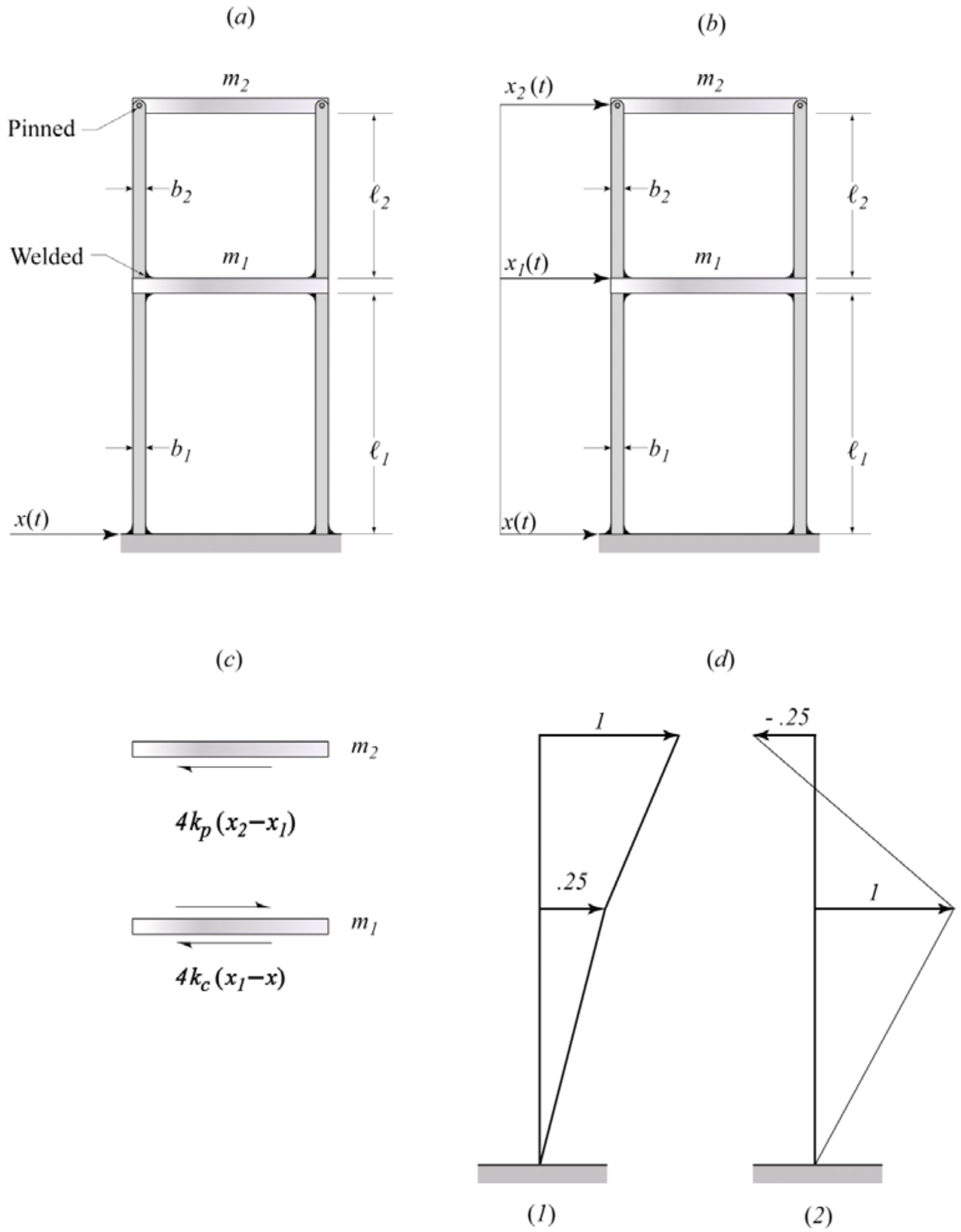


Figure XP5.5 (a) Front view of a two-story framed structure excited by base excitation, (b) Coordinates, (c) Free-body diagram for $x_1 > x > 0, x_2 > x_1$, (d) Eigenvectors

Solution. Figure XP5.5 b illustrates the coordinates $x_1(t), x_2(t)$ selected to locate the first and second floors with respect to ground. All beams connecting the foundation and the first floor are cantilevered at both ends, similar to the beam in figure 5.47 with a stiffness $k_c = 12EI_a/l^3$. The free-body diagram of figure XP5.5c was drawn assuming that the first floor has moved further than the ground $(x_1 > x)$ and defines the reaction force

$$\bar{f}_{first\,story-ground} = -4k_c(x_1 - x)\ ,\ k_c = 12EI_a/l^3\ ,$$

due to all four cantilevered beams acting at the bottom of floor 1.

Each beam connecting the floors has a cantilevered end attached to floor 1 and a pinned end attached to floor 2, similar to the pinned-end beam of figure 5.44, with a stiffness coefficient $k_p = 3EI_a/l^3$. The free-body diagram in figure XP5.5c was developed assuming that the second floor has moved further than the first floor $(x_2 > x_1)$ and provides

$$\bar{f}_{second-story} = -4k_p(x_2 - x_1)\ ,\ k_p = 3EI_a/l^3\ .$$

The negative of this force is acting at the top of floor 1. Summing forces for the two floors gives:

$$floor\,1:\; m_1\ddot{x}_1 = \sum f_{x1} = -4k_c(x_1 - x) + 4k_p(x_2 - x_1)$$
$$\Rightarrow\; m_1\ddot{x}_1 + 4(k_c + k_p)x_1 - 4k_p x_2 = 4k_c x$$

$$floor\,2:\; m_2\ddot{x}_2 = \sum f_{x2} = -4k_p(x_2 - x_1)$$
$$\Rightarrow\; m_2\ddot{x}_1 - 4_p k_p x_1 + 4k_p x_2 = 0$$

Putting these equations in matrix form gives

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + 4 \begin{bmatrix} k_c + k_p & -k_p \\ -k_p & k_p \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{i}$$
$$= \begin{Bmatrix} 4k_c x(t) \\ 0 \end{Bmatrix} .$$

This outcome is similar to Eq.(3.126) for two masses connected by springs.

Filling in the numbers gives:

$$EI_{a1} = E \times \frac{b_1 h_1^3}{12} = 2.1\times 10^{11}\,\frac{N}{m^2} \times \frac{0.1^4}{12}\,m^4 = 1.75\times 10^6\,Nm^2$$
$$EI_{a2} = E \times \frac{b_2 h_2^3}{12} = 2.1\times 10^{11}\,\frac{N}{m^2} \times \frac{(.075)^4}{12}\,m^4 = 5.54\times 10^5\,Nm^2\ .$$

Continuing, the stiffness coefficients are:

$$k_c = \frac{12EI_{a1}}{l^3} = \frac{12(1.75\times 10^6)\,Nm^2}{(6m)^3} = 97200\,\frac{N}{m}$$

$$k_p = \frac{3EI_{a2}}{l^2} = \frac{3(5.54\times 10^5\,Nm^2)}{(4m)^3} = 26000\,\frac{N}{m}\ .$$

Plugging these results into Eq.(i) gives

$$\begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 493000 & -104000 \\ -104000 & 104000 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 398000\,x(t) \\ 0 \end{Bmatrix} \quad \textbf{(ii)}$$

Substituting the assumed solution $(x_1, x_2)^T = (a_1, a_2)^T \cos\omega t$ into the homogeneous version of this equation gives

$$\begin{bmatrix} [-2000\,\omega^2 + 493000] & -104000 \\ -104000 & [-2000\,\omega^2 + 104000] \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}\cos\omega t = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(iii)

Since, $\cos\omega t \neq 0$, and a_1 and a_2 are also not zero, a nontrivial solution, for Eq.(ii) requires that the determinant of the coefficient matrix must equal zero, producing

$$4\times10^6\omega^4 - 1.194\times10^9\omega^2 + 4.0546\times10^{10} = 0$$

$$\Rightarrow\ \omega^4 - 298.5\,\omega^2 + 10110 = 0\ .$$

This characteristic equation defines the two eigenvalues and natural frequencies:

$$\omega_{n1}^2 = 38.95(rad/\sec)^2\ \Rightarrow\ \omega_{n1} = 6.24\,rad/\sec$$

$$\omega_{n2}^2 = 259.(rad/\sec)^2\ \Rightarrow\ \omega_{n2} = 16.1\,rad/\sec$$

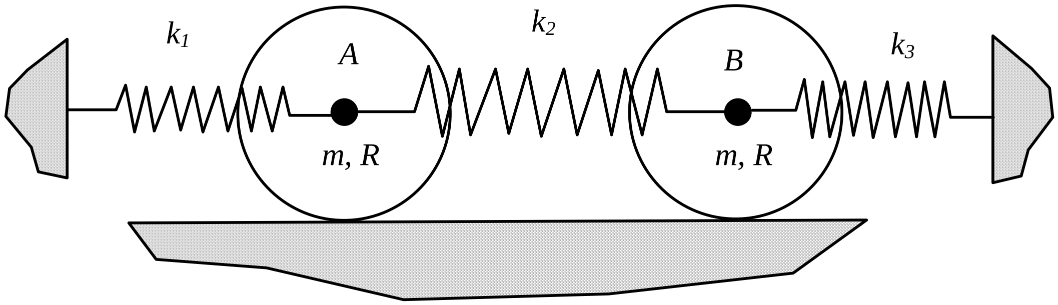
Alternately substituting $\omega_{n1}^2 = 38.95\,(rad/\sec)^2$ and $\omega_{n2}^2 = 259.5(rad/\sec)^2$ into Eq.(iii) gives the eigenvectors

$$(a)_1 = \begin{Bmatrix} 1 \\ 4.0 \end{Bmatrix},\ (a)_2 = \begin{Bmatrix} 1 \\ -.25 \end{Bmatrix}$$

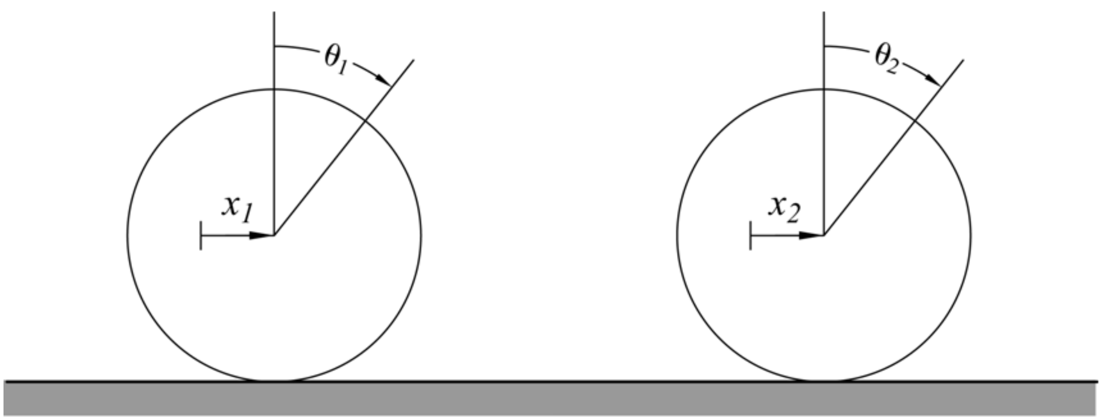
Figure XP5.5d illustrates these two eigenvectors, showing the relative motion of the two floors somewhat better that the two-mass eigenvectors of figure 3.53.

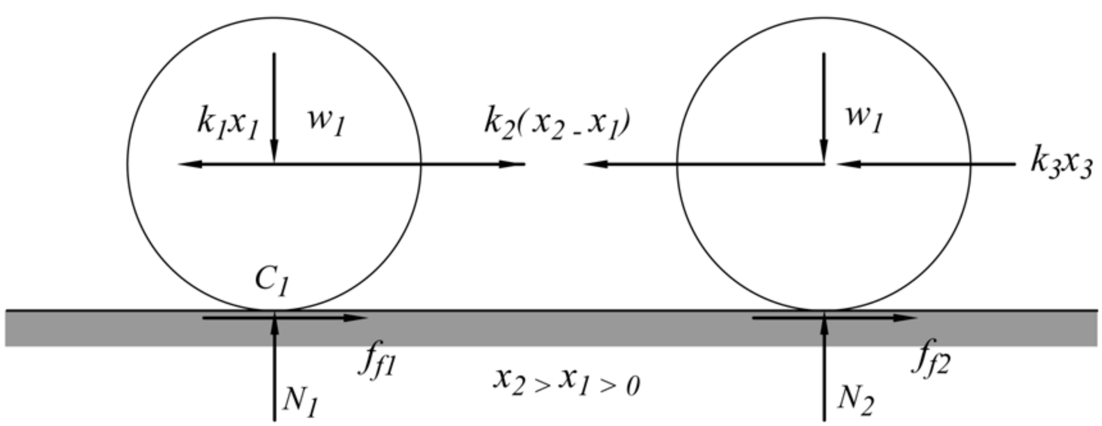
Lecture 34. 2DOF EXAMPLES

Example 1



The cylinders roll without slipping. Select coordinates, draw free-body diagrams, and derive the equations of motion.





Coordinates and free-body diagram for $x_2 > x_1$

Equations of motion, cylinder 1

$$\sum f_{x1} = m_1 \ddot{x}_1 = f_{f1} - k_1 x_1 + k_2 (x_2 - x_1)$$

$$\sum M_{g1} = I_{g1} \ddot{\theta}_1 = -f_{f1} r \quad . \tag{2}$$

Equations of motion cylinder 2

$$\sum f_{x2} = m_2 \ddot{x}_2 = f_{f2} - k_2 (x_2 - x_1) - k_3 x_2$$

$$\sum M_{g2} = I_{g2} \ddot{\theta}_2 = -f_{f2} r \quad . \tag{3}$$

Four equations, 6 unknowns $\ddot{x}_1, \ddot{\theta}_1, f_{f1}, \ddot{x}_2, \ddot{\theta}_2, f_{f2}$

Rolling-without slipping kinematic constraints:

$$r\theta_1 = x_1 \ , \ r\theta_2 = x_2 \tag{3}$$

Eliminate f_{f1} in (1), and f_{f2} in (2)

$$m_1 \ \ddot{x}_1 = -\frac{I_{g1} \ddot{\theta}_1}{r} - k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 \ \ddot{x}_2 = -\frac{I_{g2} \ddot{\theta}_2}{r} - k_2 (x_2 - x_1) - k_3 x_2 \quad .$$

Use (3) to eliminate $\ddot{\theta}_1, \ddot{\theta}_2$, and rearrange

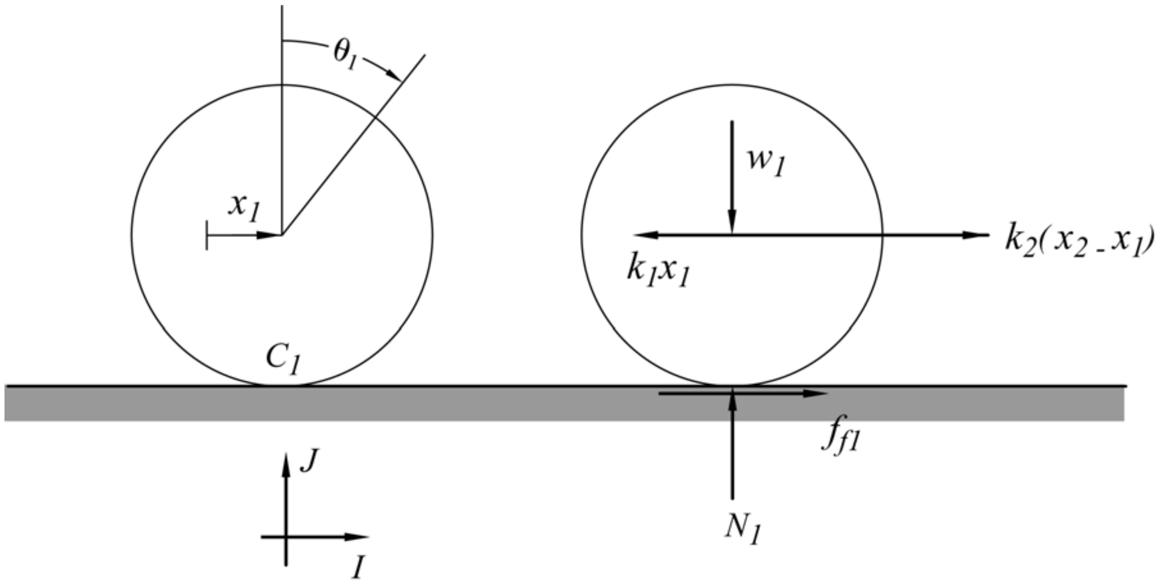
$$(m_1 + \frac{I_{g1}}{r^2})\ \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$(m_2 + \frac{I_{g2}}{r^2})\ \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0\ .$$

For $I_{g1} = m_1 r^2/2$ and $I_{g2} = m_2 r^2/2$ the matrix form is

$$\begin{bmatrix} \frac{3m_1}{2} & 0 \\ 0 & \frac{3m_2}{2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0\ .$$

Alternative development: Take moments about *C* the point of contact



General moment equation for clockwise moments

$$M_{C1} = I_{C1}\ddot{\theta}_1 - m_1(\boldsymbol{b}_{Cg} \times \ddot{\boldsymbol{R}}_C)_z$$

However,

$$\boldsymbol{b}_{Cg} = \boldsymbol{J}r \ , \ \ddot{\boldsymbol{R}}_C = \boldsymbol{J}r\dot{\theta}_1^2 \ \Rightarrow \ \boldsymbol{b}_{Cg} \times \ddot{\boldsymbol{R}}_C = 0 \ .$$

Hence,

$$M_{C1} = I_{C1}\ddot{\theta}_1 - 0 = -k_1 x_1 r + k_2(x_2 - x_1)r$$

$$or \ (\frac{mr^2}{2} + mr^2)\ddot{\theta}_1 + (k_1 + k_2)r^2\theta_1 - k_2 r^2\theta_2 = 0$$

Similarly for cylinder 2

$$M_{C2} = I_{C2}\ddot{\theta}_1 - 0 = -k_2(x_2 - x_1)r - k_3 x_2 r$$

$$or\ \left(\frac{mr^2}{2} + mr^2\right)\ddot{\theta}_2 - k_2 r^2\theta_1 + (k_2 + k_3)r^2\theta_2 = 0$$

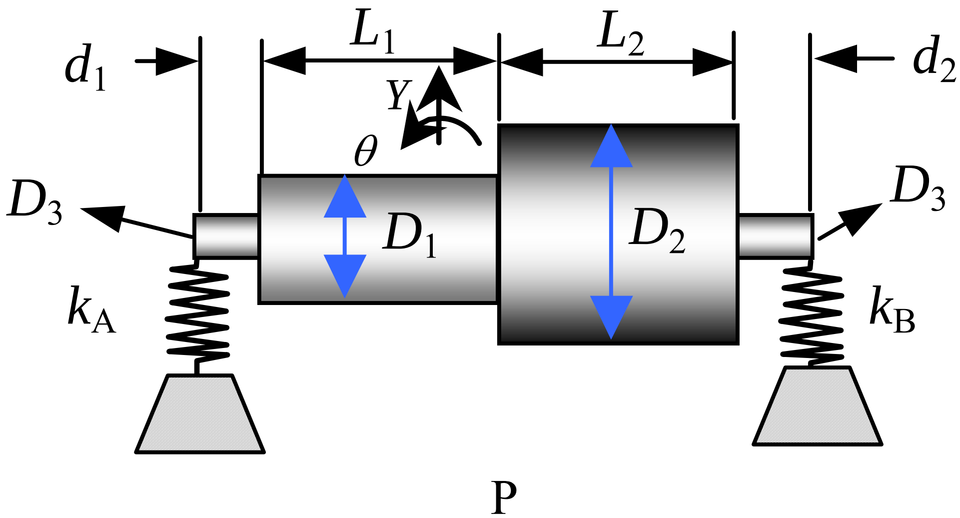
Matrix Format

$$\begin{bmatrix} \frac{3m_1 r^2}{2} & 0 \\ 0 & \frac{3m_2 r^2}{2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} (k_1 + k_2)r^2 & -k_2 r^2 \\ -k_2 r^2 & (k_2 + k_3)r^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \quad .$$

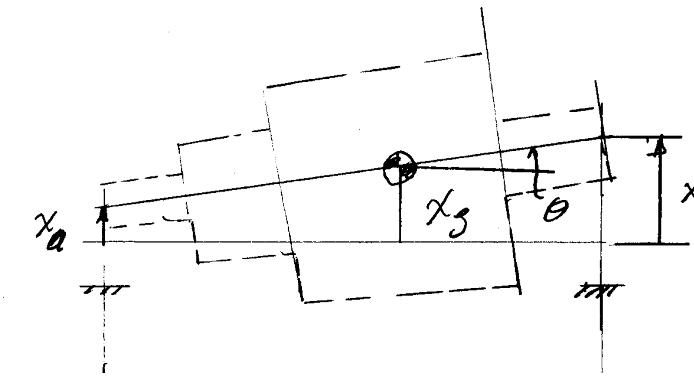
This is the same equation

Example 2

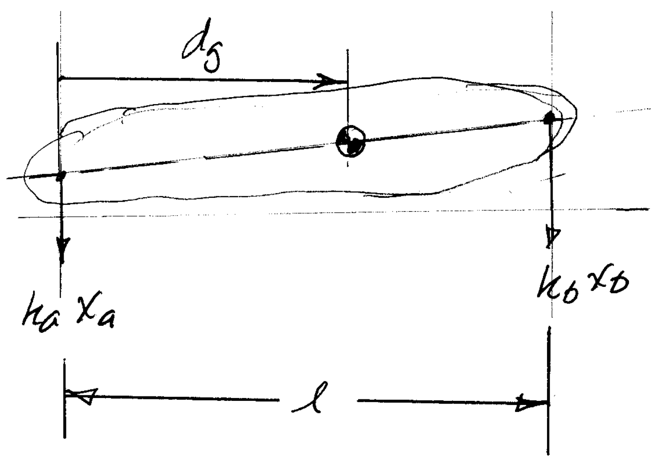


Assume small rotations, select coordinates, draw free-body diagrams, and derive the equations of motion. Gravity is vertically down, and the body is in equilibrium.

Coordinates: x_g is the vertical mass-center displacement from equilibrium. θ is the rotation angle for the body from the horizontal.



Free-body diagram for motion <u>about equilibrium</u>



Equations of motion:

$$\sum f_x = m\ddot{x}_g = -k_a x_a - k_b x_b$$

$$\sum M_g = I_g \ddot{\theta} = d_g k_a x_a - (l - d_g) k_b x_b$$

Small-angle kinematics:

$$x_a = x_g - d_g \theta \ , \ x_b = x_g + (l - d_g)\theta \ .$$

Substituting,

$$m\ddot{x}_g = -k_a(x_g - d_g\theta) - k_b[x_g + (l - d_g)\theta]$$

$$I_g\ddot{\theta} = d_g k_a(x_g - d_g\theta) - (l - d_g)k_b[x_g + (l - d_g)\theta]$$

Gathering terms,

$$m\ddot{x}_g + (k_a + k_b)x_g + [-k_a d_g + k_b(l - d_g)]\theta$$

$$I_g\ddot{\theta} + [-k_a d_g + (l - d_g)]x_g + [k_a d_g^2 + k_b(l - d_g)^2]\theta$$

Matrix Format

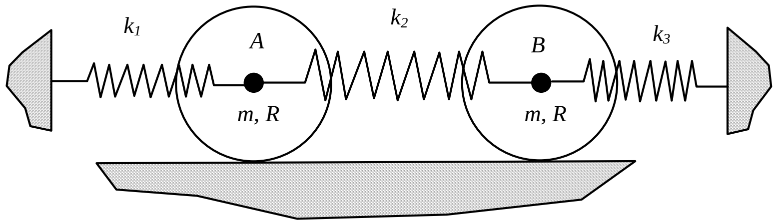
$$\begin{bmatrix} m & 0 \\ 0 & I_g \end{bmatrix} \begin{Bmatrix} \ddot{x}_g \\ \ddot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} (k_a + k_b) & -k_a d_g + k_b(l - d_g) \\ -k_a d_g + k_b(l - d_g) & [k_a d_g^2 + k_b(l - d_g)^2] \end{bmatrix} \begin{Bmatrix} x_g \\ \theta_2 \end{Bmatrix} = 0.$$

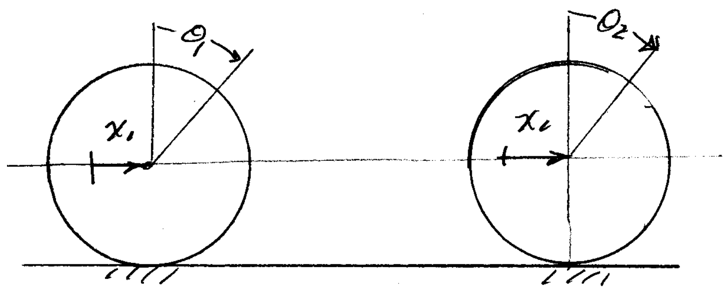
A good deal of effort is required on the homework problem to get d_g and I_g.

Lagrange's Equations

Example 1



Coordinates



Kinetic Energy

$$T = \frac{m_1}{2}\dot{x}_1^2 + \frac{I_{g1}}{2}\dot{\theta}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{I_{g1}}{2}\dot{\theta}_2^2$$

$$= (m_1 r^2 + m_1 \frac{r^2}{2})\frac{\dot{\theta}_1^2}{2} + (m_2 r^2 + m_2 \frac{r^2}{2})\frac{\dot{\theta}_2^2}{2}$$

Potential Energy

$$V = \frac{k_1}{2}x_1^2 + \frac{k_2}{2}(x_1 - x_2)^2 + \frac{k_3}{2}x_2^2$$

$$= \frac{k_1 r^2}{2}\theta_1^2 + \frac{k_2 r^2}{2}(\theta_1 - \theta_2)^2 + \frac{k_3 r^2}{2}\theta_2^2$$

Lagrangian

$$L = \frac{3m_1r^2}{2}\frac{\dot{\theta}_1^2}{2} + \frac{3m_2r^2}{2}\frac{\dot{\theta}_2^2}{2}$$
$$-[\frac{k_1r^2}{2}\theta_1^2 + \frac{k_2r^2}{2}(\theta_1 - \theta_2)^2 + \frac{k_3r^2}{2}\theta_2^2$$

EOM

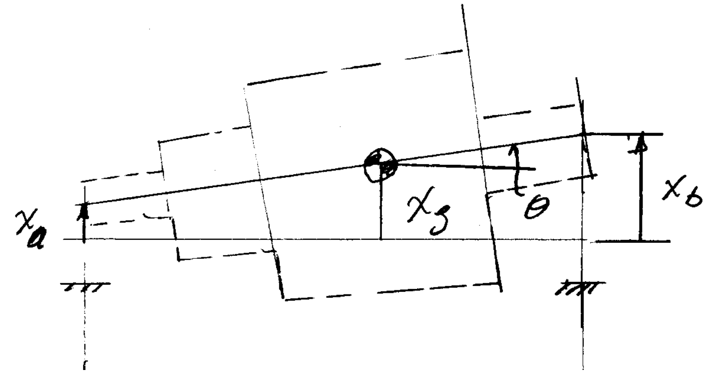
$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{3m_1r^2}{2}\dot{\theta}_1, \quad \frac{\partial L}{\partial \theta_1} = -k_1r^2\theta_1 - k_2r^2(\theta_1 - \theta_2)$$
$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{3m_2r^2}{2}\dot{\theta}_2, \quad \frac{\partial L}{\partial \theta_2} = -k_2r^2(\theta_1 - \theta_2)(-1) - k_3r^2\theta_3$$

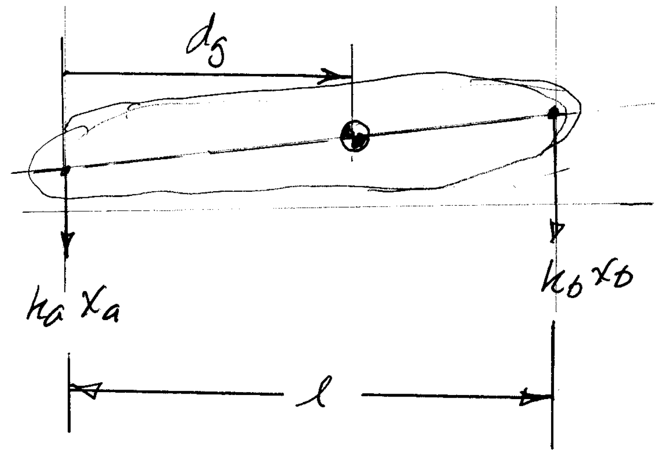
Result from $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = Q_i$

$$\frac{3m_1r^2}{2}\ddot{\theta}_1 + k_1r^2\theta_1 + k_2r^2(\theta_1 - \theta_2) = 0$$

$$3m_2r^2\ \ddot{\theta}$$

Example 2





Kinematics: $x_a = x_g - d_g\theta$, $x_b = x_g + (l - d_g)\theta$.

$$T = \frac{m}{2}\dot{x}_g^{\,2} + \frac{I_g}{2}\dot{\theta}^2$$

$$V = \frac{k_a}{2}x_a^{\,2} + \frac{k_b}{2}x_b^{\,2} = \frac{k_a}{2}(x_g - d_g\theta)^2 + \frac{k_b}{2}[x_g + (l - d_g)\theta]^2$$

$$\therefore\ L = \frac{m}{2}\dot{x}_g^{\,2} + \frac{I_g}{2}\dot{\theta}^2 - \{\frac{k_a}{2}(x_g - d_g\theta)^2 + \frac{k_b}{2}[x_g + (l - d_g)\theta]^2\}$$

Proceeding

$$\frac{\partial L}{\partial \dot{x}_g} = m\dot{x}_g,\ \ \frac{\partial L}{\partial x_g} = -k_a(x_g - d_g\theta) - k_b[x_g + (l - d_g)\theta]$$

$$\frac{\partial L}{\partial \dot{\theta}} = I_g\dot{\theta},\ \ \frac{\partial L}{\partial \theta} = -k_a(x_g - d_g\theta)(-d_g) - k_b[x_g + (l - d_g)\theta](l - d_g)$$

EOMs from $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$

$$m\ddot{x}_g + (k_a + k_b)x_g + [-k_a d_g + k_b(l - d_g)]\theta = 0$$

$$I_g\ddot{\theta} + [-k_a d_g + (l - d_g)]x_g + [k_a d_g^2 + k_b(l - d_g)^2]\theta = 0$$

Lecture 35. More 2DOF EXAMPLES

A Translating Mass with an Attached Compound Pendulum

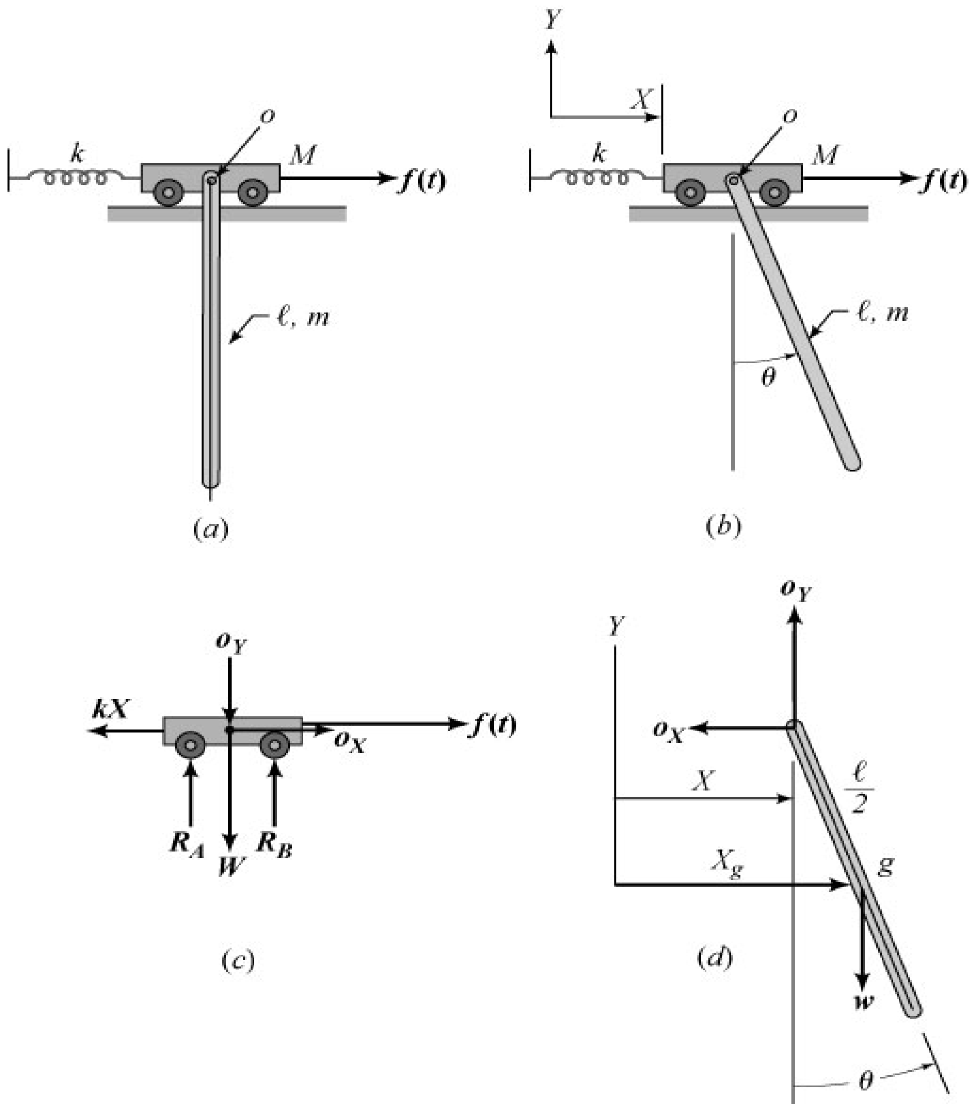


Figure 5.53 Translating cart of mass *M* supported by frictionless rollers and supporting a compound pendulum of length *l* and mass *m* . (a) Equilibrium position, (b) General position (c) Cart free-body diagram, (d) Pendulum free-body diagram.

Cart EOM

$$\Sigma f_X = f(t) + o_X - kX = M\ddot{X}$$

$$\Sigma f_Y = -o_Y - W + R_A + R_B = 0 \quad . \tag{5.201}$$

Pendulum Moment Equation

Moment about g: $\Sigma M_g = I_g \ddot{\theta}$

will draw the unknown and unwanted reaction components o_X, o_Y into the moment equation.

Moment about *o* using $M_{oz} = I_o \ddot{\theta} + m(\boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}})_z$

is quicker. An inspection of the pendulum in figure 5.53c gives:

$$\boldsymbol{b_{og}} = \frac{l}{2}(\boldsymbol{I}\sin\theta - \boldsymbol{J}\cos\theta) \ , \ \ddot{\boldsymbol{R}}_{\boldsymbol{o}} = \boldsymbol{I}\ddot{X} \quad \Rightarrow \ \boldsymbol{b_{og}} \times \ddot{\boldsymbol{R}}_{\boldsymbol{o}} = \boldsymbol{K}\frac{l}{2}\cos\theta\,\ddot{X} \ .$$

Hence, stating the moment equation about the pivot point *o* gives

$$\Sigma M_o = -w\frac{l}{2}\sin\theta = \frac{ml^2}{3}\ddot{\theta} + m\frac{l}{2}\cos\theta\,\ddot{X} \ , \tag{5.202}$$

where $ml^2/3 = I_o$ We now have two equations (the first of Eq.(5.201) and Eq.(5.202)) for the three unknowns: $\ddot{X}, \ddot{\theta}, o_X$.

The X component of the $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$ equation of the pendulum gives the last required EOM as

$$\Sigma f_X = -o_X = m\ddot{X}_g \ . \qquad \textbf{(5.203)}$$

However, this equation introduces the new unknown $\ddot{X}_g$, which can be eliminated, starting from the geometric relationship

$$X_g = X + \frac{l}{2}\sin\theta \ .$$

Differentiating this equation twice with respect to time gives

$$\ddot{X}_g = \ddot{X} + \frac{l}{2}\cos\theta\,\ddot{\theta} - \frac{l}{2}\sin\theta\,\dot{\theta}^2 \ .$$

Substituting for $\ddot{X}_g$ into Eq.(5.203) gives

$$-o_X = m\,(\ddot{X} + \frac{l}{2}\cos\theta\,\ddot{\theta} - \frac{l}{2}\sin\theta\,\dot{\theta}^2) \ .$$

Now, substituting this o_X definition into the first of Eq.(5.201) gives

$$(m+M)\ddot{X}+kX+M(\frac{l}{2}\cos\theta\,\ddot{\theta}-\frac{l}{2}\sin\theta\,\dot{\theta}^2)=f(t)\ . \qquad \textbf{(5.204)}$$

Eqs.(5.202) and (5.204) comprise the two governing equations in $\ddot{X},\ddot{\theta}$. Their matrix statement is:

$$\begin{bmatrix} \frac{ml^2}{3} & \frac{ml}{2}\cos\theta \\ \frac{ml}{2}\cos\theta & (M+m) \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{X} \end{Bmatrix} = \begin{Bmatrix} -\frac{wl}{2}\sin\theta \\ f(t)-kX+\frac{ml^2}{2}\dot{\theta}^2\sin\theta \end{Bmatrix} . \qquad \textbf{(5.205)}$$

We have now completed *Task a*.

Assuming "small" motion for this system means that second order terms in X and θ are dropped. Introducing the small angle approximations $\sin\theta\cong\theta$; $\cos\theta\cong 1$, and dropping second order and higher terms in θ and $\dot{\theta}$ yields:

$$
\begin{bmatrix} \dfrac{ml^2}{3} & \dfrac{ml}{2} \\ \dfrac{ml}{2} & (M+m) \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{X} \end{Bmatrix} + \begin{bmatrix} \dfrac{mgl}{2} & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} \theta \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} . \qquad \textbf{(5.206)}
$$

which concludes *Task b*.

APPLYING LAGRANGE'S EQUATION OF MOTION TO EXAMPLES WITH GENERALIZED COORDINATES (NO KINEMATIC CONSTRAINTS).

Coupled Cart/Pendulum

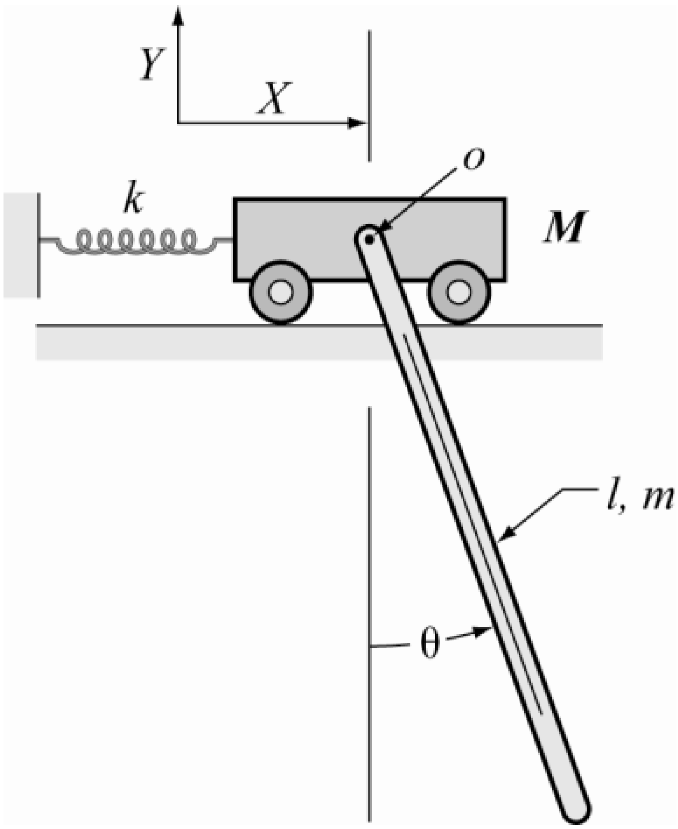


Figure 6.3 Translating cart with an attached pendulum (no external force)

This system has the two coordinates X, θ and two degrees of freedom. Hence, the two coordinates X, θ are the generalized coordinates q_i, and their derivatives $\dot{X}, \dot{\theta}$ are the generalized velocities $\dot{q}_i$ of Lagrange's equations. The following engineering task applies: *Use Lagrange's equations to derive the equations of motion.*

The kinetic energy of the cart is easily calculated as $T_{cart} = M\dot{X}^2/2$.

The kinetic energy of the pendulum follows from the general kinetic energy for planar motion of a rigid body

$$T = \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{g}}|^2}{2} + \frac{I_g\,\dot{\theta}^2}{2} \;. \tag{5.183}$$

where $\dot{\boldsymbol{R}}_{\boldsymbol{g}}$ is the velocity of the body's mass center with respect to an inertial coordinate system. The pendulum's mass center is located by

$$X_g = X + \frac{l}{2}\sin\theta \;,\; Y_g = -\frac{l}{2}\cos\theta \;.$$

Hence

$$\dot{X}_g = \dot{X} + \frac{l}{2}\cos\theta\,\dot{\theta} \;,\; \dot{Y}_g = \frac{l}{2}\sin\theta\,\dot{\theta} \;,$$

and

$$\begin{aligned} T_{pendulum} &= \frac{m}{2}[(\dot{X} + \frac{l}{2}\cos\theta\,\dot{\theta})^2 + (\frac{l}{2}\sin\theta\,\dot{\theta})^2] + \frac{1}{3}\frac{ml^2}{12}\dot{\theta}^2 \\ &= \frac{ml^2}{6}\dot{\theta}^2 + \frac{m}{2}\dot{X}^2 + \frac{ml}{2}\dot{X}\dot{\theta}\cos\theta \;. \end{aligned}$$

Hence, the system kinetic energy is

$$T = \frac{ml^2}{6}\dot{\theta}^2 + \frac{(m+M)}{2}\dot{X}^2 + \frac{ml}{2}\dot{X}\dot{\theta}\cos\theta.$$

Using a plane through the pivot point as datum for the gravity potential energy function gives $V_g = -wl/2\cos\theta$. The potential energy of the spring is $V_s = kX^2/2$; hence, the system potential energy is

$$V = V_g + V_s = -w\frac{l}{2}\cos\theta + \frac{k}{2}X^2,$$

and

$$L = T - V = \frac{ml^2}{6}\dot{\theta}^2 + \frac{(m+M)}{2}\dot{X}^2 + \frac{ml}{2}\dot{X}\dot{\theta}\cos\theta + w\frac{l}{2}\cos\theta - \frac{k}{2}X^2.$$

Proceeding with the Lagrange equations developments, the partial derivatives with respect to generalized velocities are:

$$\frac{\partial L}{\partial \dot{X}} = (m+M)\dot{X} + \frac{ml}{2}\dot{\theta}\cos\theta, \quad \frac{\partial L}{\partial \dot{\theta}} = \frac{ml^2}{3}\dot{\theta} + \frac{ml}{2}\dot{X}\cos\theta,$$

and the derivatives of these terms with respect to time are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = (m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} \ .$$

Once again, note the last terms in these derivatives.

The partial derivatives of L with respect to the generalized coordinates are

$$\frac{\partial L}{\partial X} = -kX \ , \quad \frac{\partial L}{\partial \theta} = -\frac{ml}{2}\dot{X}\dot{\theta}\sin\theta - \frac{wl}{2}\sin\theta \ .$$

By substitution, the governing equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = 0 \Rightarrow (m+M)\ddot{X} + \frac{ml}{2}\ddot{\theta}\cos\theta - \frac{ml}{2}\sin\theta\dot{\theta}^2 + kX = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{ml^2}{3}\ddot{\theta} + \frac{ml}{2}\ddot{X}\cos\theta - \frac{ml}{2}\sin\theta\dot{X}\dot{\theta} + \frac{ml}{2}\dot{X}\dot{\theta}\sin\theta + \frac{wl}{2}\sin\theta = 0 \ . \tag{6.29}$$

The right-hand terms are zero, because there are no nonconservative forces. Eqs.(6.29) are stated in matrix notation as

$$\begin{bmatrix} \frac{ml^2}{3} & \frac{ml}{2}\cos\theta \\ \frac{ml}{2}\cos\theta & (M+m) \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{X} \end{Bmatrix} = \begin{Bmatrix} -\frac{wl}{2}\sin\theta \\ -kX + \frac{ml}{2}\dot{\theta}^2\sin\theta \end{Bmatrix} \quad (6.30)$$

which coincides with Eq.(5.160) (without the external force of figure 5.38) that we derived earlier from a free-body diagram/Newtonian approach. Again, the results are obtained without recourse to free-body diagrams, and only velocities are required for the kinematics.

A Swinging Bar Supported at its End by a Cord

Figure 5.54a shows a swinging bar *AB*, supported by a cord connecting end *A* to the support point *O* The cord has length l_1 ; the bar has length l_2 and mass *m*. This system has the two degrees of freedom φ and θ. The engineering tasks for this system is: *Derive the governing differential equations of motion.*

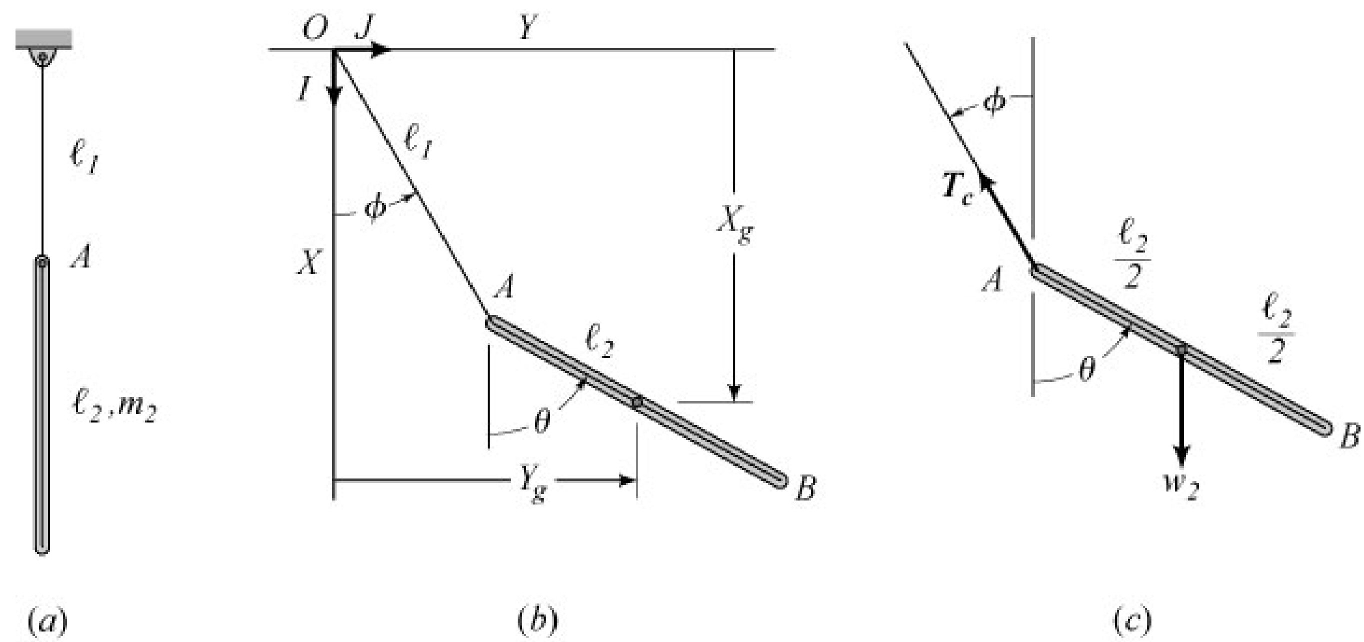


Figure 5.54 Swinging bar supported at its end by a cord. (a) Equilibrium, (b) Coordinate choices, (c) Free-body diagram

From $\Sigma \boldsymbol{f} = m\ddot{\boldsymbol{R}}_{\boldsymbol{g}}$, the force component equations are:

$$\Sigma f_X = w_2 - T_c \cos\varphi = m_2 \ddot{X}_g$$

$$\Sigma f_Y = -T_c \sin\varphi = m_2 \ddot{Y}_g \quad .$$

The acceleration components can be obtained by stating the components of $\boldsymbol{R_g}$ as:

$$X_g = l_1 \cos\varphi + \frac{l_2}{2}\cos\theta \ , \quad Y_g = l_1 \sin\varphi + \frac{l_2}{2}\sin\theta \ .$$

Differentiating these equations twice with respect to time gives

$$\begin{aligned} \ddot{X}_g &= -l_1 \cos\varphi\,\dot{\varphi}^2 - l_1 \sin\varphi\,\ddot{\varphi} - \frac{l_2}{2}\cos\theta\,\dot{\theta}^2 - \frac{l_2}{2}\sin\theta\,\ddot{\theta} \\ \ddot{Y}_g &= -l_1 \sin\varphi\,\dot{\varphi}^2 + l_1 \cos\varphi\,\ddot{\varphi} - \frac{l_2}{2}\sin\theta\,\dot{\theta}^2 + \frac{l_2}{2}\cos\theta\,\ddot{\theta} \end{aligned} \qquad (5.163)$$

Substitution gives

$$w_2 - T_c \cos\varphi = m_2(-l_1 \cos\varphi\,\dot{\varphi}^2 - l_1 \sin\varphi\,\ddot{\varphi} - \frac{l_2}{2}\cos\theta\,\dot{\theta}^2 - \frac{l_2}{2}\sin\theta\,\ddot{\theta})$$

$$-T_c \sin\varphi = m_2(-l_1 \sin\varphi\,\dot{\varphi}^2 + l_1 \cos\varphi\,\ddot{\varphi} - \frac{l_2}{2}\sin\theta\,\dot{\theta}^2 + \frac{l_2}{2}\cos\theta\,\ddot{\theta}) \quad .$$

Eliminate T_c from these equations by: (i) multiply the first by $\sin\varphi$, (ii) multiply the second by $-\cos\varphi$, and (iii) add the results to obtain

$$w_2 \sin\varphi = m_2\,[-l_1\,\ddot{\varphi} - \frac{l_2}{2}\cos(\theta-\varphi)\,\ddot{\theta} + \frac{l_2}{2}\sin(\theta-\varphi)\,\dot{\theta}^2] \quad . \tag{5.164}$$

This is the first of our required differential equations.

One could reasonably state a moment equation about either g, the mass center, or the end A. Stating the moment about A has the advantage of eliminating the reaction force T_c, and we will use the following version of the moment Eq.(5.24)

$$M_{Az} = -\frac{w_2 l_2}{2}\sin\theta = I_A\,\ddot{\theta} + m_2(\boldsymbol{b}_{Ag} \times \ddot{\boldsymbol{R}}_A)_z \quad . \tag{5.165}$$

The moment due to the weight is negative because it acts opposite to the positive counterclockwise direction of θ. The vector $\boldsymbol{b}_{Ag}$ goes from A to g and is defined by

$$\boldsymbol{b}_{Ag} = \frac{l_2}{2}(\boldsymbol{I}\cos\theta + \boldsymbol{J}\sin\theta) \tag{5.166a}$$

To complete Eq.(5.165), we need to define $\ddot{\boldsymbol{R}}_A$. The cord length is constant; hence, the radial acceleration component consists of the centrifugal-acceleration term $-\boldsymbol{\varepsilon}_r l_1 \dot{\varphi}^2$. Similarly, the circumferential acceleration term reduces to $\boldsymbol{\varepsilon}_\varphi l_1 \ddot{\varphi}$. Resolving these terms into their components along the X and Y axes gives

$$\begin{aligned} \ddot{\boldsymbol{R}}_A &= -\boldsymbol{\varepsilon}_r l_1 \dot{\varphi}^2 + \boldsymbol{\varepsilon}_\varphi l_1 \ddot{\varphi} \\ &= l_1[-\boldsymbol{I}(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) + \boldsymbol{J}(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi)] \end{aligned} \tag{5.166b}$$

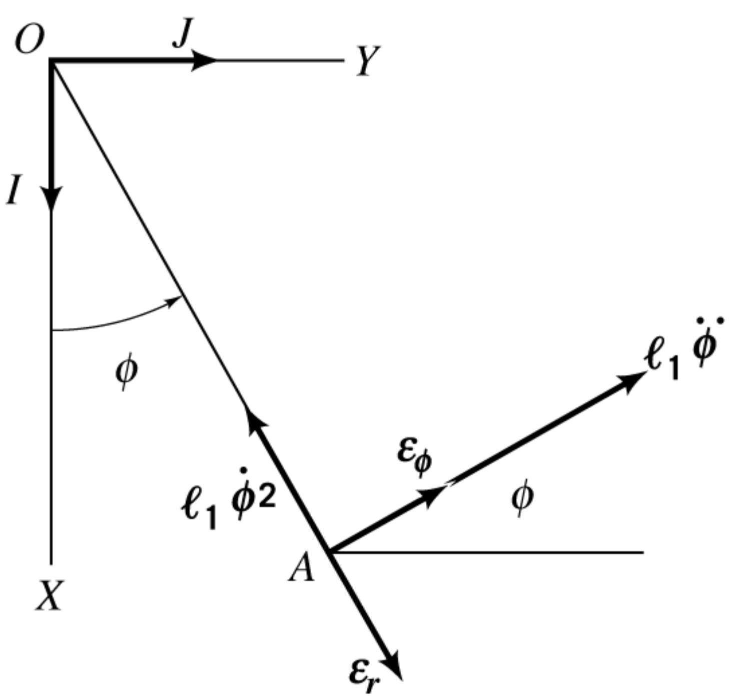


Figure 5.55 Polar kinematics for the cord to determine $\ddot{\boldsymbol{R}}_A$.

Eqs.(5.166) give

$$(\boldsymbol{b}_{Ag} \times \ddot{\boldsymbol{R}}_A)_z = l_1 \frac{l_2}{2}[\ddot{\varphi}\cos(\theta-\varphi) + \dot{\varphi}^2 \sin(\theta-\varphi)] .$$

Substituting this result into Eq.(5.165) gives

$$\frac{m_2 l_2^2}{3}\ddot{\theta} + \frac{w_2 l_2}{2}\sin\theta + \frac{m_2 l_1 l_2}{2}[\ddot{\varphi}\cos(\theta-\varphi) + \dot{\varphi}^2 \sin(\theta-\varphi)] = 0 , \tag{5.167}$$

This is the desired moment equation for the bar and is the second and last equation of motion with $I_A = m_2 l_2^2/3$. Eqs.(5.164) and (5.167) can be combined into the following matrix equation.

$$\begin{bmatrix} m_2 l_1 & \frac{m_2 l_2}{2}\cos(\theta-\varphi) \\ \frac{m_2 l_1 l_2}{2}\cos(\theta-\varphi) & \frac{m_2 l_2^2}{3} \end{bmatrix} \begin{Bmatrix} \ddot{\varphi} \\ \ddot{\theta} \end{Bmatrix}$$

$$= \begin{Bmatrix} -w_2 \sin\varphi + m_2 \frac{l_2}{2}\sin(\theta-\varphi)\dot{\theta}^2 \\ -\frac{w_2 l_2}{2}\sin\theta - \frac{m_2 l_1 l_2}{2}\dot{\varphi}^2 \sin(\theta-\varphi) \end{Bmatrix} .$$

The inertia-coupling matrix can be made symmetric by multiplying the top row by l_1. Eliminating second-order terms in θ and φ in this equation gives the linear vibration equations

$$\begin{bmatrix} m_2 l_1^2 & \dfrac{m_2 l_2 l_1}{2} \\ \dfrac{m_2 l_2 l_1}{2} & \dfrac{m_2 l_1^2}{3} \end{bmatrix} \begin{Bmatrix} \ddot{\varphi} \\ \ddot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} w_2 l_1 & 0 \\ 0 & \dfrac{w_2 l_2}{2} \end{bmatrix} \begin{Bmatrix} \varphi \\ \theta \end{Bmatrix} = 0 \quad .$$