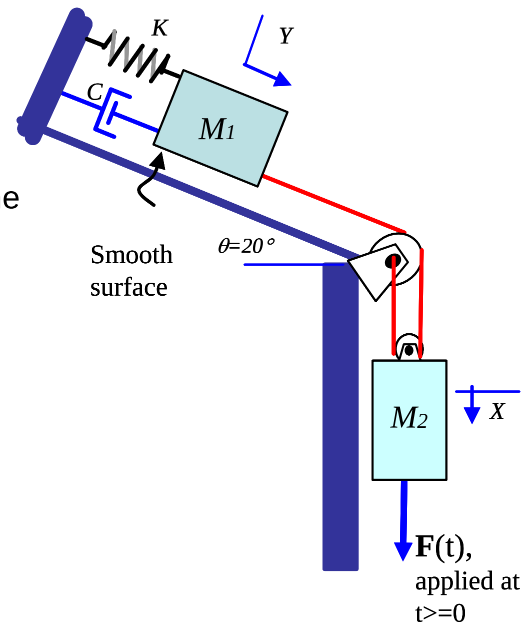
MEEN 459 - 1 DOF Example   DERIVE EOM and STEP LOAD RESPONSE

Select a coordinate system for motions
FROM static equilibrium position,
X(t) is the coordinate for motion of mass M_2 along the vertical direction, + downwards,
Y(t) is the coordinate for motion of mass M_1 along the inclined plane, + downwards and to the right



$W_1 := 5000 \cdot lbf$

$W_2 := 1000 \cdot lbf$

$K := 10^4 \cdot \frac{lbf}{in}$

$\theta := 20 \cdot \frac{\pi}{180}$ angle of inclined plane

$C := 15 \cdot lbf \cdot \frac{sec}{in}$

$M_1 := \frac{W_1}{g} = 5 \times 10^3 \, lb$

$M_2 := \frac{W_2}{g}$

Static equilibrium position defines origin of coordinates *X*, *Y* describing the motion of blocks 2 and 1, respectively.

(a) kinematic constraint - inextensible cable

The cable length is constant, thus

$l_c = l_c + 2 \cdot X - Y$ **and the kinematic constraint follows as** $Y = 2 \cdot X$ **(1)**

(b) Static deflection of spring

By definition of SEP (Static equilibrium position), i.e. system is NOT moving and without any external forces applied into the system:

cable tension $2 \cdot T = W_2$ **must hold weight 2**

Static spring force $F_s = T + W_1 \cdot \sin(\theta)$ **must hold a fraction of weight 1+ cable tension**

hence

$$F_s = \frac{W_2}{2} + W_1 \cdot \sin(\theta) = K \cdot \delta_s \qquad \textbf{(2)}$$

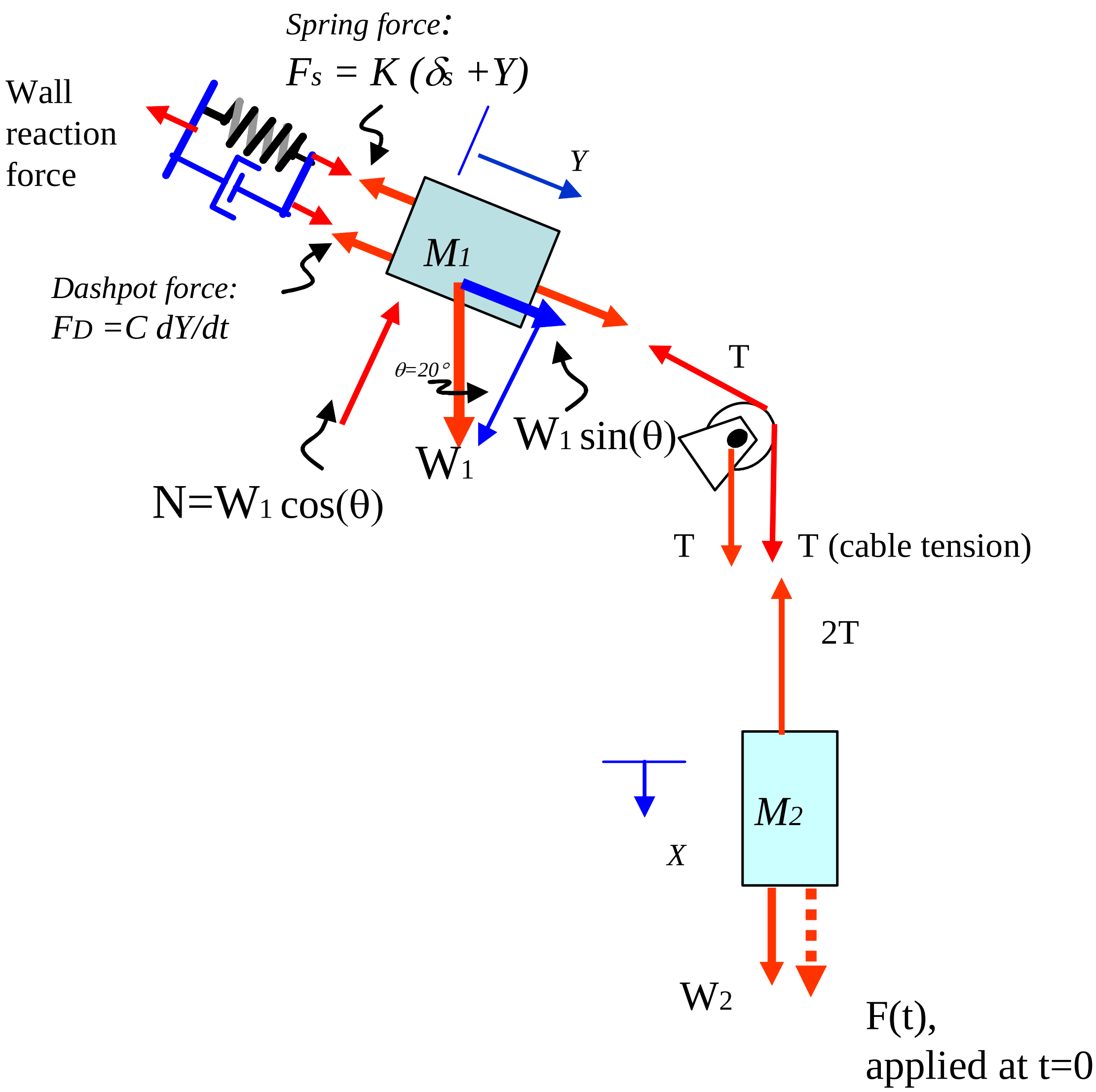
δs is the static deflection of the spring needed to hold the system (w/o motion)

$$\delta_s := \frac{\frac{W_2}{2} + W_1 \cdot \sin(\theta)}{K} \qquad \textbf{(3)}$$

$\delta_s = 0.018 \, ft$

Free Body Diagram

Assumed state of motion to draw FBDs : *X>0, Y>0*



(c) for t>0, external force F(t) is applied to block 2.

Assume state of motion with X>0, Y>0 and draw free body diagrams:

From the FBD diagrams, apply Newton's 2nd law to obtain:

BLOCK 2

$$M_2\cdot\frac{d^2}{dt^2}X = F(t) - 2\cdot T + W_2 \qquad (4)$$

BLOCK 1

$$M_1\cdot\frac{d^2}{dt^2}Y = W_1\cdot\sin(\theta) + T - F_s - F_D \qquad (5)$$

where $F_{Damper} = C\cdot\frac{d}{dt}Y$ is a viscous drag force (6)

$$F_s = (K \cdot Y + K \cdot \delta_s)$$ is the spring elastic force

(d) Derive single EOM for block motion

Note: EOM cannot contain internal forces (Tension for example). The tension is DETERMINED by the motion.

Substitute Eq. (6) into Eq. (5) and isolate the TENSION for substitution into Eq. (4):

$$M_2 \cdot \frac{d^2}{dt^2} X = F(t) + W_2 - 2 \cdot \left[M_1 \cdot \frac{d^2}{dt^2} Y + (K \cdot Y + K \cdot \delta_s) + C \cdot \frac{d}{dt} Y - W_1 \cdot \sin(\theta) \right]$$

(*)

But recall that, from SEP condition: $\frac{W_2}{2} + W_1 \cdot \sin(\theta) = K \cdot \delta_s$

Hence, Eq. (*) simplifies to

$$M_2 \cdot \frac{d^2}{dt^2} X = F(t) - 2 \cdot \left(M_1 \cdot \frac{d^2}{dt^2} Y + K \cdot Y + C \cdot \frac{d}{dt} Y \right)$$

and using the constraint **Y=2X**

$$(M_2 + 4 \cdot M_1) \cdot \frac{d^2}{dt^2} X + 4 \cdot K \cdot X + 4 \cdot C \cdot \frac{d}{dt} X = F(t)$$

(8) **final EOM**

(e) Calculate natural frequency and viscous damping ratio:

Use eq. (8) to continue wthe problem. Define equivalent physical parameters

$M_{eq} := M_2 + 4 \cdot M_1$ $\quad K_{eq} := 4 \cdot K$ $\quad C_{eq} := 4 \cdot C$

EOM: **(9)** $$M_{eq} \cdot \frac{d^2}{dt^2} X + K_{eq} \cdot X + C_{eq} \cdot \frac{d}{dt} X = F(t)$$

Define natural frequency, damping ratio and damped natural frequency:

$$\omega_n := \left(\frac{K_{eq}}{M_{eq}} \right)^{.5} = 27.118 \frac{1}{s}$$

$$f_n := \frac{\omega_n}{2 \cdot \pi} = 4.316\,\text{Hz}$$

$$\zeta := \frac{C_{eq}}{2 \cdot (K_{eq} \cdot M_{eq})^{.5}} = 0.02$$

$$\omega_d := \omega_n \cdot \left(1 - \zeta^2\right)^{0.5} = 27.113 \frac{1}{s}$$

The damped natural frequency and period of motion are:

$$f_d := \frac{\omega_d}{2 \cdot \pi} = 4.315\,\text{Hz} \qquad T_d := \frac{1}{f_d} = 0.232\,\text{s}$$

The damping ratio is not too large - motion will be oscillatory and will be damped out!

(f) Example: Applied force is constant, i.e. STEP force

Let $F_o := 10000 \cdot \text{lbf}$ Hence: $F(t) := F_o$ **(9)**

$$M_{eq} \cdot \frac{d^2}{dt^2} X + C_{eq} \cdot \frac{d}{dt} X + K_{eq} \cdot X = F(t) = F_o \quad \textbf{(8)}$$

<u>What is steady-state motion?</u>

Since F(t) is a constant, the particular solution of Eq (8) is:

$$X_p := \frac{F_o}{K_{eq}} = 0.021\,\text{ft} \quad \textbf{(10)}$$

(g) find the full transient response - dynamic motion of block

Complete solution: $X(t) = X_H + X_p$

$$X(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + X_p \quad \textbf{(11)}$$

satisfy initial conditions at t=0:

$X_o := 0 \cdot \text{ft}$ $V_o := 0 \cdot \frac{\text{ft}}{\text{sec}}$ **motion starts from rest**

at time t=0 s

$$C_1 := X_o - X_p = -0.021\,\text{ft}$$

$$C_2 := \frac{V_o + \zeta \cdot \omega_n \cdot C_1}{\omega_d} = -4.238 \times 10^{-4}\,\text{ft}$$

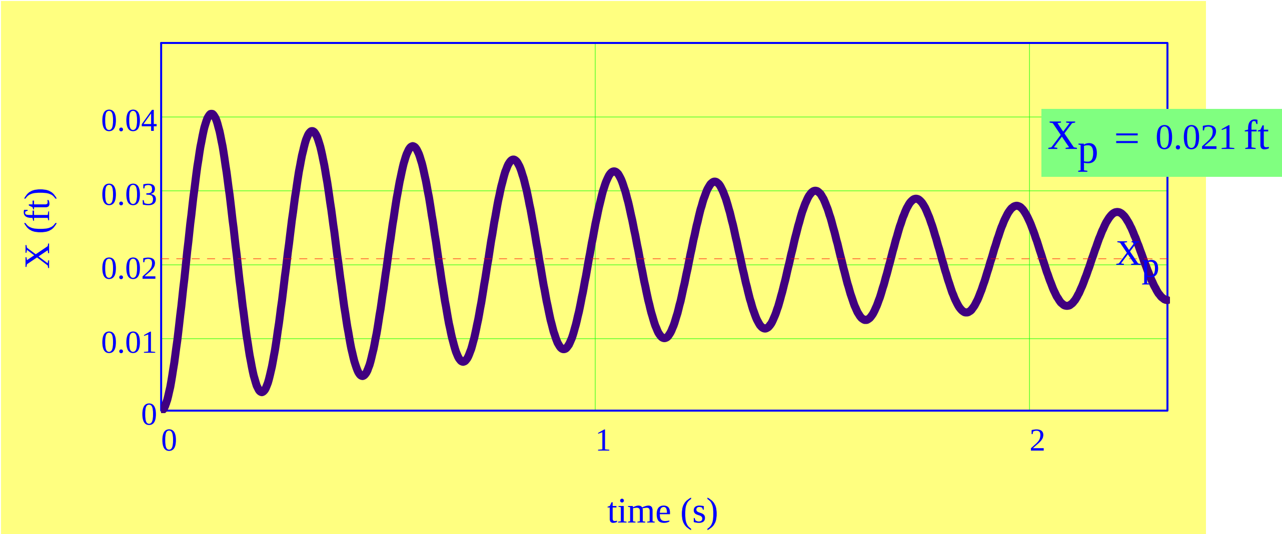
$$D_1 := -\zeta \cdot \omega_n \cdot C_1 + C_2 \cdot \omega_d = 0\,\frac{ft}{s} \qquad D_2 := -\zeta \cdot \omega_n \cdot C_2 - C_1 \cdot \omega_d = 0.565\,\frac{ft}{s}$$

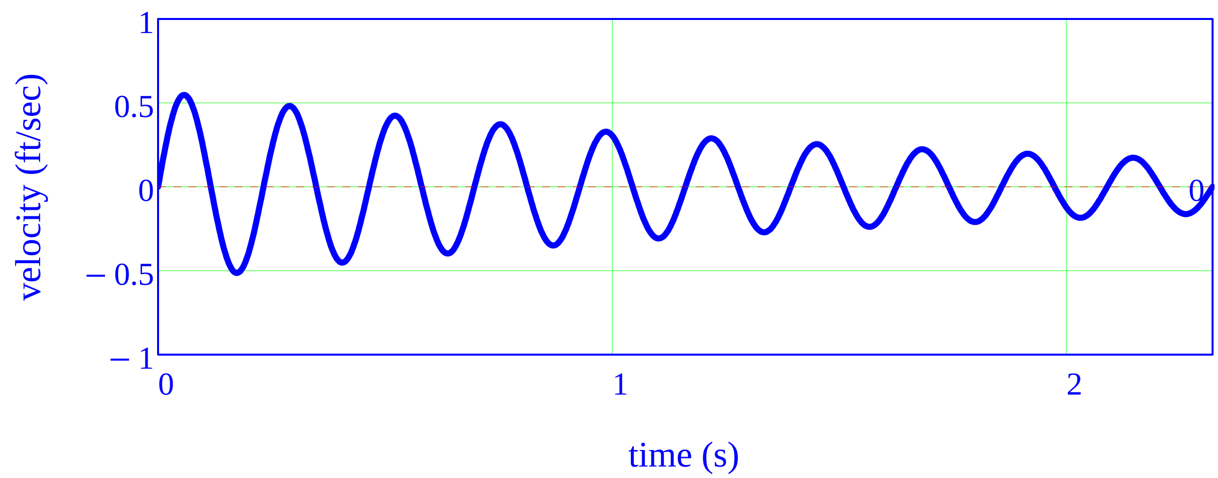
PLOT the response for times to 10 x damped period (my choice)

$$X(t) := e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + X_p$$

$$V(t) := e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos\left(\omega_d \cdot t\right) + D_2 \cdot \sin\left(\omega_d \cdot t\right)\right)$$

$$T_{max} := 10 \cdot T_d$$



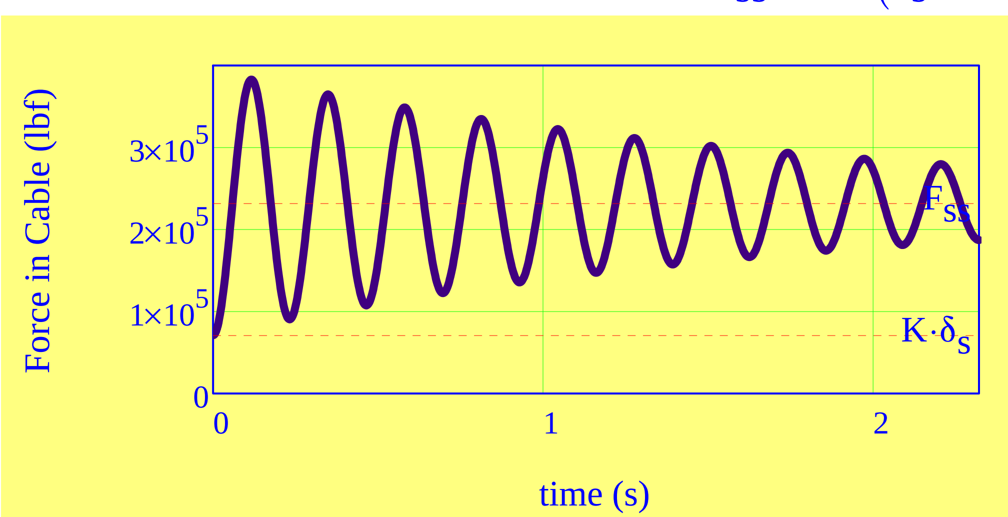


Spring (cable) force **(dynamic+static)**

$$F_s(t) := K\cdot\left(\delta_s + 2\cdot X(t)\right)$$

At S-S $F_{ss} := K\cdot\left(\delta_s + 2\cdot X_p\right) = 7.21\times 10^3\,lbf$

Steady state force magnitude



$$K\cdot\delta_s = 2.21\times 10^3\,lbf$$