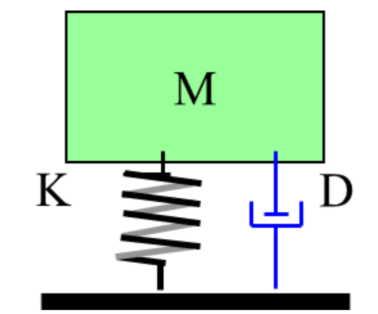
ME459/659 Notes# 1

# A Review to the Dynamic Response of SDOF Second Order Mechanical System with Viscous Damping




by
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**ME 459/659**
**Sound & Vibration Measurements**

**Reproduced from material in ME 363 and ME617**

**http://rotorlab.tamu.edu**

**ME459/659 Notes# 1a (pp. 1-31)**

# Dynamic Response of SDOF Second Order Mechanical System: Viscous Damping

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=F_{ext(t)}$$

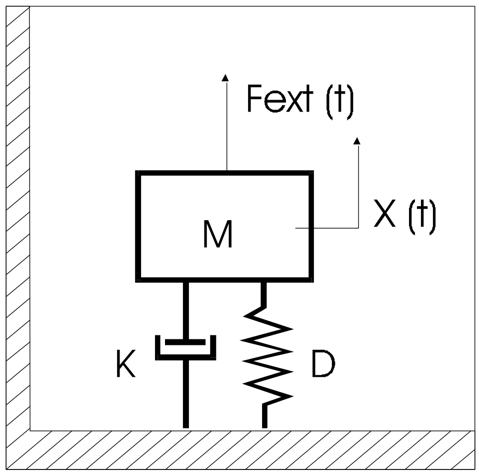
**Free Response to $F_{(t)}$ = 0 + initial conditions and**

Underdamped, Critically Damped and Overdamped Systems

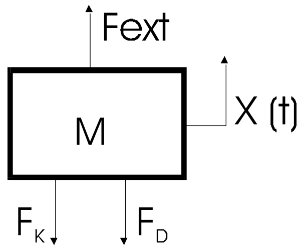
**Forced Response to a Step Loading $F_{(t)} = F_o$**

Jump to page 97 (pdf count) for ready to use formulas

# Second Order Mechanical Translational System:



No dry friction (dissipation) mechanism



Free Body Diagram

**Fundamental equation of motion** about equilibrium position ($X$=0)

$$\sum F_X = M\frac{d^2 X}{d t^2} = F_{ext(t)} - F_D - F_K$$

$$F_D = D\frac{d X}{d t}$$ : Viscous Damping Force

$$F_k = K X$$ : Elastic restoring Force

( $M$, $D$ ,$K$ ) represent the system equivalent mass, viscous damping coefficient, and stiffness coefficient, respectively.

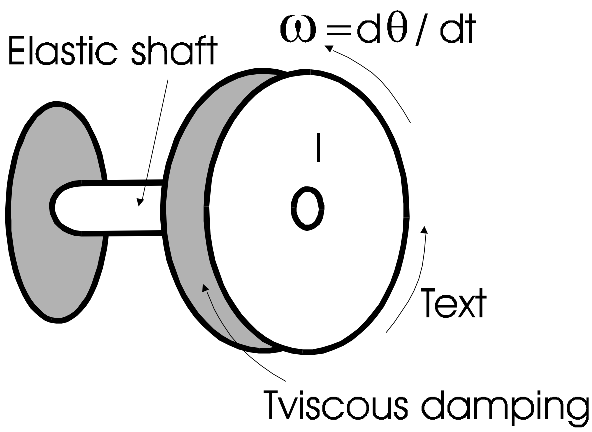
➔ equation of motion

$$M\frac{d^2 X}{d t^2} + D\frac{d X}{d t} + K X = F_{ext(t)}$$

+ **Initial Conditions** in velocity and displacement; at $t$=0:

$$X_{(0)} = X_o \text{ and } \dot{X}_{(0)} = V_o$$

# Second Order Mechanical Torsional System:




Equivalent Rotational System

**Fundamental equation of motion** about equilibrium position, $\theta=0$

$$\sum Torques = I\,\frac{d\,\omega}{d\,t} = T_{ext(t)} - T_{\theta D} - T_{\theta K}\,;\;\; \omega = \dot{\theta}$$

$T_{\theta D} = D_\theta\,\omega$ : Viscous dissipation torque

$T_{\theta K} = K_\theta\,\theta$ : Elastic restoring torque

$(I, D_\theta, K_\theta)$ are the system equivalent mass moment of inertia, rotational viscous damping coefficient, and rotational (torsional) stiffness coefficient, respectively.

Equation of motion →

$$I\,\frac{d^2\theta}{d\,t^2} + D_\theta\,\frac{d\,\theta}{d\,t} + K_\theta\,\theta = T_{ext(t)}$$

**+ Initial Conditions** in angular velocity and displacement at $t=0$:

$$\omega_{(0)} = \omega_o \quad \text{and} \quad \theta_{(0)} = \theta_o$$

# (a) **Free Response of Second Order SDOF Mechanical System**

Let the external force $F_{ext}$=0 and the system has an initial displacement $X_o$ and initial velocity $V_o$. EOM is

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=0 \tag{1}$$

Divide Eq. (1) by *M* and <u>define</u>:

$\omega_n=\sqrt{K/M}$ : natural frequency of system

$\zeta=\dfrac{D}{D_{cr}}$ : viscous damping ratio,

where $D_{cr}=2\sqrt{KM}$ is known as the **critical damping** magnitude.

With these definitions, Eqn. (1) →

$$\frac{d^2X}{dt^2}+2\zeta\omega_n\frac{dX}{dt}+\omega_n^2X=0 \tag{2}$$

The solution of the Homogeneous Second Order Ordinary Differential Equation with Constant Coefficients is of the form:

$$X(t)=Ae^{st} \tag{3}$$

where $A$ is a constant found from the initial conditions.

Substitute Eq. (3) into Eq. (2) and obtain:

$$\left(s^2+2\zeta\omega_n s+\omega_n^2\right)A=0 \tag{4}$$

*A* is not zero for a non trivial solution. Thus, Eq. (4) leads to the **CHARACTERISTIC EQUATION** of the SDOF system:

$$\left(s^2+2\zeta\,\omega_n\,s+\omega_n^2\right)=0 \qquad (5)$$

The roots of this 2nd order polynomial are:

$$s_{1,2}=-\zeta\,\omega_n\mp\omega_n\left(\zeta^2-1\right)^{1/2} \qquad (6)$$

The nature of the roots (eigenvalues) depends on the damping ratio $\zeta$ (>1 or < 1). Since there are two roots, the solution is

$$X(t)=A_1\,e^{s_1 t}+A_2\,e^{s_2 t} \qquad (7)$$

$A_1$, $A_2$ determined from the initial conditions in displacement and velocity.

From Eq. (6), differentiate three cases:

**Underdamped System:** $0<\zeta<1, \rightarrow D<D_{cr}$

**Critically Damped System:** $\zeta = 1, \rightarrow D = D_{cr}$

**Overdamped System:** $\zeta > 1, \rightarrow D > D_{cr}$

Note that $\tau=\left(1/\zeta\omega_n\right)$ has units of time; and for practical purposes, it is akin to an equivalent time constant for the second order system.

# Free Response of Undamped 2nd Order System

For an undamped system, $\zeta = 0$, i.e., a **conservative system** without viscous dissipation, the roots of the characteristic equation are imaginary:

$$s_1 = -i\,\omega_n \; ; \; s_2 = i\,\omega_n \qquad (8)$$

where $i = \sqrt{-1}$

Using the complex identity $e^{iat} = \cos(at) + i \sin(at)$, renders the **undamped response** as:

$$X(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) \qquad (9.a)$$

where $\omega_n = \sqrt{K/M}$ is the **natural frequency** of the system.

At time $t = 0$, apply the initial conditions to obtain

$$C_1 = X_0 \text{ and } C_2 = \frac{V_0}{\omega_n} \qquad (9.b)$$

Eq. (9.a) can be written as:

$$X(t) = X_M \cos(\omega_n t - \varphi) \qquad (9.c)$$

where $X_M = \sqrt{X_0^2 + \frac{V_0^2}{\omega_n^2}}$ and $\tan(\varphi) = \frac{V_0}{\omega_n X_0}$

$X_M$ is the **maximum amplitude response.**

## Notes:

In a purely conservative system ($\zeta = 0$), the motion never dies. Motion always oscillates about the equilibrium position ***X* = 0**

## Free Response of Underdamped 2nd Order System

For an underdamped system, 0 < $\zeta$ < 1, the roots are complex conjugate (real and imaginary parts), i.e.

$$s_{1,2} = -\zeta\,\omega_n \mp i\,\omega_n \left(1-\zeta^2\right)^{1/2} \qquad (10)$$

Using the complex identity $e^{iat} = \cos(at) + i\,\sin(at)$, the response is:

$$X(t) = e^{-\zeta\,\omega_n t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) \qquad (11)$$

where $\omega_d = \omega_n \left(1-\zeta^2\right)^{1/2}$ is the system damped natural frequency.

At time $t$ = 0, applying initial conditions gives

$$C_1 = X_0 \ \text{ and } \ C_2 = \frac{V_0 + \zeta\,\omega_n X_0}{\omega_d} \qquad (11.b)$$

Eqn. (11) can be written as:

$$X(t) = e^{-\zeta\,\omega_n t} X_M \cos\left(\omega_d t - \varphi\right) \qquad (11.c)$$

where $X_M = \sqrt{C_1^2 + C_2^2}$ and $\tan\left(\varphi\right) = \frac{C_2}{C_1}$

**Note** that as $t \to \infty$, $X_{(t)} \to 0$, i.e. the equilibrium position only if $\zeta$ > 0;

and $X_M$ is the **largest amplitude of response** only if $\zeta$ =0 *(no damping).*

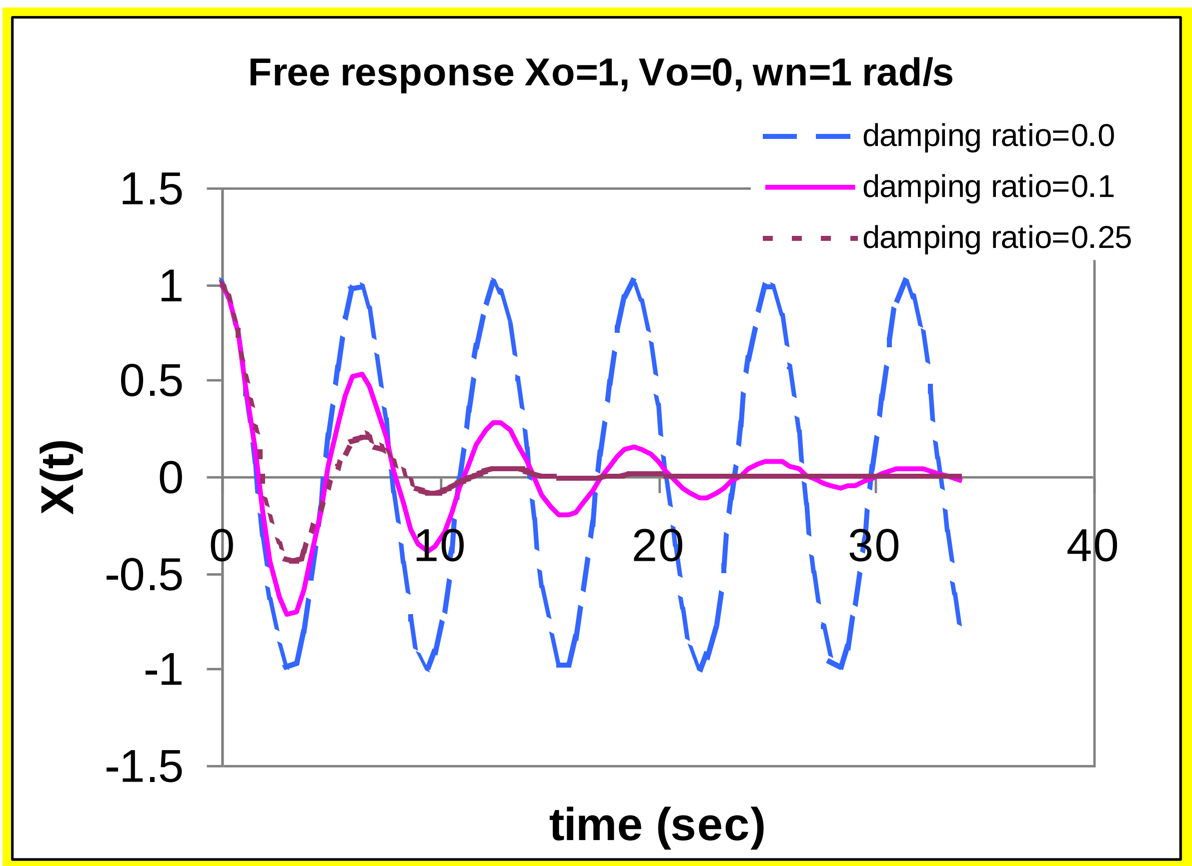
**Free Response of Underdamped 2nd Order System:**

**initial displacement only** damping ratio varies

$X_o = 1$, $V_o = 0$, $\omega_n = 1.0$ *rad/s* $\zeta = 0,\ 0.1,\ 0.25$

Motion decays exponentially for $\zeta > 0$

**Faster system response as $\zeta$ increases, i.e. faster decay towards equilibrium position *X=0***



**Free Response of Underdamper 2nd Order System:**

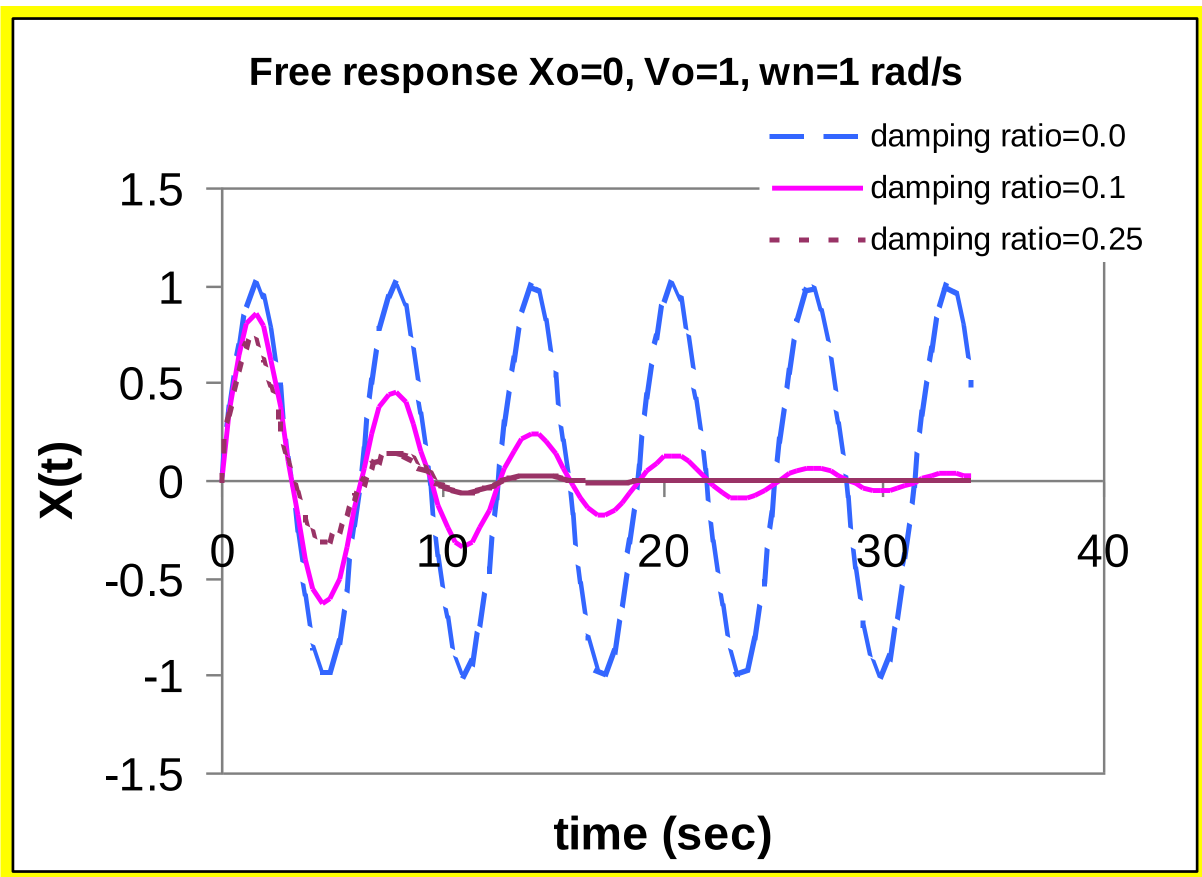
**Initial velocity only** damping varies

$X_o = 0$, $V_o = 1.0$ $\omega_n = 1.0$ rad/s; $\zeta = 0,\ 0.1,\ 0.25$

Motion decays exponentially for $\zeta > 0$

**Faster system response as $\zeta$ increases, i.e. faster decay towards equilibrium position $X$=0**

Note the initial overshoot



## Free Response of Overdamped 2nd Order System

For an overdamped system, $\zeta$ > 1, the roots of the characteristic equation are real and negative, i.e.,

$$s_1 = \omega_n \left[ -\zeta + \left(\zeta^2 - 1\right)^{1/2} \right]; \; s_2 = \omega_n \left[ -\zeta - \left(\zeta^2 - 1\right)^{1/2} \right] \tag{12}$$

The **free response** of an **overdamped** system is:

$$X(t) = e^{-\zeta \omega_n t} \left( C_1 \cosh\left(\omega_* t\right) + C_2 \sinh\left(\omega_* t\right) \right) \tag{13}$$

where $\omega_* = \omega_n \left(\zeta^2 - 1\right)^{1/2}$ has units of 1/time. Do not confuse this term with a frequency since the response of motion is NOT oscillatory.

At time $t$ = 0, apply the initial conditions to get

$$C_1 = X_0 \;\; \text{and} \;\; C_2 = \frac{V_0 + \zeta \omega_n X_0}{\omega_*} \tag{14}$$

Note that as $t \rightarrow \infty$, $X_{(t)} \rightarrow 0$, i.e. the equilibrium position.

**Note:** An **overdamped system does to oscillate**. The larger the damping ratio $\zeta$ *>1*, the longer time it takes for the system to return to its equilibrium position.

## Free Response of Critically Damped 2nd Order System

For a critically damped system, $\zeta$ = 1, the roots are real negative and identical, i.e.

$$s_1 = s_2 = -\zeta\,\omega_n \qquad (15)$$

The solution form $X(t) = A\, e^{st}$ is no longer valid. For repeated roots, the theory of ODE's dictates that the family of solutions satisfying the differential equation is

$$X(t) = e^{-\omega_n t}\left(C_1 + t\,C_2\right) \qquad (16)$$

At time $t$ = 0, applying initial conditions gives

Then $$C_1 = X_0 \text{ and } C_2 = V_0 + \omega_n X_0 \qquad (17)$$

Note that as $t \to \infty$, $X_{(t)} \to 0$, i.e. the equilibrium position.

A **critically damped system** does to oscillate, and it is the fastest to damp the response due to initial conditions.

## Free Response of 2nd order system:

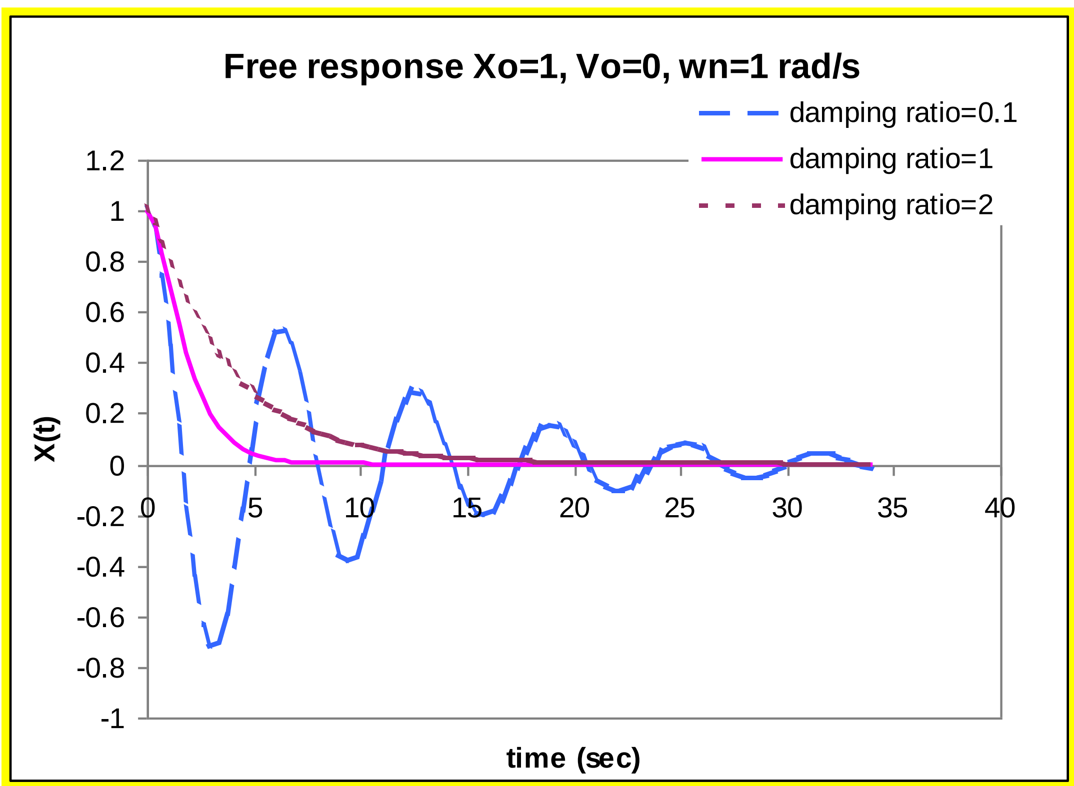
**Comparison between underdamped, critically damped and overdamped systems**

**initial displacement only**

$X_o = 1$, $V_o = 0$   $\omega_n = 1.0$ rad/s   $\zeta = 0.1, \; 1.0, \; 2.0$

**Motion decays exponentially for $\zeta > 0$**

Fastest response for $\zeta = 1$; i.e. fastest decay towards equilibrium position $X = 0$



## Free Response of 2nd order System:

Comparison between underdamped, critically damped and overdamped systems

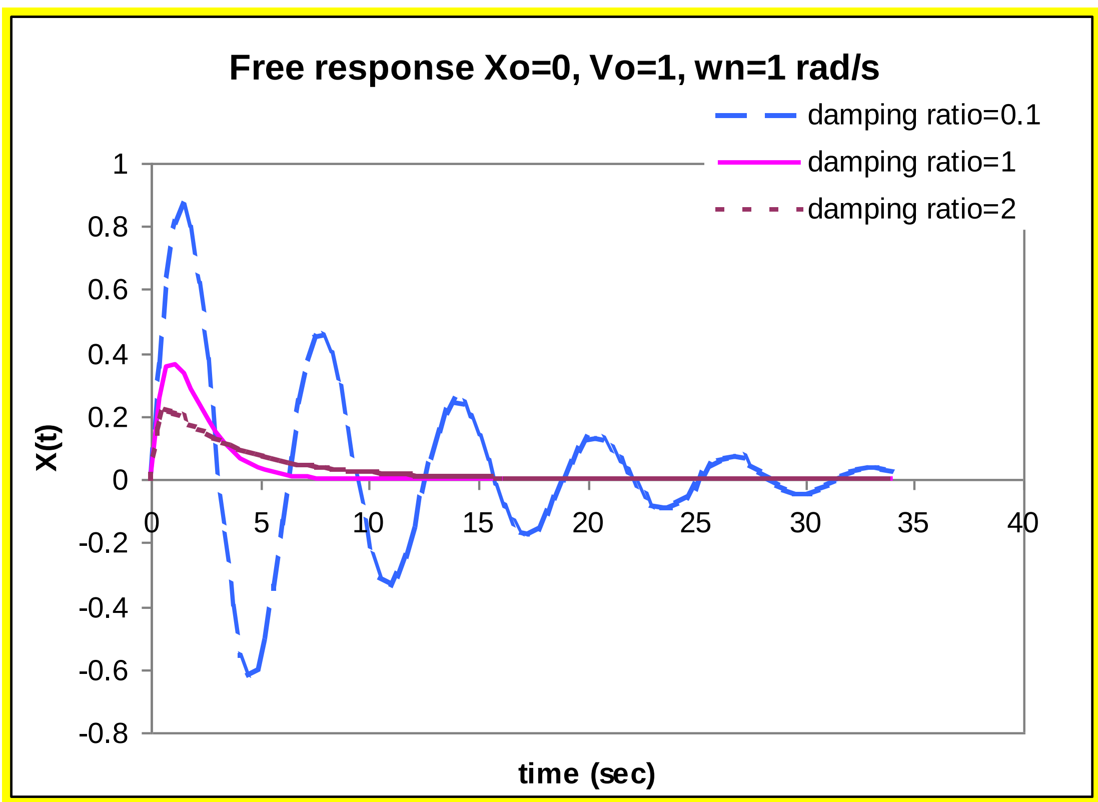
**Initial velocity only**

$X_o = 0$, $V_o = 1.0$, $\omega_n = 1.0$ rad/s $\zeta$ = 0.1, 1.0, 2.0

**Motion decays exponentially for $\zeta > 0$**

Fastest response for $\zeta = 1.0$, i.e. fastest decay towards equilibrium position *X=0.*

note initial overshoot



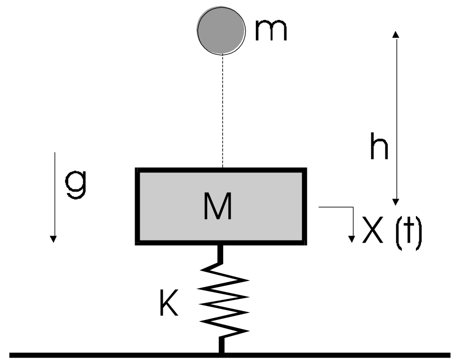
**EXAMPLE:**

A 45 gram steel ball (m) is dropped from rest through a vertical height of h=2 m. The ball impacts on a solid steel cylinder with mass M = 0.45 kg. The impact is perfectly elastic. The cylinder is supported by a soft spring with a stiffness K = 1600 N/m. The mass-spring system, initially at rest, deflects a maximum equal to $\delta$ = 12 mm, from its static equilibrium position, as a result of the impact.

**(a)** Determine the time response motion of the mass-spring system.

**(b)** Sketch the time response of the mass-spring system.

**(c)** Calculate the height to which the ball will rebound.




**(a) Conservation of linear momentum before impact = just after impact:**

$$mV_- = mV_+ + M\,\dot{x}_o \quad (1)$$

where $V_- = \sqrt{2gh}$ = 6.26 m/s is the steel ball velocity before impact

$V_+$ = velocity of ball after impact; and $\dot{x}_o$ : initial mass-spring velocity.

Mass-spring system EOM: $M\,\ddot{x} + K\,x = 0$ (2) with $\omega_n = \sqrt{\frac{K}{M}}$ = 59.62 rad/s

(from static equilibrium), the initial conditions are $x(0) = 0$ and $\dot{x}(0) = \dot{x}_o$ (3)

(2) & (3) lead to the undamped free response: $x(t) = \frac{\dot{x}_o}{\omega_n} \sin(\omega_n t) = \delta \sin(\omega_n t)$

given $\delta$ = 0.012 m as the largest deflection of the spring-mass system.

Hence, $\dot{x}_o = \delta\,\omega_n$ = 0.715 m/s

**(c) Ball velocity after impact:** from Eq. (1))

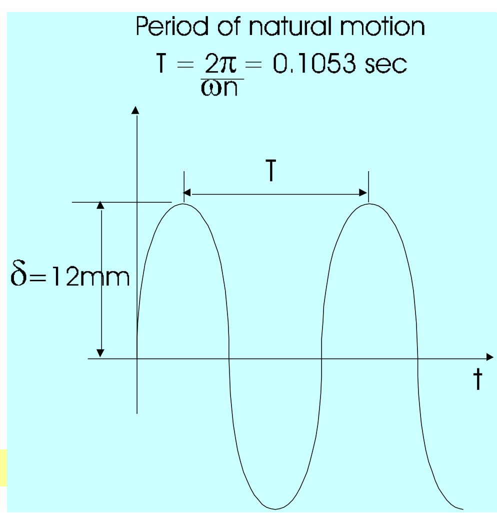
$$V_+ = V_- - \frac{M}{m}\dot{x}_o = (6.26 - 7.15)\frac{m}{s} = -0.892 \frac{m}{s}$$

(upwards)

and the height of rebound is

$$h_+ = \left[\frac{V_+^2}{2g}\right] = 41 \text{ mm}$$

**(b) Graph of motion**

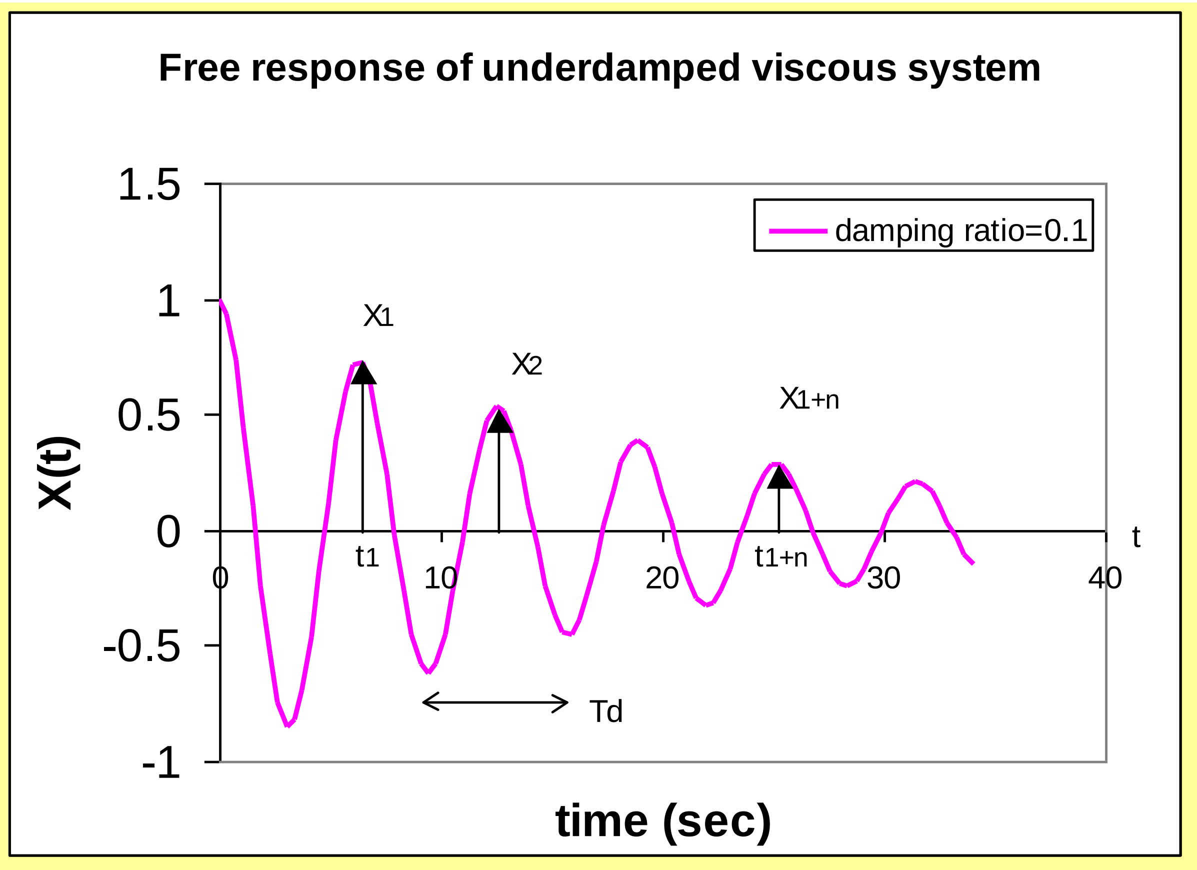


## The concept of **logarithmic decrement** for estimation of the viscous damping ratio from a free-response vibration test

The free vibration response of an **underdamped** 2nd order viscous system (*M*,*K*,*D*) due to an initial displacement $X_o$ is a decay oscillating wave with damped natural frequency $(\omega_d)$. The period of motion is $T_d = 2\pi/\omega_d$ (sec). The free vibration response is

$$x(t) = X_o\, e^{-\zeta\omega_n t} \cos\left(\omega_d t\right) \qquad (11.c)$$

where $\zeta = D/D_{cr},\ D_{cr} = 2\sqrt{KM};\ \omega_n = (K/M)^{1/2};\ \omega_d = \omega_n\left(1-\xi^2\right)^{1/2}$




Consider two peak amplitudes, say $X_1$ and $X_{1+n}$, separated by ***n*** periods of decaying motion. These peaks occur at times, $t_1$ and *(t₁+nTd)*, respectively. The system response at these two times is from Eq. (1):

$$X_1 = x(t_1) = X_o e^{-\zeta\,\omega_n t_1} \cos\,(\omega_d t_1),$$

and

$$X_{1+n} = x\left(t_{1+nT_d}\right) = X_o\, e^{-\zeta\omega_n (t_1+nT_d)}\ \cos\,(\omega_d\ t_1 + n\ \omega_d T_d)\ ,$$

Or, since $\omega_d T_d = 2\pi$.

$$X_{1+n} = X_o e^{-\zeta\,\omega_n (t_1+nT_d)}\ \cos\left(\omega_d t_1 + 2\pi\ n\right) = X_o e^{-\zeta\ \omega_n (t_1+nT_d)}\ \cos\,(\omega_d t_1) \quad (18)$$

Now the **ratio between these two peak amplitudes** is:

$$\frac{X_1}{X_{1+n}} = \frac{\left\{ X_o\ e^{-\zeta\omega_n t_1}\ \cos\ (\omega_d t_1)\right\}}{\left\{ X_o\ e^{-\zeta\omega_n (t_1+nT_d)}\ \cos\ (\omega_d t_1 + 2\pi)\ \right\}} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+nT_d)}} = e^{\zeta\omega_n (nT_d)} \quad (19)$$

Take the **natural logarithm** of the ratio above:

$$\ln\left(X_1 / X_{1+n}\right) = \xi\,\omega_n\ n\ T_d = n\ \zeta\ \omega_n \frac{2\pi}{\omega_n\left(1-\zeta^2\right)^{1/2}} = \frac{2\pi\, n\,\zeta}{\left(1-\zeta^2\right)^{1/2}} = n\cdot\delta \quad (20)$$

Define the **logarithmic decrement** as:

$$\delta = \frac{1}{n}\ln\left[\frac{X_1}{X_{1+n}}\right] = \frac{2\pi\,\zeta}{\left(1-\zeta^2\right)^{1/2}} \quad (21)$$

Thus, the ratio between peak response amplitudes determines a useful relationship to identify the damping ratio of an underdamped second order system, i.e., once the **log dec** ($\delta$) is determined then,

$$\zeta = \frac{\delta}{\left[ (2\pi)^2 + \delta^2 \right]^{1/2}} \qquad (22),$$

and for small damping ratios, $\zeta \sim \frac{\delta}{2\pi}$.

The **logarithmic decrement** method to identify viscous damping ratios should only be used if:

a) the time decay response shows an oscillatory behavior (i.e. vibration) with a clear exponential envelope, i.e. damping of viscous type,

b) the system is linear, 2nd order and **underdamped**,

c) the dynamic response is very clean, i.e. without any spurious signals such as noise or with multiple frequency components,

d) the dynamic response $X(t) \rightarrow 0$ as $t \rightarrow \infty$. Sometimes measurements are taken with some DC offset. This must be removed from your signal before processing the data.

e) Strongly recommend to use more than just two peak amplitudes separated *n* periods. **In practice**, it is more accurate to graph the magnitude of several peaks in a log scale and obtain the log-decrement $(\delta)$ as the best linear fit to the following relationship [see below Eq. (23)].

From Eq. (18),

$$X_{1+n} = X_o \cos\left(\omega_d t_1\right) e^{-\zeta\,\omega_n\left(t_1+nT_d\right)} = X_1 e^{-\xi\omega_n\left(nT_d\right)},$$

$$\text{where } X_1 = X_o \cos\left(\omega_d t_1\right) e^{-\zeta\,\omega_n\left(t_1\right)}$$

→

$$\ln(X_{1+n}) = \ln X_1 + \ln(e^{-\zeta\,\omega_n\left(nT_d\right)}) =$$

$$\ln X_1 - \zeta\,\omega_n T_d = A - n\delta, \text{ where } A = \ln X_1;$$

$$\ln(X_{1+n}) = A - n\delta \qquad (23)$$

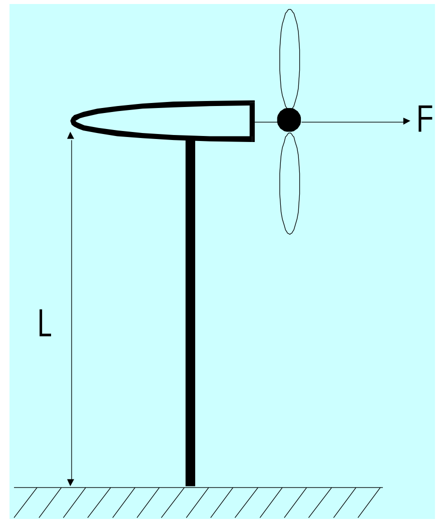
i.e., plot the natural log of the peak magnitudes versus the period numbers ($n$=1,2,...) and obtain the logarithmic decrement from a straight line curve fit. In this way you will have used more than just two peaks for your identification of damping.

Always provide the correlation number (goodness of fit = $R^2$) for the linear regression curve ($y=ax+b$), with $y=ln(X)$ and $x=n$ as variables.

**The log-dec is a most important concept widely used in the characterization of damping in a mechanical system (structure, pipe system, spinning rotor, etc) as it gives a quick estimation of the available damping in a system.**

**In rotating machinery, API specifications provide the minimum value of logdec a machine should have to warrant its acceptance in a shop test as well as safe (and efficient) operation in the film.**

## EXAMPLE




A wind turbine is modeled as a concentrated mass (the turbine) atop a weightless elastic tower of height L. To determine the dynamic properties of the system, a large crane is brought alongside the tower and a lateral force F=200 lb is exerted along the turbine axis as shown. This causes a horizontal displacement of 1.0 in.

The cable attaching the turbine to the crane is instantaneously severed, and the resulting free vibration of the turbine is recorded. At the end of two complete cycles (periods) of motion, the time is 1.25 sec and the motion amplitude is 0.64 in.

From the data above determine:

**(a)** equivalent stiffness *K* (lb/in)

**(b)** damping ratio $\zeta$

**(c)** undamped natural frequency $\omega_n$ (rad/s)

**(d)** equivalent mass of system (lb-s$^2$/in)

**a)** $$K = \frac{\text{static force}}{\text{static deflection}} = \frac{200\ lb}{1.0\ \text{in}} = 200\ \text{lb/in}$$

**b)** Use **log dec** to find the viscous damping ratio

| cycle | amplitude | time |
|---|---|---|
| 0 | 1.0 in | 0.0 sec |
| 2 | 0.64 in | 1.25 sec |

$$\delta = \frac{1}{n}\ \ln\left(\frac{x_o}{x_2}\right) = \frac{1}{2}\ln\left(\frac{1.0}{0.64}\right) = 0.2231$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-{}^2}}\ ; \qquad \zeta = \frac{\delta}{\sqrt{\delta^2+4\pi^2}} \simeq \frac{\delta}{2\pi} = 0.035$$

underdamped system with 3.5% of critical damping.

**c)** Damped period of motion, $T_d = \frac{1.25}{2}\frac{\text{s}}{\text{cyc}} = 0.625$ **sec/cyc**

Damped natural frequency, $\omega_d = \frac{2\pi}{T_d} = 10.053 \frac{rad}{\text{sec}}$

Natural frequency, $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \mathbf{10.059} \frac{\mathbf{rad}}{\mathbf{sec}}$

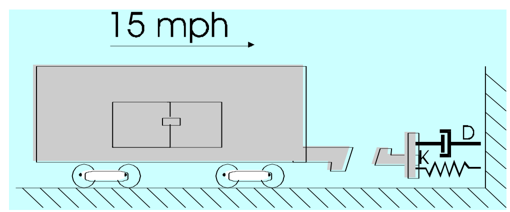
**d)** Equivalent mass of system:

from $\omega_n = \sqrt{K/M}$

$$M = \frac{K}{\omega_n^{\ 2}} = \frac{200\ lb/\text{in}}{10.059^2 \ \ ^1/_{\text{sec}^2}}$$ **= 1.976 lb/ sec$^2$/in**

**EXAMPLE**

A loaded railroad car weighing 35,000 lb is rolling at a constant speed of 15 mph when it couples with a spring and dashpot bumper system. If the recorded displacement-time curve of the loaded railroad car after coupling is as shown, determine



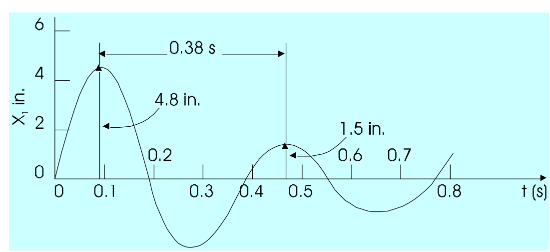

**(a)** the logarithmic decrement $\delta$

**(b)** the damping ratio $\zeta$

**(c)** the natural frequency $\omega_n$ (rad/sec)

**(d)** the spring constant $K$ of the bumper system (lb/in)

**(e)** the damping ratio $\zeta$ of the system when the railroad car
is empty. The unloaded railroad car weighs 8,000 lbs.




**(a)** logarithmic decrement $\delta = \ell n\left(\frac{x_o}{x_1}\right) = \ell n\left(\frac{4.8}{1.5}\right) = 1.1631$

**(b)** damping ratio

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \frac{\delta}{\left[(2\pi)^2 + \delta^2\right]^{1/2}}$$

$\zeta$ **≡ 0.1820**

**(c)** Damped natural period: $T_d$ **= 0.38 sec.**

and frequency

$$\omega_d = \frac{2\pi}{T_d} = 16.53\frac{rad}{\sec} = \omega_n\sqrt{1-\zeta^2}$$

The natural frequency is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 16.816\frac{rad}{\sec}$$

**(d)** Bumper stiffness,

$$K = \omega_n^2\ \mathrm{M_{car}} = 16.816^2\frac{1}{\sec^2}\left[\frac{35{,}000\ \ \mathrm{lb}}{386.4\ \mathrm{in/sec^2}}\right] = K = 25612.5\frac{lb}{in}$$

**(e)** Damping ratio when car is full:

$$\zeta = \frac{D}{2\sqrt{K\ \mathrm{M_{full}}}} = 0.182$$

Note that the physical damping coefficient ($D$) does not change whether car is loaded or not, but $\zeta$ does change.

__**Damping ratio when car is empty**__

$$\zeta_e = \frac{D}{2\sqrt{K\ M_{empty}}}$$

The ratio

$$\frac{\zeta}{\zeta_e} = \sqrt{\frac{M_{empty}}{M_{full}}} \Rightarrow \zeta_e = \zeta\sqrt{\frac{M_{full}}{M_{empty}}} = 0.182\left[\frac{35{,}000}{8{,}000}\right]^{1/2}$$

$\zeta_e$ **≡ 0.381**

**INSERT examples of identification of system damping ratio**

# Identification of parameters from transient response

LSA(c) 2012

The figure below shows the free response (amplitude vs. time) of a simple mechanical structure. Prior tests determined the system equivalent stiffness Ke=1000 lbf/in.

From the measurements determine:

a) damped period of motion Td (s)

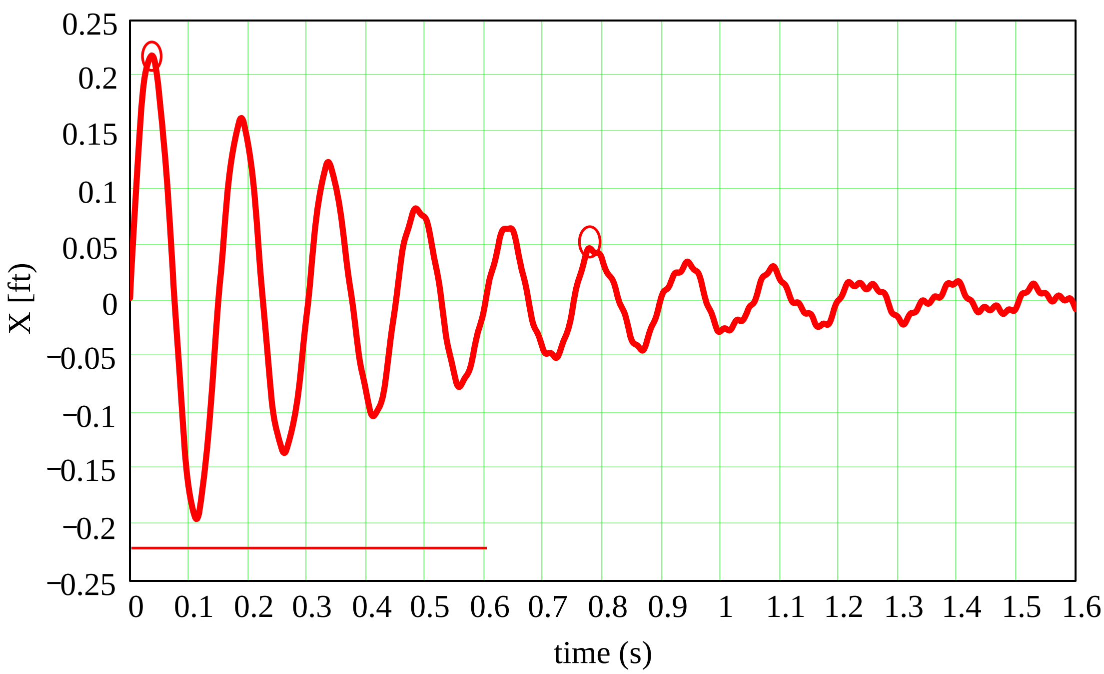
b) damped natural frequency $\omega_d$ (rad/s)

c) Using the log dec ($\delta$) concept, estimate the system damping ratio.

d) the system equivalent mass Me (lbm)

e) the system equivalent damping coefficient De (lbf.s/in)

**DISPLACEMENT (ft) vs time (sec)**




(a) Determine damped period of motion: $T_d := \frac{0.6 \cdot sec}{4}$ from 4 periods of damped motion

$T_d = 0.15\,sec$

(b) Determine damped natural frequency: $\omega_d := \frac{2 \cdot \pi}{T_d}$

$\omega_d = 41.888\,\frac{rad}{sec}$

(c) Determine damping ratio from log-dec:

Select two amplitudes of motion (well spaced) and count number od periods in between

$X_o := 0.23 \cdot ft$ after $n := 5$ periods $X_n := 0.05 \cdot ft$

**Log-dec** is derived from ratio: $\delta := \frac{1}{n} \cdot \ln\left(\frac{X_o}{X_n}\right)$

1

$\delta = 0.305$

from log-dec formula $\delta = \frac{2\cdot\pi\cdot\xi}{\left(1-\xi^2\right)^{0.5}}$ $\xi := \frac{\delta}{\left(4\cdot\pi^2+\delta^2\right)^{.5}}$ $\xi = 0.049$

Note that approximate formula: $\frac{\delta}{2\cdot\pi} = 0.049$ is a very good **estimation** of damping ratio

(d) damped natural frequency:

$$\omega_n := \frac{\omega_d}{\left(1-\xi^2\right)^{0.5}} \qquad \omega_n = 41.937\,\frac{rad}{sec}$$

a little higher than the damped frequency (recall damping ratio is small)

(e) system equivalent mass:

Static tests conducted on the structure show its stiffness to be $K = 1\times10^3\,\frac{lbf}{in}$

and from the equation for natural frequency, the equivalent system mass is

$$M_e := \frac{K}{\omega_n^{\,2}} \qquad M_e = 219.526\,lb$$

(f) system damping coefficient:

$$D_e := \xi\cdot2\cdot\left(K\cdot M_e\right)^{0.5} \qquad D_e = 2.314\,lbf\cdot\frac{sec}{in}$$

**Note:**
**Actual values of parameters are** $M = 220\,lb$ $C = 2.4\,lbf\cdot\frac{sec}{in}$ $\zeta = 0.05$

# Identification of parameters from transient response

LSA(c) 2012

The figure below shows the free response (amplitude vs. time) of a simple mechanical structure. Prior tests determined the system equivalent stiffness Ke=1000 lbf/in.

From the measurements determine:

a) damped period of motion Td (s)

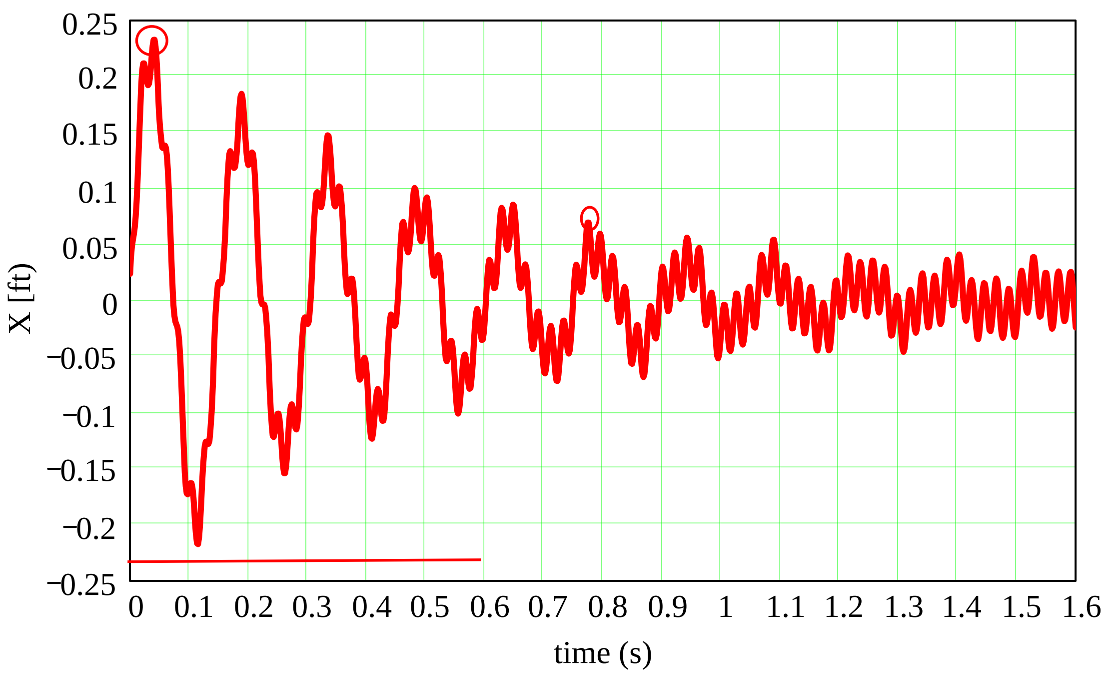
b) damped natural frequency $\omega_d$ (rad/s)

c) Using the log dec ($\delta$) concept, estimate the system damping ratio.

d) the system equivalent mass Me (lbm)

e) the system equivalent damping coefficient De (lbf.s/in)

**DISPLACEMENT (ft) vs time (sec)**




(a) Determine damped period of motion: $T_d := \frac{0.6 \cdot sec}{4}$ from 4 periods of damped motion

$T_d = 0.15\,sec$

(b) Determine damped natural frequency: $\omega_d := \frac{2 \cdot \pi}{T_d}$

$\omega_d = 41.888\,\frac{rad}{sec}$

(c) Determine damping ratio from log-dec:

Select two amplitudes of motion (well spaced) and count number od periods in between

$X_o := 0.23 \cdot ft$ after $n := 5$ periods $X_n := 0.0776 \cdot ft$

**Log-dec** is derived from ratio: $\delta := \frac{1}{n} \cdot \ln\left(\frac{X_o}{X_n}\right)$

$\delta = 0.217$

from log-dec formula $\delta = \frac{2\cdot\pi\cdot\xi}{\left(1-\xi^2\right)^{0.5}}$ $\xi := \frac{\delta}{\left(4\cdot\pi^2+\delta^2\right)^{.5}}$ $\xi = 0.035$

Note that approximate formula: $\frac{\delta}{2\cdot\pi} = 0.035$ is a very good **estimation** of damping ratio

(d) damped natural frequency:

$$\omega_n := \frac{\omega_d}{\left(1-\xi^2\right)^{0.5}} \qquad \omega_n = 41.913\,\frac{rad}{sec}$$

a little higher than the damped frequency (recall damping ratio is small)

(e) system equivalent mass:

Static tests conducted on the structure show its stiffness to be $K = 1\times10^3\,\frac{lbf}{in}$

and from the equation for natural frequency, the equivalent system mass is

$$M_e := \frac{K}{\omega_n^{\,2}} \qquad M_e = 219.781\,lb$$

(f) system damping coefficient:

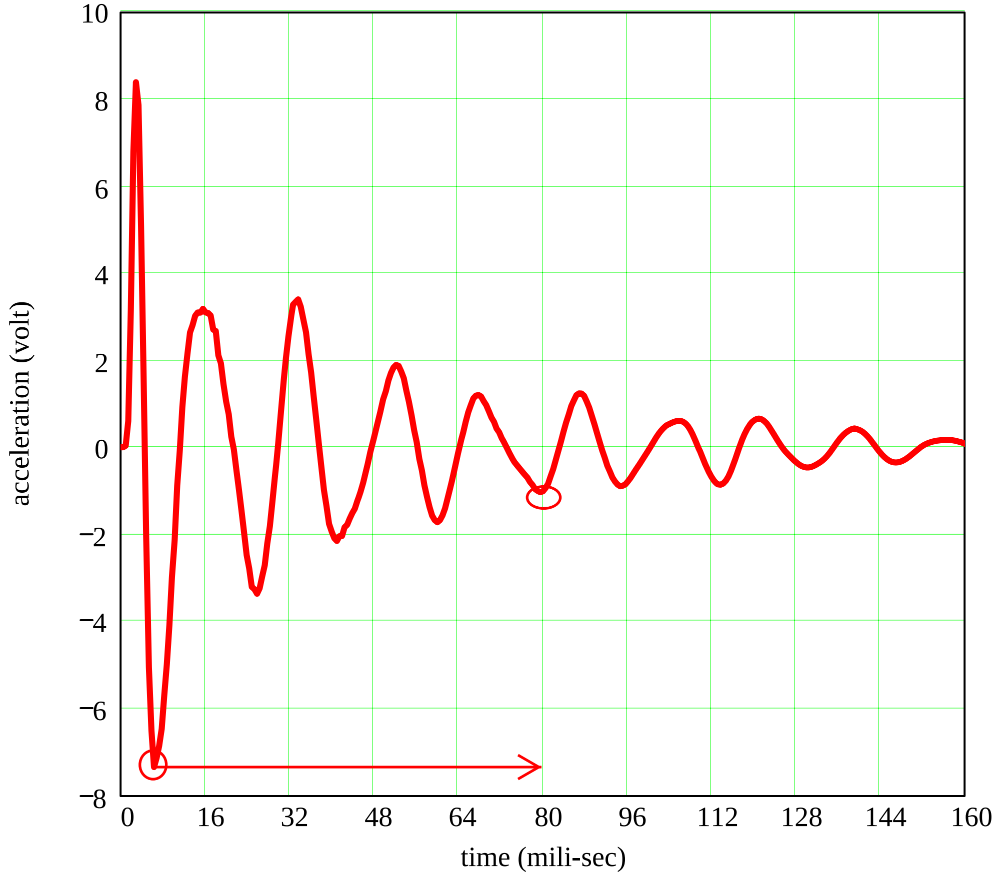
$$D_e := \xi\cdot2\cdot\left(K\cdot M_e\right)^{0.5} \qquad D_e = 1.649\,lbf\cdot\frac{sec}{in}$$

**Note:**
**Actual values of parameters are**

$M = 220\,lb$ $C = 2.4\,lbf\cdot\frac{sec}{in}$ $\zeta = 0.05$

Note large difference in damping - WHY?

# Measurements of acceleration due to impact load




(a) Determine damped period of motion: $T_d := \frac{80 - 7}{4 \cdot 1000} \cdot sec$

from 4 periods of damped motion

$T_d = 0.02\,sec$

(b) Determine damped natural frequency: $\omega_d := \frac{2 \cdot \pi}{T_d}$

$\omega_d = 344.28\,\frac{rad}{sec}$

(c) Determine damping ratio from log-dec:

Select two amplitudes of motion (well spaced) and count number od periods in between

$A_o := -7.34$ after $n := 4$ periods $A_n := -1.03$

**Log-dec** is derived from ratio: $\delta := \frac{1}{n} \cdot \ln\left(\frac{A_o}{A_n}\right)$

$\delta = 0.49$

from log-dec formula

$$\delta = \frac{2\cdot\pi\cdot\xi}{\left(1-\xi^2\right)^{0.5}}$$

$$\xi := \frac{\delta}{\left(4\cdot\pi^2+\delta^2\right)^{.5}}$$

$$\xi = 0.08$$

Note that approximate formula: $\frac{\delta}{2\cdot\pi} = 0.08$ is a very good **estimation** of damping ratio

(d) Determine damped natural frequency:

$$\omega_n := \frac{\omega_d}{\left(1-\xi^2\right)^{0.5}}$$

$$\omega_n = 345.33\,\frac{rad}{sec}$$

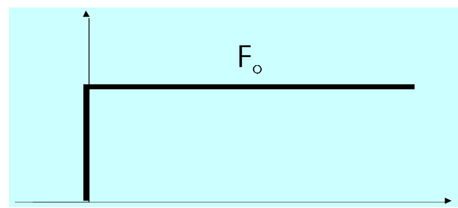
a little higher than the damped frequency (recall damping ratio is small)

## (b) Forced Response of 2nd Order Mechanical System

### Step Force or Constant Force

A force with constant magnitude $F_o$ is suddenly applied at $t$=0. Besides the system had initial displacement $X_0$ and velocity $V_0$. EOM is:

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=F_o \qquad (24)$$




Using

$\omega_n=\sqrt{K/M}$ : natural frequency of system

$\zeta=\frac{D}{D_{cr}}$ : damping ratio, where $D_{cr}=2\sqrt{KM}$ = critical damping

Eqn. (1) becomes:

$$\frac{d^2X}{dt^2}+2\zeta\omega_n\frac{dX}{dt}+\omega_n^2X=\frac{F_o}{M}=\frac{F_o}{K}\omega_n^2 \qquad (25)$$

The solution of the ODE is (homogenous + particular):

$$X(t)=X_H+X_P=A_1e^{s_1t}+A_2e^{s_2t}+\left(F_o/K\right) \qquad (26)$$

Where $A's$ are constants found from the initial conditions and $X_P=F_o/K$ is the particular solution for the step load.

Note: $X_{SS}=F_o/K$ is equivalent to the static displacement if the force is applied very slowly.

The roots of the characteristic Eq. for the system are

$$s_{1,2} = -\zeta\,\omega_n \mp \omega_n\left(\zeta^2 - 1\right)^{1/2} \tag{6}$$

From Eq. (6), differentiate three cases:

**Underdamped System:** $0 < \zeta < 1, \rightarrow D < D_{cr}$

**Critically Damped System:** $\zeta = 1, \rightarrow D = D_{cr}$

**Overdamped System:** $\zeta > 1, \rightarrow D > D_{cr}$

## Step Forced Response of Underdamped 2nd Order System

For an **underdamped** system, $0 < \zeta < 1$, the roots are complex conjugate ( real and imaginary parts), i.e.

$$s_{1,2} = -\zeta\,\omega_n \mp i\,\omega_n\left(1-\zeta^2\right)^{1/2}$$

The response is:

$$X(t) = e^{-\zeta\,\omega_n t}\left(C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right) + X_{ss} \tag{27}$$

where $\omega_d = \omega_n\left(1-\zeta^2\right)^{1/2}$ is the system damped natural frequency.

and $X_{ss} = F_o / K$

At time $t = 0$, applying the initial conditions gives

$$C_1 = \left(X_0 - X_{ss}\right) \text{ and } C_2 = \frac{V_0 + \zeta\,\omega_n\,C_1}{\omega_d} \tag{28}$$

Note that as $t \rightarrow \infty$, $X_{(t)} \rightarrow X_{ss} = F_o/K$ for $\zeta > 0$,

**i.e. the system response reaches the steady state (static) equilibrium position.**

The largest the damping ratio $\zeta$ ($\rightarrow 1$) , the fastest the motion will damp to reach the static position $X_{ss.}$ In addition, the period of damped natural motion will increase, i.e., $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \left(1-\zeta^2\right)^{1/2}} \rightarrow \infty$

## Step Forced Response of Undamped 2nd Order System:

For an undamped system, i.e., a conservative system, $\zeta$ **= 0**, and the response is just

$$X(t) = \left(C_1 \cos\left(\omega_n t\right) + C_2 \sin\left(\omega_n t\right)\right) + X_{ss} \qquad (29)$$

with $X_{ss} = F_o/K$, $C_1 = (X_0 - X_{ss})$ *and* $C_2 = V_0/\omega_n$ (30)

**if** the initial displacement and velocity are null, i.e. $\boldsymbol{X_0 = V_0 = 0}$, then

$$X(t) = X_{ss}\left(1-\cos\left(\omega_n t\right)\right) \qquad (31)$$

**Note** that as $t \rightarrow \infty$, $X_{(t)}$ does not approach $X_{ss}$ for $\zeta = 0$.

The **system oscillates forever a**bout the static equilibrium position $X_{ss}$ and, the maximum displacement is 2 $X_{ss}$, i.e. twice the static displacement *($F_0/K$).*

The observation reveals the great difference in response for a force slowly applied when compared to one suddenly applied. The difference explains **why *things* break** when sudden efforts act on a system.

**Forced Step Response of Underdamped Second Order System:**

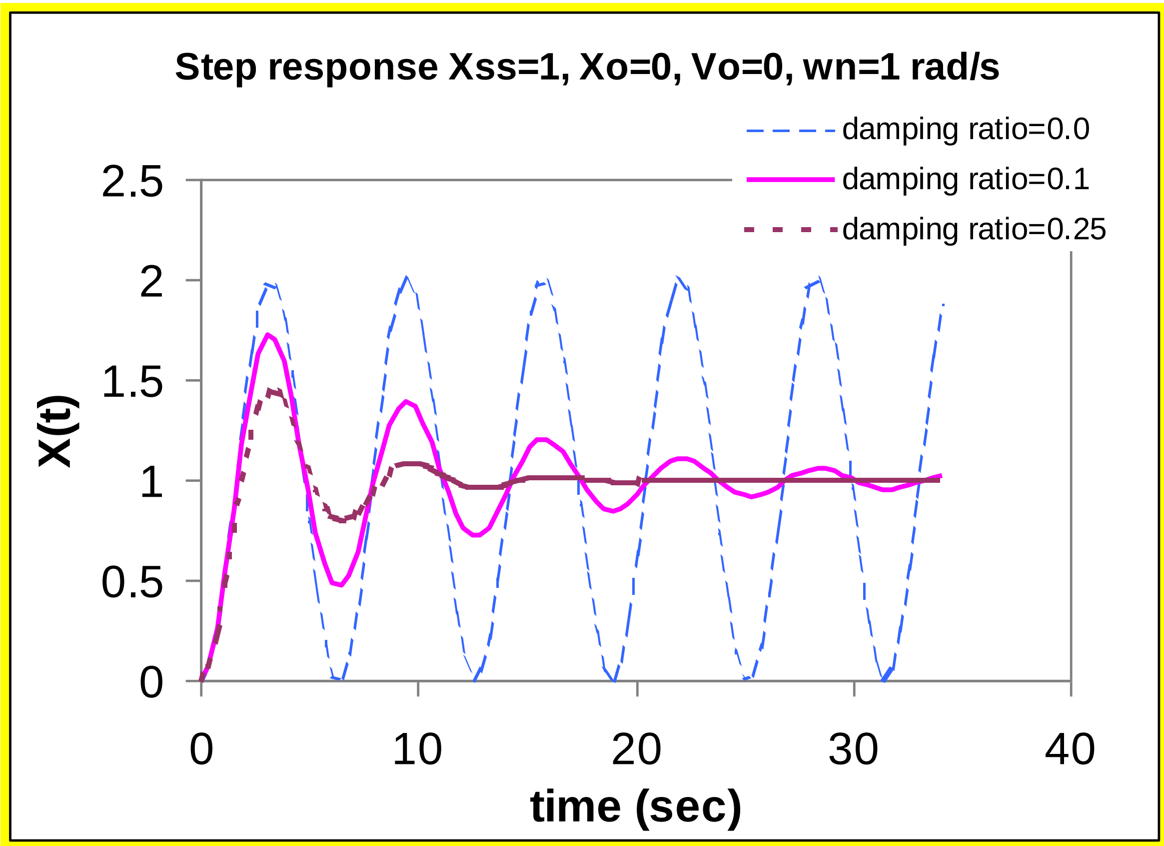
**damping ratio varies**

$X_o = 0$, $V_o = 0$, $\omega_n = 1.0$ rad/s $\zeta = 0,\ 0.1,\ 0.25$

**zero initial conditions**

$F_o/K = X_{SS} = 1$;

**faster response as $\zeta$ increases;** i.e. as $t \to \infty$, $X \to X_{SS}$ for $\zeta > 0$



## Forced Step Response of Overdamped 2nd Order System

For an **overdamped** system, $\zeta > 1$, the roots of the characteristic eqn. are real and negative, i.e.

$$s_1 = \omega_n \left[ -\zeta + \left( \zeta^2 - 1 \right)^{1/2} \right];\ s_2 = \omega_n \left[ -\zeta - \left( \zeta^2 - 1 \right)^{1/2} \right] \tag{32}$$

The **forced response of an overdamped** system is:

$$X(t) = e^{-\zeta \omega_n t} \left( C_1 \cosh(\omega_* t) + C_2 \sinh(\omega_* t) \right) + X_{ss} \tag{33}$$

where $\omega_* = \omega_n \left( \zeta^2 - 1 \right)^{1/2}$. Response or motion is NOT oscillatory.

and

$$C_1 = \left( X_0 - X_{ss} \right) \text{ and } C_2 = \frac{V_0 + \zeta \omega_n C_1}{\omega_*} \tag{34}$$

Note that as $t \to \infty$, $X_{(t)} \to X_{SS} = F_o/K$ for $\zeta > 1$, i.e. the steady-state (static) equilibrium position.

**An overdamped system does not oscillate (or vibrate).**

The larger the damping ratio $\zeta$, the longer time it takes the system to reach its final equilibrium position $X_{SS}$.

## Forced Step Response of Critically Damped System

For a critically damped system, $\zeta$ = *1*, the roots are real negative and identical, i.e.

$$s_1 = s_2 = -\zeta\,\omega_n \qquad (35)$$

The step-forced response for a critically damped system is

$$X(t) = e^{-\omega_n t}\left(C_1 + t\,C_2\right) + X_{ss} \qquad (36)$$

with

$$C_1 = \left(X_0 - X_{ss}\right) \text{ and } C_2 = V_0 + \omega_n C_1 \qquad (37)$$

Note that as $t \rightarrow \infty$, $X_{(t)} \rightarrow X_{ss} = F_o/K$ for $\zeta$ > 1, i.e. the steady-state (static) equilibrium position.

A **critically damped system** does not oscillate and it is the fastest to reach the steady-state value $X_{ss}$.

**Forced Step Response of Second Order System:**

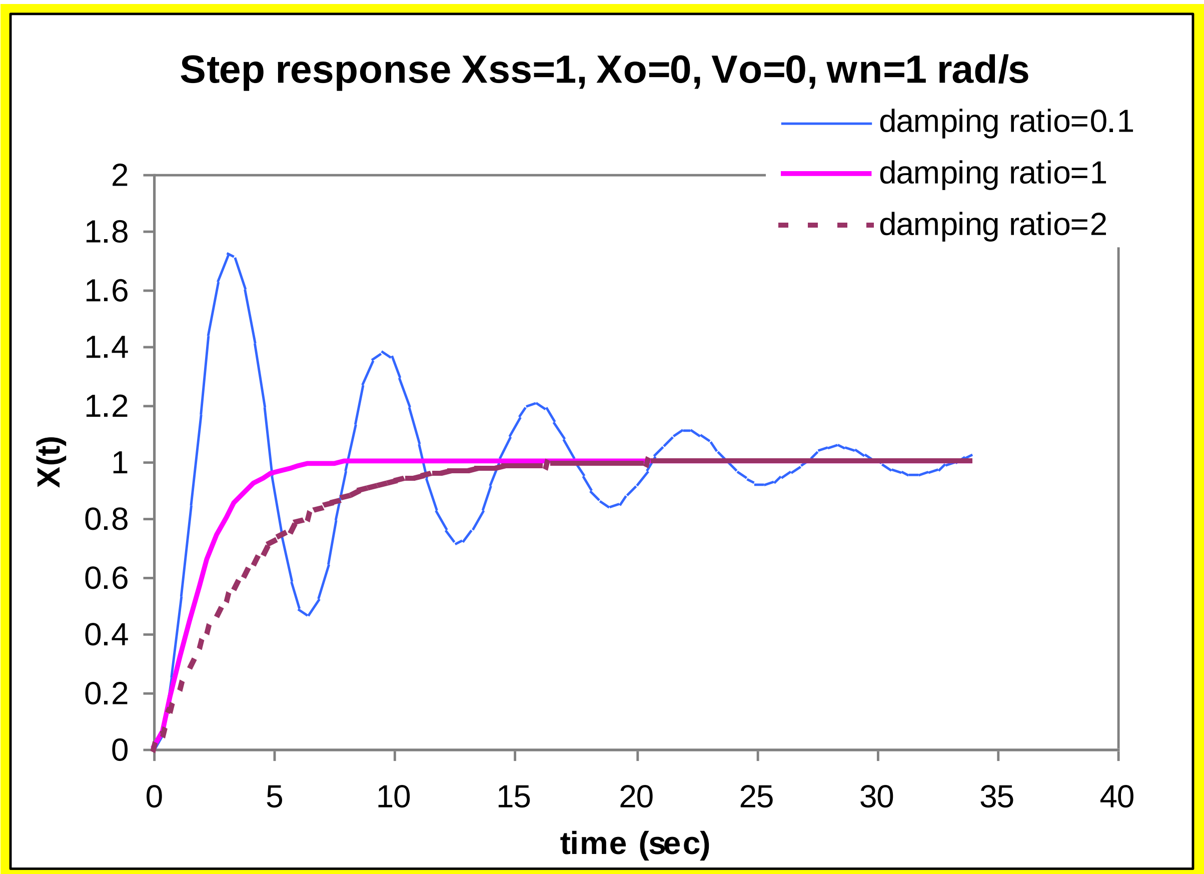
**Comparison of Underdamped, Critically Damped and Overdamped system responses**

$X_o = 0$, $V_o = 0$, $\omega_n = 1.0$ *rad/s* $\zeta = 0.1, 1.0, 2.0$

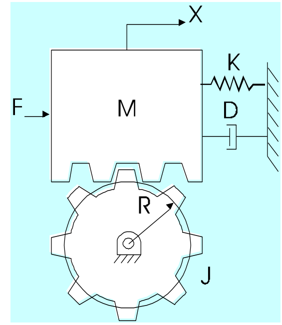
**zero initial conditions**

$F_o/K = X_{SS} = 1$; (magnitude of s-s response)

Fastest response for $\zeta = 1$. As $t \rightarrow \infty$, $X \rightarrow X_{SS}$ for $\zeta > 0$



## EXAMPLE:



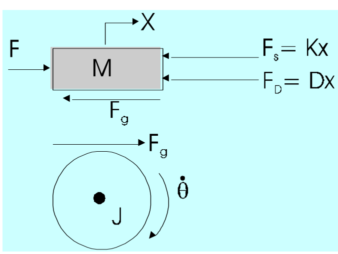

The equation describing the motion and initial conditions for the system shown are:

$$(M + I/R^2)\ddot{X} + D\,\dot{X} + K\,X = F\,, X(0) = \dot{X}(0) = 0$$

Given $M$=2.0 kg, $I$=0.01 kg-m$^2$, $D$=7.2 N.s/m, $K$=27.0 N/m, $R$=0.1 m; and $F$ =5.4 N (a step force),

**a)** Derive the differential equation of motion for the system (as given above).
**b)** Find the system natural frequency and damping ratio
**c)** Sketch the dynamic response of the system $X(t)$
**d)** Find the steady-state value of the response $X_{s-s}$.

**(a)** **Using free body diagrams**: Note that θ= X/R is a kinematic constraint.
The EOM's are:




$$M\,\ddot{X} = F - K\,X - D\,\dot{X} - F_G \qquad (1)$$

$$I\,\ddot{\theta} = F_G \cdot R \qquad (2)$$

Then from (2) $F_G = I\,\dfrac{\ddot{\theta}}{R} = I\,\dfrac{\ddot{X}}{R^2}$ (3) ;

(3) into (1) gives

$$\left(M + \frac{I}{R^2}\right)\ddot{X} + D\,\dot{X} + K\,X = F \qquad (4)$$

or **Using the Mechanical Energy Method**:

(system kinetic energy): $T = \frac{1}{2}\,M\,\dot{X}^2 + \frac{1}{2}\,I\,\dot{\theta}^2 = \frac{1}{2}\left[M + \frac{I}{R^2}\right]\dot{X}^2$ (5)

(system potential energy): $V = \frac{1}{2}\,K\,X^2$

(viscous dissipated energy) $E_D = \int D\,\dot{X}^2\, dt$, and External work: $W = \int F\, dX$

Derive identical Eqn. of motion (4) from $\frac{d}{dt}\left(T + V + E_d - W\right) = 0$ (6)

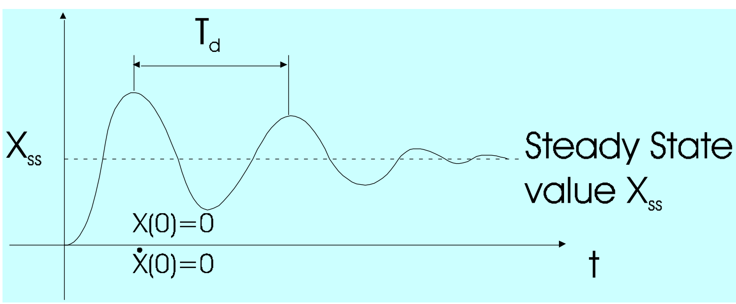
**(b)** define $M_{eq} = M + \frac{I}{R^2} = 3$ Kg, $K$ = 27 N/m, $D$ = 7.2 N.s/m

and calculate the system natural frequency and viscous damping ratio:

$$\omega_n = \left[\frac{K}{M_{eq}}\right]^{1/2} = 3\ \frac{\text{rad}}{\text{sec}};\ \xi = \frac{D}{2\sqrt{K\ M}} = 0.4,$$ underdamped system

$\omega_d = \omega_n \sqrt{1-\xi^2} = 2.75\frac{rad}{\sec}$, and $T_d = \frac{2\pi}{\omega_d} = 2.28$ sec is the damped period of motion

**(c)** The step response of an underdamped system with I.C.'s $X(0) = \dot{X}(0) = 0$ is:




**(d)** At steady-state, no motion occurs, $X = X_{SS}$, and $\dot{X} = 0,\ \ddot{X} = 0$

Then

$$X_{ss} = \frac{F}{K} = \frac{5.4\text{ N}}{27\frac{\text{N}}{\text{m}}} \qquad X_{ss} = 0.2\text{ m}$$

$$X(t) = X_{ss}\left[1 - e^{-\zeta\,\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\sin(\omega_d t)\right)\right]$$

ME459/659 Notes# 1b (pp. 32-46)

# Dynamic Response of SDOF Second Order Mechanical System: Viscous Damping

$$M\ddot{X} + D\dot{X} + KX = F_{(t)}$$

**Periodic Forced Response to**

$F_{(t)} = F_o \sin(\Omega t)$ **and** $F_{(t)} = M\,u\,\Omega^2 \sin(\Omega t)$

→

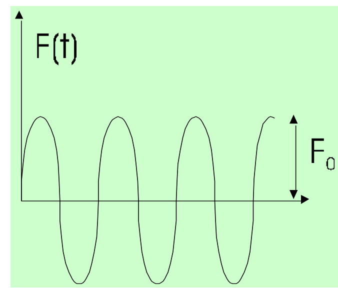
**Frequency Response Function of Second Order Systems**

## (c) Forced response of 2nd order mechanical system to a periodic force excitation

Let the external force be **PERIODIC** with frequency $\Omega$ (period $T=2\pi/\Omega$) and amplitude $F_o$ . The EOM is:

$$M\frac{d^2 X}{dt^2}+D\frac{dX}{dt}+K X=F_o \sin\left(\Omega t\right) \quad (38)$$

with initial conditions $\dot{X}_{(0)}=V_o$ and $X_{(0)}=X_o$




The solution of the non-homogeneous ODE (38) is

$$X(t)=X_H+X_P=A_1 e^{s_1 t}+A_2 e^{s_2 t}+C_c \cos\left(\Omega t\right)+C_s \sin\left(\Omega t\right) \quad (39)$$

where ***X**$_H$* is the solution to the homogeneous form of (38) and such that *($s_1$, $s_2$)* satisfy the characteristic equation of the system:

$$\left(s^2+2\zeta\,\omega_n\,s+\omega_n^2\right)=0$$

The roots of this 2nd order polynomial are: $s_{1,2}=-\zeta\,\omega_n \mp \omega_n\left(\zeta^2-1\right)^{1/2}$ (*)

where $\omega_n=\sqrt{K/M}$ is the natural frequency, and $\zeta=\dfrac{D}{D_{cr}}$ is the viscous damping ratio. Recall $D_{cr}=2\sqrt{K\,M}$ is the critical damping coefficient.

The value of damping ratio determines whether the system is underdamped ($\zeta$ <1), critically damped ($\zeta$ =1), overdamped ($\zeta$ >1).

## Response of 2nd Order Mechanical System to a Periodic Loading:

And the particular solution is

$$X_P = C_c \cos(\Omega t) + C_s \sin(\Omega t) \quad (40)$$

Substitution of Eq. (40) into Eq. (38), and after some algebra, gives

$$\begin{bmatrix} \{1-\Omega^2 M/K\} & \Omega\, D/K \\ -\Omega\, D/K & \{1-\Omega^2 M/K\} \end{bmatrix} \begin{Bmatrix} C_c \\ C_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_o/K \end{Bmatrix} \quad (41a)$$

Substitute above $\omega_n = \sqrt{K/M}$ ; $\frac{D}{K} = \frac{D}{K}\frac{\zeta\sqrt{K\,M}}{D} = \frac{\zeta}{1}\sqrt{\frac{M}{K}} = \frac{\zeta}{\omega_n}$

and $X_{ss} = F_o/K$, **a "pseudo" static displacement** →

$$\begin{bmatrix} \{1-\Omega^2/\omega_n^2\} & \Omega\, \zeta/\omega_n \\ -\Omega\, \zeta/\omega_n & \{1-\Omega^2/\omega_n^2\} \end{bmatrix} \begin{Bmatrix} C_c \\ C_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ X_{ss} \end{Bmatrix} \quad (41b)$$

**D**efine the **frequency ratio** $f = \Omega/\omega_n$ (42)

that relates the (external) excitation frequency *(Ω)* to the *natural frequency of the system (ω<sub>n</sub>);* i.e. when

$\Omega \ll \omega_n \rightarrow f \ll 1$, the system operates ***below*** its natural frequency

$\Omega \gg \omega_n \rightarrow f \gg 1$, the system operates ***above*** its natural frequency

With this definition, write Eq. (41b) as:

$$\begin{bmatrix} \{1-f^2\} & f\,\zeta \\ -f\,\zeta & \{1-f^2\} \end{bmatrix} \begin{Bmatrix} C_c \\ C_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ X_{ss} \end{Bmatrix} \qquad (43)$$

Solve Eq. (50) for the coefficients $C_s$ and $C_c$:

$$C_c = X_{ss} \frac{-2\zeta f}{\left(1-f^2\right)^2 + \left(2\zeta f\right)^2};\; C_s = X_{ss} \frac{\left(1-f^2\right)^2}{\left(1-f^2\right)^2 + \left(2\zeta f\right)^2} \qquad (44)$$

**Response of 2nd Order Mechanical System to a Periodic Load:**

For an **underdamped** system, $0 < \zeta < 1$, the homogeneous solution (free response) is

$$X_{H(t)} = e^{-\zeta\,\omega_n t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) \qquad (45)$$

where $\omega_d = \omega_n \left(1-\zeta^2\right)^{1/2}$ is the system damped natural frequency.

By superposition, the complete response is $X(t) = X_H + X_P$ =

$$X_{(t)} = e^{-\zeta\,\omega_n t}\left(C_1 \cos\left(\omega_d t\right) + C_2 \sin\left(\omega_d t\right)\right) + C_c \cos\left(\Omega t\right) + C_s \sin\left(\Omega t\right) \qquad (46)$$

*with* $C_s$ and $C_c$ ← Eq. (44).

At time $t$ = 0, apply the initial conditions and obtain

$$C_1 = \left(X_0 - C_c\right) \text{ and } C_2 = \frac{V_0 + \zeta\,\omega_n C_1 - \Omega C_s}{\omega_d} \qquad (47)$$

As long as $\zeta$ **> 0**, the homogeneous solution (also known as the **TRANSIENT or free response)** will die as time elapses, i.e., for $t$ >>0 then $X_H$ → 0.

Thus, after all transients have passed, the dynamic response of the system is just the particular response $X_{P(t)}$ .

## Steady State – Periodic Forced Response of Underdamped System

The **steady-state** (or **quasi-stationary**) response is given by:

$$X_{(t)} \approx C_c \cos(\Omega t) + C_s \sin(\Omega t) = C \sin(\Omega t - \varphi) \qquad (48)$$

where *($C_s$, $C_c$)* ← Eq. (44). Define $C_s = C\cos(\varphi);\ C_c = -C\sin(\varphi)$; where $\varphi$ is a **phase angle**; then

$$\tan(\varphi) = \frac{-C_c}{C_s} = \frac{2\,\zeta\,f}{\left(1-f^2\right)} \qquad (49)$$

and $C = \sqrt{C_S^2 + C_C^2} = X_{ss}\ A$, where

$$A = \frac{1}{\sqrt{\left(1-f^2\right)^2 + \left(2\zeta\,f\right)^2}}; \qquad (50)$$

= ***amplitude ratio***

(a dimensionless quantity)

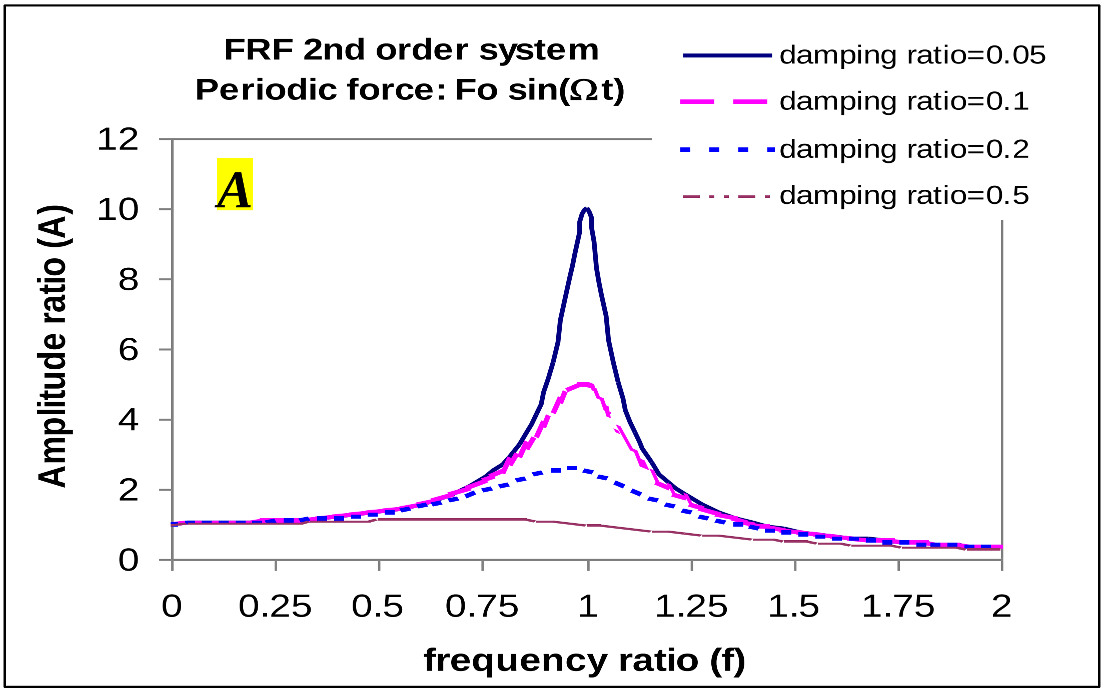
Thus, the steady-state system response is just

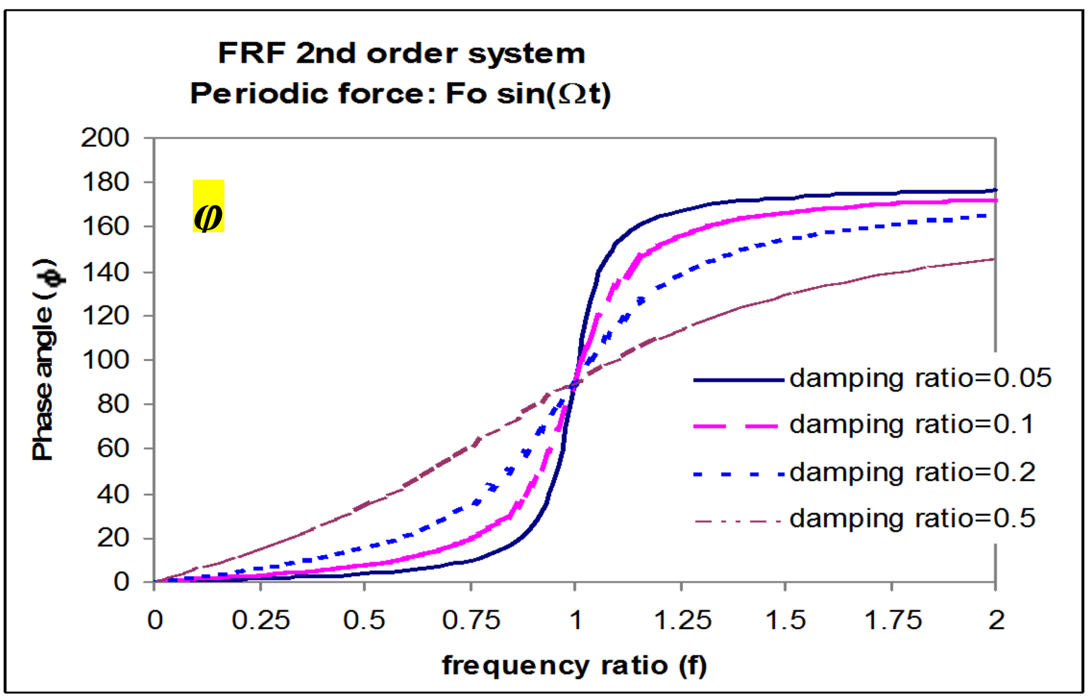
$$X_{(t)} = X_{ss}\ A\ \sin(\Omega t - \varphi) \qquad (51)$$

**Note there "steady-state" implies that, for excitation with a constant frequency $\omega$, the system response amplitude *C* and the phase angle $\varphi$ are constant or time invariant**

## Amplitude and Phase Lag of Periodic Force Response

$$X_{(t)} = X_{ss}\, A \sin\left(\Omega t - \varphi\right) \qquad \text{for} \qquad F_{(t)} = F_o \sin\left(\Omega t\right)$$



## Regimes of Dynamic Operation:

$\Omega \ll \omega_n \rightarrow f \ll 1$, the system operates ***below*** its natural frequency

$$\left(1-f^2\right)\rightarrow 1; \left(2\zeta f\right)\rightarrow 0 \quad \Rightarrow A\rightarrow 1 \;\; \varphi \rightarrow 0$$

$$X(t)\rightarrow X_{ss}\sin\left(\Omega t\right)$$

i.e. similar to the "static" response

$\Omega = \omega_n \rightarrow f = 1$, the system is excited ***at*** its natural frequency

$$\left(1-f^2\right)\rightarrow 0; \quad \Rightarrow A\rightarrow \frac{1}{2\zeta} \; ; \; \varphi \rightarrow \frac{\pi}{2} \;\; \left(90^\circ\right)$$

$$X(t)\rightarrow \frac{X_{ss}}{2\zeta}\sin\left(\Omega t-\frac{\pi}{2}\right)$$

if $\zeta < 0.5$, the amplitude ratio $A > 1$ and a **resonance** is said to occur.

$\Omega \gg \omega_n \rightarrow f \gg 1$, the system operates ***above*** its natural frequency

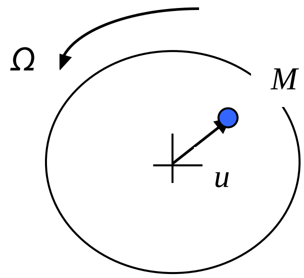
$$\left(1-f^2\right) << 1; \left(2\zeta f\right)\gg 0 \quad \Rightarrow A\rightarrow 0 \;\; \varphi \rightarrow \pi \;\; \left(180^\circ\right)$$

$$X(t)\rightarrow X_{ss}\,A\sin\left(\Omega t-\pi\right)=-X_{ss}\,A\sin\left(\Omega t\right)$$

$\boldsymbol{A}$ <<< 1, i.e. very small,

## Steady State – Periodic Forced Response of 2nd Order system: Imbalance Load

Imbalance loads are typically found in rotating machinery. In operation, due to inevitable wear, material build ups or assembly faults, the center of mass of the rotating machine does not coincide with the center of rotation (spin). Let the center of mass be located a distance (***u***) from the spin center, and thus, the load due to the imbalance is a centrifugal "force" with magnitude $F_o = M\,u\,\Omega^2$ and rotating with the same frequency as the rotor speed (***Ω***). This force excites the system and induces vibration[1]. Note that the imbalance force is proportional to the frequency[2] and grows rapidly with shaft speed.




Ne that although in practice the offset distance (***u***) is very small (a few mil), the system response or amplitude of vibration can be quite large affecting the performance and integrity of the rotor assembly.

For example if the rotating shaft & disk has a small imbalance mass (***m***) located at a radius (***r***) from the spin center, then it is easy to determine that the center of mass offset (***u***) is approximately equal to ***(m r/M).*** Note that ***u<<r.***

$$u\left(M+m\right)=r\,m \quad \rightarrow\; u \simeq \frac{r\,m}{M}$$

[1] The current analysis only describes vibration along direction *X*. In actuality, the imbalance force induces vibrations along the planes (*X*,*Y*) and the rotor whirls in an orbit around the center of rotor spinning. For isotropic systems, the motion in the *X* plane is identical to that in the *Y* plane but out of phase by 90°.

The equation of motion for the system with an imbalance force is

$$M\frac{d^2 X}{d t^2}+D\frac{d X}{d t}+K X=F_o \sin(\Omega t)=M u \Omega^2 \sin(\Omega t)$$

then $$X_{ss}=\frac{F_o}{K}=\frac{M u \Omega^2}{K}=u\frac{\Omega^2}{\omega_n^2}=u f$$

with $f=\Omega/\omega_n$ . The "steady-state" system response is

$$X_{(t)}=X_{ss} A \sin(\Omega t-\varphi)=u\left(\Omega^2 A\right) \sin(\Omega t-\varphi)$$

$$X_{(t)}=u B \sin(\Omega t-\varphi) \quad (52)$$

where $\varphi$ is a **phase angle**, $\tan(\varphi)=\frac{2\zeta f}{\left(1-f^2\right)}$, and

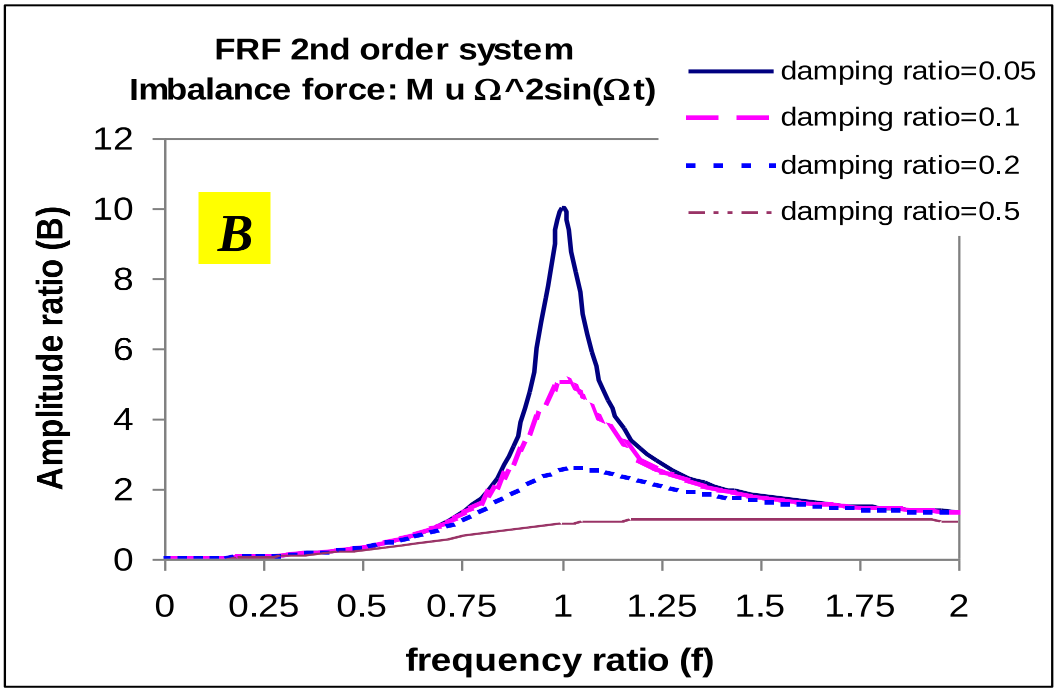
$$B=\frac{f^2}{\sqrt{\left(1-f^2\right)^2+\left(2\zeta f\right)^2}} \quad (53)$$

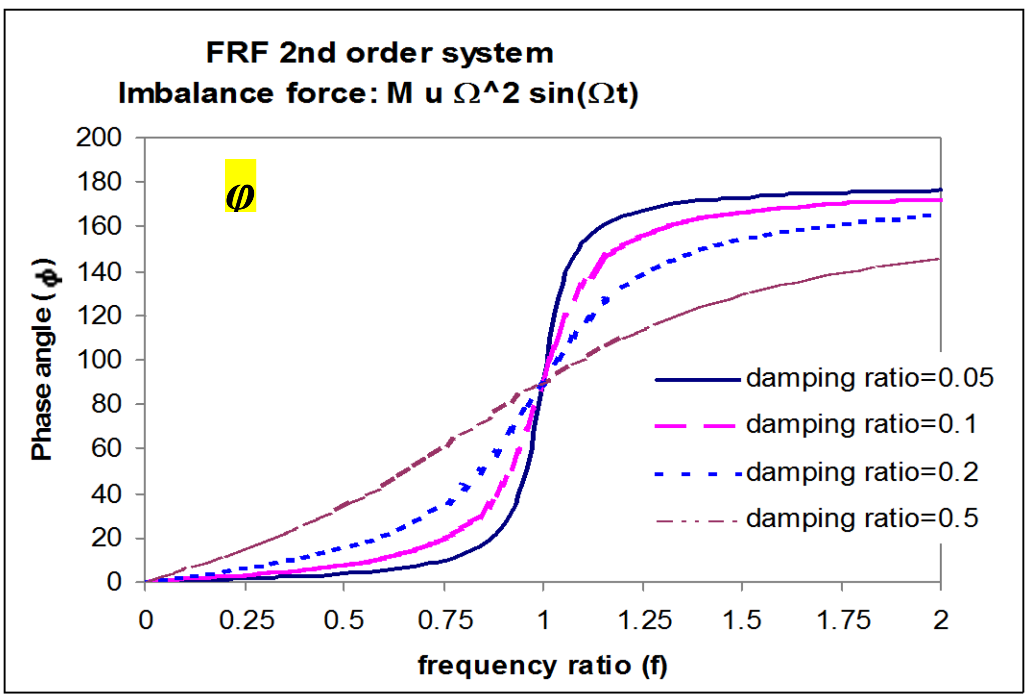
is the ***amplitude ratio*** (dimensionless quantity).

| Educational video – watch UNBALANCE RESPONSE demonstration |
|---|
| https://www.youtube.com/watch?v=R2hO--TIjjA |

## Amplitude B and phase lag ($\varphi$) for response to an imbalance ($u$)

$$X_{(t)} = u\,B\,\sin(\omega t - \varphi) \qquad \text{for} \qquad F_{(t)} = M\,u\,\omega^2\,\sin(\omega t)$$



## Regimes of Dynamic Operation:

$\Omega \ll \omega_n \rightarrow f \ll 1$, rotor speed is ***below*** its natural frequency

$$\left(1-f^2\right)\rightarrow 1;\ \left(2\zeta f\right)\rightarrow 0 \quad \Rightarrow B\rightarrow f^2 \approx 0 \quad \text{and } \varphi \rightarrow 0$$

$$X(t)\rightarrow u\, f^2 \sin\left(\Omega t\right)\rightarrow 0$$

i.e. little motion or response

$\Omega = \omega_n \rightarrow f = 1$, rotor speed coincides with natural frequency

$$\left(1-f^2\right)\rightarrow 0; \quad \Rightarrow B\rightarrow \frac{1}{2\zeta}\ ;\ \varphi \rightarrow \frac{\pi}{2} \ \left(90^\circ\right)$$

$$X(t)\rightarrow \frac{u}{2\zeta}\sin\left(\Omega t-\frac{\pi}{2}\right)=-\frac{u}{2\zeta}\cos\left(\Omega t\right)$$

if $\zeta$ < 0.5, amplitude ratio $B$ > 1 and a **resonance** is said to occur. Damping is needed to survive passage through a natural frequency **(critical speed).**

$\Omega \gg \omega_n \rightarrow f \gg 1$, the rotor speed is ***above*** its natural frequency

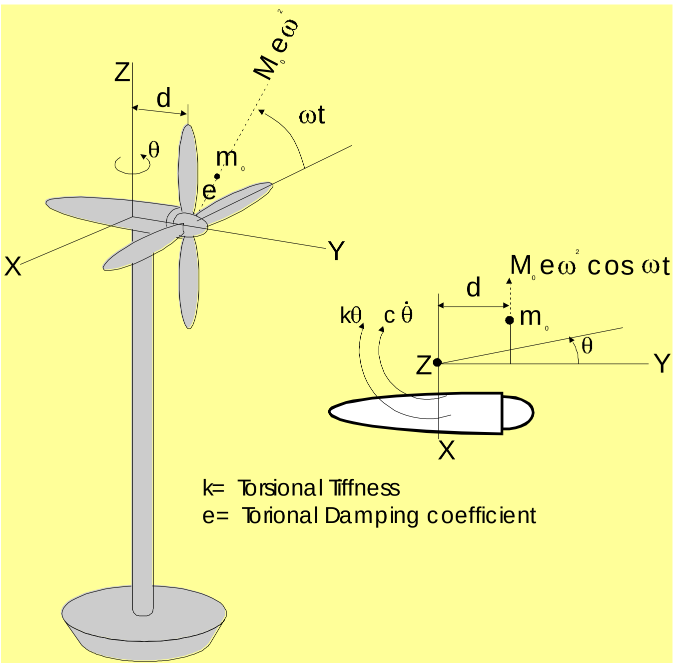
$$\frac{\left(1-f^2\right)}{f^2}\rightarrow -1;\ \left(\frac{2\zeta f}{f^2}\right)\rightarrow 0 \ \Rightarrow B\rightarrow 1 \quad \varphi \rightarrow \pi \ \left(180^\circ\right)$$

$$X(t)\rightarrow u\sin\left(\Omega t-\pi\right)=-u\sin\left(\Omega t\right)$$

$\boldsymbol{B}$ ~ 1, at high frequency operation, the maximum amplitude of vibration ($X_{max}$) equals the unbalance displacement (*u).*

**Note:** API demands that the operating speed of a rotor system is not too close to the natural frequency to avoid too large amplitudes that could endanger the life of the system. A rotor speed cannot operate (but for very short times) 20% above and below the natural frequency.

## EXAMPLE:




A cantilevered steel pole supports a small wind turbine. The pole torsional stiffness is **K** (N.m/rad) with a rotational damping coefficient **C** (N.m.s/rad).

The four-blade turbine rotating assembly has mass $\mathbf{m_o}$ , and its center of gravity is displaced distance **e** [m] from the axis of rotation of the assembly.

$\mathbf{I_z}$ (kg.m$^2$) is the mass moment of inertia about the *z* axis of the complete turbine, including rotor assembly, housing pod, and contents.

The total mass of the system is **m** (kg). The plane in which the blades rotate is located a distance **d** (m) from the *z* axis as shown.

For a complete analysis of the vibration characteristics of the turbine system, determine:

**a)** Equation of motion of torsional vibration system about z axis.
**b)** The steady-state torsional response $\theta_{(t)}$ (after all transients die out).
**c)** For system parameter values of k=98,670 N.m/rad, $I_z$=25 kg.m$^2$, C = 157 N.m.s/rad, and $m_O$ = 8 kg, e = 1 cm, d = 30 cm, present graphs showing the response amplitude (in rads) and phase angle as the turbine speed (due to wind power variations) changes from 100 rpm to 1,200 rpm.
**d)** From the results in (c), at what turbine speed should the largest vibration occur and what is its magnitude?
**e)** Provide a design recommendation or change so as to reduce this maximum vibration amplitude value to half the original value.

Neglect any effect of the mass and bending of the pole on the torsional response, as well as any gyroscopic effects.

### (a) derive the drive torque and EOM

The torque or moment induced by the mass imbalance is

$$T_{(t)} = d \times F_u = \underbrace{m_o\, e\, d\, \omega^2}_{T(\omega)} \cos(\omega t),$$ i.e., a function of frequency

The equation for torsional motions of the turbine-pole system is:

$$I_z \ddot{\theta} + C\dot{\theta} + K\theta = m_o e\, d\ \omega^2 \cos\ \omega t = T_{(\omega)} \cos\left(\omega t\right) \quad \text{(e.1)}$$

Note that all terms in the EOM represent moments or torques.

### (b) The steady state periodic forced response of the system:

$$\theta(t) = \frac{\theta_{ss}}{\left[\left(1-f^2\right)^2 + \left(2\ \zeta\ f\right)^2\right]^{1/2}} \cos\left(\omega t - \phi\right) \quad \text{(e.2)}$$

but $$\theta_{ss} = \frac{T_{(\omega)}}{K} = \frac{m_o\ ed\ \omega^2}{K}\left(\frac{I_z}{I_z}\right) = \frac{m_o\ ed}{I_z}\ f^2 \quad \text{(e.3)}$$

with $$f = \frac{\omega}{\omega_n};\ \omega_n = \sqrt{\frac{K}{I_z}}\ ;\ \zeta = \frac{C}{2\sqrt{K I_z}},\ \text{and}\ \ \phi = \tan^{-1}\left(\frac{2\zeta f}{1-f^2}\right) \quad \text{(e.4)}$$

(e.3) in (**e.2**) leads to

$$\theta(t) = \frac{m_o\ ed}{I_z}\ \frac{f^2}{\left[\left(1-f^2\right)^2 + \left(2\ \zeta\ f\right)^2\right]^{1/2}} \cos\ (\omega t - \phi) \quad \text{(e.5)}$$

Let $$\theta_\infty = \frac{m_o\, e\, d}{I_z} \quad \text{(e.6)} \qquad B = \frac{f^2}{\left[\left(1-f^2\right)^2 + \left(2\zeta\ f\right)^2\right]^{1/2}} \quad \text{(e.7)}$$

and rewrite (e.5) as: $$\theta(t) = \theta_\infty\ B \cos\ (\omega t - \phi) \quad \text{(e.8)}$$

**(c) for the given physical values of the system parameters:**

$K$ = 98,670 N.m/rad

$I_z$ = 25 kg · m²

$C$ = 157. N.m.s/rad

$$\omega_n = \sqrt{\frac{K}{I_Z}} = 62.82 \ \frac{\text{rad}}{\text{sec}}$$

$$\zeta = \frac{C}{2\sqrt{K I_Z}} = 0.05$$

$$\theta_\infty = \frac{m_o e d}{I_Z} = \frac{8 \cdot 0.01 \cdot 0.3}{25} = \frac{0.024}{25} = 96\ 10^{-5} \text{rad}$$

And the turbine speed varies from 100 rpm to 1,200 rpm, i.e.

$\omega$= rpm $\pi$/30 = 10.47 rad/s to 125.66 rad/s, i.e.

$f = \dfrac{\omega}{\omega_n}$ = 0.167 to 2.00,

thus indicating the system will operate through resonance.

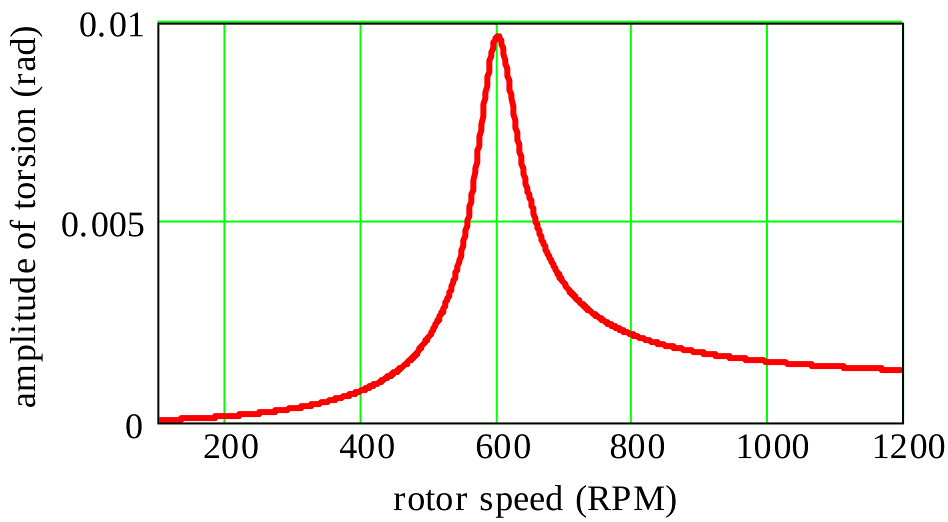
Hence, the angular response is $\theta(t) = \left(96.4\ 10^{-5}\ \text{rad}\right) \cdot B \cos\ (\omega t - \phi)$

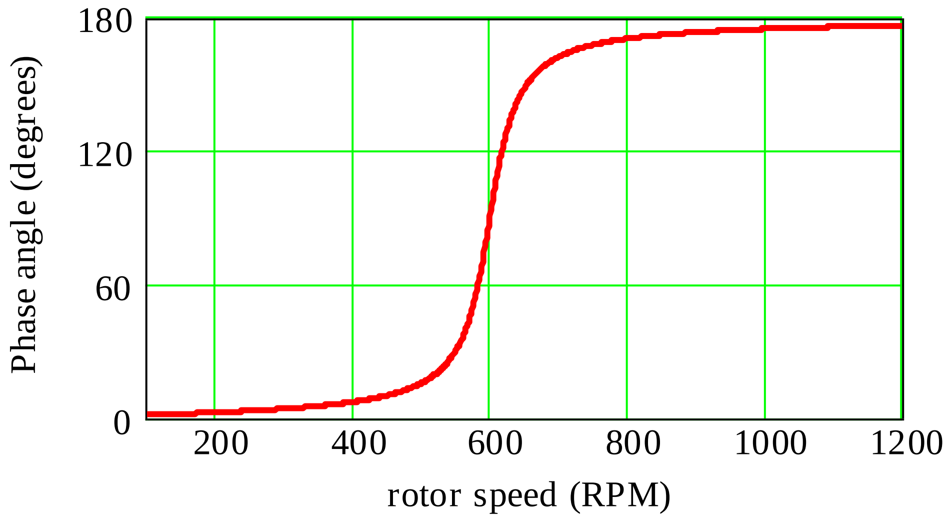
**(d) Maximum amplitude of response**:
since $\zeta$ << 1, the maximum amplitude of motion will occur when the turbine speed coincides with the natural frequency of the torsional system,

i.e., at $f = 1$ → $B \approx {}^{1}\!/_{2\zeta}$ and $\theta(t) = \theta_\infty \cdot \dfrac{1}{2\zeta} \cos\left(\omega t - \dfrac{\pi}{2}\right)$

the magnitude is **θMAX =** $\theta_{\max} = \dfrac{\theta_\infty}{2\xi} = 0.964 \text{ x } 10^{-2} \text{ rad}$**, i.e. 10 times larger than** $\theta_\infty$.

The graphs below show the **amplitude** (radians) and **phase angle** (degrees) of the polo twist vs. turbine speed (**RPM**)




**(e) Design change: DOUBLE DAMPING but first BALANCE ROTOR!**

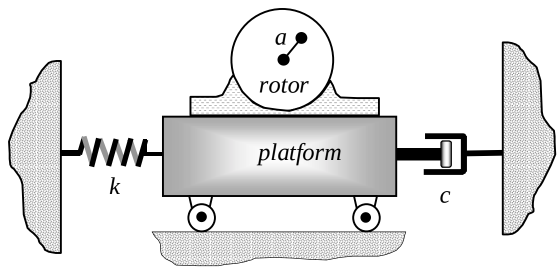
**MEEN 363 - QUIZ 3** Names:

The rotor of an electric generator weights 750 lb and is attached to a platform weighing 7750 lb. The motor has an imbalance eccentricity (a) of 2.5 mils. The platform can be modeled as shown in the figure. The equivalent stiffness (K) of the platform is 1 million-lb/in, and the equivalent damping is C=100 lb-s/in. The operating speed of the generator is 1800 rpm.

(a) determine the response of the platform (amplitude and phase) at the operating speed.

(b) determine the response of the platform (amplitude and phase) if the rotor spins with a speed coinciding with the system natural frequency.

(c) if the platform stiffness is increased by 25%, determine the allowable amount of imbalance (a) that will give the same amplitude of motion as determined in (a). Assume that the mass of the platform and the damping do not change appreciably by performing the stiffening.




**KEY:** System excitation due to rotating imbalance

Given $K := 10^6 \cdot \frac{lbf}{in}$ $M_{rotor} := 750 \cdot lb$ $M_{platform} := 7750 \cdot lb$

$M := M_{rotor} + M_{platform}$ $C := 100 \cdot lbf \cdot \frac{sec}{in}$

$M = 8.5 \times 10^3 \, lb$

calculate the system natural frequency and damping ratio:

$$\omega_n := \left(\frac{K}{M}\right)^{.5} \qquad \zeta := \frac{C}{2 \cdot M \cdot \omega_n}$$

$\omega_n = 213.125 \, \frac{rad}{s}$ $\zeta = 0.011$ little damping

and operating frequency ratio (r) for rotor speed: $RPM := 1800$

$$\omega := RPM \cdot \frac{2 \cdot \pi}{60} \cdot \frac{rad}{s} \qquad \omega = 188.496 \, \frac{rad}{s} \qquad r := \frac{\omega}{\omega_n} \qquad r = 0.884$$

The system response (amplitude and phase) for imbalance excitation are: $a := 2.5 \cdot 10^{-3} \cdot in$

$$Y_{op}(r) := a \cdot \frac{M_{rotor}}{M} \cdot \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \Psi(r) := atan\left(\frac{-2 \cdot \zeta \cdot r}{1 - r^2}\right)$$

Thus, at $r = 0.884$ <1

$Y_{op}(r) = 7.894 \times 10^{-4}\,in$  $\Psi(r)\cdot\frac{180}{\pi} = -4.947$ [degrees]

Let: $Y_{oper} := Y_{op}(r)$

**(b) If the rotor should spin with a speed coinciding with the system natural frequency,**

$r := 1$

$\omega := \omega_n$

$RPM := \omega\cdot\frac{60}{2\cdot\pi}\cdot\frac{s}{rad}$  $RPM = 2.035 \times 10^{3}$

the system response is

$Y_{op}(1) = 0.01\,in$  $\Psi := -90$ degrees  $\frac{Y_{op}(1)}{Y_{oper}} = 13.111$

also determined from:

$$a\cdot\frac{M_{rotor}}{M}\cdot\frac{1}{2\cdot\zeta} = 0.01\,in$$

**(c) If the platform increases K by 25%,** $K_{original} := K$

$K := 1.25\cdot K$  to maintain $Y_{oper} = 7.894 \times 10^{-4}\,in$

calculate the NEW system natural frequency and damping ratio:

$$\omega_n := \left(\frac{K}{M}\right)^{.5} \qquad \zeta := \frac{C}{2\cdot M\cdot\omega_n}$$

$\omega_n = 238.281\,\frac{rad}{s}$  $\zeta = 9.531 \times 10^{-3}$ small change in damping ratio

and operating frequency ratio (r) for rotor speed: $RPM := 1800$

$\omega := RPM\cdot\frac{2\cdot\pi}{60}\cdot\frac{rad}{s}$  $\omega = 188.496\,\frac{rad}{s}$  $r := \frac{\omega}{\omega_n}$  $r = 0.791$

from relationship:

$$Y_{op}(r) := a\cdot\frac{M_{rotor}}{M}\cdot\frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}$$

determine the (new) allowable imbalance:

$$a := \frac{Y_{oper}}{\left[\frac{M_{rotor}}{M}\cdot\frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}\right]}$$

$a = 5.354 \times 10^{-3}\,in$

i.e. ~ twice as original imbalance (eccentricity) displacement,

# Example: system response due to multiple frequency inputs

Consider a 2nd order system described by the following EOM

L San Andres (c) 2008

$$M\cdot\frac{d^2}{dt^2}Y + C\cdot\frac{d}{dt}Y + K\cdot Y = K\cdot z(t)$$

where

$$z(t) := a_1\cdot\cos(\omega_1\cdot t) + a_2\cdot\sin(\omega_2\cdot t) + a_3\cdot\cos(\omega_3\cdot t)$$

is an external excitation displacement function

**Find the forced response of the system, i.e, find Y(t)**

Given the system parameters $M := 100\cdot kg$ $\quad K := 10^6\cdot\frac{N}{m}$ $\quad \zeta := 0.10$

calculate natural frequency and physical damping

$$\omega_n := \left(\frac{K}{M}\right)^{0.5} \qquad f_n := \frac{\omega_n}{2\cdot\pi} \qquad f_n = 15.915\,Hz$$

$$C := 2\cdot M\cdot\omega_n\cdot\zeta \qquad C = 2\times 10^3\,s\frac{N}{m}$$

$$\omega_d := \omega_n\cdot(1-\zeta^2)^{.5} \qquad f_d := \frac{\omega_d}{2\cdot\pi} \qquad T_d := \frac{1}{f_d} \qquad T_d = 0.063\,s$$

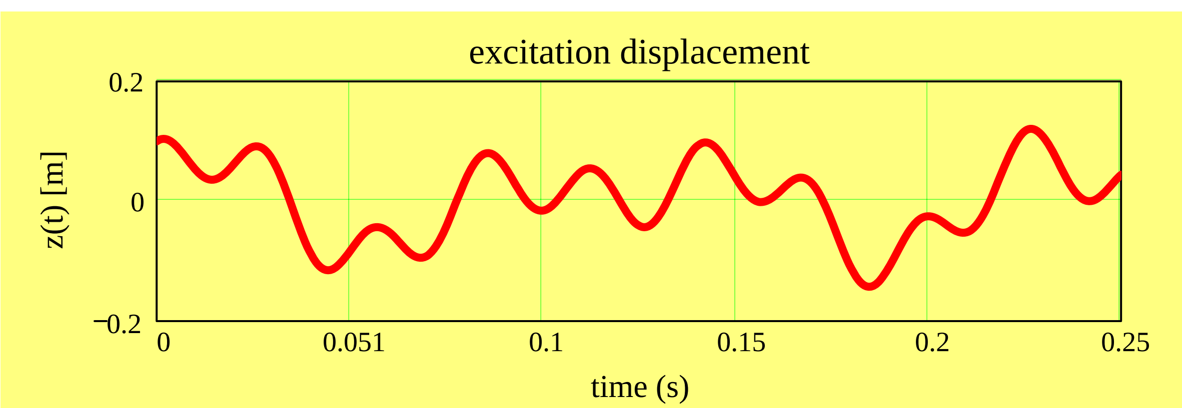
damped natural period

Set frequencies and amplitudes of the excitation z(t) are:

$$a_1 := 0.05\cdot m \qquad a_2 := 0.05\cdot m \qquad a_3 := 0.05\cdot m$$

$$\omega_1 := \frac{\omega_n}{2} \qquad \omega_2 := \omega_n\cdot\frac{9}{10} \qquad \omega_3 := 2.2\cdot\omega_n$$

assemble: $z(t) := a_1\cdot\cos(\omega_1\cdot t) + a_2\cdot\sin(\omega_2\cdot t) + a_3\cdot\cos(\omega_3\cdot t)$




4 periods of damped natural motion

SYSTEM RESPONSE is: $Y(t) := a_i \cdot H \cdot \cos\left(\omega_i \cdot t + \phi_i\right)$

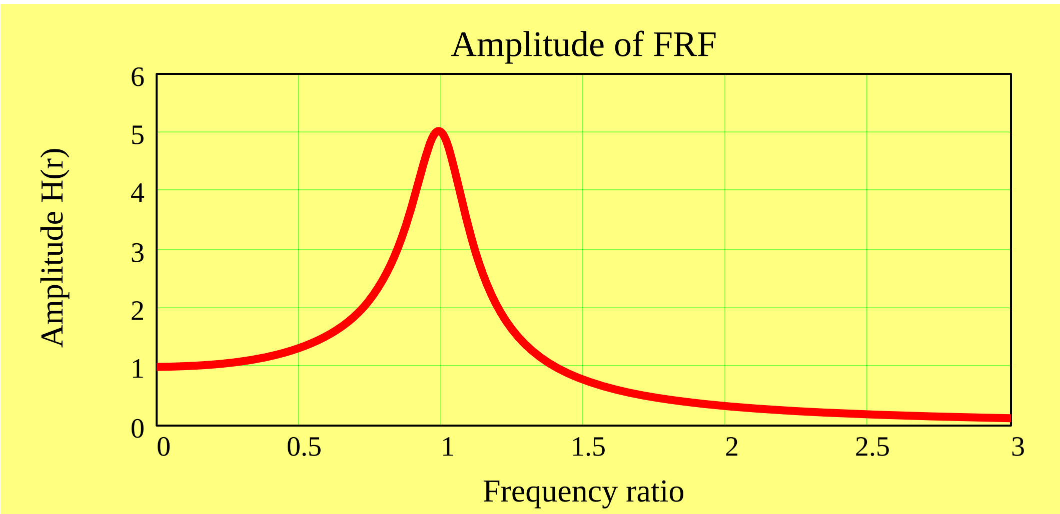
The system frequency response function: amplitude and phase angle are

$$H(r) := \frac{1}{\left[\left(1 - r^2\right)^2 + \left(2 \cdot \zeta \cdot r\right)^2\right]^{.5}}$$

$$\phi(r) := \begin{array}{|l} \phi \leftarrow -\mathrm{atan}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right) \\ \phi \leftarrow \phi - \pi \quad \text{if } r > 1 \\ \text{return } \phi \end{array}$$

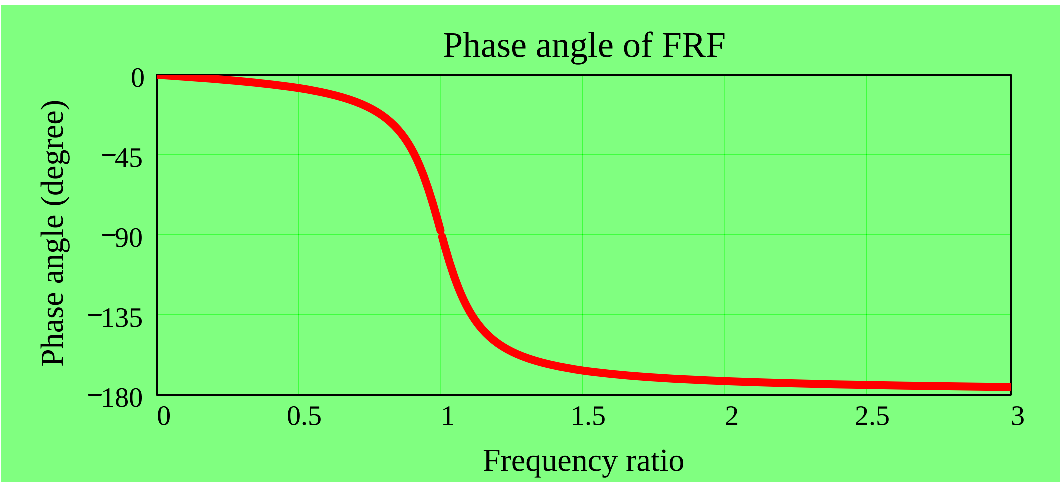
graphs of frequency response function

**Amplitude and phase lag** as a function of $r = \frac{\omega}{\omega_n}$ frequency ratio




$Q - \text{factor}$

$$\frac{1}{2 \cdot \zeta} = 5$$



**The response of the system is given by the superposition of individual responses, i.e**

$$Y(t) := Y_1 \cdot \cos(\omega_1 \cdot t + \phi_1) + Y_2 \cdot \sin(\omega_2 \cdot t + \phi_2) + Y_3 \cdot \cos(\omega_3 \cdot t + \phi_3)$$

where

**for first excitation:** $r_1 := \frac{\omega_1}{\omega_n}$ $H(r_1) = 1.322$ $\phi_1 := \phi(r_1)$ $r_1 = 0.5$

$Y_1 := \frac{K}{K} \cdot a_1 \cdot H(r_1)$ $Y_1 = 0.066\,m$ $\phi_1 \cdot \frac{180}{\pi} = -7.595$ degrees

**for second excitation:** $r_2 := \frac{\omega_2}{\omega_n}$ $H(r_2) = 3.821$ $\phi_2 := \phi(r_2)$ $r_2 = 0.9$

$Y_2 := a_2 \cdot H(r_2)$ $Y_2 = 0.191\,m$ $\phi_2 \cdot \frac{180}{\pi} = -43.452$ degrees

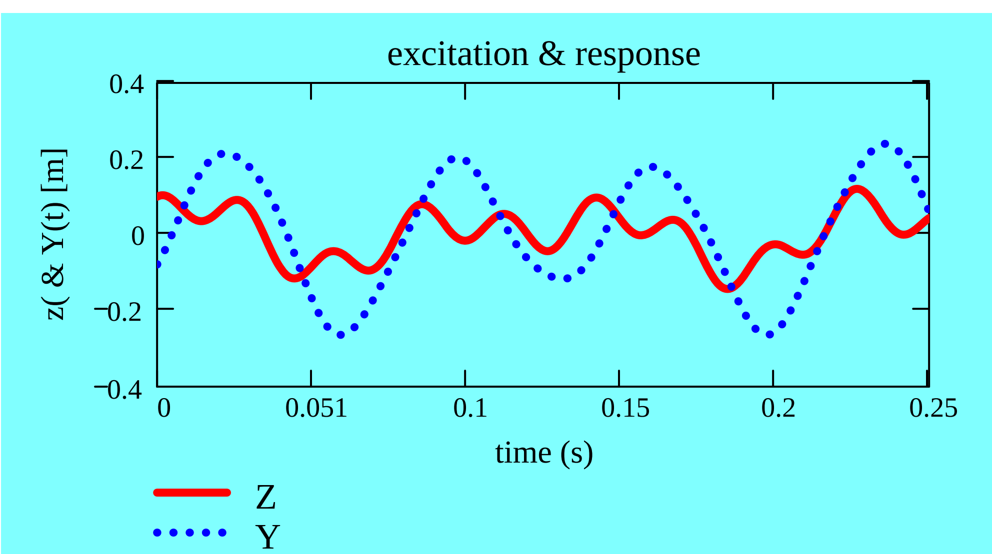
**for third excitation:** $r_3 := \frac{\omega_3}{\omega_n}$ $H(r_3) = 0.259$ $\phi_3 := \phi(r_3)$ $r_3 = 2.2$

$Y_3 := a_3 \cdot H(r_3)$ $Y_3 = 0.013\,m$ $\phi_3 \cdot \frac{180}{\pi} = -173.463$ degrees

Assemble physical response:

$$Y(t) := Y_1 \cdot \cos(\omega_1 \cdot t + \phi_1) + Y_2 \cdot \sin(\omega_2 \cdot t + \phi_2) + Y_3 \cdot \cos(\omega_3 \cdot t + \phi_3)$$

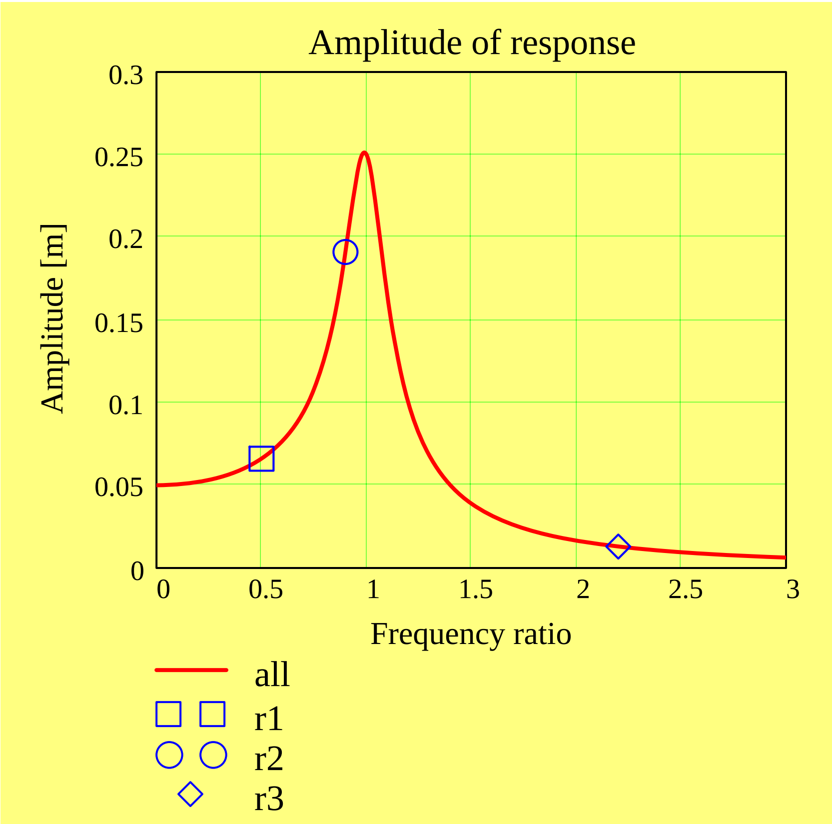
Now graph the response Y(t) and the excitation z(t):




**Note**: The response Y shows little motion at the highest excitation frequency (ω3).
There is an obvious amplification of motion with second frequency (ω2 ~ ωn).

To understand better, let's plot the actual FRF:

$T_d = 0.063\,s$




$a_1 = 0.05\,m$

$r_1 = 0.5$ $\quad Y_1 = 0.066\,m$

$r_2 = 0.9$ $\quad Y_2 = 0.191\,m$

$r_3 = 2.2$

$Y_3 = 0.013\,m$

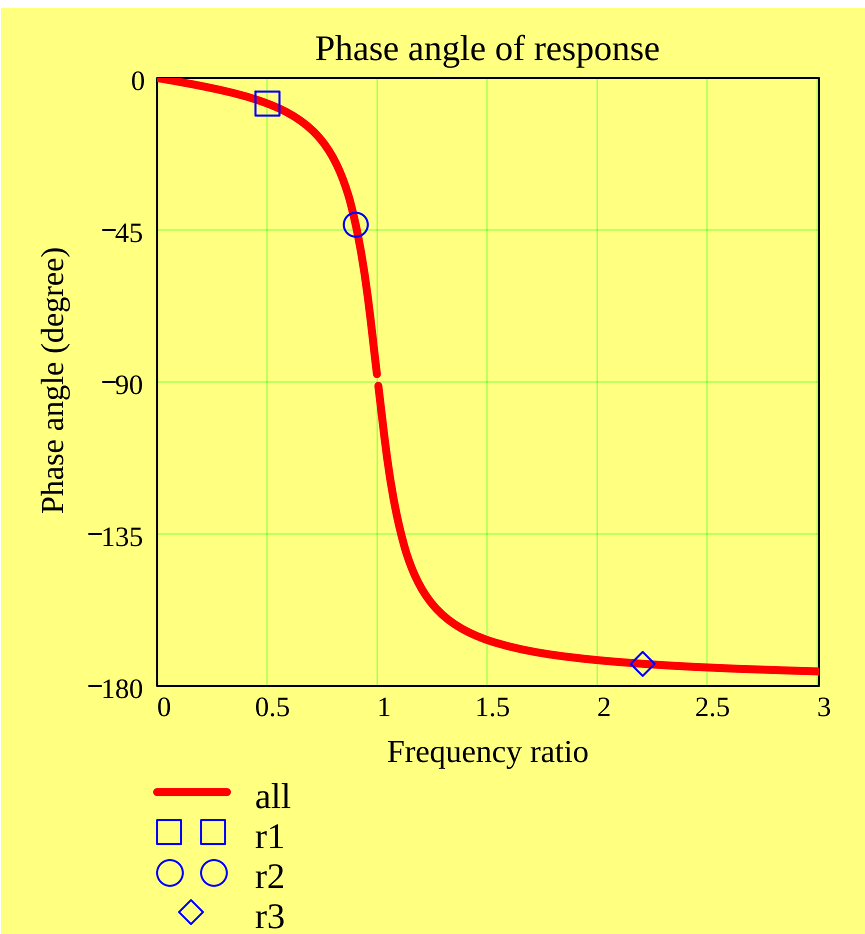
Note how response amplitude for largest frequency is largely attenuated

$Y_3 < a_3$

while amplitudes for first two frequencies are amplified, in particular for ω2 which is close to the natural frequency

$$\frac{Y_2}{a_2} = 3.821$$

$$\frac{Y_1}{a_1} = 1.322$$



**ME459/659 Notes# 1c (pp. 47-55)**

# Dynamic Response of SDOF Second Order Mechanical System: Viscous Damping

$$M\ddot{X} + D\dot{X} + KX = F_{(t)}$$

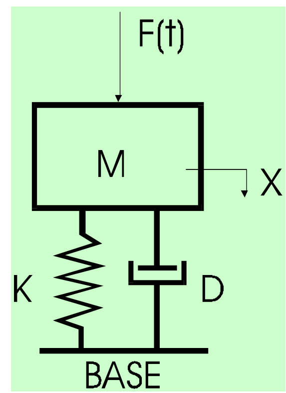
**Transmissibility: Forces transmitted to base or foundation**

**Response to Periodic Motion of Base or Support**

## TRANSMISSIBILITY: transmitted force to base or foundation

Needed to calculate stresses on the structural supports as well as to verify the isolation characteristics of the system from its base or support frame.

The EOM for a SDOF (*K,D.M*) system excited by a periodic force of constant magnitude $F_o$ and frequency ($\Omega$) is:



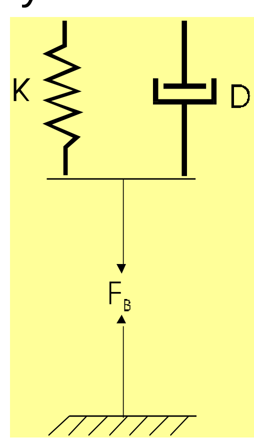

$$M\frac{d^2X}{dt^2}+D\frac{dX}{dt}+KX=F_o\sin(\Omega t) \tag{54}$$

with solution $$X(t)=X_{ss}\,A\sin(\Omega t-\varphi) \tag{55}$$

where

$$X_{ss}=\frac{F_o}{K}\quad;\ A=\frac{1}{\left[(1-f^2)+(2\zeta f)^2\right]^{1/2}};\quad \varphi=\tan^{-1}\left(\frac{2\zeta f}{1-f^2}\right) \tag{56}$$

with $f=\Omega/\omega_n$ as the ratio of the excitation frequency to the system natural frequency.




The **dynamic force transmitted to the base** or foundation is

$$F_B=F_D+F_K=D\dot{X}+KX \tag{57}$$

Substitution of Eq. (55) into Eq. (57) gives,

$$F_B = F_o\, A\left[\ \sin\left(\Omega t-\varphi\right)+2\zeta\, f\ \cos\left(\Omega t-\varphi\right)\right] \qquad (58)$$

Let $\cos(\alpha)=\dfrac{1}{\sqrt{1+(2\zeta\, f\,)^2}}$ ; $\sin(\alpha)=\dfrac{2\zeta\, f}{\sqrt{1+(2\zeta f\,)^2}}$

And after manipulation rewrite Eq. (58) as:

$$F_B = F_o\, A_T\ \sin\left(\Omega t-\phi_T\right) \qquad (59)$$

$$A_T = \frac{\left[1+\left(2\,\zeta\, f\right)^2\right]^{1/2}}{\left[\left(1-f^2\right)+\left(2\,\zeta\, f\right)^2\right]^{1/2}};\quad \varphi_T = \varphi+\alpha;$$

where (60)

$$\varphi=\tan^{-1}\left(\frac{2\,\zeta\, f}{1-f^2}\right);\quad \alpha=\tan^{-1}\left(2\,\zeta\, f\right)$$

The **transmissibility (T)** is the ratio of force transmitted to base or foundation $|F_B|$ to the (input) excitation force $|F_0 \sin(\Omega t)|$, i.e.,

$$T=\left|\frac{F_B}{F_o}\right| \equiv A_T = \frac{\sqrt{1+(2\zeta\, f)^2}}{\sqrt{(1-f^2)^2+(2\zeta\, f)^2}} \qquad (61)$$

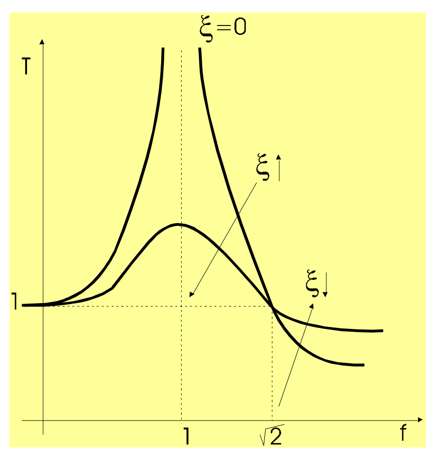
**Regimes of operation:**

**at low frequencies:**

$$\Omega \ll \omega_n \rightarrow f\rightarrow 0 \Rightarrow A_T = 1$$

**at high frequencies:**

$$\Omega \gg \omega_n \rightarrow f\rightarrow \infty \Rightarrow A_T = 2\zeta / f$$



**at resonance:** $\Omega = \omega_n \rightarrow f = 1 \Rightarrow A_T = \dfrac{\sqrt{1+4\zeta^2}}{2\zeta}$

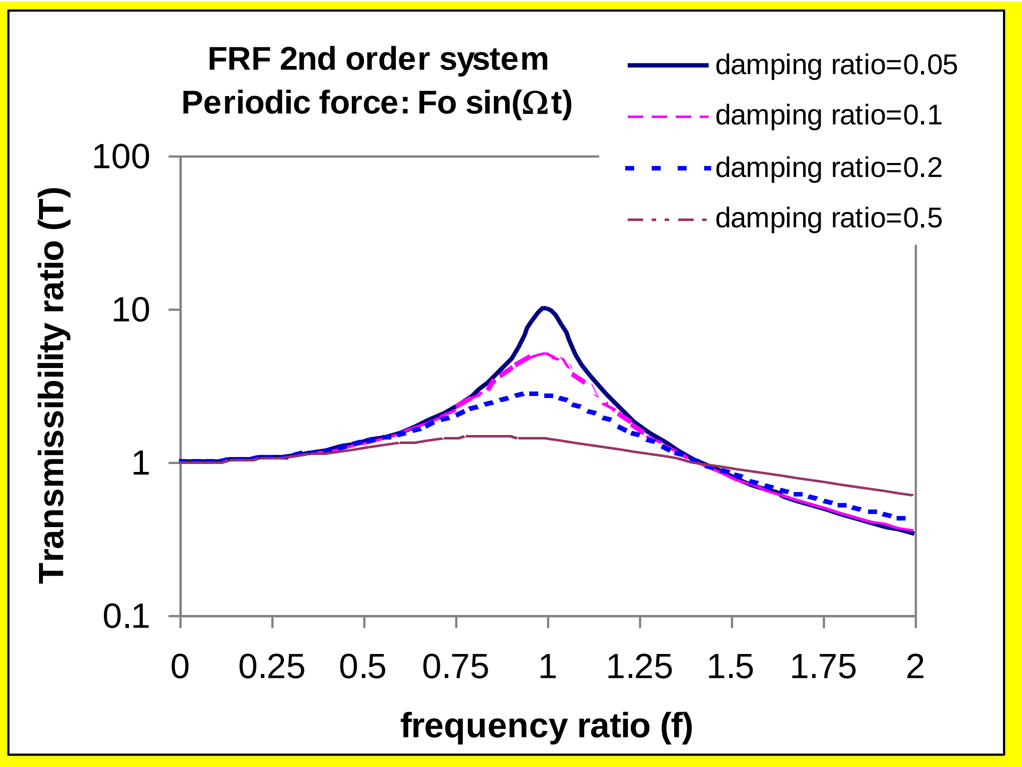
## NOTES:

At low frequencies, $f < \sqrt{2}$, the transmitted force (to base) is larger than external force, i.e. *T*>1

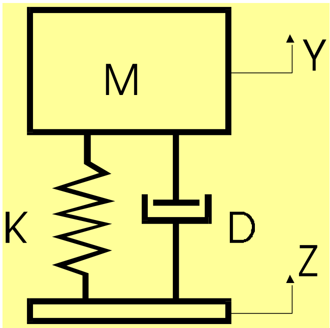
At $f = \sqrt{2}$, the system shows the same transmissibility, independent of the damping coefficient or $\zeta$..

Operation above $f > \sqrt{2}$ determines the lowest transmitted forces, i.e. mechanical system is **ISOLATED** from base (foundation). A desirable operating condition

When operation is at large frequencies, $f > \sqrt{2}$ , viscous damping causes transmitted forces to be larger than w/o damping. Damping is NOT desirable for operation at high frequencies.



# Response to Periodic Motion of Base or Support




Consider the motion of a (*K, D,M*) system with its base (or support) moving with a known periodic displacement $Z_{(t)} = b \cos(\Omega t)$.

The dynamic response of this system is of particular importance for the correct design and performance of vehicle suspension systems. Response to earthquake excitations as well.

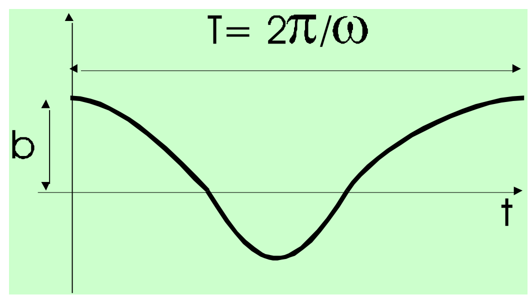
For motions about a static equilibrium, $Y=Z=0$. The EOM is:

$$M\ddot{Y} + K(Y-Z) + D(\dot{Y}-\dot{Z}) = 0 \qquad (62a) \text{ or}$$

$$M\ddot{Y} + D\dot{Y} + KY = KZ + D\dot{Z} \qquad (62b)$$

Since $Z = b\cos(\Omega t)$ is prescribed, then

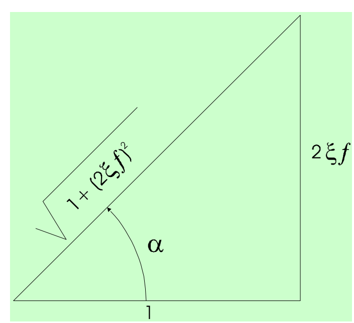
$$\dot{Z} = -b\,\Omega \sin(\Omega t)$$




Substitution of *Z* and d*Z*/dt into eqn. (62b) gives

$$M\ddot{Y} + D\dot{Y} + KY = (K\,b)\left[\cos(\Omega t) - 2\zeta f\,\sin(\Omega t)\right] \qquad (63)$$

Let : $\cos(\alpha) = \dfrac{1}{\sqrt{1+(2\zeta f)^2}}$ ; $\sin(\alpha) = \dfrac{2\zeta f}{\sqrt{1+(2\zeta f)^2}}$




and write Eq.(63) as:

$$M\ddot{Y} + D\dot{Y} + KY = K\,b\sqrt{1+(2\zeta f)^2}\,\cos(\Omega t - \alpha) = F_o\cos(\Omega t - \alpha) \qquad (64)$$

After all transients die due to damping, the system **periodic steady-state response** or motion is:

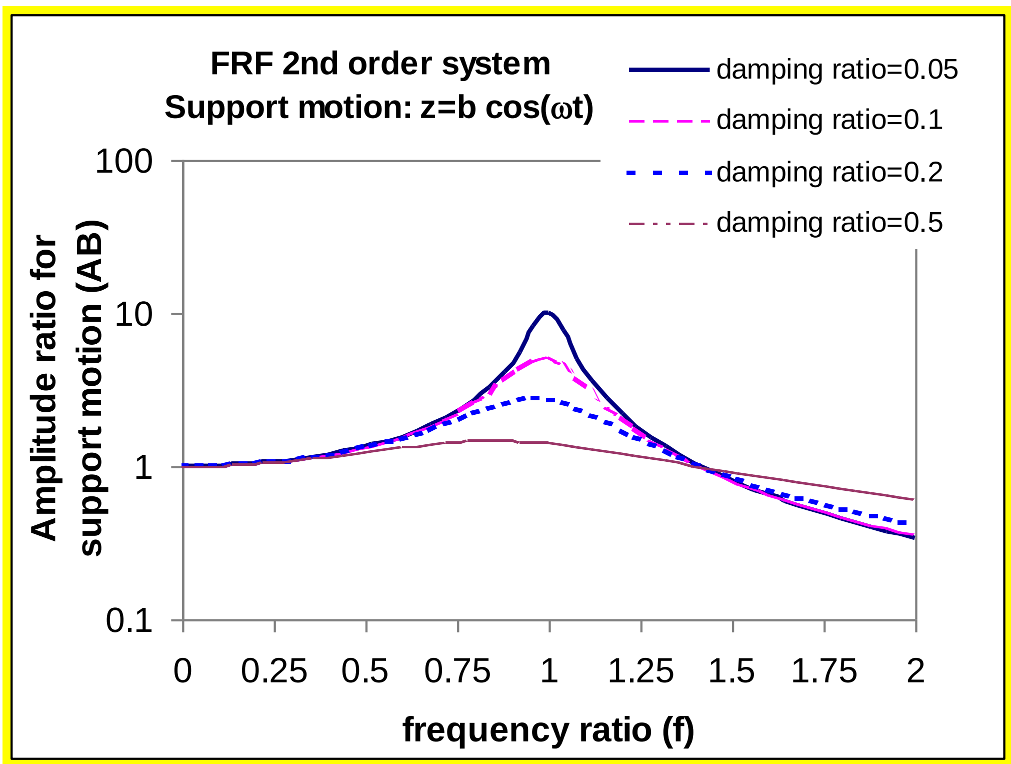
$$Y_{(t)} = b\,A_B\;\cos\left(\omega t - \varphi_B\right) \qquad (65)$$

$$A_B = \frac{\left[1+\left(2\,\zeta\,f\right)^2\right]^{1/2}}{\left[\left(1-f^2\right)+\left(2\,\zeta\,f\right)^2\right]^{1/2}};\quad \varphi_B = \varphi + \alpha;$$

where (66)

$$\varphi = \tan^{-1}\left(\frac{2\,\zeta\,f}{1-f^2}\right);\quad \alpha = \tan^{-1}\left(2\,\zeta\,f\right)$$

**NOTE that $A_B$ is identical to the amplitude of FRF for transmitted force, i.e. the transmissibility ratio**



## EXAMPLE:

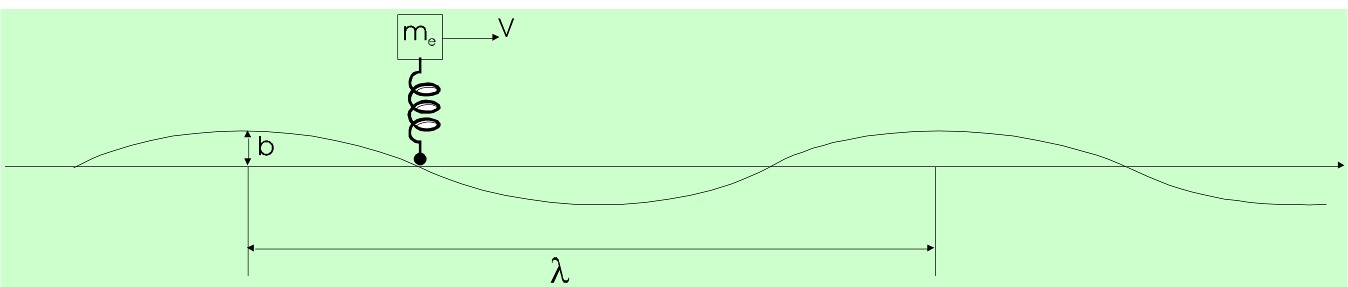
A 3000 lb (empty) automobile with a 10' wheel-base has wheels which weigh 70 lb each (with tires). Each tire has an effective stiffness (contact patch to ground) of 1000 lb/in. A static test is done in which 5 passengers of total weight 800 lb climb inside and the car is found to sag (depress toward the ground) by 2".

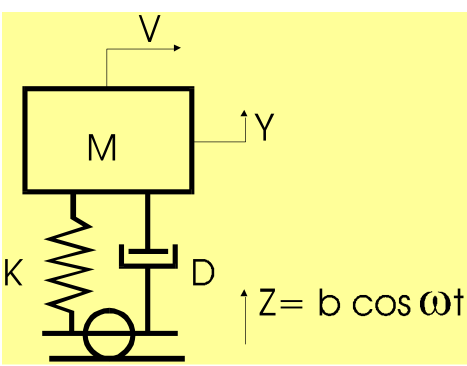
(a) From the standpoint of the passenger comfort, what is the worst wavelength (in feet) [sine wave road] which the car (with all 5 passengers) could encounter at 65 mph?

(b) For the worst case in (a) above, what percent of critical damping is required to keep the absolute amplitude of the vertical heaving oscillations less than ½ of the amplitude of the undulated road?

(c) What is the viscous damping coefficient required for the shock absorber on each wheel (assume they are all the same) to produce the damping calculated in (b) above? Give the physical units of your answer.

(d) State which modes of vibration you have neglected in this analysis and give justifications for doing so.




The wavelength is equal to $\lambda = v\,T$, with $T$ as the period of motion. And the frequency **(ω)** of the forced motion is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi\, v}{\lambda}$$

The system mass is:

$$M_{eq} = \frac{W}{g} = \frac{(3000 + 800 - 4x70)}{g}$$

$$K = \frac{800 \text{ lb}}{2 \text{ in}} = 400 \frac{\text{lbf}}{\text{in}}; \quad M_{eq} = \frac{3{,}520 \text{ lbf}}{386.4 \text{ in/sec}^2} = 9.12 \frac{lbf \text{ sec}^2}{in}$$

$$\omega_n = \left(\frac{K}{M_{eq}}\right)^{1/2} = 6.2\frac{rad}{sec}\ (1.05\ Hz)$$

**(a)** For passenger comfort, the worst wavelength (in feet) which the car could encounter at 65 mph is when the excitation frequency coincides with the system natural frequency, i.e. $\omega = \omega_n$. Thus from

$\omega = \frac{2\pi}{T} = \frac{2\pi v}{\lambda} = \omega_n$ . Then

$$\lambda = \frac{2\pi v}{\omega_n} = \frac{2\pi\ 65\ mph \cdot \left(\frac{5,280\ ft/mile}{3,600\ \sec/hour}\right)}{6.62\ \frac{rad}{\sec}} = \lambda = 90.47 \text{ ft} = (0.0171 \text{ miles})$$

**(b)** For the worst case what percent of critical damping is required to keep the absolute amplitude of the vertical heaving oscillations less than ½ of the amplitude of the undulated road? i.e. What value of damping ratio (ζ) makes $\frac{|Y|}{b} = \frac{1}{2}$ at $\omega = \omega_n$ ?

Recall that at $\omega = \omega_n$, the amplitude of the support FRF is from eqn. (77):

$$A_B = \left[\frac{1+(2\zeta)^2}{(2\zeta)^2}\right]^{1/2} \Rightarrow \frac{1}{2} \quad ? \quad \frac{1}{4\zeta^2} = -\frac{3}{4}$$

The solution indicates that the damping ratio (ζ) is imaginary! This is clearly impossible. Note that the amplification ratio $\boldsymbol{A}_B > 1$ at $\boldsymbol{f} = 1$, i.e. the amplitude of motion **|Y|** for the system will always be larger than the amplitude of the base excitation (***b***), regardless of the amount of damping.

**(c)** What is the viscous damping coefficient required for the shock absorber on each wheel (assume they are all the same) to produce the damping calculated in (b) above?

No value of viscous damping ratio (ζ) is available to reduce the amplitude of motion. However, if there should be one value, then

$$\zeta = \frac{D_{eq}}{2\,M_{eq}\,\omega_n} = \frac{4D}{2\,M_{eq}\,\omega_n}\ ;\ \text{then} \rightarrow D = \frac{1}{2}\,\zeta\,M_{eq}\,\omega_n \quad \left[\frac{\text{lb}_f}{\text{in/sec}}\right]$$

**(d)** State which modes of vibration you have neglected in this analysis and give justifications for doing so.

Heaving (up & down) motion is the most important mode and the one we have studied. In this example, pitching motion is not important because the road wavelength ¬ is large. We have also neglected yawing which is not important if the car cg is low.

One important mode to consider is the one related to "**tire bouncing**", i.e. the tires have a mass and spring coefficient of their own, and therefore, its natural frequency is given by

$$\omega_{n_{tire}} = \sqrt{\frac{1,000}{70/386.4}} = 74.25 \frac{rad}{\sec}$$

However, the car bouncing natural frequency is 6.62 rad/sec is much lower than the tire natural frequency, i.e.

$$\omega_{n\,car} = 6.62 \text{ rad/s} \ll \omega_{n\,tire} = 74.25 \text{ rad/sec}$$

Therefore, it is reasonable to neglect the "tire" bouncing mode since its frequency is so high that it can not be excited by the road wavelength specified.

ME459/659 Notes# 1d (pp. 56-60)

# Dynamic Response of SDOF Second Order Mechanical System: Viscous Damping

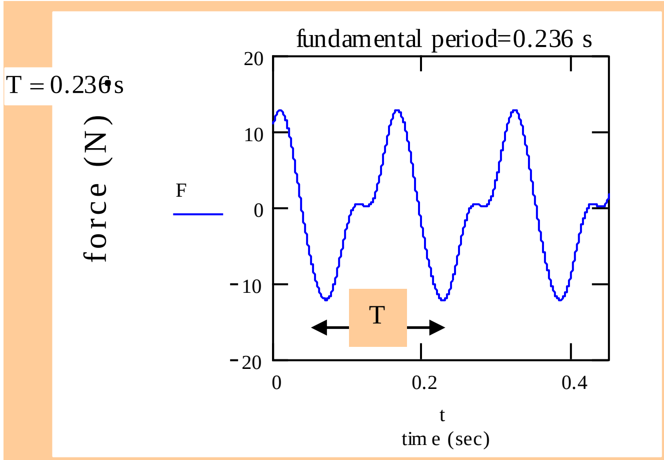
$$M\,\ddot{X} + D\,\dot{X} + K\,X = F_{(t)}$$

## DYNAMIC RESPONSE OF A SDOF SYSTEM TO ARBITRARY PERIODIC LOADS

# DYNAMIC RESPONSE OF SDOF SYSTEM TO ARBITRARY PERIODIC LOADS

## Fourier Series

Forces acting on structures are frequently periodic or can be approximated closely by superposition of periodic loads.




As illustrated →, the **function $F_{(t)}$ is periodic but not harmonic**.

Any periodic function, however, can be represented by a convergent series of harmonic functions whose frequencies are integer multiples of a certain fundamental frequency **Ω**.

The integer multiples are called ***harmonics***. The series of harmonic functions is known as a **FOURIER SERIES**, written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t) \qquad (67)$$

with $F_{(t+T)} = F_{(t)}$ and $T=2\pi/\Omega$ is the fundamental period. $a_n$, $b_n$ are the coefficients of the $n_{th}$ harmonic, and related to $F_{(t)}$ by the following formulas

$$a_n = \frac{2}{T}\int_t^{t+T} F(t)\cos(n\Omega t)dt, \quad n = 0,1,2,\ldots\ldots\infty$$

(68)

$$b_n = \frac{2}{T}\int_t^{t+T} F(t)\sin(n\Omega t)dt, \quad n = 1,2,\ldots\ldots.\infty$$

each representing a measure of the participation of the harmonic content of *cos(nΩt)* and *sin(nΩt)*, respectively. All the $a_0$, $b_m$, $c_m$ have the units of force.

**Note** that ($\frac{1}{2} a_0$) is the **period averaged** magnitude of $F_{(t)}$ .

In practice, $F_{(t)}$ can be approximated by a relatively small number of terms. Some useful simplifications arise when

If $F_{(t)}$ is an EVEN fn., i.e., $F_{(t)} = F_{(-t)}$ then, $b_n = 0$ for all *n*

If $F_{(t)}$ is an ODD fn., i.e., $F_{(t)} = -F_{(-t)}$ then, $a_n = 0$ *for* all *n*

The Fourier series representation, Eq. (67), can also be written as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\Omega t - \beta_n) \tag{69}$$

where $c_n = \left(a_n^2 + b_n^2\right)^{1/2}$ and $\beta_n = \tan^{-1}\left(b_n / a_n\right), \quad n = 1,2,\ldots\ldots\infty$

are the magnitude and phase angle, respectively, of the *nth* harmonic frequency (*n*Ω).

# RESPONSE OF UNDAMPED SYSTEM

For an **undamped SDOF system**, the **steady state response** (w/o the transient solution) produced by each sine and cosine term in the harmonic load (Fourier series) is

$$Xs_{m(t)} = \frac{b_m/K}{1-f_m^{\,2}}\sin\left(m\Omega t\right),\quad Xc_{m(t)} = \frac{a_m/K}{1-f_m^{\,2}}\cos\left(m\Omega t\right) \qquad m=1,2, \quad (70)$$

where $f_m = m\Omega/\omega_n,\quad \omega_n = \sqrt{K/M}$. For the constant force $a_0$, the s-s response is simply $X_0 = \frac{1}{2}\frac{a_0}{K}$.

Using the **principle of superposition**, gives the system response as the sum of the individual components:

$$X_{(t)} = \frac{1}{K}\left(\frac{a_0}{2} + \sum_{m=1}^{\infty}\frac{1}{\left(1-f_m^{\,2}\right)}\left[a_m\cos(m\Omega t) + b_m\sin(m\Omega t)\right]\right) \quad (71)$$

Note when $m\Omega = \omega_n$ , i.e., there is a harmonic frequency equal to the natural frequency of the system, then the system response will (theoretically) be **UNBOUNDED** (the system will fail!).

## PERIODIC FORCED RESPONSE OF A DAMPED SDOF

In a damped SDOF system, the **steady-state response** produced by each sine and cosine term in the harmonic load series is

$$Xc_{m(t)} = \frac{a_m}{K} \frac{\left[\left(1-f_m^{\,2}\right)\cos(m\Omega\, t) + \left(2\zeta\, f_m\right)\,\sin(m\Omega\, t)\right]}{\left[\left(1-f_m^{\,2}\right)^2 + \left(2\zeta\, f_m\right)^2\right]} \qquad (72a)$$

$$Xs_{m(t)} = \frac{b_m}{K} \frac{\left[-\left(2\zeta\, f_m\right)\,\cos(m\Omega\, t) + \left(1-f_m^{\,2}\right)\sin(m\Omega\, t)\right]}{\left[\left(1-f_m^{\,2}\right)^2 + \left(2\zeta\, f_m\right)^2\right]} \qquad (72b)$$

Superposition gives the total system response as

$$X(t) = \frac{a_0}{2K} + \frac{1}{K}\sum_{m=1}^{\infty}\left[\frac{a_m\left(1-f_m^2\right) - 2\zeta\, f_m\, b_m}{\left(1-f_m^2\right)^2 + \left(2\zeta\, f_m\right)^2}\cos(m\Omega\, t)\right] + \frac{1}{K}\sum_{m=1}^{\infty}\left[\frac{b_m\left(1-f_m^2\right) + 2\zeta\, f_m\, a_m}{\left(1-f_m^2\right)^2 + \left(2\zeta\, f_m\right)^2}\sin(m\Omega\, t)\right] \qquad (73)$$

o r,

$$X(t) = \frac{a_0}{2K} + \frac{1}{K}\sum_{m=1}^{\infty}\left[\frac{c_m}{\left(1-f_m^2\right)^2 + \left(2\zeta\, f_m\right)^2}\cos(m\Omega t - \gamma_m)\right] \qquad (74)$$

where, $f_m = {}^{m\Omega}\!/_{\omega_n}$; $\omega_n = \sqrt{K/M}$

and $c_m = \sqrt{a_m^{\,2} + b_m^{\,2}}$ ; $\gamma_m = \tan^{-1}\dfrac{\left\{b_m\left(1-f_m^{\,2}\right) + 2\zeta\, f_m\, a_m\right\}}{\left\{a_m\left(1-f_m^{\,2}\right) - 2\zeta\, f_m\, b_m\right\}}$, $m = 1, 2, \ldots\, \infty$

# Example: system response due to periodic function

Consider a 2nd order system described by the following EOM

L San Andres (c) 2008

ORIGIN := 1

$$M\cdot\frac{d^2}{dt^2}Y + C\cdot\frac{d}{dt}Y + K\cdot Y = K\cdot z(t)$$

where z (t) is an external periodic excitation function

Given the system parameters $M := 100\cdot kg$ $K := 10^6\cdot\frac{N}{m}$ $\zeta := 0.10$

calculate natural frequency and physical damping

$$\omega_n := \left(\frac{K}{M}\right)^{0.5} \qquad f_n := \frac{\omega_n}{2\cdot\pi} \qquad f_n = 15.915\,Hz$$

$$C := 2\cdot M\cdot\omega_n\cdot\zeta \qquad C = 2\times10^3\,s\frac{N}{m}$$

$$\omega_d := \omega_n\cdot\left(1-\zeta^2\right)^{.5} \qquad f_d := \frac{\omega_d}{2\cdot\pi} \qquad T_d := \frac{1}{f_d} \qquad T_d = 0.063\,s$$

damped natural period

Define periodic excitation function:

**Example - square wave**

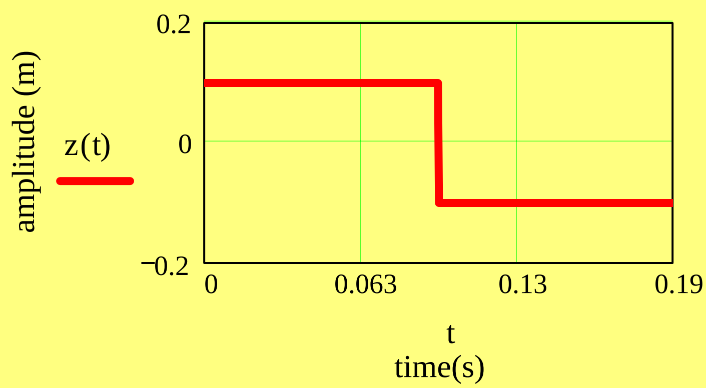
$$T := \frac{T_d}{.33333} \qquad z_o := 0.1\cdot m$$

$$z(t) := \begin{array}{|l} amp \leftarrow z_o \ \text{ if } \ t < \frac{T}{2} \\ amp \leftarrow -z_o \ \text{ if } \ t > \frac{T}{2} \\ amp \end{array}$$

$$\Omega := \frac{2\cdot\pi}{T}$$ fundamental frequency

$$\frac{\Omega}{\omega_d} = 0.333$$

$N_F := 7$ number of Fourier coefficients




## Find Fourier Series coefficients for excitation z(t)

mean value $a_0 := \frac{1}{T}\cdot\int_0^T z(t)\,dt$ $a_0 = 0\,m$

$j := 1 .. N_F$

coefs of cos & sin

$$a_j := \frac{2}{T} \cdot \int_0^T z(t) \cdot \cos(j \cdot \Omega \cdot t)\, dt \qquad b_j := \frac{2}{T} \cdot \int_0^T z(t) \cdot \sin(j \cdot \Omega \cdot t)\, dt$$

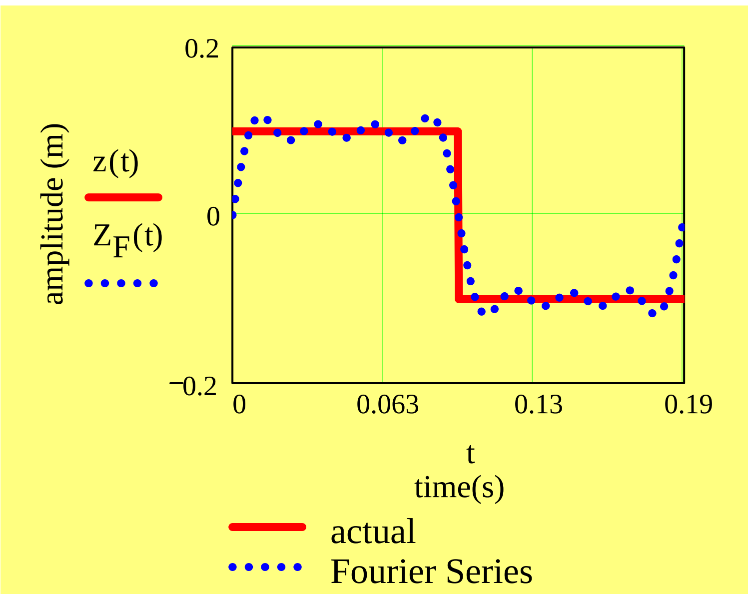
$a^T = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)\, m$ $\qquad b^T = (0.127 \;\; 0 \;\; 0.042 \;\; 0 \;\; 0.025 \;\; 0 \;\; 0.018)\, m$

Build z(t) as a Fourier series

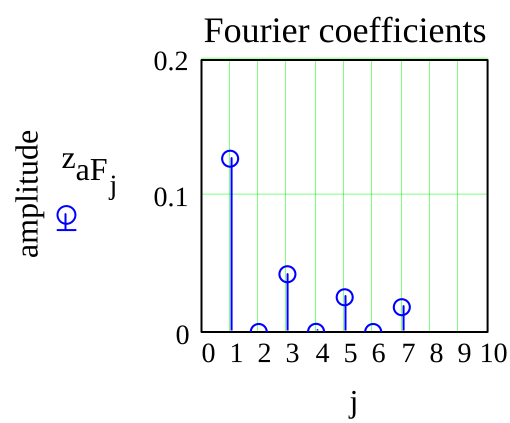
$$Z_F(t) := a_0 + \sum_{j=1}^{N_F} \left( a_j \cdot \cos(j \cdot \Omega \cdot t) + b_j \cdot \sin(j \cdot \Omega \cdot t) \right)$$

$$z_{aF_j} := \left[ (a_j)^2 + (b_j)^2 \right]^{.5}$$

Amplitude




$N_F = 7$




# Find the forced response of the system, i.e, find Y(t)

SYSTEM RESPONSE is:

$$Y(t) := Y_o + \sum_{m=1}^{N_F} \left[ \left( Yc_m \cdot \cos(m \cdot \Omega \cdot t) + Ys_m \cdot \sin(m \cdot \Omega \cdot t) \right) \right]$$

$$Y_o := a_0 \cdot \frac{K}{K}$$

$m := 1 .. N_F$

(a) set frequency ratio $\quad f_m := \dfrac{m \cdot \Omega}{\omega_n}$

(b) build denominator

$$den_m := \left[1 - (f_m)^2\right]^2 + (2 \cdot \zeta \cdot f_m)^2$$

(c) build coefficient of cos()

$$Yc_m := \frac{K}{K} \cdot \frac{\left[a_m \cdot \left[1 - (f_m)^2\right] - 2 \cdot \zeta \cdot f_m \cdot b_m\right]}{den_m}$$

(d) build coefficient of sin()

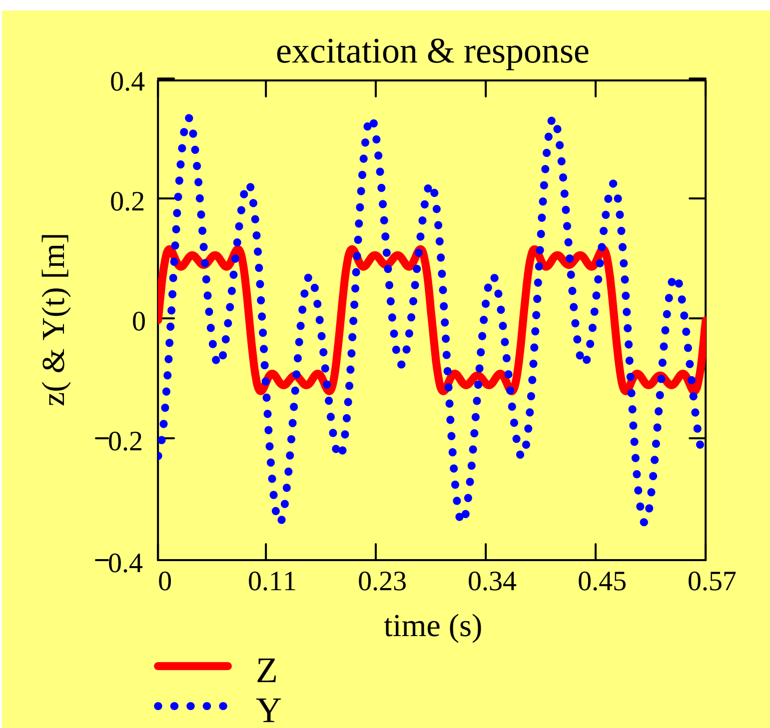
$$Ys_m := \frac{K}{K} \cdot \frac{\left[b_m \cdot \left[1 - (f_m)^2\right] + 2 \cdot \zeta \cdot f_m \cdot a_m\right]}{den_m}$$

(e) for graph of components

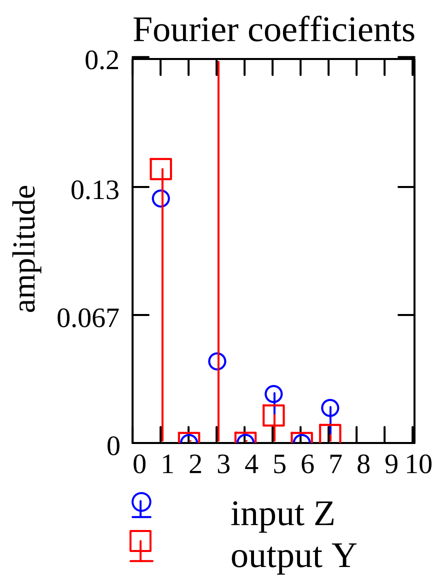
$$Y_{F_m} := \left[(Yc_m)^2 + (Ys_m)^2\right]^{.5}$$

$$Y(t) := Y_o + \sum_{m=1}^{N_F} \left(Yc_m \cdot \cos(m \cdot \Omega \cdot t) + Ys_m \cdot \sin(m \cdot \Omega \cdot t)\right)$$

Now graph the response Y(t) and the excitation (Fourier) z(t):



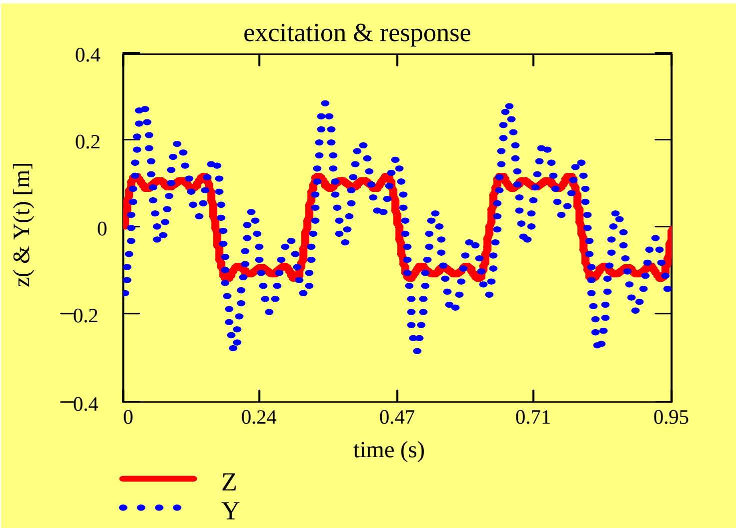

$$\frac{T}{T_d} = 3 \qquad \frac{\Omega}{\omega_n} = 0.332$$



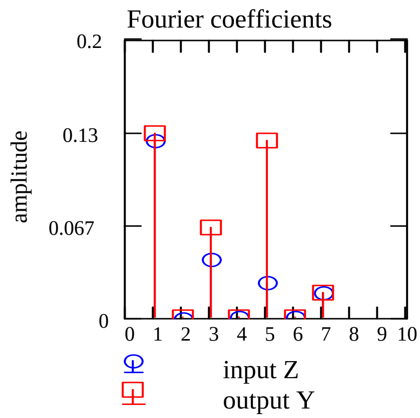

3 periods of fundamental excitation motion

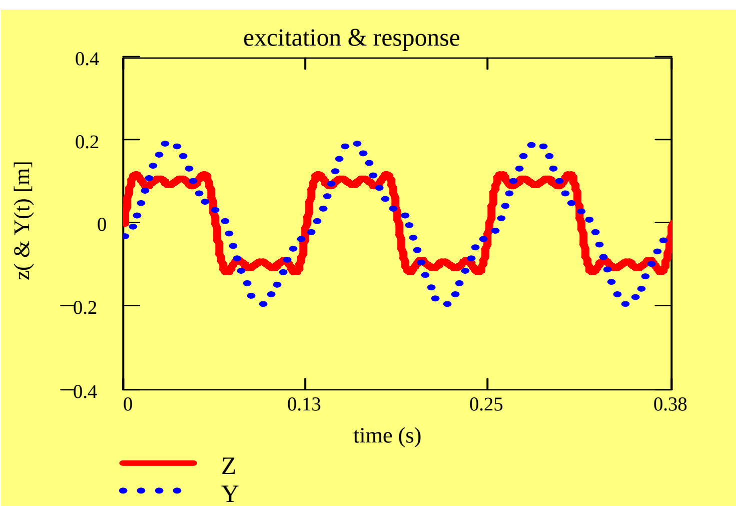
**Note**: obtain response for inputs with increasing frequencies (periods decrease)

==================================================

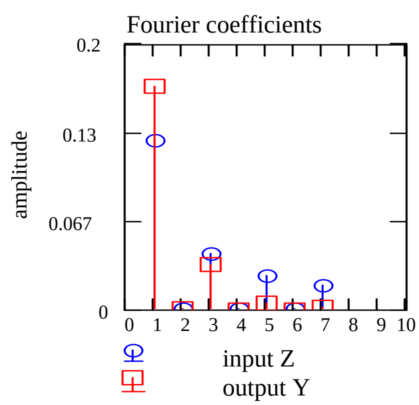


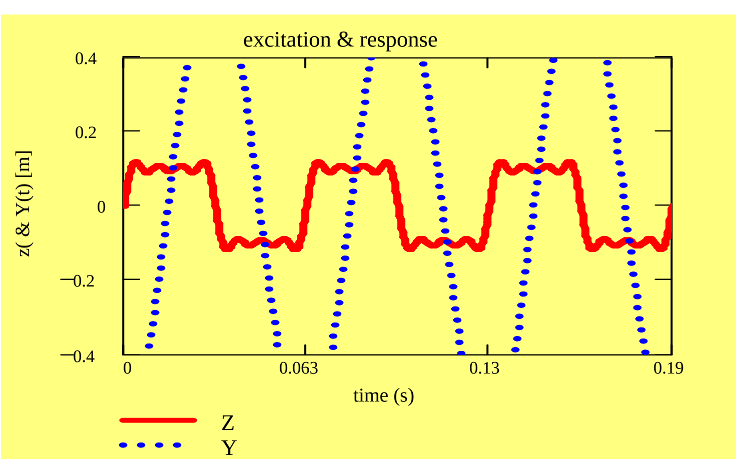

$$\frac{T}{T_d} = 5 \qquad \frac{\Omega}{\omega_n} = 0.199$$



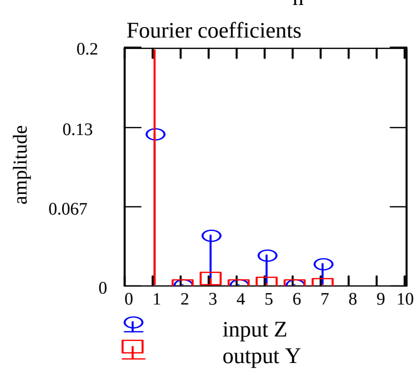



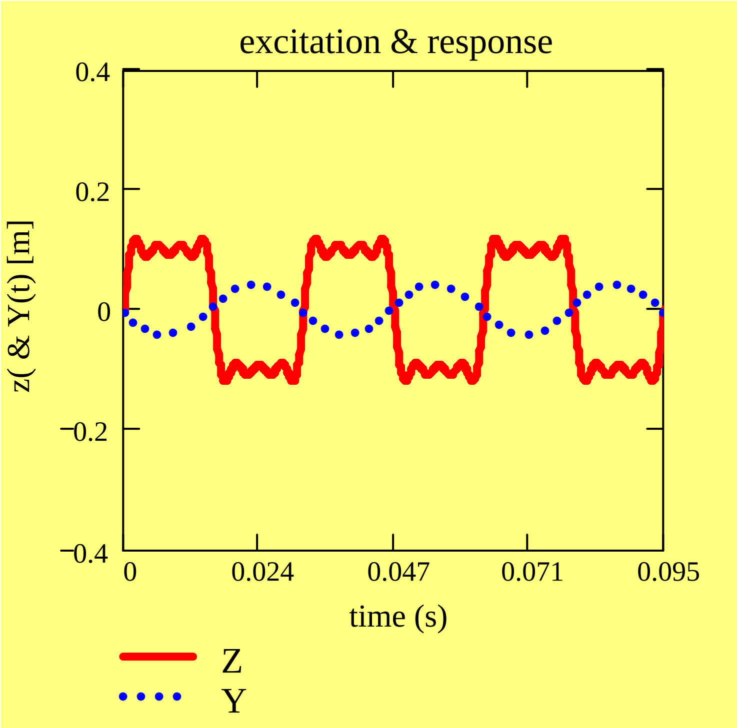

$$\frac{T}{T_d} = 2 \qquad \frac{\Omega}{\omega_n} = 0.497$$



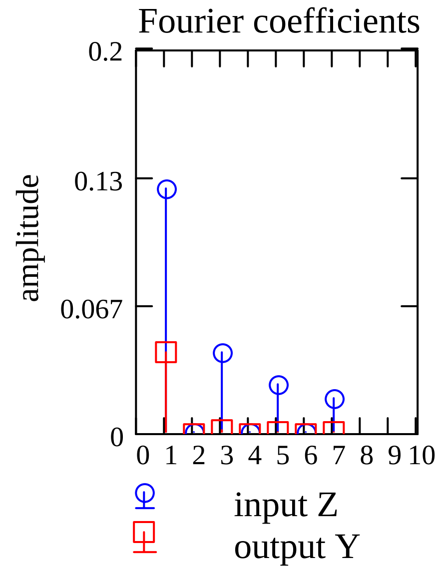




$$\frac{T}{T_d} = 1 \qquad \frac{\Omega}{\omega_n} = 0.995$$

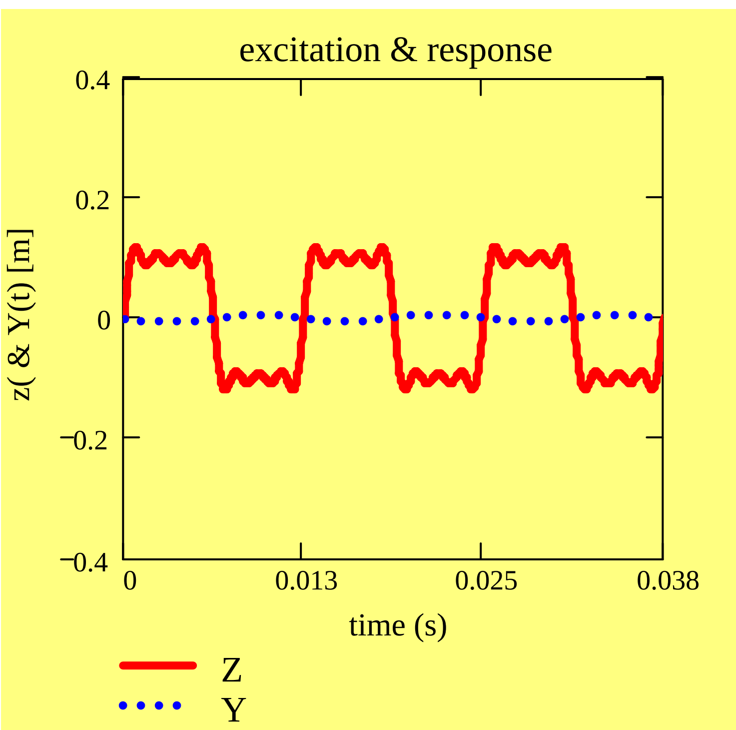




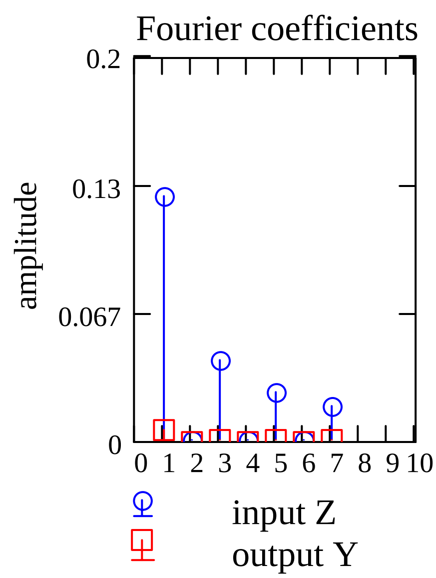

$$\frac{T}{T_d} = 0.5 \qquad \frac{\Omega}{\omega_n} = 1.99$$




================================================ fastest Z (smallest period)




$$\frac{T}{T_d} = 0.2 \qquad \frac{\Omega}{\omega_n} = 4.975$$



# Example: system response due to periodic function

Consider a 2nd order system described by the following EOM

L San Andres (c) 2013

$$M\cdot\frac{d^2}{dt^2}Y + C\cdot\frac{d}{dt}Y + K\cdot Y = K\cdot z(t)$$

ORIGIN := 1

where z (t) is an external periodic excitation displacement function

## Given the system parameters

$M := 100\cdot kg$ $K := 10^6\cdot\frac{N}{m}$ $\zeta := 0.05$

calculate natural frequency and physical damping

$\omega_n := \left(\frac{K}{M}\right)^{0.5}$ $f_n := \frac{\omega_n}{2\cdot\pi}$ $f_n = 15.915\,Hz$

$C := 2\cdot M\cdot\omega_n\cdot\zeta$ $C = 1\times10^3\,s\frac{N}{m}$

$\omega_d := \omega_n\cdot\left(1-\zeta^2\right)^{.5}$ $f_d := \frac{\omega_d}{2\cdot\pi}$ $T_d := \frac{1}{f_d}$ $T_d = 0.063\,s$

damped natural period

## Define periodic excitation function:

**Example - triangular wave**

$z_{mean} := 0.00\cdot m$

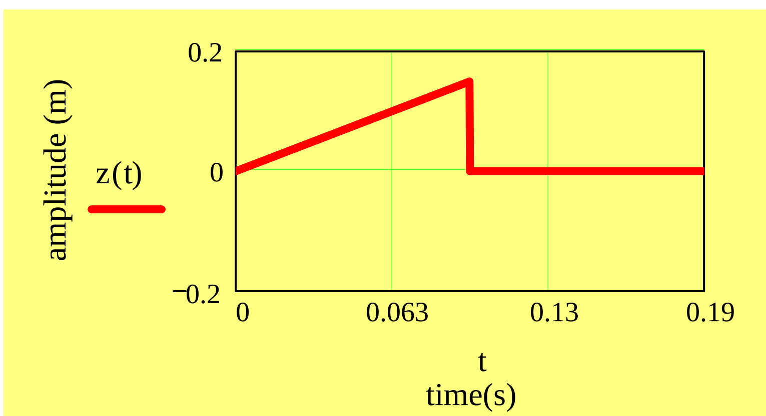
$T := \frac{T_d}{0.333}$ $z_o := 0.1\cdot m$

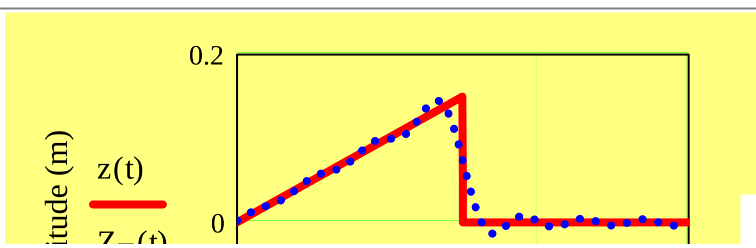
$$z(t) := \begin{array}{|l} amp \leftarrow z_o\cdot t\cdot\frac{3}{T} \text{ if } t < \frac{T}{2} \\ amp \leftarrow 0 \text{ if } t > \frac{T}{2} \\ amp + z_{mean} \end{array}$$

$\Omega := \frac{2\cdot\pi}{T}$ fundamental frequency

$\frac{\Omega}{\omega_d} = 0.333$

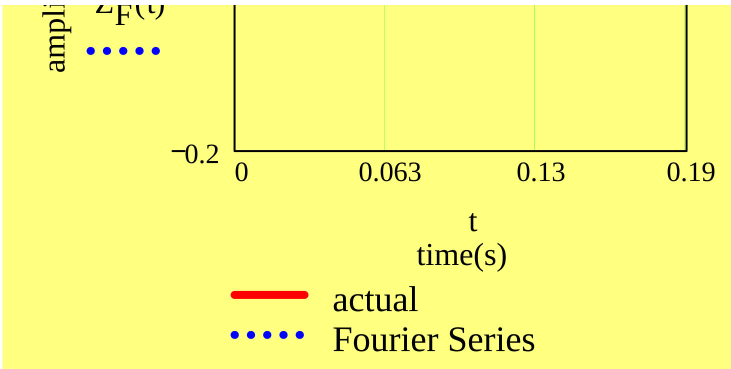
$N_F := 7$ number of Fourier coefficients

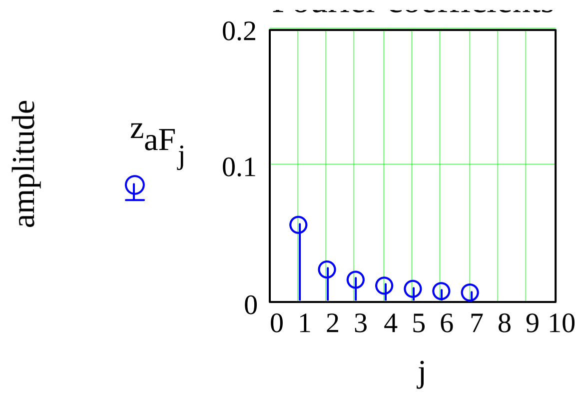


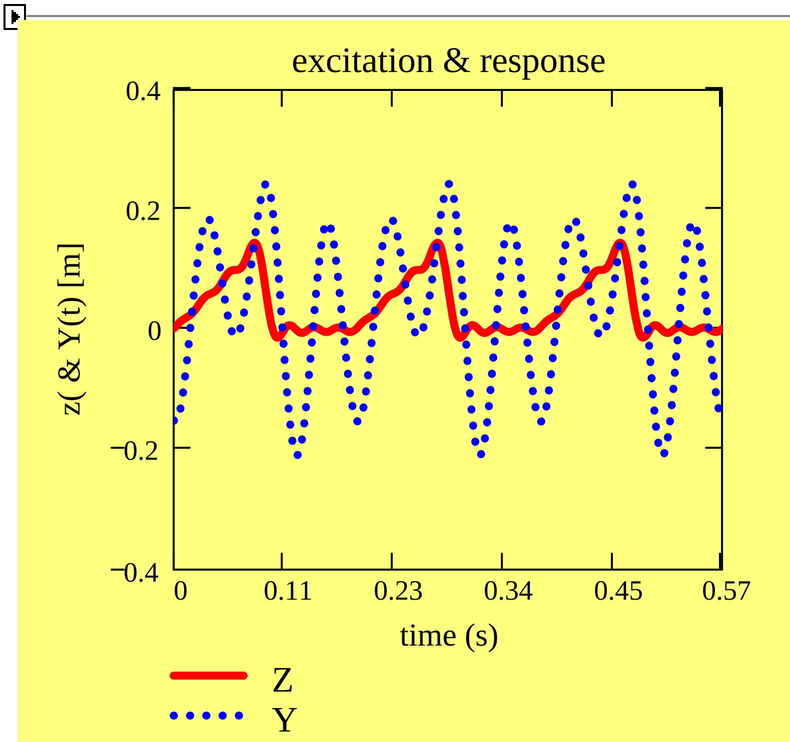




$N_F = 7$

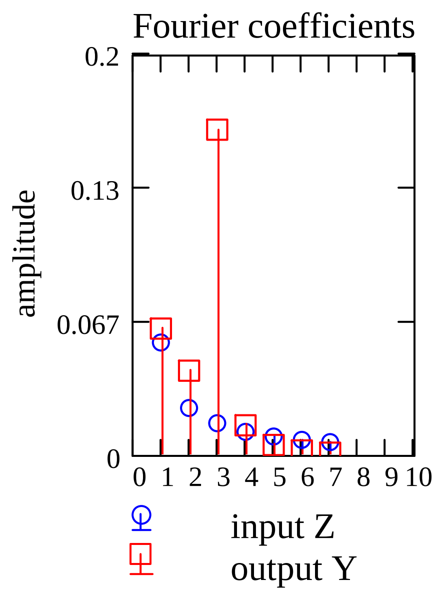
Fourier coefficients




$$\frac{T}{T_d} = 3.003 \qquad \frac{\Omega}{\omega_n} = 0.333$$




3 periods of fundamental excitation motion

**Note**: obtain response for inputs with increasing frequencies (periods decrease)

Notes

# Important design issues and engineering applications of SDOF system Frequency response Functions

The following descriptions show typical questions related to the design and dynamic performance of a second-order mechanical system operating under the action of an external force of periodic nature, i.e. $F_{(t)}=F_o \cos(\Omega t)$ *or* $F_{(t)}=F_o \sin(\Omega t)$

The system EOM is: $M\,\ddot{X} + D\,\dot{X} + K\,X = F_o \cos\left(\Omega t\right)$

Recall that the system response is governed by its parameters, i.e. stiffness ***(K)***, mass ***(M)*** and viscous damping ***(D)*** coefficients. These parameters determine the fundamental natural frequency, $\omega_n = \sqrt{K/M}$, and viscous damping ratio, $\zeta = D/D_c$, with $D_c = 2\sqrt{KM}$

In all design cases below, let $r=(\Omega/\omega_n)$ as the frequency ratio. This ratio (excitation frequency/system natural frequency) largely determines the system periodic forced performance.

## PROBLEM TYPE 1

Consider a system excited by a periodic force of magnitude $\boldsymbol{F_o}$ with external frequency $\boldsymbol{\Omega}$.

a) Determine the damping ratio $\zeta$ needed such that the amplitude of motion does not ever exceed (say) twice the displacement ($X_s=F_o/K$) for operation at a frequency (say) 20% above the natural frequency of the system ($\Omega=1.2\omega_n$).

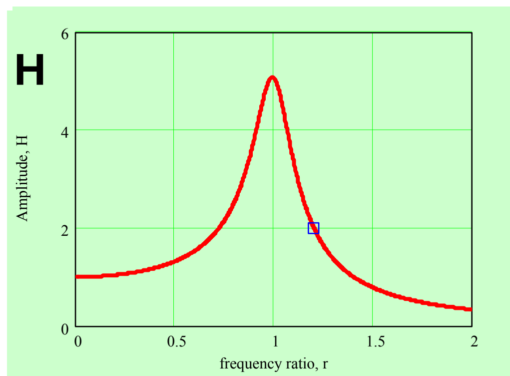
b) With the result of (a), determine the amplitude of motion for operation with an excitation frequency coinciding with the system natural frequency. Is this response the maximum ever expected? Explain.

Recall that system periodic response is

$$X(t)=X_s H_{(r)}\cos(\Omega t+\psi)$$

**Solution.** From the amplitude of FRF

$$\left|\frac{X}{X_s}\right|=H_{(r)}=\frac{1}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$$




Set $r=r_a=1.2$ and $|X/X_s|=H_a=2$.

Find the damping ratio $\zeta$ from the algebraic equation:

$$H_a^2\left(\left(1-r_a^2\right)^2+(2\zeta r_a)^2\right)=1$$

$$(2\zeta r_a)^2=\frac{1}{H_a^2}-\left(1-r_a^2\right)^2$$

$$\Rightarrow \zeta=\frac{1}{2r_a}\left[\frac{1}{H_a^2}-\left(1-r_a^2\right)^2\right]^{1/2}=0.099$$

Finally, calculate the viscous damping coefficient $D=\zeta D_c$

For excitation at the natural frequency, i.e., at resonance, then $r=1$, $|X/X_s|=1/(2\zeta)=Q$. *Thus* $|X|=Q\,X_s$

The maximum amplitude of motion does not necessarily occur at $r$=1. In actuality, the magnitude of the frequency ratio ($r_*$) which maximizes the response, $\left( \partial \left| \frac{X}{X_s} \right| \Big/ \partial r \right) = 0$, is (after some algebraic manipulation):

$$r_* = \sqrt{\left(1 - 2\zeta^2\right)}; \text{ and } \left|\frac{X}{X_s}\right|_{\max} = \frac{1}{2\zeta}\frac{1}{\sqrt{\left(1-\zeta^2\right)}}$$

Corrected 2/19/13

Note that for small values of damping $\left|\frac{X}{X_s}\right|_{\max} \approx \frac{1}{2\zeta}$

# PROBLEM TYPE 2

Consider a system exited by an imbalance ($u$), giving an amplitude of force excitation equal to $F_o = M\,u\,\boldsymbol{\Omega}^2$. Recall that $u = m\,e/M$, where $m$ is the imbalance mass and $e$ is its radial location

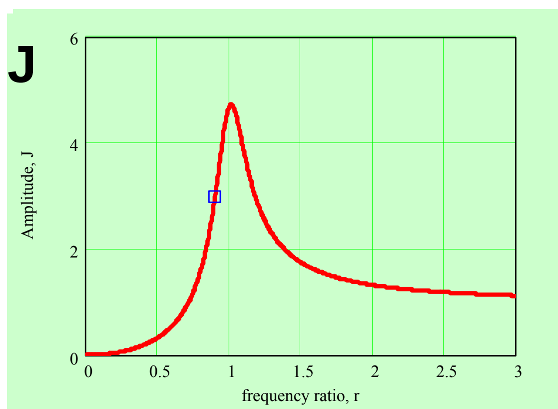
$$M\,\ddot{X} + D\,\dot{X} + K\,X = M\,u\,\Omega^2 \cos(\Omega t)$$

Recall that system periodic response is

$$X(t) = u\,J_{(r)} \cos(\Omega t + \psi)$$

a) What is the value of damping $\zeta$ necessary so that the system response never exceeds (say) three times the imbalance ***u*** for operation at a frequency (say) 10% below the natural frequency of the system ($\Omega = 0.9\omega_n$).

b) With the result of (a), determine the amplitude of motion for operation with an excitation frequency coinciding with the system natural frequency.




**Solution** From the fundamental FRF amplitude ratio

$$\left|\frac{X}{u}\right| = J_{(r)} = \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta\,r)^2}}$$

Set $r = 0.9$ and $|X/u| = J_a = 3$. Calculate the damping ratio $\zeta$ from the algebraic equation.

$$\Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{r_a^4}{J_a^2} - \left(1 - r_a^2\right)^2\right]^{1/2} = 0.107$$

Finally, calculate the viscous damping coefficient, $D = \zeta D_c$.

Note that for forced operation with frequency = natural frequency, i.e., at resonance,

$r=1$, $|X/u| = 1/(2\zeta) = Q$. *Thus* $|X| = Q\,u$

The maximum amplitude of motion does not occur at $r=1$. The value of frequency ratio ($r_*$) which maximizes the response is obtained from

$$\left(\frac{\partial \left|\frac{X}{u}\right|}{\partial r}\right) = 0 \text{ then}$$

$$r_* = \frac{1}{\sqrt{\left(1 - 2\zeta^2\right)}}; \text{ and } \left|\frac{X}{u}\right|_{\max} = \frac{1}{2\zeta}\frac{1}{\sqrt{\left(1 - \zeta^2\right)}}$$ corrected 2/19/13

Note that for small values of damping $\left|\frac{X}{u}\right|_{\max} \approx \frac{1}{2\zeta}$

# PROBLEM TYPE 3

Consider a system excited by a periodic force of magnitude $F_o$ and frequency $\Omega$. Assume that the spring and dashpot connect to ground.

a) Determine the damping ratio needed such that the **transmitted force** to ground does not ever exceed (say) two times the input force for operation at a frequency (say) = 75% of natural frequency $(\Omega=0.75\omega_n)$.

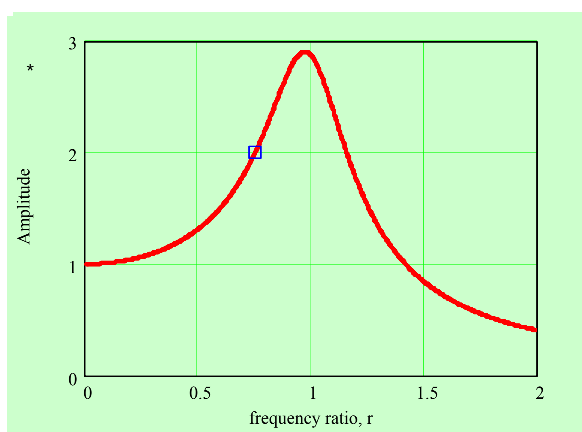
b) With the result of (a), determine the transmitted force to ground if the excitation frequency coincides with the system natural frequency. Is this the maximum transmissibility ever?

c) Provide a value of frequency such that the transmitted force is less than the applied force, irrespective of the damping in the system.

**Solution:** From the fundamental FRF amplitude for a base force excitation $F_{transmitted} = K\,X + D\,\dot{X}$

$$\left|\frac{F_{transmitted}}{F_o}\right| = A_{T(r)} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

Set $A_T$=2 and $r$=0.75, and find the damping ratio $\zeta$.




$$\Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{1-A_T^2\left(1-r_a^2\right)^2}{A_T^2-1}\right]^{½}$$

=0.186

Finally, calculate the viscous damping coefficient $D = \zeta D_c$

**At resonance**, $r=1$, $A_T = \frac{\left[1+(2\zeta)^2\right]^{.5}}{2\zeta}$. Then calculate the magnitude of the transmitted force.

Again, the maximum transmissibility occurs at a frequency $f_*$ which satisfies $\left( \partial A_T / \partial r \right) = 0$. Perform the derivation and find a closed form solution.

Recall that operation at frequencies $r \geq \sqrt{2}$, i.e. for $\Omega \geq 1.414 \omega_n$, (41 % above the natural frequency) determines transmitted forces that are lower than the applied force (i.e. an **effective structural isolation** is achieved).

## PROBLEM TYPE 4

Consider a system excited by a periodic force with magnitude $F_o = M\, a_{cc}$ (for example) and frequency $\Omega$.

a) Determine the damping ratio $\zeta$ needed such that the maximum acceleration in the system does not exceed (say) **4 g**'s for operation at a frequency (say) 30% above the natural frequency of the system ($\Omega=1.3\omega_n$).

b) With the result of (a), determine the system acceleration for operation with an excitation frequency coinciding with the system natural frequency. Explain your result

Recall the periodic response is $X(t) = X_s H_{(r)} \cos(\Omega t + \psi)$, then the acceleration of the system is

$$\ddot{X}(t) = -\Omega^2 X_s H_{(r)} \cos(\Omega t + \psi) = -\Omega^2 X(t)$$

**Solution:** From the amplitude of FRF

$$\left|\frac{\ddot{X}}{F_o / K}\right| = \frac{\omega_n^2 r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}} \rightarrow \left|\frac{\ddot{X}}{F_o / M}\right| = \frac{r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}}$$

Follow a similar procedure as in other problems above.

## OTHER PROBLEMS

Think of similar problems and questions related to system dynamic performance.

In particular, you may also "cook up" similar questions related to the dynamic response of **first-order systems** (mechanical, thermal, electrical, etc). $M\dot{V} + DV = F_o \cos(\Omega t)$

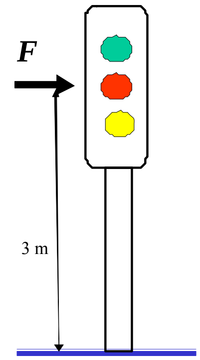
**Luis San Andrés - MEEN 363/617 instructor**

**The following worked problems should teach you how to apply the frequency response function to resolve issues and to design many mechanical systems**

## P2. Periodic forced response of a SDOF mechanical system. DESIGN COMPONENT

The signal lights for a rail may be modeled as a 176 lb mass mounted 3 m above the ground of an elastic post. The natural frequency of the system is measured to be 12.2 Hz. Wind buffet generates a horizontal harmonic force at 12 Hz. The light filaments will break if their peak accelerations exceed *15g*. Determine the maximum acceptable force amplitude $|F|$ when the damping ratio $\zeta$=0.0 and 0.01.

Full grade requires you to explain the solution procedure with due attention to physical details




The excitation force is periodic, say F(t)=Fo sin($\omega$t). then the system response will also be periodic, Y(t), with same frequency as excitation. Assuming steady state conditions:

## STEADY RESPONSE of M-K-C system to PERIODIC Force with frequency $\omega$

**Case: periodic force of constant magnitude** Define operating frequency ratio: $r = \dfrac{\omega}{\omega_n}$

$$F(t) = F_o \cdot \sin(\omega \cdot t)$$

System periodic response: $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t + \Psi)$ (1)

where:

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{-2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

From (1), the acceleration is $a(t) = -\omega^2 \cdot Y(t) = A \cdot \sin(\omega(t + \Psi - 180))$

the magnitude of acceleration is $A = \dfrac{F_o}{K_e} \cdot \dfrac{\omega^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$ or $A = \dfrac{F_o}{M_e} \cdot \dfrac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$

hence, define $A_{max} := 10 \cdot g$ maximum allowed acceleration of filament

$$HZ := 2 \cdot \pi \cdot \frac{1}{s}$$

system mass $M_e := 150 \cdot lb$

$f_n := 12 \cdot HZ$ natural frequency

$f := 11.5 \cdot HZ$ excitation frequency due to wind buffets

Let $r_o := \dfrac{f}{f_n}$ $r_o = 0.958$ close to natural frequency

The maximum force allowed equals

$$F_{max}(r, \zeta) := A_{max} \cdot M_e \cdot \frac{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}{r^2}$$

without any damping $F_{max}(r_o, 0) = 133.27\,lbf$ Note the importance of damping that leads to a substantial increase in force allowed

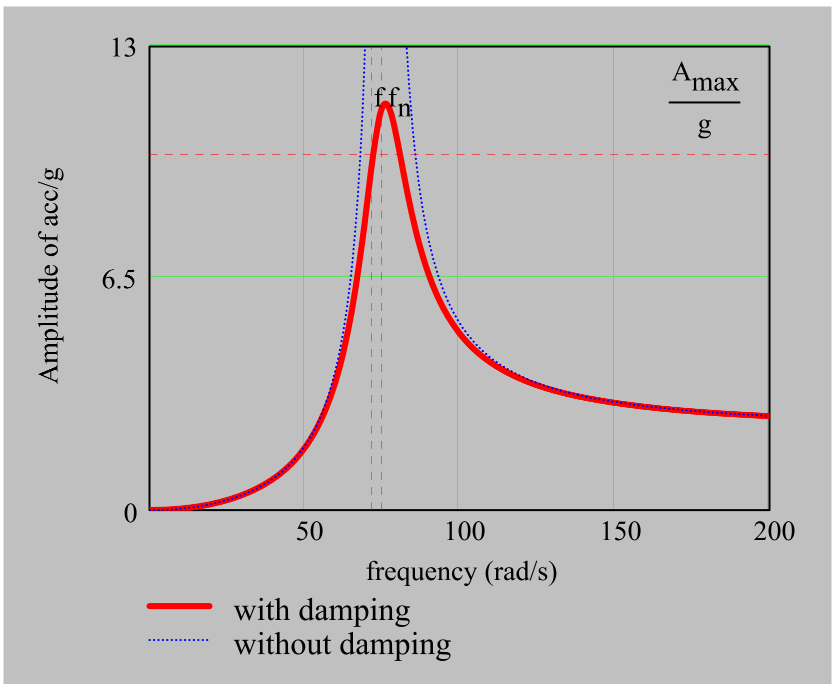
with damping $\xi := 0.1$ $F_{max}(r_o, \xi) = 340.231\,lbf$

$$\frac{F_{max}(r_o, \xi)}{F_{max}(r_o, 0)} = 2.553$$

For the force found the amplitude of acceleration is

$F_o := F_{max}(r_o, \xi)$

$$A(r, \zeta) := \frac{F_o}{M_e} \cdot \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}}$$




$f_n = 75.398 \frac{1}{s}$

$\frac{1}{2 \cdot \xi} = 5$

GRAPH NOT FOR EXAM

SInce $r_o$~1, a simpler enginering formula gives $A_{max} \cdot M_e \cdot 2 \cdot \xi = 300\,lbf$

which gives a very good estimation of the maximum wind force allowed

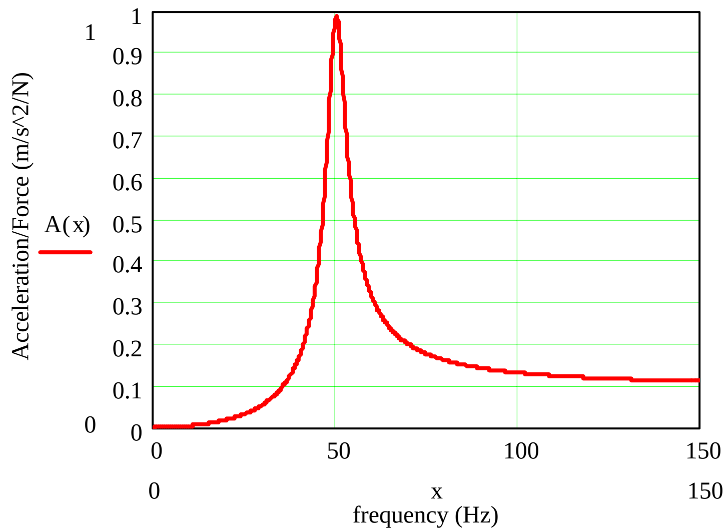
b) a system with damping ξ=0.1 will produce a 255 % increase in allowable force
Hence, the rail lightsystem will be more reliable, lasting longer.

$$\frac{\quad}{F_{max}(r_o, 0)} = 2.553$$

c) Posts are usually hollow for the cables to be routed. These posts have layers of elastomeric material (~rubber-like) inside to increase their structural damping. Modern posts are wound up fro composites that integrate damping layers. Clearly, adding a "true" dashpot is not cost-effective

**EXAMPLE - EXAM 2 TYPE:**

Dynamic measurements were conducted on a mechanical system to determine its FRF (frequency response function). Forcing functions with multiple frequencies were exerted on the system and a digital signal analyzer (FFT) recorded the magnitude of the ACCELERATION/FORCE ([m/s2]/N) Frequency Response Function, as shown below. From the recorded data determine the system parameters, i.e. natural frequency (wn:rad/s) and damping ratio (z), and system stiffness (K:N/m), mass (M:kg), and viscous damping coefficient (C:N.s/m).
Explain procedure of ANALYSIS/INTERPRETATION of test data for full credit.




Test data showing amplitude of (acceleration/force)

Magnitude of FRF for mechanical system

**Solution:**

Recall that for an imposed external force of periodic form: $F(t) = F_o \cdot \sin(\omega t)$ [1]

the system response Y(t) is given by: $Y(t) = Y_{op} \cdot \sin(\omega t + \psi)$ [2]

where the amplitude of motion (Yop) and phase angle (Y) are defined as:

$$Y_{op}(r) = \frac{\frac{F_o}{K}}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \quad [3a] \qquad \Psi = -\operatorname{atan}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right) \quad [3b]$$

with $r = \frac{\omega}{\omega_n}$ [4]

from [2], we find that the acceleration is given by:

$$a_Y(t) = -\omega^2 \cdot Y_{op} \cdot \sin(\omega t + \psi) = a_{op} \cdot \sin(\omega t + \Psi - 180) \quad [5]$$

where:

$$a_{op}(r) = \frac{\frac{F_o}{M} \cdot r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \quad [6]$$

since: $\frac{F_o}{K} \cdot \omega^2 = \frac{F_o}{M} \cdot \frac{\omega^2}{\omega_n^2} = \frac{F_o}{K} \cdot r^2$

thus, the magnitude of amplitude of acceleration over force amplitude follows as:

$$\frac{a_{op}(r)}{F_o} = \frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}\cdot\frac{1}{M} \qquad [7]$$

The units of this expression are 1/kg = $\dfrac{m}{\frac{s^2}{N}}$

For excitation at very high frequencies, r>>1.0 $\quad \dfrac{1}{M} \leftarrow \dfrac{a_{op}(r)}{F_o}$

From the graph (test data): $\quad \dfrac{1}{M} = 0.1\cdot\left(\dfrac{m}{s^2\cdot N}\right)$ $\quad$ Thus $\quad M := 10\cdot kg$

The system appears to have little damping, i.e. amplitude of FRF around a frequency of 50 Hz is rather large and varying rapidly over a narrow frequency range.

Thus, take the natural frequency as $\quad f_n := 50\cdot Hz$

expressed in rad/s as: $\quad \omega_n := f_n\cdot 2\cdot\pi \qquad \omega_n = 314.159\,\dfrac{rad}{s}$

We can estimate the stiffness (K) from the fundamental relationship:

$$K := \omega_n{}^2\cdot M \qquad K = 9.87\times 10^5\,\frac{N}{m}$$

for excitation at the natural frequency (r=1), the ratio of amplitude of acceleration to force reduces to

$$\frac{a_{op}(1)}{f_o} = \frac{1}{2\cdot M\cdot\zeta}$$

from the graph (test data), the ratio is approximately equal to one (1/kg). Thus. the damping ratio is determined as

$$\zeta := \frac{1}{2\cdot M\cdot\left(\frac{1.0}{kg}\right)} \qquad \zeta = 0.05$$

That is, the system has a damping ratio equal to 5%. This result could have also been easily obtained by studying the ratio of (amplitude at the natural frequency divided by the amplitude at very high frequency, i.e.)

$$\frac{1}{2\cdot\zeta} = \frac{1}{0.1} = 10$$

Once the damping ratio is obtained, the damping coefficient can be easily determined from the formula:

$$C := \zeta\cdot 2\cdot M\cdot\omega_n \qquad C = 314.159\,N\cdot\frac{s}{m}$$

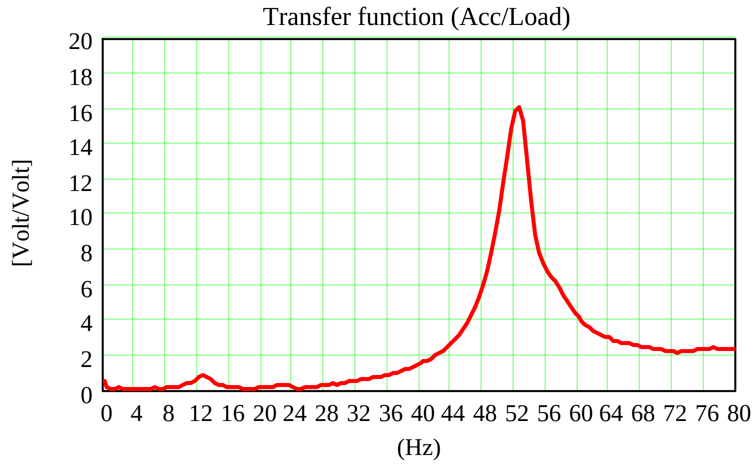
The number of calculations is minimal. One needs to interpret correctly the test data results, however.

# P1: Identification of parameters from FRF

The graph below shows the amplitude of ACCELERANCE (acceleration/force) versus frequency (Hz) for shaker load tests conducted on a rotor at the Laboratory. In the measurements, a force transducer with a gain of 0.87milliVolt/lbf and a piezoelectric accelerometer with a gain of 0.101 Volt/g were used. The graph shows the amplitude of the accelerance function in units of [Volt/Volt].

a) Determine the appropriate relation to convert the results given into appropriate physical units. [10]

b) Estimate the natural frequency [rad/s] and damping ratio ($\zeta$) of the system, the stiffness K [$lb_f$/in], mass M [lb] and viscous damping D [$lb_f$.sec/in] coefficients. [20]

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**(a) Graph shows ratio of acceleration in volts to force in volts. To convert amplitudes in graph to proper physical units:**

$$Gain\_loadcell := 0.87\cdot 10^{-3}\cdot\frac{volt}{lbf} \qquad Gain\_accel := 101\cdot 10^{-3}\cdot\frac{volt}{g}$$

$$Convert\_VV\_to\_physical := \frac{1\cdot volt}{Gain\_accel}\cdot\frac{1}{\left(\frac{1\cdot volt}{Gain\_loadcell}\right)} \qquad g = 32.174\frac{ft}{s^2}$$

$$Convert\_VV\_to\_physical = 0.277\frac{ft}{sec^2\cdot lbf}$$ OR $$Convert\_VV\_to\_physical = 3.326\frac{in}{sec^2\cdot lbf}$$

**(b) the amplitude of FRF for accelerance function equals**:

$$\frac{a}{F} = \frac{\omega^2}{\left[\left(K - M\cdot\omega^2\right)^2 + (C\cdot\omega)^2\right]^{.5}}$$

b.1) the system inertia or mass (M) is determined from the accelerance function at high frequency, i.e.

$$\left|\frac{a}{F}\right| = \frac{1}{M}$$

From graph, at frequency of 80 Hz:

$$a\_F := 2.1\cdot\frac{volt}{volt}\cdot Convert\_VV\_to\_physical$$

$$M := \frac{1}{a\_F} \qquad M = 55.282\,lb$$

**Engineering considerations**

* narrow zone for peak vibration means little damping,

** from graph the natural frequency is about

$f_n := 52 \cdot Hz$

Thus, $\omega_n := f_n \cdot 2 \cdot \pi$ $\omega_n = 326.726 \frac{rad}{sec}$

b.2) Thus the system equivalent mass equals: $K := M \omega_n^2$ $K = 1.528 \times 10^4 \frac{lbf}{in}$

b.3) at the natural frequency, the magnitude of the acceleration/force function is inversely proportional to the damping coefficient, i.e.at 52 Hz graph shows

$$\left| \frac{a}{F} \right| = \frac{\omega_n}{D}$$

$$a\_F := 16 \cdot \frac{volt}{volt} \cdot Convert\_VV\_to\_physical$$

$$D := \frac{\omega_n}{a\_F} \qquad D = 6.14\, lbf \cdot \frac{sec}{in}$$

b.4) Damping ratio follows from

$$\zeta := \frac{D}{2 \cdot (K \cdot M)^{.5}} \qquad \zeta = 0.066$$

damping ratio is small justifying assumption of $\omega_n = \omega_d$ $\left(1 - \zeta^2\right)^{.5} = 0.998$

Another way to obtain the damping is by using the Q-factor. In this case, it is equal to the amplitude at the natural frequency divided by the amplitude at high frequency, i.e.

$$Q := \frac{16}{2.1} \qquad Q = 7.619$$

$$\zeta := \frac{1}{2 \cdot Q} \qquad \zeta = 0.066$$

$$D := \zeta \cdot 2 \cdot (K \cdot M)^{.5}$$

$$D = 6.14\, lbf \cdot \frac{sec}{in}$$

# MEEN 363/459/617/659: Response of SDOF mechanical system

EOM: $M_e \ddot{y} + K_e y + C_e \dot{y} = F(t)$

Luis San Andres (c)

Given a mechanical system with equivalent parameters (Me) mass, (Ke) stiffness, (Ce) viscous damping coefficient.

define natural frequency and damping ratio as: $\omega_n = \sqrt{\frac{K_e}{M_e}}$ $\quad \zeta = \frac{C_e}{2\cdot M_e\cdot\omega_n} = \frac{C_e}{2\cdot\sqrt{K_e\cdot M_e}} = \frac{C_e\cdot\omega_n}{2\cdot K_e}$

## TRANSIENT RESPONSE system to STEP Force F(t)=Fo

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = F_o$$

### Underdamped system only, $\zeta<1$

+ initial conditions $Y_0 = Y(0)$ $\quad V_0 = \frac{d}{dt}Y$ at t=0

system response is: $Y(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right) + Y_{ss}$

where $Y_{ss} = \frac{F_o}{K_e}$ $\quad C_1 = Y_0 - Y_{ss}$ $\quad C_2 = \frac{\left[V_0 + \zeta\cdot\omega_n\cdot(Y_0 - Y_{ss})\right]}{\omega_d}$ and $\omega_d = \omega_n\cdot\sqrt{1-\zeta^2}$

damped natural frequency

for $\zeta>=1$: see page 2)

**LOG DEC ($\delta$):** formula to estimate damping ratio ($\zeta$) from a free response

$$\delta = \frac{1}{n}\cdot\ln\left(\frac{Ao}{An}\right) = \frac{2\pi\cdot\zeta}{\sqrt{1-\zeta^2}} \qquad \zeta = \frac{\delta}{\sqrt{4\cdot\pi^2+\delta^2}}$$

Ao and An are peak motion amplitudes separated by n periods

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## TRANSIENT RESPONSE system to Force F(t)=A+Bt

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = A + B\cdot t$$

$$Y = a + b\cdot t + e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right)$$

for $\zeta < 1$

$a = \left(A - C_e\cdot\frac{B}{K_e}\right)\cdot\frac{1}{K_e}$ ; $b = \frac{B}{K_e}$ ; $\quad C_1 = Y_0 - a$ $\quad C_2\cdot\omega_d = V_0 - b + \zeta\cdot\omega_n\cdot(Y_0 - a)$

## Transient response of **overdamped system**, step force Fo=constant $\zeta>1$

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = F_o$$

+ initial conditions $Y_0 = Y(0)$ $\quad V_0 = \frac{d}{dt}Y$ at t=0

$Y(t) = A_1\cdot e^{s_1\cdot t} + A_2\cdot e^{s_2\cdot t} + Y_{ss}$ where: $s_1 = \omega_n\cdot\left(-\zeta + \sqrt{\zeta^2-1}\right)$ $\quad s_2 = \omega_n\cdot\left(-\zeta - \sqrt{\zeta^2-1}\right)$

$Y_{ss} = \frac{F_o}{K_e}$ $\quad A_1 + A_2 = Y_0 - Y_{ss}$ $\quad (A_1\cdot s_1 + A_2\cdot s_2) = V_0$ **Solve for A1 and A2** $\quad$ s1, s2 < 0

## Transient response of **critically damped system**, step force Fo=constant $\zeta=1$

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = F_o$$

+ initial conditions $Y_0 = Y(0)$ $\quad V_0 = \frac{d}{dt}Y$ at t=0

$Y(t) = e^{s\cdot t}\cdot(A_1 + t\cdot A_2) + Y_{ss}$ where: $s = -\omega_n$

$Y_{ss} = \frac{F_o}{K_e}$ $\quad A_1 = Y_0 - Y_{ss}$ $\quad (A_1\cdot s + A_2) = V_0$

## STEADY RESPONSE of system to PERIODIC LOADS with frequency $\omega$

### Case: periodic force of constant magnitude

$F(t) = F_o \cdot \sin(\omega \cdot t)$

Define operating frequency ratio:

$$r = \frac{\omega}{\omega_n}$$

System periodic response: $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t - \Psi)$

where:

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to 180deg

### Case: base motion of constant amplitude

$$M_e \cdot \frac{d^2}{dt^2} Y + C_e \cdot \frac{d}{dt} Y + K_e \cdot Y = K \cdot Y_B + C_e \cdot \frac{d}{dt} Y_B$$

where

$Y_B(t) = A \cdot \sin(\omega \cdot t)$

System periodic response:

$$Y(t) = A \cdot G(r) \cdot \sin(\omega \cdot t - \Psi - \phi)$$

where:

$$G(r) = \left[\frac{1 + (2 \cdot \zeta \cdot r)^2}{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2}\right]^{.5} \qquad \tan(\Psi + \phi) = \frac{2 \cdot \zeta \cdot r^3}{1 + 4 \cdot \zeta^2 - r^2}$$

### Case: response to mass imbalance

$F(t) = m \cdot e \cdot \omega^2 \cdot \sin(\omega \cdot t)$

u=imbalance (offset center of mass) displacement

System periodic response: $Y(t) = e \cdot \frac{m}{M_e} J(r) \cdot \sin(\omega \cdot t - \Psi)$

$M_e = M + m$

$m \cdot e = M_e \cdot u$

$$J(r) = \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

## OTHER USEFUL formulas: (program them in a calculator)

### Underdamped system $\zeta$<1: step force response

constants in formulas

Given

$$Y(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(C_1 \cdot \cos\left(\omega_d \cdot t\right) + C_2 \cdot \sin\left(\omega_d \cdot t\right)\right) + Y_{ss}$$

**find velocity** $V(t) = \frac{d}{dt} Y$

$$V(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(D_1 \cdot \cos\left(\omega_d \cdot t\right) + D_2 \cdot \sin\left(\omega_d \cdot t\right)\right)$$

where: $D_1 = \left(-\zeta \cdot \omega_n \cdot C_1\right) + C_2 \cdot \omega_d \qquad D_2 = \left(-\zeta \cdot \omega_n \cdot C_2\right) - C_1 \cdot \omega_d$

**find acceleration**

$a(t) = \frac{d}{dt} V$

$$a(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(E_1 \cdot \cos\left(\omega_d \cdot t\right) + E_2 \cdot \sin\left(\omega_d \cdot t\right)\right)$$

where: $E_1 = \left(-\zeta \cdot \omega_n \cdot D_1\right) + D_2 \cdot \omega_d \qquad E_2 = \left(-\zeta \cdot \omega_n \cdot D_2\right) - D_1 \cdot \omega_d$

## A case your memory should retain forever

## NO DAMPING, C=0 Ns/m

**TRANSIENT RESPONSE of M-K system to STEP Force F(t)=Fo**

Undamped system, $\zeta=0$

$$M_e \cdot \frac{d^2}{dt^2}Y + K_e \cdot Y = F_o$$

+ initial conditions $Y_0 = Y(0)$ $V_0 = \frac{d}{dt}Y$ at t=0

response is:

$$Y(t) = \left(C_1 \cdot \cos(\omega_n \cdot t) + C_2 \cdot \sin(\omega_n \cdot t)\right) + Y_{ss}$$

where $Y_{ss} = \frac{F_o}{K_e}$ $C_1 = Y_0 - Y_{ss}$ $C_2 = \frac{V_0}{\omega_n}$

MOTION never dies since there is no dissipation action (no damping)

and velociy and acceleration:

$$V(t) = \frac{d}{dt}Y \qquad V(t) = \left(D_1 \cdot \cos(\omega_n \cdot t) + D_2 \cdot \sin(\omega_n \cdot t)\right)$$

$D_1 = C_2 \cdot \omega_n$ $D_1 = -C_1 \cdot \omega_n$

$$a(t) = \frac{d}{dt}V \qquad a(t) = \left(E_1 \cdot \cos(\omega_n \cdot t) + E_2 \cdot \sin(\omega_n \cdot t)\right)$$

$E_1 = -C_1 \cdot \omega_n^2$ $E_2 = -C_2 \cdot \omega_n^2$

Note that the velocity and acceleration superimpose a cos & a sin functions. Thus, the maximum values of velocity and acceleration equal

$$V_{max} = \omega_n \cdot \sqrt{C_1^2 + C_2^2}$$

$$a_{max} = \omega_n^2 \cdot \sqrt{C_1^2 + C_2^2} = \omega_n \cdot V_{max}$$

since

$Y_{ss} = \frac{F_o}{K_e}$ $C_1 = Y_0 - Y_{ss}$ $C_2 = \frac{V_0}{\omega_n}$

$$a_{max} = \omega_n^2 \cdot \sqrt{(Y_0 - Y_{ss})^2 + \left(\frac{V_0}{\omega_n}\right)^2}$$

Note: the function $x(t) = a \cdot \cos(\omega \cdot t) + b \cdot \sin(\omega \cdot t)$ can be written as

$$x(t) = c \cdot \cos(\omega \cdot t - \phi)$$

where $c = \sqrt{a^2 + b^2}$ $\tan(\phi) = \frac{b}{a}$

## OTHER important information

given a function $f(t)$ find its maximum value

The maximum or mimimum values are obtained from $\frac{d}{dt}f = 0$

For example, for the underdamped response, $\zeta$<1, the system response for a step load is

$$Y(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right) + Y_{ss}$$

where $Y_{ss} = \frac{F_o}{K_e}$ $\quad C_1 = Y_0 - Y_{ss}$ $\quad C_2 = \frac{\left[V_0 + \zeta\cdot\omega_n\cdot(Y_0 - Y_{ss})\right]}{\omega_d}$ and $\omega_d = \omega_n\cdot\sqrt{1-\zeta^2}$

damped natural frequency

**when does Y(t) peak (max or min) ?**

from the formulas sheet $\quad V(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(D_1\cdot\cos(\omega_d\cdot t) + D_2\cdot\sin(\omega_d\cdot t)\right)$

$$D_1 = (-\zeta\cdot\omega_n\cdot C_1) + C_2\cdot\omega_d \qquad D_2 = (-\zeta\cdot\omega_n\cdot C_2) - C_1\cdot\omega_d$$

A peak value occurs at time t=$\tau$ when dY/dt=V=0, i.e.

$$0 = e^{-\zeta\cdot\omega_n\cdot\tau}\cdot\left(D_1\cdot\cos(\omega_d\cdot\tau) + D_2\cdot\sin(\omega_d\cdot\tau)\right)$$

$e^{-\zeta\cdot\omega_n\cdot\tau} \neq 0$ for most times; hence $\quad 0 = D_1\cdot\cos(\omega_d\cdot\tau) + D_2\cdot\sin(\omega_d\cdot\tau)$

$\tan(\omega_d\cdot\tau) = \frac{-D_1}{D_2}$ $\qquad$ solve this equation to find $\tau$

there are an infinite # of time values ($\tau$) satisfying the equation above. Select the lowest $\tau$ as this will probaly will give you the largest peak.

## Example:

$$M_e\cdot\frac{d^2}{dt^2}Y + C_e\cdot\frac{d}{dt}Y + K_e\cdot Y = A + B\cdot t$$

Obtain constants **C1** and **C2** for case of force F(t)=A+Bt - underdamped response

Given $Y(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(C_1\cdot\cos(\omega_d\cdot t) + C_2\cdot\sin(\omega_d\cdot t)\right) + a + b\cdot t$

$V(t) = \frac{d}{dt}Y$ $\qquad V(t) = e^{-\zeta\cdot\omega_n\cdot t}\cdot\left(D_1\cdot\cos(\omega_d\cdot t) + D_2\cdot\sin(\omega_d\cdot t)\right) + b$

$$D_1 = (-\zeta\cdot\omega_n\cdot C_1) + C_2\cdot\omega_d \qquad D_2 = (-\zeta\cdot\omega_n\cdot C_2) - C_1\cdot\omega_d$$

at t=0, $\quad Y_0 = Y(0) \quad V_0 = \frac{d}{dt}Y$ at t=0

satisfy initial conditions: $\quad Y_0 = Y(0) \qquad Y_0 = C_1 + a \qquad C_1 = Y_0 - a$

$$V_0 = \frac{d}{dt}Y \qquad V_0 = D_1 + b = (-\zeta\cdot\omega_n\cdot C_1) + C_2\cdot\omega_d + b$$

$$V_0 = \left[-\zeta\cdot\omega_n\cdot(Y_0 - a)\right] + C_2\cdot\omega_d + b \qquad C_2 = \frac{V_0 - b + \zeta\cdot\omega_n\cdot(Y_0 - a)}{\omega_d}$$

## FREQUENCY RESPONSE FUNCTIONS for PERIODIC LOAD with frequency $\omega$

**Case: periodic force of constant magnitude**

$$M_e \cdot \frac{d^2}{dt^2}Y + C_e \cdot \frac{d}{dt}Y + K_e \cdot Y = F_o \cdot \sin(\omega \cdot t)$$

Define operating frequency ratio: $r = \frac{\omega}{\omega_n}$

System periodic responses:

**Displacement:** $Y(t) = \delta s \cdot H(r) \cdot \sin(\omega \cdot t - \Psi)$

**velocity:** $V(t) = \frac{d}{dt}Y = \delta s \cdot \omega \cdot H(r) \cdot \sin(\omega \cdot t - \Psi)$

**acceleration:** $a(t) = \frac{d^2}{dt^2}Y = -\delta s \cdot \omega^2 \cdot H(r) \cdot \sin(\omega \cdot t - \Psi) = -\omega^2 \cdot Y$

where:

harmonic response

$$\delta_s = \frac{F_o}{K_e} \qquad H(r) = \frac{1}{\left[\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2\right]^{.5}} \qquad \tan(\Psi) = \frac{2 \cdot \zeta \cdot r}{1 - r^2}$$

care with angle, range: 0 to -180deg

Define **dimensionless amplitudes of frequency response function for**

**displacement** $\frac{|Y|}{\delta_s} = H(r)$

**velocity** $\frac{|V| \cdot \omega_n}{\delta_s} = r \cdot H(r)$

**acceleration** $\frac{|a| \cdot \omega_n^2}{\delta_s} = r^2 \cdot H(r) = J(r)$

# GRAPHS for AMPLITUDE OF TRANSFER functions

## Constant amplitude force

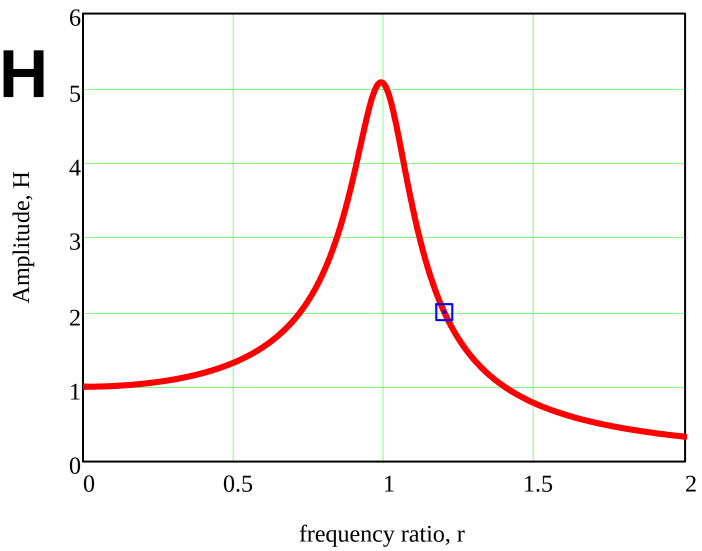
$$H(r,\zeta) := \frac{1}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}$$

Given desired H and freq r

**formula for damping ratio** $H_a := 2$ $r_a := 1.2$

$$\zeta := \frac{1}{r_a\cdot 2}\cdot\left[\frac{1}{H_a^{\,2}} - \left(1-r_a^{\,2}\right)^2\right]^{.5}$$ $\zeta = 0.099$

**H**




$H(r_a,\zeta) = 2$

$\frac{1}{2\cdot\zeta} = 5.053$

$H(1,\zeta) = 5.053$

==========================================================================

## Acceleration or Imbalance

$$J(r,\zeta) := \frac{r^2}{\left[\left(1-r^2\right)^2 + (2\cdot\zeta\cdot r)^2\right]^{.5}}$$

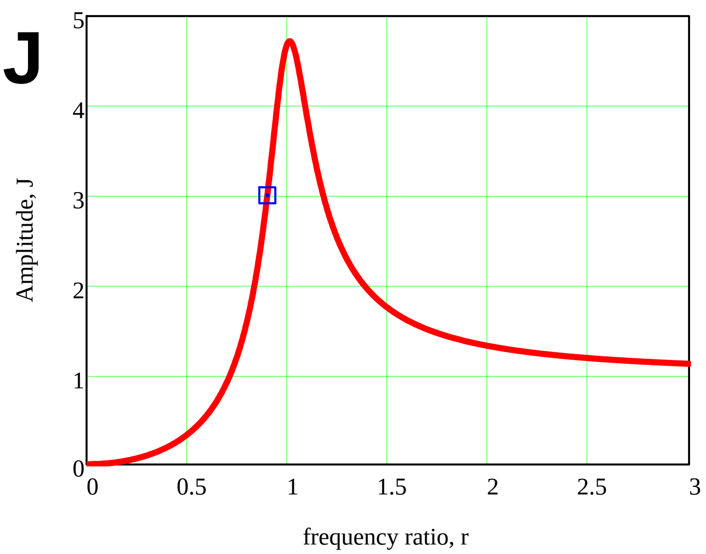
Given desired J and freq r

**formula for damping ratio** $J_a := 3$ $r_a := 0.9$

$$\zeta := \frac{1}{r_a\cdot 2}\cdot\left[\frac{r_a^{\,4}}{J_a^{\,2}} - \left(1-r_a^{\,2}\right)^2\right]^{.5}$$

$\zeta = 0.107$

**J**




$J(r_a,\zeta) = 3$

$\frac{1}{2\cdot\zeta} = 4.692$

$J(1,\zeta) = 4.692$

==========================================================================================

# Transmitted force to base or foundation

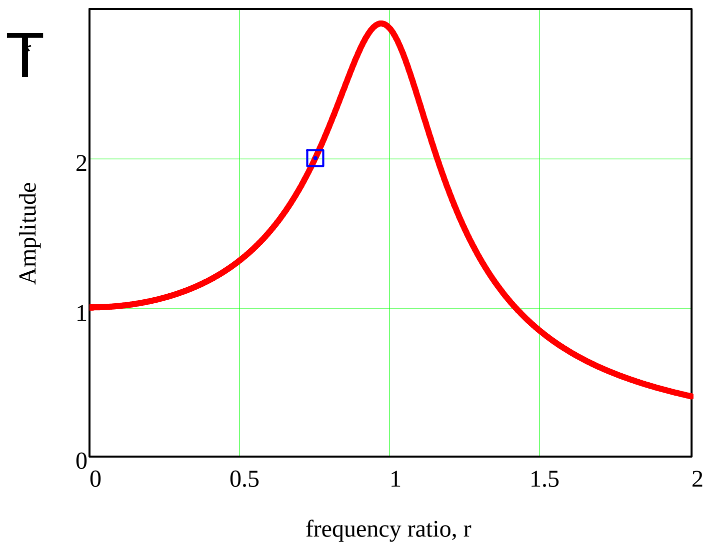
Given desired T and freq r

$r_a := 0.75 \quad T_a := 2.0$

$$T(r,\zeta) := \frac{\left[1+(2\cdot\zeta\cdot r)^2\right]^{0.5}}{\left[\left(1-r^2\right)^2+(2\cdot\zeta\cdot r)^2\right]^{.5}}$$

$T\left(r_a,0\right) = 2.286$

$$\zeta := \frac{1}{r_a\cdot 2}\cdot\left[\frac{\left[\left(1-r_a^2\right)^2\cdot T_a^2-1\right]}{1-T_a^2}\right]^{.5}$$




$\zeta = 0.186$

$T\left(r_a,\zeta\right) = 2$

$T(1,\zeta) = 2.864$

$$\frac{\left[1+(2\cdot\zeta)^2\right]^{.5}}{2\cdot\zeta} = 2.864$$