ME459 – Notes 10

Sound Propagation & Wave Equation

Wikipedia:

Sound is "(a) oscillation in pressure, stress, particle displacement, particle velocity, etc., propagated in a medium with internal forces: elastic or viscous, or the superposition of such propagated oscillation. (b) Auditory sensation evoked by the oscillation in (a).

Sound can be viewed as a wave motion in air or other elastic media. In this case, sound is a stimulus. Sound can also be viewed as an excitation of the hearing mechanism that results in the perception of sound. In this case, sound is a sensation.

Physics of sound

Sound can propagate through a medium such as air, water and solids as longitudinal (compression) waves, and also as a transverse wave in solids (alternate shear stress).

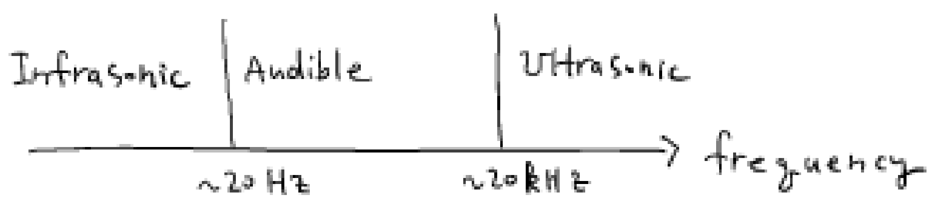
The sound waves are generated by a sound source, such as the vibrating diaphragm of a stereo speaker. The source creates oscillations in the surrounding medium. As the source continues to vibrate the medium, the vibrations propagate away from the source at the speed of sound, thus forming **the sound wave:** **At a fixed distance** from the source, the pressure, velocity, and displacement of the medium vary in time. **At an instant in time**, the pressure, velocity, and displacement vary in space. Note that the particles of the medium do not travel with the sound wave. The vibrations of particles in the gas or liquid transport the vibrations, while the *average* spatial position of the particles over time does not change.

During propagation, waves can be reflected, refracted, or attenuated by the medium

Sound waves are characterized by these generic properties:

- Frequency, or its inverse, wavelength
- Amplitude, sound pressure or intensity
- Speed of sound (*c*)
- Direction

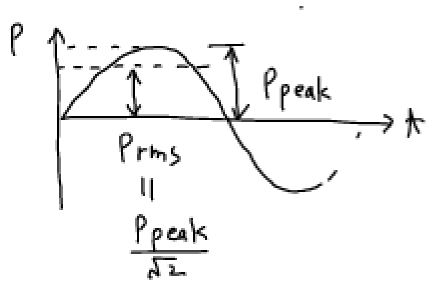
Sound that is perceptible by humans has frequencies from about 20 Hz to 20,000 Hz. In air at standard temperature and pressure, the corresponding wavelengths of sound waves range from 17 m to 17 mm. Sometimes speed and direction are combined as a velocity vector; wave number and direction are combined as a wave vector.



Sound measurement is performed with microphones. Do note that sound (as seen next) is a perturbation of pressure (in time and space) in a stationary (calm) medium.

To measure sound level or magnitude one typically uses a **decibel** **(db) =** 1/10 bel.

A human's perceived sound level is proportional to the log scale of sound pressure (*p'*).



$$L_p = 20\log_{10}\left(\frac{p_{RMS}}{p_{ref}}\right)$$

since

$$L_p = 10\log_{10}\left(\left(\frac{p_{RMS}}{p_{ref}}\right)^2\right)$$

as the power in a sound wave is ~ pressure^2.

The threshold sound pressure of human's hearing at 1 kHz is

$p_{ref} = 20\ 10^{-6}$ Pa

If the **sound <u>power</u> drops** to ½ of the reference, then

$$L_p = 10\log_{10}\left(p_{RMS}^2 \Big/ p_{ref}^2\right) = 10\log_{10}(0.5) = 10*(-.3) = -3dB$$

While if the **sound power doubles**, then

$$L_p = 10\log_{10}\left(p_{RMS}^2 \Big/ p_{ref}^2\right) = 10\log_{10}(2) = 10*(+.3) = +3dB$$

Also 80 dB means → 10,000 larger than reference

$$8 = \log_{10}\left(\left(\frac{p_{RMS}}{p_{ref}}\right)^2\right) \rightarrow \left(\frac{p_{RMS}}{p_{ref}}\right)^2 = 10^8 \rightarrow p_{RMS} = 10^4\, p_{ref} = 0.01Pa$$

While 120 dB means → 1 million larger than reference

$$6 = \log_{10}\left(\frac{p_{RMS}}{p_{ref}}\right) \rightarrow \left(\frac{p_{RMS}}{p_{ref}}\right)^1 = 10^6 \rightarrow p_{RMS} = 10^6\, p_{ref} = 1Pa$$

LOUD!

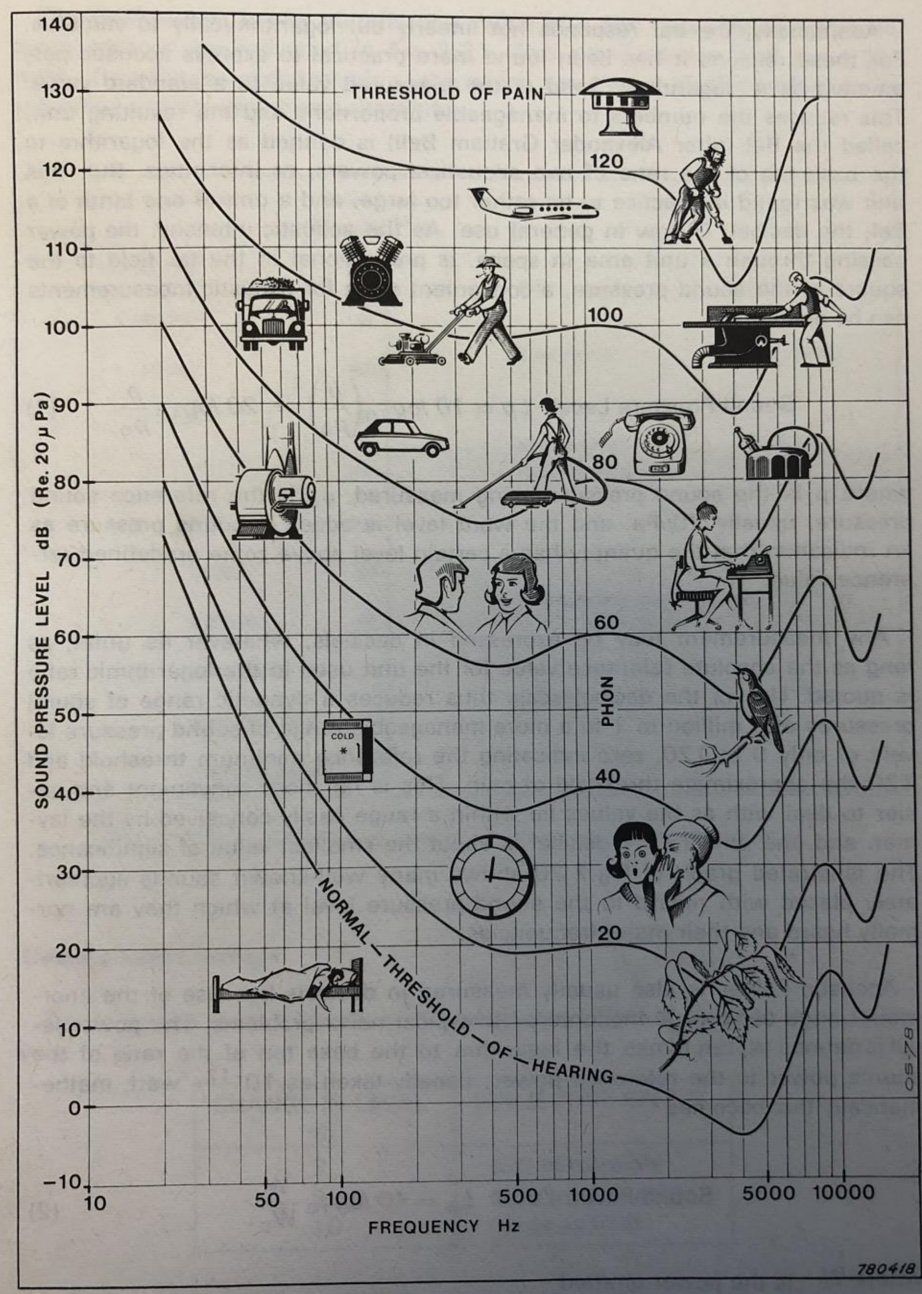


Fig.2.20. Typical sound pressure levels of common noise sources

Typical sound pressure levels of common noise sources
(copied from Acoustic Noise Measurements – Bruel & Kjaer)

(Based on handwritten Lecture Notes 18-19 by Dr. Joe Kim)

Mathematical analysis of sound propagation and acoustic pressure fluctuations

considers

a) **Inviscid media** (gas has no viscosity)

b) **Adiabatic process** – no thermal energy exchange between gas (air) particles → a constant entropy process.

The **equation of state** for an ideal gas is:

$$P = \rho R_G T \tag{1}$$

where P and T are the gas absolute pressure and absolute temperature, R_G is the gas constant and γ is the ratio of specific heats ($=C_p/C_v$).

The media (gas) is considered stagnant (with zero mean velocity) and the pressure is equal to

$$P = P_a + p' \rightarrow \ \rho = \rho_a + \rho' \tag{2}$$

Where the sub index a stands for ambient condition and the ' stands for a fluctuation (or perturbation) about the mean value.

Conservation of mass

The equation for conservation of mass in a compressible fluid is

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \tag{3}$$

where $\mathbf{v}=(v_x, v_x, v_z)$ is the acoustic velocity of a particle in a static medium. Above ∇ is the divergence operator.

Substitute $\rho = \rho_a + \rho'(<< \rho_a)$ into Eq. (3) to obtain the linearized equation

$$\frac{\partial \rho'}{\partial t} + \rho_a \nabla \bullet (\mathbf{v}) = 0 \qquad (4)$$

Here, a small term like ($\rho'\mathbf{v}$) is neglected.

Conservation of momentum

For an inviscid fluid (no viscosity), the momentum equations of a fluid reduce to

$$\frac{D(\rho \mathbf{v})}{Dt} = \frac{\partial(\rho \mathbf{v})}{\partial t} + \mathbf{v}\nabla \bullet (\rho \mathbf{v}) = -\underline{\nabla} P \qquad (5)$$

where $\underline{\nabla}$ is the gradient operator. Ignoring advection of fluid motion and linearization of the temporal change in momentum leads to **Euler's equation**:

$$\rho_a \frac{\partial \mathbf{v}}{\partial t} = -\underline{\nabla}\, p' \qquad (6a)$$

These are three independent equations written in the Cartesian coordinate system as

$$\rho_a \frac{\partial v_x}{\partial t} = -\frac{\partial p'}{\partial x};\ \rho_a \frac{\partial v_y}{\partial t} = -\frac{\partial p'}{\partial y};\rho_a \frac{\partial v_z}{\partial t} = -\frac{\partial p'}{\partial z} \qquad (6b)$$

which shows the particle acceleration (change in momentum) is proportional to the gradient of the perturbed (dynamic) pressure along that direction.

Now, derive the conservation of mass Eq. (4) with respect to time to obtain

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_a \frac{\partial}{\partial t} \nabla \bullet (\mathbf{v}) = 0 \rightarrow \quad \frac{\partial^2 \rho'}{\partial t^2} + \rho_a \nabla \bullet \left(\frac{\partial}{\partial t} \mathbf{v} \right) = 0 \qquad (7)$$

And substitute Euler's Equation into Eq. (7) to obtain

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla \bullet \left(\underline{\nabla} p' \right) = 0 \qquad (8a)$$

And recall $\nabla \bullet \left(\underline{\nabla} \right) = \nabla^2$ is the Laplacian operator. Hence, Eq. (8a) becomes

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p' \qquad (8b)$$

Changes in pressure and density define the sound speed c_o as

$$\frac{p'}{\rho'} = \gamma R_G T = c_o^2 \qquad (9)$$

For air (molecular weight=29) R_G=286.7 J/(kg K) and γ =1.4, and at a temperature of 300 K (~27 C), the **speed of sound c_o =**347 m/s

Substitute Eq. (9) into Eq. (8b) to obtain

$$\frac{1}{c_o^2} \frac{\partial^2 p'}{\partial t^2} = \nabla^2 p' = \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2} \qquad (10a)$$

This is the **WAVE Equation** or the e**quation of propagation of acoustic pressure** in Cartesian coordinates (x,y,z).

In polar coordinates (r, θ, z):

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2}=\nabla^2 p'=\frac{\partial^2 p'}{r^2\partial\theta^2}+\frac{\partial^2 p'}{\partial r^2}+\frac{1}{r}\frac{\partial p'}{\partial r}+\frac{\partial^2 p'}{\partial z^2} \qquad (10b)$$

The wave Eq. (10) is solved in the domain with specified particular boundary conditions in the closure of the domain as well as with initial conditions.

Once the pressure field *p'* is obtained, the velocity field (**v**) follows from Euler's Eqn.

$$\frac{\partial \mathbf{v}}{\partial t}=-\frac{1}{\rho_a}\underline{\nabla}\, p' \qquad (6)$$

Harmonic wave propagation

The solution of the wave equation is of the general form

$$p'_{(x,y,z,t)}=\left(A_{x+}e^{i\kappa_x x}+A_{x-}e^{-i\kappa_x x}\right)\left(A_{y+}e^{i\kappa_y y}+A_{y-}e^{-i\kappa_y y}\right)\left(A_{z+}e^{i\kappa_z z}+A_{z-}e^{-i\kappa_z z}\right)\left(A_{t+}e^{i\omega t}+A_{t-}e^{-i\omega t}\right) \qquad (11)$$

where i is the imaginary unit. Above κ is a characteristic 1/length=**wave number** and ω is a 1/time=**frequency** scale.

(w/o **reflective** boundaries) Let

$$p'_{(x,y,z,t)}=A\left(e^{i\kappa_x x}\right)\left(e^{i\kappa_y y}\right)\left(e^{i\kappa_z z}\right)\left(e^{-i\omega t}\right) \qquad (12)$$

Substitute into Eq. 10(a) to obtain

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2}=\nabla^2 p'\rightarrow \qquad \frac{\omega^2}{c_o^2}=\kappa_x^2+\kappa_y^2+\kappa_z^2=\kappa^2 \qquad (*)$$

where the **wave number** (κ) is a vector: $\vec{\kappa}=\kappa_x\hat{i}+\kappa_y\hat{j}+\kappa_z\hat{k}$.

Similarly for the velocity vector, let $\mathbf{v} = \mathbf{v}'_{(x,y,z)}\, e^{-i\omega t}$ and sub into Euler's equation $\rho_a \frac{\partial \mathbf{v}}{\partial t} = -\underline{\nabla}\, p'$ to obtain

$$\mathbf{v}' = +\frac{1}{i\omega\rho_a}\underline{\nabla}\, p' = +\frac{1}{i c_o \kappa \rho_a}\underline{\nabla}\, p' \qquad (13)$$

Since $\omega = c_o \kappa$.

Note above the relationship between frequency (ω) and wave number (κ). For example, given c_o =347 m/s, then

Freq	ω	$\kappa = \omega / c_o$	$\lambda = 2\pi/\kappa$
Hz	rad/s	1/m	m
20	125.6	0.36	17.34
500	3,142	9.05	0.68
5,000	31,420	90.5	0.068
20,000	125,700	362	0.0174

Recall sound that is perceptible by humans has frequencies from about 20 Hz to 20,000 Hz. The higher the frequency is, the smaller is the wave length (λ) [full period]

WAVE PROPAGATION IN A DUCT

For sound propagation along the *x*-direction[1] (only), the equations are

$$\frac{\partial^2 p'}{\partial t^2} = c_o^2 \frac{\partial^2 p'}{\partial x^2} \quad \text{(15a) and} \quad \rho_a \frac{\partial v_x}{\partial t} = -\frac{\partial p'}{\partial x} \qquad (15b)$$

Above $c_o = \sqrt{\gamma R_G T}$ [m/s] is the sound speed.

The solution of **PDE** is of the form $p'_{(x,t)} = \phi_{(x)} v_{(t)}$ (16)

Sub into Eq. (15a) to get

$$\frac{\ddot{v}_{(t)}}{v_{(t)}} = c_o^2 \frac{\phi''_{(x)}}{\phi_{(x)}} = -\omega^2 \qquad (17)$$

or

$$\ddot{v}_{(t)} + \omega^2 v = 0 \;\&\; \phi''_{(x)} + \kappa^2 \phi_{(x)} = 0 \qquad (18)$$

where

$$\kappa = \omega / c_o \qquad (19)$$

ω **is a frequency** [1/s] & κ is the <u>inverse</u> of the **wave length** λ [m].

The solution of the ODEs (18) is (for example)

$$v_{(t)} = C_t \cos(\omega t) + S_t \sin(\omega t) = D_t\, e^{-i\omega t} \qquad (20a)$$

$$\phi_{(x)} = C_x \cos(\kappa x) + S_x \sin(\kappa x) = A e^{i\kappa x} + B e^{-i\kappa x} \qquad (20b)$$

Hence,

$$p'_{(x,t)} = \phi_{(x)} v_{(t)} = \left[A e^{i\kappa x} + B e^{-i\kappa x} \right] e^{-i\omega t} \qquad (21a)$$

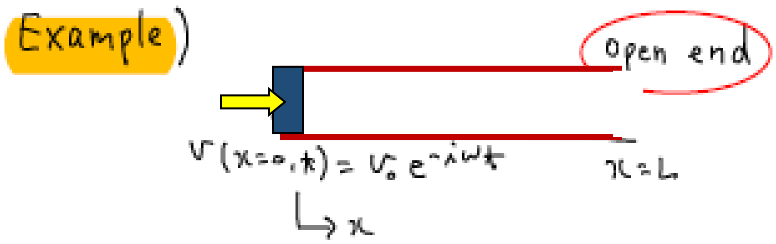
And the propagation speed is

$$v_x = + \frac{1}{i c_o \kappa \rho_a} \frac{\partial p'}{\partial x} = \frac{1}{c_o \rho_a} \left[A e^{ix} - B e^{-ix} \right] e^{-i\omega t} \qquad (21b)$$

[1] Note the wave equation is identical to that derived for vibrations of a taut string (see Lecture Notes 3).

The coefficients (A, B) are determined by satisfying the boundary conditions for the specific duct configuration.

Example 1. Duct with one open end



At $x=0$, **specified velocity**

$$v_{x(0,t)} = v_o e^{-i\omega t} = \phi_{(0)} v_{(t)} \Rightarrow \frac{\left[Ae^{i\kappa x} - Be^{-i\kappa x}\right]_{x=0}}{c_o\,\rho_a} = \frac{[A-B]}{c_o\,\rho_a} = v_o \quad (22a)$$

At $x=L$, **open end** → no pressure perturbation (a pressure release condition)

$$p'_{(L,t)} = 0 = \phi_{(L)} v_{(t)} \Rightarrow \phi_{(L)} = 0 = \left[Ae^{i\kappa L} + Be^{-i\kappa L}\right] \quad (22b)$$

Hence, the two equations for finding A and B are

$$\begin{bmatrix} 1 & -1 \\ e^{i\kappa L} & e^{-i\kappa L} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} \rho_a c_o \\ 0 \end{Bmatrix} v_o \quad (23)$$

with $\Delta = e^{i\kappa L} + e^{-i\kappa L} = 2\cos(\kappa L)$ (24)

recall: $e^{+ia} = \cos(a) + i\sin(a);\ e^{-ia} = \cos(a) - i\sin(a)$

then

$$A = \frac{1}{\Delta}(\rho_a c_o) v_o e^{-i\kappa L};\ B = \frac{-1}{\Delta}(\rho_a c_o) v_o e^{i\kappa L} \quad (25)$$

Sub Eq. (25) into Eq. (21a) $\boxed{p'_{(x,t)} = \left[A e^{i\kappa x} + B e^{-i\kappa x} \right] e^{-i\omega t}}$ to get

$$p'_{(x,t)} = \left(A e^{i\kappa x} + B e^{-i\kappa x} \right) e^{i\omega t} = \frac{\rho_a c_o v_o}{\Delta} \left[e^{i\kappa(x-L)} - e^{-i\kappa(x-L)} \right] e^{-i\omega t}$$

Since $\begin{bmatrix} e^{+ia} = \cos(a) + i\sin(\text{a}) \\ - \\ e^{-ia} = \cos(a) - i\sin(\text{a}) \end{bmatrix} \rightarrow 2i\sin(a)$

Then →
$$p'_{(x,t)} = i\rho_a c_o \frac{\sin\left[\kappa(x-L)\right]}{\cos(\kappa \text{L})} \left(v_o e^{-i\omega t} \right) \qquad (26)$$

Similarly, the propagation speed is

$$v_x = + \frac{1}{i c_o \kappa \rho_a} \frac{\partial p'}{\partial x} = \frac{\cos\left[\kappa(x-L)\right]}{\cos(\kappa \text{L})} \left(v_o e^{-i\omega t} \right) \qquad (27)$$

where the wave number $\kappa = \omega / c_o$.

Natural frequencies and mode shapes

From the system of Eqns. (23), note the **characteristic equation** is $\Delta = 2\cos(\kappa L) = 0 \rightarrow$

$$\cos(\kappa L) = 0 \qquad (28)$$

having an infinite number of solutions. The wave numbers and natural frequencies are

$$\kappa_n = \frac{(2n-1)}{2L}\pi \rightarrow \omega_n = \kappa_n c_o = \frac{(2n-1)}{2}\pi \frac{c_o}{L}; \quad _{n=1,2.....} \qquad (29)$$

Associated to each natural frequency, the **mode shapes** are (see Eqn. (26)):

$$\psi_n = \sin\left(\kappa_n \{x - L\}\right) \quad {}_{n=1,2,...} \tag{30}$$

or

$$\psi_1 = \sin\left(\frac{\pi}{2}\frac{\{x-L\}}{L}\right); \quad \psi_2 = \sin\left(\frac{3\pi}{2}\frac{\{x-L\}}{L}\right); \quad \psi_3 = \sin\left(\frac{5\pi}{2}\frac{\{x-L\}}{L}\right)$$

shown below.

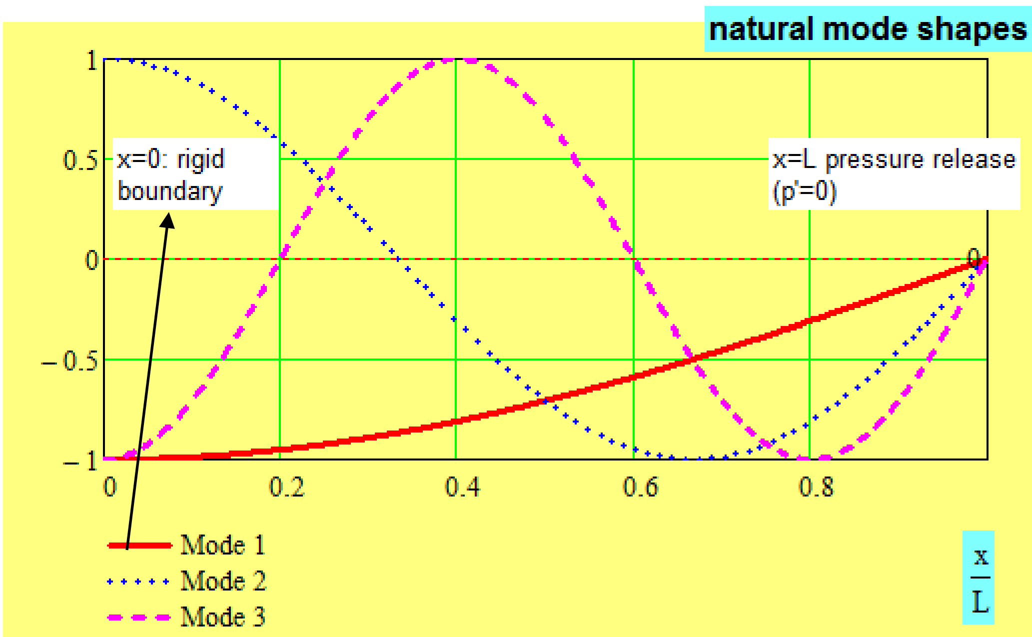


Fig 3. Natural modes shapes ψ(x) for duct with one end open

Acoustic impedance

Recall the definition of an impedance

$$Z = \text{effort/flow} \tag{31}$$

In an electrical system → *Z* = voltage/current = *V/I*; while

in a mechanical system → *Z*=Force/velocity *or* Torque/angular speed.

In an acoustic system, the impedance Z is the ratio between pressure (p') and acoustic velocity (v_x).

$$Z = \frac{p'}{v_x} = i\rho_a c_o \frac{\sin\left[\kappa\left(x-L\right)\right]}{\cos\left[\kappa\left(x-L\right)\right]} = i\rho_a c_o \tan\left[\kappa\left(x-L\right)\right] \qquad (32)$$

At the inlet of the duct, x=0, the acoustic impedance is

$$Z_{(x=0)} = -i\rho_a c_o \tan\left[\kappa L\right] \qquad (33)$$

The relationship is imaginary, hence **reactive** (not adding energy)

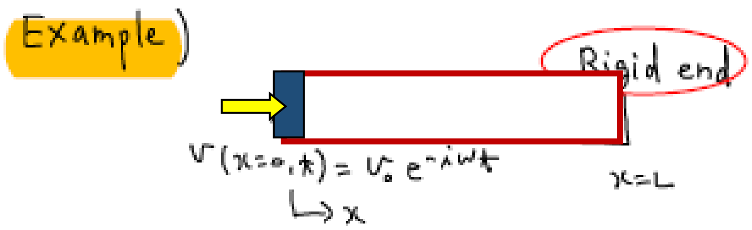
Recall in an electric system, with a resistor element $Z = V/I \rightarrow R$ (real #) > 0 dissipates energy as the voltage and current are in phase (power = $I^2 R$).

Not so for a capacitance (C) or an inductance (L) where $V=1/C\ q$ (charge) and $V = L\ dI/dt$, with $I=dq/dt$ (the temporal change in charge q)

These two elements are conservative, i.e. storing energy, and with the power in a full cycle = 0.

Example 2. Duct with a closed (rigid) end

(organ pipe)



At x=0, **specified velocity**

$$v_{x(0,t)} = v_o e^{-i\omega t} = \phi_{(0)} v_{(t)} \Rightarrow \frac{\left[A e^{i\kappa x} - B e^{-i\kappa x} \right]_{x=0}}{c_o \, \rho_a} = \frac{\left[A - B \right]}{c_o \, \rho_a} = v_o \quad (34a)$$

At x=L, **closed end** → no velocity

$$v_{x(L,t)} = 0 = \phi_{(L)} v_{(t)} \Rightarrow \frac{\left[A e^{i\kappa L} - B e^{-i\kappa L} \right]}{c_o \, \rho_a} = 0 \qquad (34b)$$

Hence, find A and B from

$$\begin{bmatrix} 1 & -1 \\ e^{i\kappa L} & -e^{-i\kappa L} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} \rho_a c_o \\ 0 \end{Bmatrix} v_o \qquad (35)$$

with
$$\Delta = e^{i\kappa L} - e^{-i\kappa L} = 2i \sin(\kappa L) \qquad (36)$$

then
$$A = \frac{-1}{\Delta} \left(\rho_a c_o \right) v_o e^{-i\kappa L}; \; B = \frac{-1}{\Delta} \left(\rho_a c_o \right) v_o e^{i\kappa L} \qquad (37)$$

Sub Eq. (37) into $\boxed{p'_{(x,t)} = \left[A e^{i\kappa x} + B e^{-i\kappa x} \right] e^{-i\omega t}}$ to get

$$p'_{(x,t)} = \frac{-\rho_a c_o v_o}{2i\sin(\kappa \mathrm{L})}\left[e^{i\kappa(x-L)} + e^{-i\kappa(x-L)}\right]e^{-i\omega t} =$$

Then →
$$p'_{(x,t)} = i\rho_a c_o \frac{\cos\left[\kappa\left(x-L\right)\right]}{\sin(\kappa \mathrm{L})} v_o e^{-i\omega t} \quad (38)$$

And the propagation speed is

$$v_x = +\frac{1}{i c_o \kappa \rho_a}\frac{\partial p'}{\partial x} = \frac{-\sin\left[\kappa\left(x-L\right)\right]}{\cos(\kappa \mathrm{L})}\left(v_o e^{-i\omega t}\right) \quad (39)$$

where the wave number $\kappa = \omega / c_o$.

Natural frequencies and mode shapes

The **characteristic equation** is $\Delta = 2i\sin(\kappa L) = 0$ (39)

having an infinite number of solutions. The wave numbers and natural frequencies are

$$\kappa_n = \frac{n\pi}{L} \rightarrow \omega_n = \kappa_n c_o = n\pi \frac{c_o}{L}; \quad {}_{n=0,1,2.....} \quad (40)$$

Associated to each natural frequency, the **mode shapes** are (see Eqn. (38):

$$\psi_n = \cos\left(\kappa_n \left\{x - L\right\}\right) \quad {}_{n=0,1,2,...} \quad (41)$$

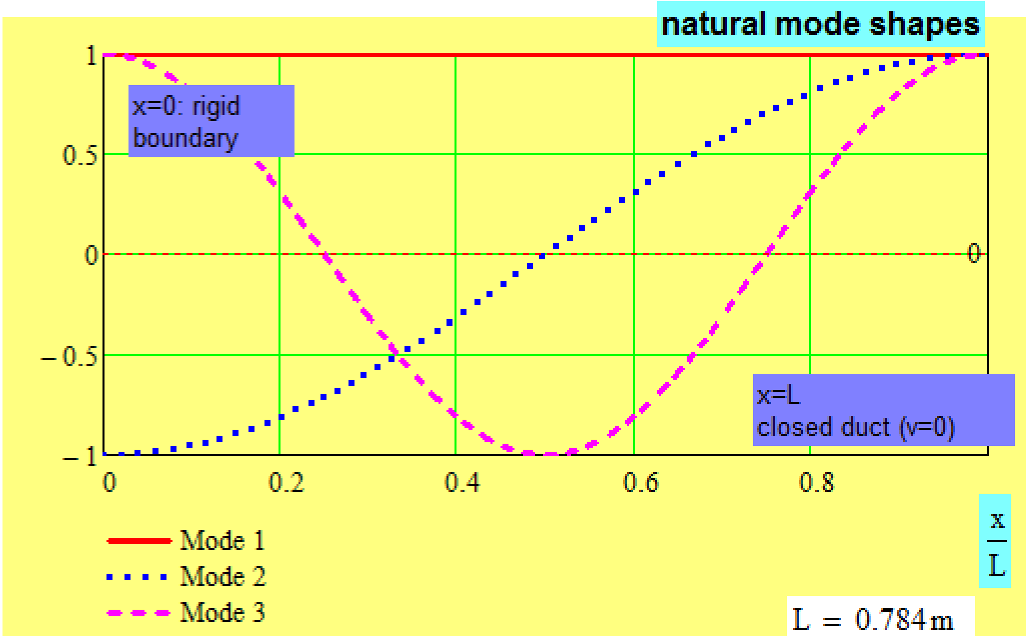


Fig 3. Natural modes shapes ψ(x) for duct with one end closed

Acoustic impedance

The acoustic impedance is

$$Z = \frac{p'}{v_x} = -i\rho_a c_o \frac{\cos\left[\kappa\left(x-L\right)\right]}{\sin\left[\kappa\left(x-L\right)\right]} = i\rho_a c_o \left\{\tan\left[\kappa\left(x-L\right)\right]\right\}^{-1} \quad (42)$$

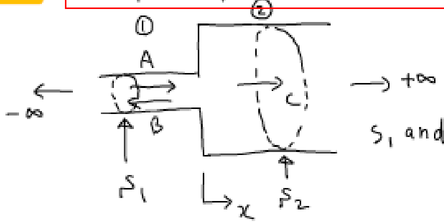
At the inlet of the duct, x=0,

$$Z_{(x=0)} = -i\rho_a c_o \cot\left[\kappa L\right] \quad (33)$$

The relationship is imaginary, hence **reactive** (not dissipating or adding energy).

OTHER Example (from Dr. Joe Kim)

Example) Simple expansion (or contraction) duct



S_1 and S_2 : cross-sectional areas

Section 1 $(x \le 0)$

$$p_1(x,t) = (A e^{ikx} + B e^{-ikx}) e^{-i\omega t}$$

$$v_1(x,t) = \frac{1}{\rho_o c_o}(A e^{ikx} - B e^{-ikx}) e^{-i\omega t}$$

Section 2 $(x \ge 0)$

$$p_2(x,t) = C e^{ikx}$$

$$v_2(x,t) = \frac{1}{\rho_o c_o} C e^{ikx}$$

Boundary conditions at $x = 0$

$$p_1(x=0) = p_2(x=0) : \quad A + B = C \qquad — ①$$

$$S_1 v_1(x=0) = S_2 v_2(x=0) : \quad A - B = \frac{S_2}{S_1} C \qquad — ②$$

$\frac{S_2}{S_1}$ = r : Area ratio

Define

$R = \frac{B}{A}$: Reflect wave ratio

$T = \frac{C}{A}$: Transmitted wave ratio

Eqs. ① and ②

$$1 + R = T$$

$$1 - R = rT$$

$$\Rightarrow T = \frac{2}{1+r}$$

$$R = T - 1 = \frac{2-1-r}{1+r} = \frac{1-r}{1+r}$$

Limit Cases

when $r = 1$ (no cross-section change)

$T = 1$

$R = 0$ (no reflected wave)

when $r = \infty$ (acoustic pressure release condition)

$T = 0$

$R = -1$

when $r = 0$ (Rigid termination)

$T = 2$

$R = 1$

Example 3. More on acoustics of organ pipe (duct with one end closed)

Watch videos on web site too

MUSIC Notes - equal tempered scale

In equal temperament, the octave is divided into equal parts on the logarithmic scale.

12 tone scale A4 A3 =1/2 A4 A5 = 2A4

"A4"	"La4"	440
"A#4"	"La#4"	466.16
"B4"	"Si4"	493.88
"C5"	"Do5"	523.25
"C#5"	"Do#5"	554.37
"D5"	"Re5"	587.33
"D#5"	"Re#5"	622.25
"E5"	"Mi5"	659.25
"F5"	"Fa5"	698.46
"F#5"	"Fa#5"	739.99
"G5"	"Sol5"	783.99
"G#5"	"Sol#5"	830.61

$f_s :=$ (vector above) $\cdot Hz$

$\frac{f_s}{2} =$ Hz

	1
1	220
2	233.08
3	246.94
4	261.625
5	277.185
6	293.665
7	311.125
8	329.625
9	349.23
10	369.995
11	391.995
12	415.305

$f_s \cdot 2 =$ Hz

	1
1	880
2	932.32
3	987.76
4	$1.046 \cdot 10^3$
5	$1.109 \cdot 10^3$
6	$1.175 \cdot 10^3$
7	$1.244 \cdot 10^3$
8	$1.319 \cdot 10^3$
9	$1.397 \cdot 10^3$
10	$1.48 \cdot 10^3$
11	$1.568 \cdot 10^3$
12	$1.661 \cdot 10^3$

$\kappa := \frac{2 \cdot \pi \cdot f_s}{c_o} =$ $\frac{1}{m}$

	1
1	7.967
2	8.441
3	8.943
4	9.475
5	10.038
6	10.635
7	11.267
8	11.937
9	12.647
10	13.399
11	14.196
12	15.04

wave length for A4

$\lambda := \frac{c_o}{f_s} =$ cm

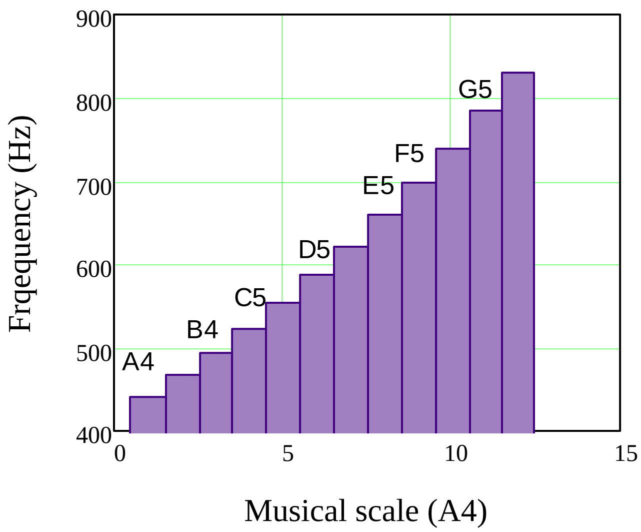
	1	
1	78.864	"A4"
2	74.438	"A#4"
3	70.26	"B4"
4	66.316	"C5"
5	62.594	"C#5"
6	59.081	"D5"
7	55.765	"D#5"
8	52.636	"E5"
9	49.681	"F5"
10	46.893	"F#5"
11	44.261	"G5"
12	41.777	"G#5"

$n := 12$ $j := 1..n-1$

$rat_j := \frac{f_{s_{j+1}}}{f_{s_j}}$ **ratio between notes = 1.059**

An equal temperament is a musical temperament, or a system of tuning, in which the frequency interval between every pair of adjacent notes has the same ratio. In other words, the ratios of the frequencies of any adjacent pair of notes is the same

$\max(rat) = 1.059$ $\min(rat) = 1.059$ $\left(rat_1\right)^{12} = 2$ $j := 1..n$



$$\begin{pmatrix} "A4" \\ "A\#4" \\ "B4" \\ "C5" \\ "C\#5" \\ "D5" \\ "D\#5" \\ "E5" \\ "F5" \\ "F\#5" \\ "G5" \\ "G\#5" \end{pmatrix}$$

Example Acoustic Vibrations - ORGAN PIPE

PHYSICAL Parameters for gas

$MW := 29$ molecular weight

$R_G := \frac{8314.34}{MW} \cdot \frac{J}{kg \cdot K} = 286.701 \frac{m^2}{K \cdot s^2}$ gas constant

$\gamma := 1.4$ ratio of specific heats

$T := 300 \cdot K$ temperature

$P_a := 1 \cdot bar$

$c_o := \left(\gamma \cdot R_G \cdot T\right)^{0.5} = 347.008 \frac{m}{s}$ sound speed

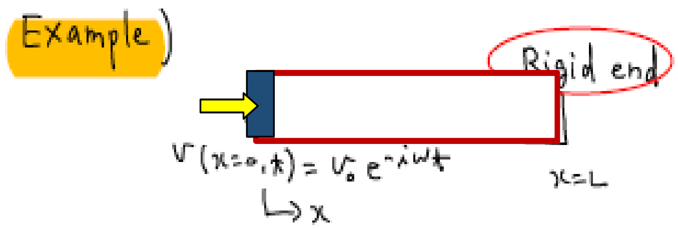
pipe length

$L := 78.41 \cdot cm$

exact length for A4

CLOSED DUCT

$\rho_a := \frac{P_a}{R_G \cdot T} = 1.163 \frac{kg}{m^3}$



(a) natural frequencies & mode shapes

using separation of variables, $p(x,t) = \phi(x) \cdot v(t)$ (1) with $\kappa = \omega \cdot c_o^{.5}$

Substitute into the field Eq. (0) to obtain the following two ODEs:

$$\frac{d^2}{dx^2}\phi + k^2 \cdot \phi = 0 \quad (2a)$$

$$\frac{d^2}{dt^2}v + \omega^2 \cdot v = 0 \quad (2b)$$

The solution to the ODEs is

$$\phi(x) = A_x \cdot \cos(\kappa \cdot x) + B_x \cdot \sin(\kappa \cdot x) \quad (3a)$$

$$v(t) = A_t \cdot \cos(\omega \cdot t) + B_t \cdot \sin(\omega \cdot t) \quad (3b)$$

Satisfy the **boundary conditions.**

At left end x=0, dp/dx=0 (zero velocity).

$$\frac{d\phi}{dx} = -A_x \cdot \kappa \cdot \sin(\kappa \cdot 0) + B_x \cdot \kappa \cdot \cos(\kappa \cdot 0) = 0$$

then $B_X = 0$

and $\phi(x) = \cos(\kappa \cdot x)$ (4) is the equation for the **shape funcion.**

At the right end, x=L, the duct is closed, hence p'=0 (zero velocity)

at x=L $\frac{d\phi}{dx} = \sin(\kappa \cdot L) = 0$ (5a)

$j := 0 .. n$

Set $T := 4000 \cdot N$ example

= **characteristic equation.** The roots are $\kappa_j := \frac{j \cdot \pi}{L}$

$\kappa^T = (0 \quad 4.007 \quad 8.013 \quad 12.02) \frac{1}{m}$

And thus, the **natural frequencies** are $\omega_{n_j} := c_o \cdot \kappa_j$ rad/s

The longer the pipe is, the lower the natural frequency

$$\frac{\omega_n{}^T}{2\cdot\pi} = (0 \quad 221.278 \quad 442.556 \quad 663.834)\cdot Hz$$

A3 A4 A5

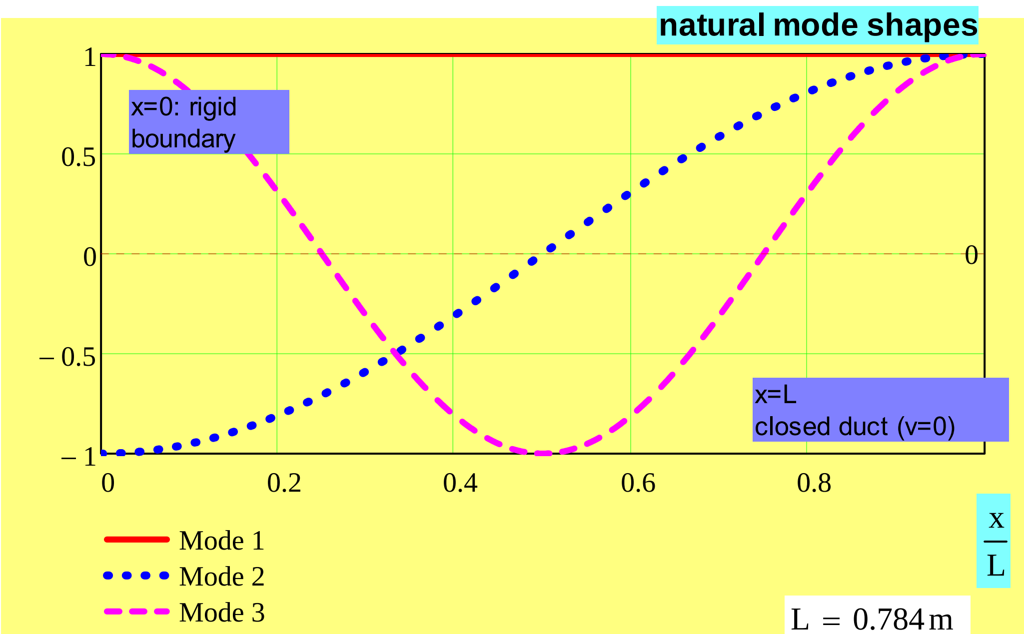
rigid body mode + hamonics of first A3

$j := 1 .. n$

The mode shape functions are **CLOSED DUCT**

$\lambda_j := \frac{2\pi}{\kappa_j}$ wave lengths $\blacksquare := \frac{2\cdot L}{j}$

$\phi1(x) := \cos(0)$ $\phi2(x) := \cos[\pi(x-1)]$ $\phi3(x) := \cos[2\cdot\pi(x-1)]$



$L = 0.784\,m$

$$\lambda = \begin{pmatrix} 0 \\ 1.568 \\ 0.784 \\ 0.523 \end{pmatrix} m$$

$$f_n := \frac{\omega_n}{2\cdot\pi} = \begin{pmatrix} 0 \\ 221.278 \\ 442.556 \\ 663.834 \end{pmatrix} Hz$$

$\frac{\lambda_2}{L} = 1$ (mode 3)

Solution for a specific velocity and frequency

$L = 0.784\,m$

$\rho_a = 1.163 \frac{kg}{m^3}$ $\quad c_o = 347.008 \frac{m}{s}$

recall natural frequencies

$f_n^T = (0 \;\; 221.278 \;\; 442.556 \;\; 663.834) \cdot Hz$

Set a velocity perturbation at x=0

$v_o := 1 \cdot \frac{m}{s}$ $\quad f := 430 \cdot Hz$

pitch lower than A4

build pressure and velocity waves

$\omega := f \cdot 2 \cdot \pi = 2.702 \times 10^3 \frac{1}{s}$

$\kappa := \frac{\omega}{c_o} = 7.786 \frac{1}{m}$

period $T := \frac{1}{f} = 2.326 \cdot ms$

wave length $\lambda := \frac{2\pi}{\kappa} = 0.807\,m$

$p\_ := \rho_a \cdot c_o \cdot v_o = 403.449\,Pa$ constant for excited pressure

from lecture notes 10

$$p(x,t) := p\_ \cdot \frac{\cos[\kappa \cdot (x - L)]}{\sin(\kappa \cdot L)} \cdot i \cdot e^{-i \cdot \omega \cdot t}$$

$$v(x,t) := -v_o \cdot \left[\frac{\sin[\kappa \cdot (x - L)]}{\sin(\kappa \cdot L)} \cdot e^{-i \cdot \omega \cdot t} \right]$$

take real part & divide by constant value p_ and vo

$$p(x,t) := 1 \cdot \frac{\cos[\kappa \cdot (x - L)]}{\sin(\kappa \cdot L)} \cdot \sin(\omega \cdot t)$$

$$v(x,t) := 1 \cdot \left[\frac{\sin[\kappa \cdot (L - x)]}{\sin(\kappa \cdot L)} \cdot \cos(\omega \cdot t) \right]$$

Note pressure is 90 away out of phase with velocity

$$power(x,t) := p(x,t) \cdot v(x,t)$$

$\kappa \cdot L = 6.105$ $\quad \frac{\lambda}{L} = 1.029$

< 2π

$\sin(\kappa \cdot L) = -0.177$ near resonance

$$S_{max} := \frac{1}{\sin(\kappa \cdot L)} = -5.64$$

For time-varying solution

$N\_periods := 2$ for analysis

$t_{max} := T \cdot N\_periods$

$N := 30$ total number of frames or steps

$\Delta t := \frac{t_{max}}{N} = 1.55 \times 10^{-4}\,s$ time step

$k := 10$ FRAME

$t\_k := k \cdot \Delta t$

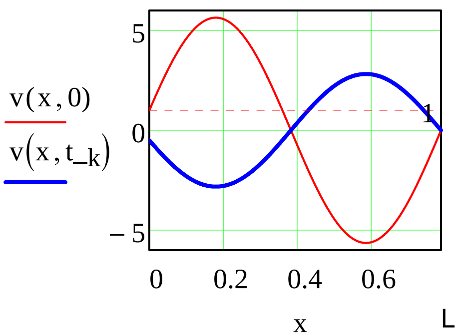
$S_{max} := 6$

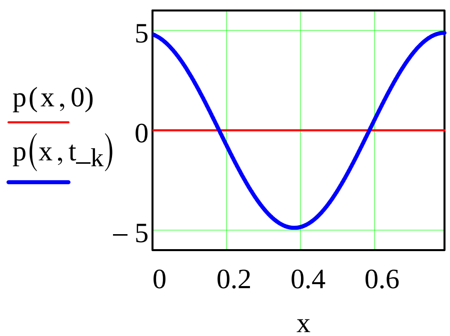
BUILD: pressure &velocity waves vs time

$f = 430 \cdot Hz$ $T = 2.326\,ms$ $L = 0.784\,m$

$\lambda = 0.807\,m$

velocity **pressure**

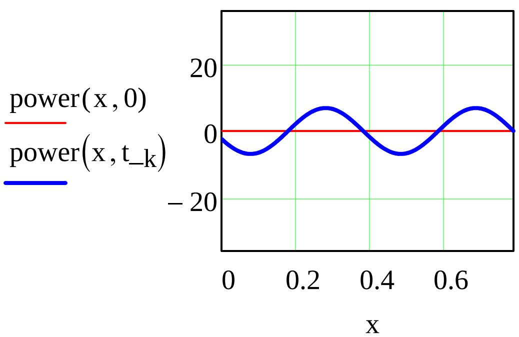




$\frac{t\_k}{T} = 0.667$

$\frac{\lambda}{L} = 1.029$

power



$p\_ = 403.449\,Pa$

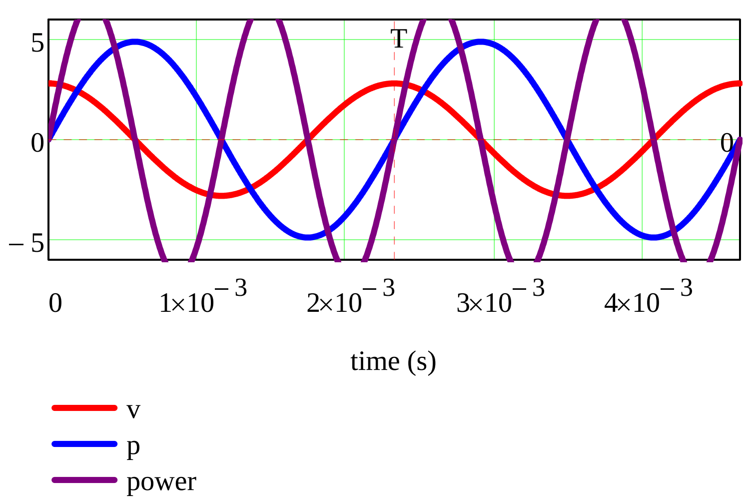
$\frac{1}{\sin(\kappa \cdot L)} = -5.64$

$f_n^{\,T} = (0 \;\; 221.278 \;\; 442.556 \;\; 663.834) \cdot Hz$

Waves of velocity, pressure and power vs. time at

$xx := L \cdot \frac{0.4}{1} = 31.364\,cm$

$T = 2.326 \times 10^{-3}\,s$



$f = 430 \frac{1}{s}$

PRESSURE AMPLITUDE FREQUENCY RESPONSE at varios spatial locations x

$\frac{p}{p\_}$

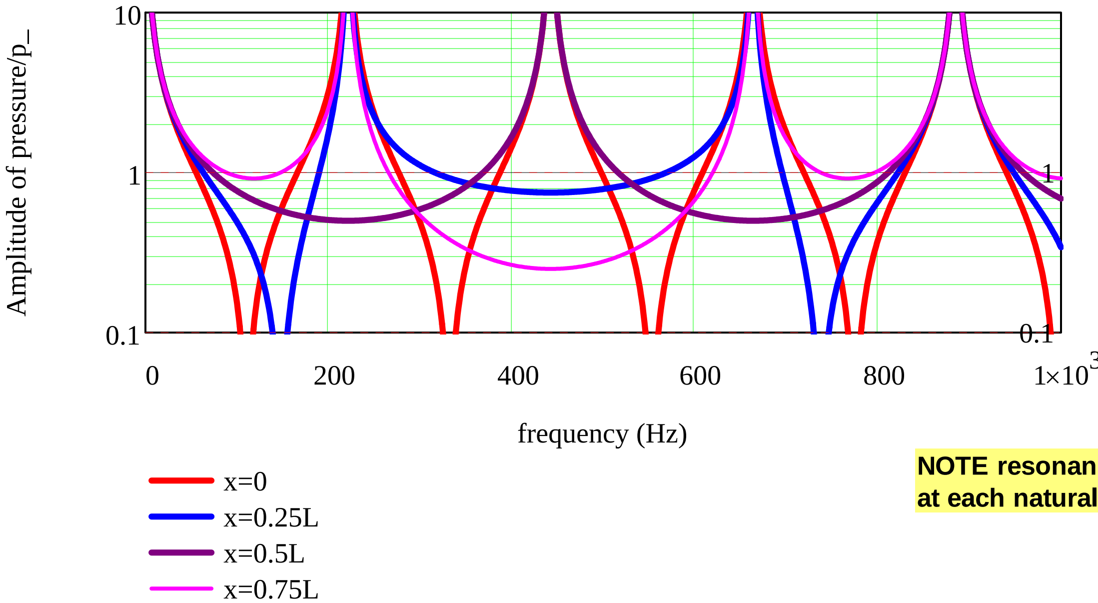
$$p_F(x,\omega) := \left| 1\cdot\frac{\cos\left[\frac{\omega}{c_o}\cdot(x-L)\right]}{\sin\left(\frac{\omega}{c_o}\cdot L\right)} \right|$$

graph $f_{max} := 1000\cdot H$

$f_n^T = (0\ \ 221.278\ \ 442.556\ \ 663.834)\ \frac{1}{s}$

natural frequencies:

$p\_ = 403.449\,Pa$



NOTE resonances at each natural frequency.

VELOCITY FREQUENCY RESPONSE at varios spatial locations x

$$\frac{v}{v_o}$$

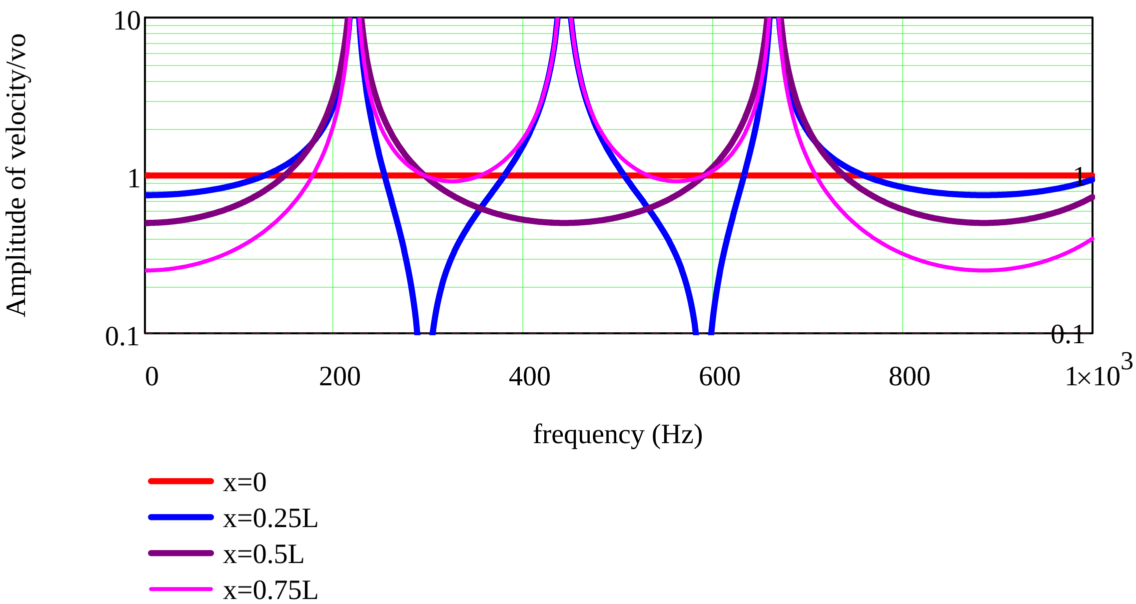
$$v_F(x,\omega) := \left| 1 \cdot \frac{\sin\left[\frac{\omega}{c_o}\cdot(L-x)\right]}{\sin\left(\frac{\omega}{c_o}\cdot L\right)} \right|$$

natural frequencies:

$$f_n^T = (0 \quad 221.278 \quad 442.556 \quad 663.834)\,\frac{1}{s}$$

NOTE resonances at each natural frequency

$$v_o = 1\,\frac{m}{s}$$



More notes from Dr. Joe Kim

Sound intensity level

Sound Intensity Level (SIL), $\boxed{L_I = 10\log_{10}\left(\frac{I}{I_{ref}}\right)}$

Why $I_{ref} = 10^{-12}$ W? $\quad I_{ref} = 10^{-12}$ watt/m²

⇒ In an infinite duct with only positive-propagating, plane wave

$$P = A e^{ikr} e^{-i\omega t}$$

$$v_x = \frac{1}{\rho_o c_o} A e^{ikr} e^{-i\omega t}$$

$$I_x = \frac{1}{2} Re[p^* v] = \frac{1}{2\rho_o c_o} |A|^2$$

$$L_I = 10\log_{10}\left(\frac{I_x}{I_{ref}}\right) = 10\log_{10}\left(\frac{\frac{1}{2\rho_o c_o}|A|^2}{I_{ref}}\right)$$

$$= 20\log_{10}\left(\frac{\frac{|A|}{\sqrt{2}}}{\sqrt{\rho_o c_o I_{ref}}}\right)$$

By setting $L_I = L_p$

$$\sqrt{\rho_o c_o I_{ref}} = P_{ref} \Rightarrow I_{ref} = 9.72\times 10^{-13} \text{ Watt/m}^2$$

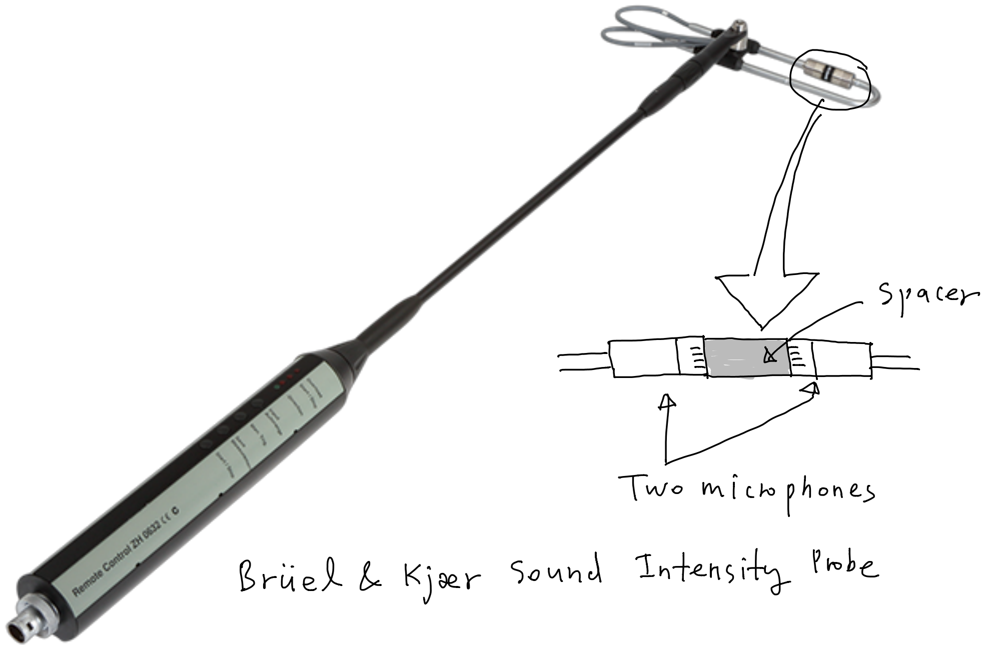
If $\rho_o = 1.21$ kg/m³ and $C_o = 343$ m/s

$\therefore I_{ref} = 10^{-12}$ watt/m²

Dr Joe Kim
NOTES 20

Sound Intensity Probe

To measure sound power per unit area.



Brüel & Kjær Sound Intensity Probe

$$\vec{I} = \frac{1}{2}\mathrm{Re}\left[p^* \vec{v}\right] = \frac{1}{2}\mathrm{Re}\left[p\,\vec{v}^*\right] \quad (*)$$: Definition

$\vec{I}$ ← Sound intensity : acoustic power per unit area

Recall in HW #2

$$\underline{P} = \frac{1}{2}\mathrm{Re}\left[\underline{F}^* \underline{V}\right] = \frac{1}{2}\mathrm{Re}\left[F V^*\right]$$

$\underline{P}$: power, $\underline{F}$: Force, $\underline{V}$: velocity

$p = \frac{F}{S}$: pressure (S: area)

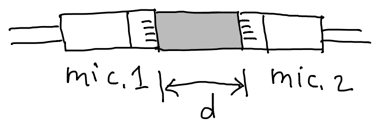
For the measurement of the sound intensity, p can be measured with a microphone and $\vec{v}$ with various sensors.

Here, $\vec{v}$ is measured with two microphones based on Euler's equation.

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}$$

Finite difference Approximation (FDA)

$$\Rightarrow v = \frac{1}{i\rho_0 c_0 k}\frac{\partial p}{\partial x} \approx \frac{1}{i\rho_0 c_0 k}\frac{p_2 - p_1}{d} \quad \text{— ①}$$



$$p = \frac{p_1 + p_2}{2} \quad \text{— ②}$$

① and ② → (★)

$$I_x \cong \frac{1}{2}\,\mathrm{Re}\left[\left(\frac{p_1+p_2}{2}\right)^* \left(\frac{1}{i\rho_0 c_0 k}\frac{p_2-p_1}{d}\right)\right]$$

$$= \frac{1}{4\rho_0 c_0 k d}\,\mathrm{Re}\left[-i\left(p_1^* p_2 - p_1^* p_1 + p_2^* p_2 - p_2^* p_1\right)\right]$$

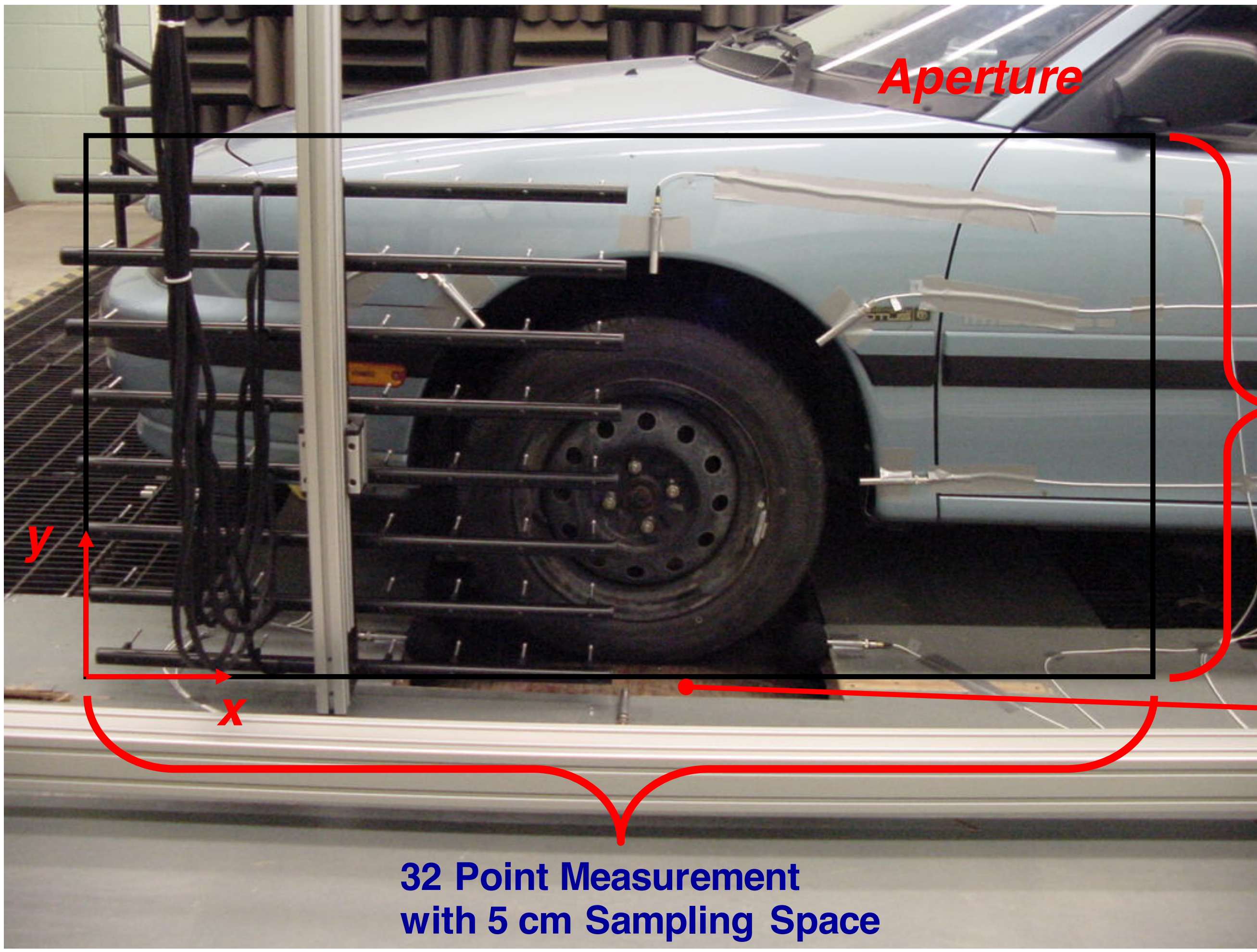
Define

complex numbers ⇒ $S_{12} = p_1^* p_2$: cross-spectrum; $S_{21} = p_2^* p_1 = S_{12}^*$

real numbers ⇒ $S_{11} = p_1^* p_1 = |p_1|^2$: Auto-spectrum; $S_{22} = p_2^* p_2$

$$\Rightarrow \boxed{I_x = \frac{1}{2\rho_0 c_0 k d}\,\mathrm{Im}[S_{12}]}$$

Rolling Tire at 21 mph



Tire driven by a roller

8 reference microphones

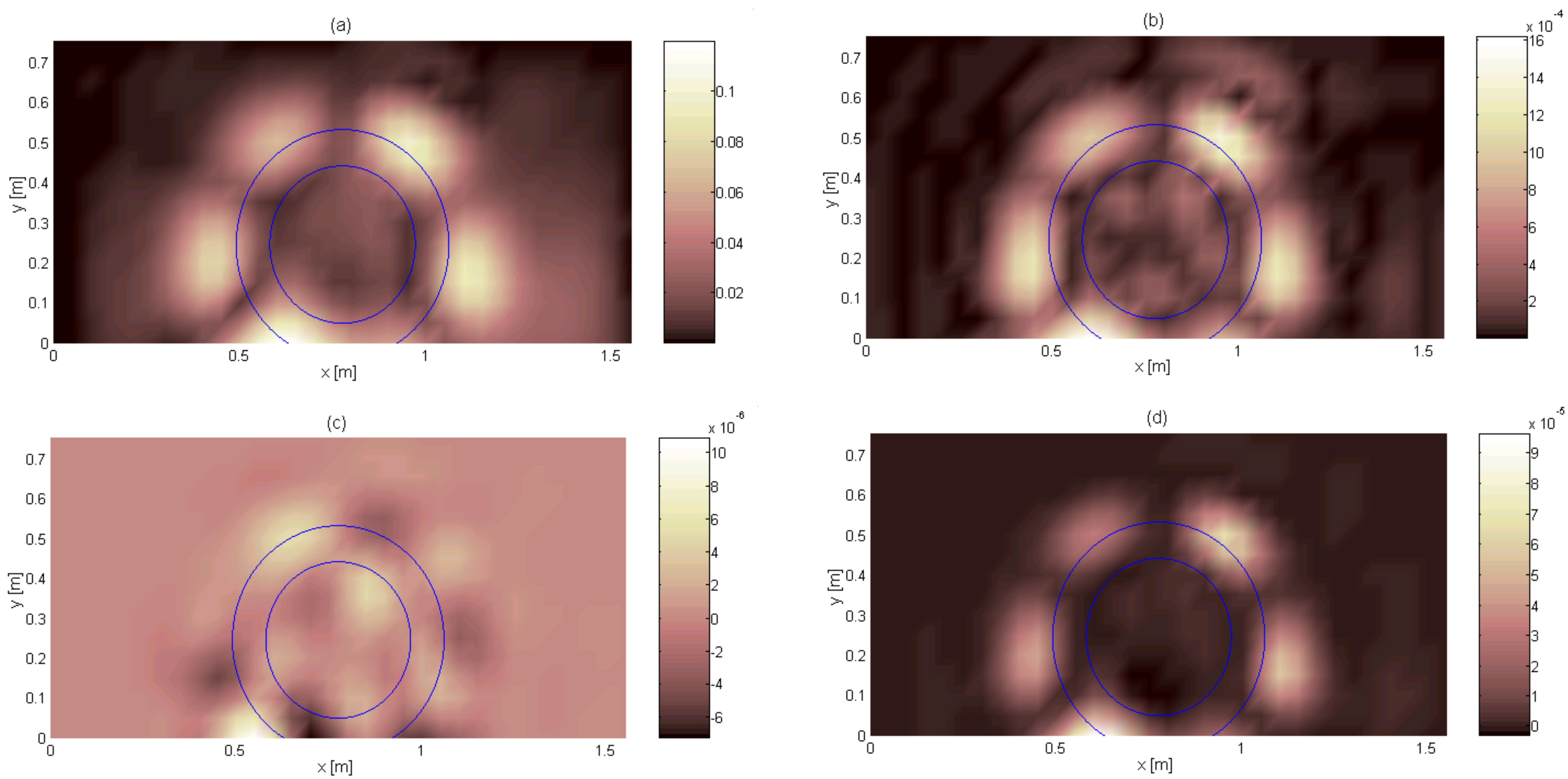
8 by 8 scanning microphone array (10 cm spacing)

Hologram height: 6 cm

16 Point Measurement with 5 cm Sampling Space

Smooth Roller Surface

Rolling Tire: NAH Results at 21 mph (128 Hz, n = 3)



Projected sound fields on source plane ($z = 0$) at 128 Hz: (a) acoustic pressure, (b) particle velocity in z-direction, (c) active sound intensity in z-direction, and (d) reactive sound intensity in z-direction.