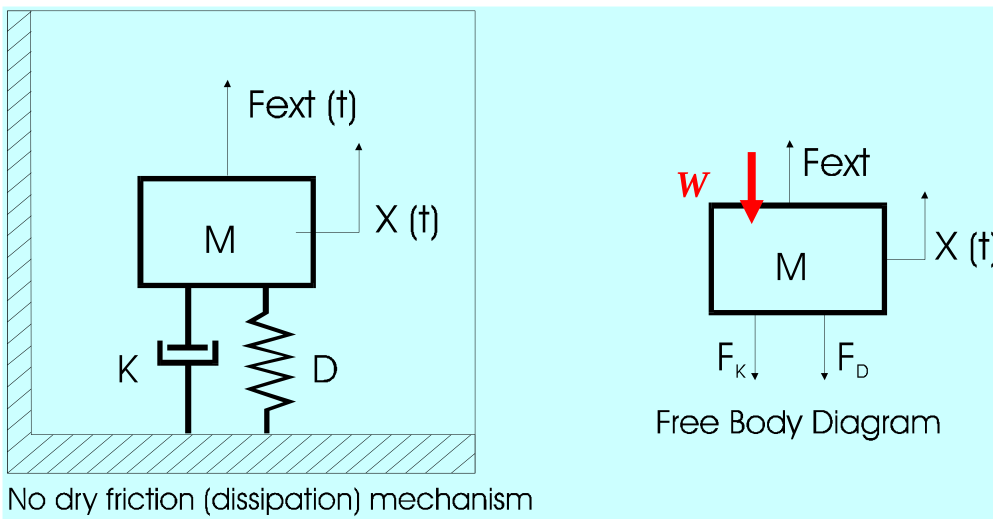
Appendix A: Conservation of Mechanical Energy = Conservation of Linear Momentum

Consider the motion of a 2nd order mechanical system comprised of the fundamental mechanical elements: inertia or mass (M), stiffness (K), and viscous damping coefficient, (D). The **Principle of Conservation of Linear Momentum** (Newton's 2nd Law of Motion) leads to the following 2nd order differential equation:

$$M\,\ddot{X} + D\,\dot{X} + K\,X = F_{(t)} \qquad (1)$$

where the coordinate $X_{(t)}$ describes the system motion. X has its origin at the system **static equilibrium position** (SEP).



In the free body diagram above,

$F_{(t)}=F_{ext}$ is the external force acting on the system,

$F_k = -K\,Y = -K\left(X-\delta_s\right)$ is the reaction force from the spring.

$\delta_s=W/K$ represents the static deflection . $Y=\left(X-\delta_s\right)$ is the total deflection of the spring from its unstretched position.

$F_D = -D\,\dot{X}$ is the reaction force from the dashpot element.

(1) is recast as $$M\,\ddot{X}+D\,\dot{X}+K\,(X-\delta_s)=F_{(t)}-W \quad (1)$$

Now, integrate this Eq. (1) between two arbitrary displacements $X_1=X_{(t_1)},\ X_2=X_{(t_2)}$ occurring at times t_1 and t_2, respectively At these times the system velocities are $\dot{X}_1=\dot{X}_{(t_1)},\ \dot{X}_2=\dot{X}_{(t_2)}$, respectively.

The process gives:

$$\int_{X_1}^{X_2} M\,\ddot{X}\,dX+\int_{X_1}^{X_2} D\,\dot{X}\,dX+\int_{X_1}^{X_2} K\left(X-\delta_s\right)dX=\int_{X_1}^{X_2}\left(F_{(t)}-W\right)dX \quad (2a)$$

Since $Y=\left(X-\delta_s\right)$ then $dY=dX$, then write Eq. (2a) as

$$\int_{X_1}^{X_2} M\,\ddot{X}\,dX+\int_{X_1}^{X_2} D\,\dot{X}\,dX+\int_{Y_1}^{Y_2} K\,Y\,dY=\int_{X_1}^{X_2}\left(F_{(t)}-W\right)dX \quad (2b)$$

The acceleration and velocity are $\ddot{X}=\dfrac{d\,\dot{X}}{d\,t},\ \dot{X}=\dfrac{d\,X}{d\,t}$, respectively. Using these definitions, write Eq. (2b) as:

$$\int_{t_1}^{t_2} M\,\frac{d\,\dot{X}}{dt}\frac{dX}{d\,t}\,dt+\int_{t_1}^{t_2} D\,\dot{X}\,\frac{dX}{d\,t}\,dt+\int_{Y_1}^{Y_2} K\,d\left(\tfrac{1}{2}Y^2\right)=\int_{X_1}^{X_2} F_{(t)}\,dX-\int_{X_1}^{X_2} W\,dX$$

or,

$$\int_{t_1}^{t_2} M\,\frac{d\,\dot{X}}{dt}\,\dot{X}\,dt+\int_{t_1}^{t_2} D\,\dot{X}\,\dot{X}\,dt+\int_{Y_1}^{Y_2} K\,d\left(\tfrac{1}{2}Y^2\right)+\int_{X_1}^{X_2} W\,dX=\int_{X_1}^{X_2} F_{(t)}\,dX$$

$$\int_{\dot{X}_1}^{\dot{X}_2} M\, d\left(\tfrac{1}{2}\dot{X}^2\right) + \int_{t_1}^{t_2} D\,\dot{X}\,\dot{X}\; dt + K\left(\tfrac{1}{2}Y^2\right]_{Y_1}^{Y_2} + W\left(X_2 - X_1\right) = \int_{X_1}^{X_2} F_{(t)}\, dX \quad (3)$$

and since *(M, K, D)* are constant parameters, express Eq. (3) as:

$$\tfrac{1}{2}M\,(\dot{X}_2^2 - \dot{X}_1^2) + \int_{t_1}^{t_2} D\,\dot{X}^2\; dt + \tfrac{1}{2}K\,(Y_2^2 - Y_1^2) + W\,(X_2 - X_1) = \int_{X_1}^{X_2} F_{(t)}\, dX \quad (4)$$

Let's recognize several of the terms in the equation above. These are known as

Change in kinetic energy,

$$T_2 - T_1 = \tfrac{1}{2}M\,\dot{X}_2^2 - \tfrac{1}{2}M\,\dot{X}_1^2 \qquad (5.a)$$

Change in potential energy (elastic strain and gravitational)

$$V_2 - V_1 = \tfrac{1}{2}K\,Y_2^2 - \tfrac{1}{2}K\,Y_1^2 + W\,X_2 - W\,X_1 \qquad (5.b)$$

Total work from external force input into the system,

$$W_{1-2} = \int_{X_1}^{X_2} F_{(t)}\, dX \qquad (5.c)$$

Set $P_v = D\,\dot{X}^2$ as the viscous power dissipation, Then, **the dissipated viscous energy (removed from system)** is,

$$E_{v_{1-2}} = \int_{t_1}^{t_2} D\,\dot{X}^2\; dt = \int_{t_1}^{t_2} P_v\; dt \qquad (5.d)$$

With these definitions, write Eq. (4) as

$$\left(T_2 - T_1\right) + \left(V_2 - V_1\right) + E_{v_{1-2}} = W_{1-2} \qquad (6)$$

That is, the **change in (kinetic energy + potential energy) + the viscous dissipated energy = External work**. This is also known as the **Principle of Conservation of Mechanical Energy (PCME).**

Note that Eq. (1) and Eq. (6) are **NOT** independent. They actually represent the same physical law. Note also that Eq. (6) is not to be mistaken with the first-law of thermodynamics since it does not account for heat flows and/or changes in temperature.

One can particularize Eqn. (6) for the initial time t_0 with initial displacement and velocities given as $(X_0, \dot{X}_0)$, and at an arbitrary time (*t*) with displacements and velocities equal to $(X_{(t)}, \dot{X}_{(t)})$, respectively, i.e., Thus, the **PCME** states

$$\left(T_{(t)} + V_{(t)}\right) = W_{(0\to t)} - E_{v(0\to t)} + \left(T_0 + V_0\right) \qquad (7)$$

where $\left(T_0 + V_0\right)$ is the initial state of energy for the system at time *t*=0 s. Eqn. (7a) is also written as

$$\tfrac{1}{2} M\,\dot{X}_{(t)}^2 + \tfrac{1}{2} K\,Y_{(t)}^2 + W\,X = \int_{X_0}^{X_{(t)}} F_{(t)}\,dX \;-\int_{t_0}^{t} D\,\dot{X}^2\;dt \;+\tfrac{1}{2} M\,\dot{X}_0^2 + \tfrac{1}{2} K\,Y_0^2 + W\,X_0 \quad (8)$$

Taking the time derivative of Eq. (7) gives

$$\frac{d}{d}\left(T_{(t)} + V_{(t)}\right) = \frac{d\,W}{dt} - \frac{dE_v}{dt} = \wp_{ext} - \wp_v \qquad (9)$$

where $\wp_{ext}, \wp_v$ are the mechanical power from external forces acting on the system and the power dissipated by a viscous-type forces, respectively.

Work with Eq. (8) to obtain

$$\frac{d}{dt}\left[\tfrac{1}{2}M\,\dot{X}_{(t)}^2+\tfrac{1}{2}K\,Y_{(t)}^2+W\,X=\int_{X_0}^{X(t)}F_{(t)}\,dX-\int_{t_0}^{t}D\,\dot{X}^2\,dt\;+\tfrac{1}{2}M\,\dot{X}_0^2+\tfrac{1}{2}K\,Y_0^2+W\,X_0\right]$$

$$\tfrac{2}{2}M\,\dot{X}_{(t)}\frac{d\dot{X}_{(t)}}{dt}+\tfrac{2}{2}K\,Y_{(t)}\frac{dY_{(t)}}{dt}+W\frac{dX_{(t)}}{dt}=F_{(t)}\frac{dX_{(t)}}{dt}-D\,\dot{X}^2 \quad (10)$$

Recall that the derivative of an integral function is just the integrand.

To obtain

$$M\,\dot{X}_{(t)}\,\ddot{X}_{(t)}+K\,Y_{(t)}\,\dot{Y}_{(t)}+W\,\dot{X}_{(t)}\;=F_{(t)}\,\dot{X}-D\,\dot{X}^2 \quad (11a)$$

Since $Y=\left(X-\delta_s\right)$ and $\dot{Y}=\dot{X}$, Eq. (11) becomes

$$\dot{X}_{(t)}\left(M\,\ddot{X}_{(t)}+K\left[X_{(t)}-\delta_s\right]+W\;\right)=F_{(t)}\,\dot{X}-D\,\dot{X}^2$$

Canceling the static load balance terms, $W=K\,\delta_s$, and factoring out the velocity, obtain

$$\left[M\ddot{X}_{(t)}+K\,X_{(t)}+D\dot{X}\right]\,\dot{X}_{(t)}=F_{(t)}\,\dot{X}_{(t)} \quad (11)$$

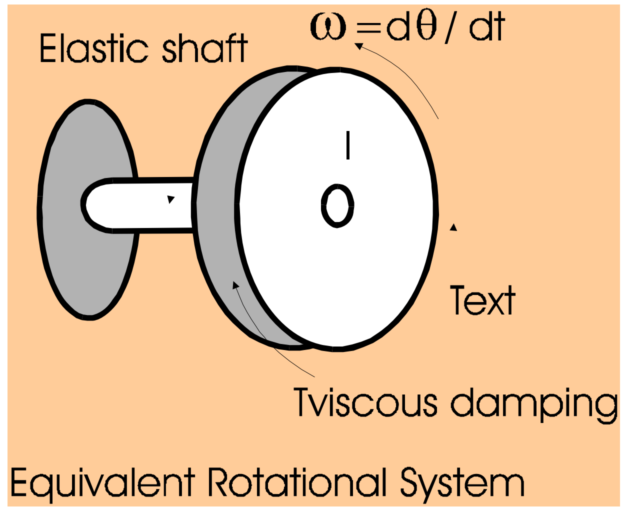
Since for most times the system velocity is different from zero, i.e., $\dot{X}_{(t)}\neq 0$; that is, the system is moving; then

$$M\,\ddot{X}+D\,\dot{X}+K\,X=F_{(t)} \quad (1)$$

i.e., the original equation derived from Newton's Law (**conservation of linear momentum)**.

Suggestion/recommended work:

Rework the problem for a rotational (torsional) mechanical system and show the equivalence of conservation of mechanical energy to the principle of angular momentum, i.e. start with the following Eqn.



$$I\ddot{\theta} + D_\theta \dot{\theta} + K_\theta \theta = T_{(t)}$$

where (I, D_θ, K_θ) are the equivalent mass moment of inertia, rotational viscous damping and stiffness coefficients, $T_{(t)}=T_{ext}$ is an applied external moment or torque, and $\theta(t)$ is the angular displacement of the rotational system.