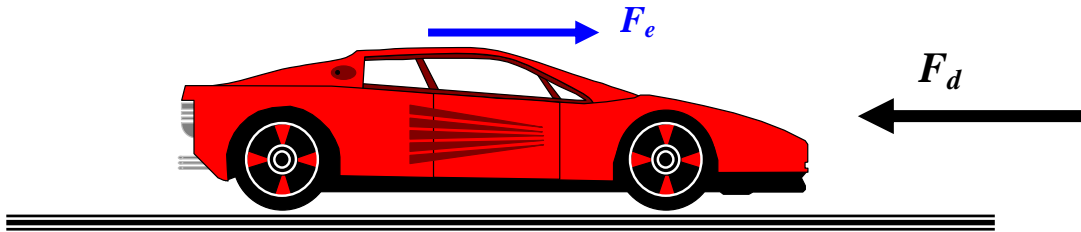


Appendix B. LINEARIZATION

Linearization is a tool that allows us to use linear models to approximate the performance of nonlinear systems. Linear models are desirable because there are many powerful tools to analyze and design linear systems, whereas there are very few general tools available for analysis and design of nonlinear systems.

The linearization technique will be explained using an example. The example is the speed control/analysis of a car (could be a boat, airplane, rocket, cycle, etc.). The main forces acting on the car are aerodynamic drag, F_d , and the driving force F_e , produced by the engine and drive train and exerted at the rear wheels. The free body diagram (FBD) is:



The equation of motion is

$$\sum F_x = F_e - F_d = M_{eq} \dot{v} \quad (1)$$

where

$F_e = k\theta$, driving force from engine.

k = gain (a constant),

θ = throttle angle (controlled by driver),

$F_d = \frac{1}{2} \rho C_d A v^2$, drag force

ρ = air density,

C_d = drag coefficient,

A = projected frontal area of car,

v = car speed,

M_{eq} = equivalent mass of car.

Thus, the equation of motion (EOM) becomes

$$k\theta - \frac{1}{2} (\rho C_d A) v^2 = M_{eq} \dot{v} \quad (2)$$

This is a 1st order nonlinear ODE for the velocity (v). We wish **to linearize** the EOM about some constant “operating point” denoted by v^* . The throttle angle corresponding to a steady state speed v^* is θ^* , i.e.

$$k\theta^* - \frac{1}{2} \rho C_d A v^{*2} = 0 \quad (3)$$

at θ^* and v^* , i.e. the thrust force balances the drag force, so $\dot{v} = 0$

Define δv and $\delta \theta$ as **small changes in velocity and throttle angle**,

$$\begin{aligned} v &= v^* + \delta v \quad \Rightarrow \quad \delta v = v - v^* \\ \theta &= \theta^* + \delta \theta \quad \Rightarrow \quad \delta \theta = \theta - \theta^* \end{aligned} \quad (4)$$

For example, if $v^* = 60$ mph = constant, and at time t_1 , $v(t_1) = 72$ mph, then $\delta v(t_1) = 12$ mph. This is simply a change in independent variable from $v(t)$ to $\delta v(t)$.

Expand F_e and F_d in **Taylor series** about θ^* and v^* and drop nonlinear terms since these are second order, that is small,

$$F_e = k\theta = F_e \Big|_{\theta^*} + \frac{\partial F_e}{\partial \theta} \Big|_{\theta^*} \delta \theta + \frac{1}{2} \frac{\partial^2 F_e}{\partial \theta^2} \Big|_{\theta^*} \delta \theta^2$$

$$F_e \cong k \theta^* + k \delta \theta \quad (5)$$

$$F_d = \frac{1}{2} \rho C_d A v^2 = F_d \Big|_{v^*} + \frac{\partial F_d}{\partial v} \Big|_{v^*} \delta v + 0 (\delta v^2)$$

$$F_d \approx \frac{1}{2} \rho C_d A v^{*2} + (\rho C_d A v^*) \delta v \quad (6)$$

Substitution of the forces into the EOM (1) gives

$$\sum F_X = F_e - F_d = M_{eq} \dot{v}$$

$$\left[k\theta^* + k\delta\theta \right] - \left[\frac{1}{2} \rho C_d A v^{*2} + (\rho C_d A v^*) \delta v \right] = M_{eq} \frac{d}{dt} [v^* + \delta v]$$

and rearranging

$$\left\{ k\theta^* - \frac{1}{2} \rho C_d A v^{*2} \right\} + \langle k\delta\theta - (\rho C_d A v^*) \delta v \rangle = M_{eq} \frac{d}{dt} (v^*) + M_{eq} \frac{d}{dt} (\delta v)$$

Since v^*, θ^* define the steady state operating point, **the first term in curly brackets { } is zero** (see Eq. 3). The resulting linearized equation of motion is:

$$k\delta\theta - (\rho C_d A v^*) \delta v = M_{eq} \delta \dot{v} \quad (7)$$

where k is the slope of the curve of F_e vs θ at the operating point θ^* , and $(\rho C_d A v^*)$ is the slope of the curve of F_d vs v at the operating point v^* , i.e.,

$$k = \left(\frac{\partial F_e}{\partial \theta} \right)_{\theta^*}; \quad k_d = \left(\frac{\partial F_d}{\partial v} \right)_{v^*} = (\rho C_d A v^*) \quad (8)$$

Hence, Eq. (7) becomes

$$M_{eq} \delta \dot{v} + k_d \delta v = k \delta \theta \quad (9)$$

Note that if using some other curve fit other than a Taylor series about a steady state operating point, the term in curly brackets would not be zero (it would be a constant). By using a Taylor series about a steady state operating point, the method eliminates this constant forcing term in the linearized ODE.

Use engineering judgement to select the appropriate operating point for each problem (situation)