APPENDIX C. DERIVATION OF EQUATIONS OF MOTION FOR MULTIPLE DEGREE OF FREEDOM SYSTEM

Consider a linear mechanical system with *n*-independent degrees of freedom. Let $\mathbf{x} = \left\{ x_{1_{(t)}}, x_{2_{(t)}}, x_{3_{(t)}}, \ldots .x_{n_{(t)}} \right\}^{\mathbf{T}}$ be the independent coordinates describing the motion of the system about the equilibrium position, and with $\mathbf{F} = \left\{ F_{1_{(t)}}, F_{2_{(t)}}, F_{3_{(t)}}, \ldots .F_{n_{(t)}} \right\}^{\mathbf{T}}$ as the set of external forces applied at each degree of freedom.

The kinetic (T) and potential energy (V) of the system are written as,

$$T = \tfrac{1}{2} \dot{\mathbf{x}}^{\mathbf{T}} \mathbf{M} \, \dot{\mathbf{x}}, \quad V = \tfrac{1}{2} \mathbf{x}^{\mathbf{T}} \mathbf{K} \, \mathbf{x} \tag{1}$$

where $\dot{\mathbf{x}} = \left\{ \dot{x}_1 \, \dot{x}_2 \, \dot{x}_3 \ldots \ldots . \dot{x}_n \right\}^T$ is the vector of velocities. $\mathbf{M} = \{ m_{i,j} \}_{i,j=1,n}$ and $\mathbf{K} = \{ k_{i,j} \}_{i,j=1,n}$ are the *(n x n)* matrices of generalized inertia (mass) and stiffness coefficients, respectively. The elements of these matrices are constant coefficients.

Note that energies are scalar functions, i.e. $T = T^{\mathrm{T}}$ and $V = V^{\mathrm{T}}$. Eq. (1) above is correct only if the stiffness and mass matrices are symmetric. That is, from

$$V = \tfrac{1}{2} \mathbf{x}^{\mathbf{T}} \mathbf{K} \, \mathbf{x}$$

$$\rightarrow \quad V^T = \tfrac{1}{2} \left(\mathbf{x}^{\mathbf{T}} \mathbf{K} \, \mathbf{x} \right)^{\mathbf{T}} = \tfrac{1}{2} \left(\mathbf{K} \, \mathbf{x} \right)^{\mathbf{T}} \left(\mathbf{x}^{\mathbf{T}} \right)^{\mathbf{T}} = \mathbf{x}^{\mathbf{T}} \mathbf{K}^{\mathbf{T}} \, \mathbf{x} \tag{2a}$$

Above $(\mathbf{A}^{\mathbf{T}} \, \mathbf{B})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}} \mathbf{A}$, where **A** and **B** are general matrices.

Thus
$$V - V^T = 0 = \tfrac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\,\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{K}^{\mathbf{T}}\,\mathbf{x} = \tfrac{1}{2}\mathbf{x}^{\mathbf{T}}\left(\mathbf{K} - \mathbf{K}^{\mathbf{T}}\right)\mathbf{x}$$
$$\rightarrow \mathbf{K} = \mathbf{K}^{\mathbf{T}} \tag{2b}$$
similarly, $\mathbf{M}=\mathbf{M}^{\mathbf{T}}$

The **viscous power dissipation** and **viscous dissipated energy** are of the form

$$P_v = \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{D}\,\dot{\mathbf{x}} \;\;\rightarrow E_v = \int_0^t P_v dt \tag{3}$$

where $\mathbf{D} = \{D_{i,j}\}_{i,j=1,n}$ is a matrix of constant damping coefficients. Above, $\mathbf{F_D} = \mathbf{D}\,\dot{\mathbf{x}}$ is a vector of viscous damping (reaction) forces

The work performed by external forces is ,

$$W = \int \mathbf{dx}^{\mathbf{T}} \cdot \mathbf{F}_{(t)} \tag{4}$$

Note that $\mathbf{dx}^{\mathbf{T}}\mathbf{F} = dx_1 F_1 + dx_2 F_2 + ...dx_n F_n$ $= \mathrm{d}W$ is the differential of work exerted by the external forces on the system.

The **principle of conservation of mechanical energy (PCME)** establishes that for any instant of time,

$$T + V + E_v = W + T_0 + V_0 \tag{5}$$

where $T_0 = \tfrac{1}{2}\dot{\mathbf{x}}_0{}^{\mathbf{T}}\mathbf{M}\,\dot{\mathbf{x}}_0, \quad V_0 = \tfrac{1}{2}\mathbf{x}_0{}^{\mathbf{T}}\mathbf{K}\,\mathbf{x}_0$, with $\left\{\mathbf{x}_0, \dot{\mathbf{x}}_0\right\}$ as the initial state of the system.

Now, take the time derivative of Eq. (5) – the PCME- to obtain

$$\frac{d}{dt}\left(T+V+E_v-W\right)=0 \tag{6}$$

Using the definitions in Eq. (1)

$$\begin{aligned}
\frac{dT}{dt} &= \frac{d}{dt}\left(\tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\dot{\mathbf{x}}\right)= \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\ddot{\mathbf{x}}+\tfrac{1}{2}\ddot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\dot{\mathbf{x}}=\tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\ddot{\mathbf{x}}+\tfrac{1}{2}\left(\ddot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\dot{\mathbf{x}}\right)^{\mathbf{T}} \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\ddot{\mathbf{x}}+\tfrac{1}{2}\left(\mathbf{M}\,\dot{\mathbf{x}}\right)^{\mathbf{T}}\left(\ddot{\mathbf{x}}^{\mathbf{T}}\right)^{\mathbf{T}}= \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}\,\ddot{\mathbf{x}}+\tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{M}^{\mathbf{T}}\ddot{\mathbf{x}} \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{M}+\mathbf{M}^{\mathbf{T}}\right)\ddot{\mathbf{x}} \quad \leftarrow \mathbf{M}=\mathbf{M}^{\mathbf{T}};
\end{aligned} \tag{7}$$

$$\frac{dT}{dt}=\dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{M}\ddot{\mathbf{x}}\right)$$

where $\ddot{\mathbf{x}}=\left\{\ddot{x}_1\,\ddot{x}_2\,\ddot{x}_3\,.......\ddot{x}_n\right\}^T$ is a vector of accelerations; and

$$\begin{aligned}
\frac{dV}{dt} &= \frac{d}{dt}\left(\tfrac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\,\mathbf{x}\right)= \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}\,\mathbf{x}+\tfrac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{K}\,\dot{\mathbf{x}}=\tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}\,\mathbf{x}+\tfrac{1}{2}\left(\mathbf{x}^{\mathbf{T}}\mathbf{K}\,\dot{\mathbf{x}}\right)^{\mathbf{T}} \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}\,\mathbf{x}+\tfrac{1}{2}\left(\mathbf{K}\,\dot{\mathbf{x}}\right)^{\mathbf{T}}\left(\mathbf{x}^{\mathbf{T}}\right)^{\mathbf{T}}= \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}\,\mathbf{x}+\tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}^{\mathbf{T}}\mathbf{x} \\
&= \tfrac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{K}+\mathbf{K}^{\mathbf{T}}\right)\ddot{\mathbf{x}} \quad \leftarrow \mathbf{K}=\mathbf{K}^{\mathbf{T}};
\end{aligned} \tag{8}$$

$$\frac{dV}{dt}=\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{K}\mathbf{x}$$

In addition, $$\frac{d\,E_v}{dt} = \frac{d}{dt}\int_0^t P_v\,dt = P_v = \dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{D}\,\dot{\mathbf{x}}\right) \qquad (9)$$

and $$\frac{d\,W}{dt} = \frac{d}{dt}\int_0^t \mathbf{dx}^{\mathbf{T}}\,\mathbf{F} = \frac{\mathbf{dx}^{\mathbf{T}}}{dt}\mathbf{F} = \dot{\mathbf{x}}^{\mathbf{T}}\,\mathbf{F} \qquad (10)$$

Substitution of Eqs. (8-10) into Eq. (7) gives

$$\dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{M}\ddot{\mathbf{x}}\right) + \dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{K}\mathbf{x}\right) + \dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{D}\dot{\mathbf{x}}\right) - \dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{F}\right) = 0$$

Or

$$\dot{\mathbf{x}}^{\mathbf{T}}\left(\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{D}\dot{\mathbf{x}} - \mathbf{F}\right) = 0$$

And since for most times $\dot{\mathbf{x}} \neq \mathbf{0}$, then

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{D}\dot{\mathbf{x}} - \mathbf{F} = \mathbf{0} \qquad (11)$$

The difficulty in using this approach is to devise a simple method to establish ALL the elements in the system parameter matrices **M, K, D**. The use of the Lagrangian Method is particularly useful in this case.

Derivation of equations of motion using Lagrange's approach[1]

Consider a mechanical system with n-independent degrees of freedom, and where $\{x_i, \dot{x}_i\}_{i=1,....n}$ are the generalized coordinates and velocities for each degree of freedom in the system. The work performed on the system by external generalized forces is

$$W = \int \left(F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_4 + ...F_n\, dx_n \right) = \int \sum_{i=1}^{n} F_i\, dx_i \qquad (12)$$

Here the term generalized denotes that the product of a generalized displacement, say x_i , and the generalized effort, F_i , produces units of work [N.m]. For example if $x_2=\theta$ denotes an angular coordinate, then the effort F_2 must correspond to a moment or torque.

Let the **total kinetic energy** and **potential energy** of the n-dof mechanical system be given by the generic expressions

$$\begin{aligned} T &= f\left\{ \dot{x}_1\, \dot{x}_2\, \dot{x}_3 ....\dot{x}_n\, , x_1, x_2\, ,....., x_n\, , t\right\} \\ V &= g\left\{ x_1, x_2\, ,....., x_n\, , t\right\} \end{aligned} \qquad (13)$$

The **kinetic energy** above is a function of the generalized displacements, velocities and time, while the **potential energy** in a **conservative system** is only a function of the generalized displacements and time.

The **viscous dissipated power** is a general function of the velocities, i.e.,

[1] Sources Meirovitch, L., Analytical Methods in Vibrations, pp. 30-50, and San Andrés, L., Vibrations Class Notes, 1996.

$$P_v = P_v\left\{\dot{x}_1\,\dot{x}_2\,\dot{x}_3\,.......\dot{x}_n\right\} \tag{14}$$

The *n*-equations of motion for the system are derived using the Lagrangian approach[2], i.e.,

$$\frac{\partial}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} + \tfrac{1}{2}\frac{\partial P_v}{\partial \dot{x}_i} = F_i \qquad i=1,2,....n \tag{15}$$

Once you have performed the derivatives above for each coordinate, $i=1,\ldots n$, the resulting equations are of the form:

$$\begin{aligned}
&m_{11}\,\ddot{x}_1 + ... + m_{1n}\,\ddot{x}_n + d_{11}\,\dot{x}_1 + ... + d_{1n}\,\dot{x}_n + k_{11}\,x_1 + ... + k_{1n}\,x_n = F_1\\
&m_{21}\,\ddot{x}_1 + ... + m_{2n}\,\ddot{x}_n + d_{21}\,\dot{x}_1 + ... + d_{2n}\,\dot{x}_n + k_{21}\,x_1 + ... + k_{2n}\,x_n = F_2\\
&....\\
&m_{n1}\,\ddot{x}_1 + ... + m_{nn}\,\ddot{x}_n + d_{n1}\,\dot{x}_1 + ... + d_{nn}\,\dot{x}_n + k_{n1}\,x_1 + ... + k_{nn}\,x_n = F_n
\end{aligned} \tag{16}$$

or written in matrix form as

$$\mathbf{M\ddot{x}} + \mathbf{Kx} + \mathbf{D\dot{x}} = \mathbf{F} \qquad (17)=(11)$$

[2] A later lecture will demonstrate the derivation of the Lagrangian Equations